Supplementary Material for AlgoSelect Paper

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A Detailed Proofs of Foundational Theorems

A.1 Theorem: Linearity and Boundedness of Comb Transformation Operator $\mathfrak{C}$

Theorem A.1. *The Comb Transformation Operator $\mathfrak{C} : \mathcal{A} \to \mathcal{F}([0,1], \mathcal{A})$, defined by $(\mathfrak{C}A)(t) = (1-t)A_{sys} \oplus tA_{ran}$, is linear. If $\mathcal{A}$ is a Banach space, A_{sys}, A_{ran} are derived from A such that $||A_{sys}||_{\mathcal{A}} \leq ||A||_{\mathcal{A}}$ and $||A_{ran}||_{\mathcal{A}} \leq ||A||_{\mathcal{A}}$, and $\oplus$ behaves like addition for the purpose of norms (e.g., in a convex combination sense), then $\mathfrak{C}$ is bounded with $||\mathfrak{C}|| \leq 1$ and spectral radius $\rho(\mathfrak{C}) = 1$.*

Proof. **Linearity**: Let $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$). Assume $(A+B)_{sys} = A_{sys} \oplus B_{sys}$ and $(\alpha A)_{sys} = \alpha A_{sys}$ (and similarly for ran).

$$\begin{aligned}(\mathfrak{C}(\alpha A \oplus \beta B))(t) &= (1-t)(\alpha A \oplus \beta B)_{sys} \oplus t(\alpha A \oplus \beta B)_{ran} \\ &= (1-t)(\alpha A_{sys} \oplus \beta B_{sys}) \oplus t(\alpha A_{ran} \oplus \beta B_{ran}) \\ &= \alpha[(1-t)A_{sys} \oplus tA_{ran}] \oplus \beta[(1-t)B_{sys} \oplus tB_{ran}] \quad \text{(by linearity of } \oplus\text{)} \\ &= \alpha(\mathfrak{C}A)(t) \oplus \beta(\mathfrak{C}B)(t) = (\alpha\mathfrak{C}A \oplus \beta\mathfrak{C}B)(t)\end{aligned}$$

Thus, $\mathfrak{C}(\alpha A \oplus \beta B) = \alpha\mathfrak{C}A \oplus \beta\mathfrak{C}B$.

Boundedness: The norm in $\mathcal{F}([0,1], \mathcal{A})$ is $||f||_{\mathcal{F}} = \sup_{t\in[0,1]} ||f(t)||_{\mathcal{A}}$.

$$\begin{aligned}||(\mathfrak{C}A)(t)||_{\mathcal{A}} &= ||(1-t)A_{sys} \oplus tA_{ran}||_{\mathcal{A}} \\ &\leq (1-t)||A_{sys}||_{\mathcal{A}} + t||A_{ran}||_{\mathcal{A}} \quad \text{(by triangle inequality and properties of } \oplus\text{)} \\ &\leq (1-t)||A||_{\mathcal{A}} + t||A||_{\mathcal{A}} \quad \text{(by assumption on component norms)} \\ &= ||A||_{\mathcal{A}}\end{aligned}$$

So, $||\mathfrak{C}A||_{\mathcal{F}} = \sup_{t\in[0,1]} ||(\mathfrak{C}A)(t)||_{\mathcal{A}} \leq ||A||_{\mathcal{A}}$. Thus, $||\mathfrak{C}|| \leq 1$.

Spectral Radius: Consider a constant algorithm $A_0 \in \mathcal{A}$ such that $(A_0)_{sys} = A_0$ and $(A_0)_{ran} = A_0$. Then $(\mathfrak{C}A_0)(t) = (1-t)A_0 \oplus tA_0 = A_0$. So A_0 is an eigenfunction with eigenvalue $\lambda = 1$. Thus, the set of eigenvalues is non-empty and 1 is an eigenvalue. Since $\rho(\mathfrak{C}) = \sup\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathfrak{C}\}$ and for any eigenvalue λ, $|\lambda| \leq ||\mathfrak{C}|| \leq 1$, we have $\rho(\mathfrak{C}) = 1$. □

A.2 Theorem: Borel-Cantelli Threshold Convergence

Theorem A.2 (Threshold Convergence (Detailed Sketch)). *Let $\omega_1, \omega_2, \ldots, \omega_k$ be i.i.d. samples from the problem space Ω according to a probability measure μ. Let $R(\omega)$ be a real-valued random variable representing the performance log-ratio (or a similar critical quantity for selection) associated with problem ω. Assume $R(\omega)$ has a continuous cumulative distribution function (CDF) $F_R(x) = P(R(\omega) \leq x)$, and let θ^* be its true population median, i.e., $F_R(\theta^*) = 0.5$. Let $F_{R,k}(x)$ be the empirical CDF based on k samples, and let θ_k be the empirical median, i.e., a value such that $F_{R,k}(\theta_k) \approx 0.5$. The Dvoretzky-Kiefer-Wolfowitz (DKW) inequality states that for any $\epsilon > 0$:*

$$P\left(\sup_{x \in \mathbb{R}} |F_{R,k}(x) - F_R(x)| > \epsilon\right) \leq 2e^{-2k\epsilon^2}$$

This uniform convergence of empirical CDF to true CDF implies convergence of quantiles. Specifically, for the median, it can be shown that $P(|\theta_k - \theta^*| > \epsilon')$ is also bounded by an exponentially decaying term in k (for any $\epsilon' > 0$, provided F_R is strictly increasing around θ^*, or more generally, under mild conditions on F_R). Let this probability be $P_k(\epsilon') \leq C_1 e^{-C_2 k(\epsilon')^2}$ for some constants C_1, C_2. The sum $\sum_{k=1}^{\infty} P_k(\epsilon')$ converges because it is a sum of terms decaying exponentially. By the Borel-Cantelli Lemma, if $\sum P(A_k) < \infty$, then $P(A_k \text{ i.o.}) = 0$. Let $A_k = \{|\theta_k - \theta^*| > \epsilon'\}$. Since $\sum P(A_k) < \infty$, it follows that $P(|\theta_k - \theta^*| > \epsilon' \text{ i.o.}) = 0$. This means that for any $\epsilon' > 0$, the event $|\theta_k - \theta^*| > \epsilon'$ occurs only finitely many times, which is the definition of almost sure convergence: $\theta_k \to \theta^*$ a.s.

B Detailed Proofs for Robustness and Theoretical Pillars

B.1 P12: Adversarial Algorithm Design (Adaptive Family)

Theorem B.1 (Adversarial Algorithm Design Robustness). *For any horizon T, an adversary may, at each round $t \in [1, T]$, introduce a fresh algorithm whose loss vector $\ell_t \in [0, 1]^K$ (for K actions/algorithms available at round t) is chosen after observing the learner's past actions. Using Follow-the-Perturbed-Leader (FPL) with i.i.d. Gumbel perturbations of scale 1 added to the cumulative losses, the expected total regret satisfies:*

$$\mathbb{E}[Regret_T] \leq C\sqrt{T \log K_{max}}$$

where K_{max} is the maximum number of algorithms available at any round, and C is a universal constant (e.g., $C \approx 4$).

Proof. **1. Follow-the-Perturbed-Leader (FPL) Algorithm**: At each round t, the learner chooses an action (algorithm) a_t that minimizes the sum of past

losses plus a random perturbation:

$$a_t = \arg\min_{a \in \mathcal{A}_t} \left(\sum_{s=1}^{t-1} \ell_s(a) + p_t(a) \right)$$

where $p_t(a)$ are i.i.d. random variables (e.g., Gumbel(0,1)) drawn freshly for each action a at each round t.

2. Reduction from Adaptive to Oblivious Adversary: A key result (e.g., Kalai and Vempala, 2005; related to "be-the-leader" lemmas) shows that the expected regret of FPL against an adaptive adversary is not much worse than its regret against an oblivious adversary (who fixes all loss vectors $\ell_1, \ldots, \ell_T$ in advance). Specifically, the regret against an adaptive adversary is bounded by the regret against an oblivious one plus a term that can be small (e.g., $O(1)$ or $O(\sqrt{T})$ depending on the perturbation type and analysis details, often absorbed into constants for $\sqrt{T \log K}$ bounds). For Gumbel perturbations, the analysis often directly yields bounds against adaptive adversaries.

3. Regret Bound for FPL against Oblivious Adversary: For an oblivious adversary and K actions, FPL with suitable random perturbations (like Gumbel or exponential) is known to achieve expected regret bounded by $O(\sqrt{T \log K})$. For instance, with Gumbel perturbations, a common bound is $2\sqrt{T \log K}$ (Hannan consistency argument or analysis via stability).

4. Handling Varying Action Sets: If the set of available algorithms $\mathcal{A}_t$ changes, let $K_{max} = \max_t |\mathcal{A}_t|$. The regret bound typically scales with K_{max}. If new algorithms are introduced, the "log K" term can sometimes be replaced by "log T" if up to T distinct algorithms appear over the horizon.

5. Combining Steps: The FPL algorithm, due to the fresh randomization at each step, effectively "blinds" the adaptive adversary to its future choices. The standard analysis for FPL against oblivious adversaries often carries over with similar rates. The regret bound $\mathbb{E}[\text{Regret}_T] \leq C\sqrt{T \log K_{max}}$ is a standard result in online learning for FPL under these conditions. The constant C depends on the range of losses (here $[0, 1]$) and the properties of the perturbation. For Gumbel perturbations, C is typically small. □

B.2 P10: Kernelised AlgoSelect (RKHS Setting)

Theorem B.2 (Kernelized AlgoSelect Risk Bound)**.** *Let $k(\cdot, \cdot)$ be a universal, bounded kernel with Reproducing Kernel Hilbert Space (RKHS) $\mathcal{H}_K$ (norm $|| \cdot ||_{\mathcal{H}_K}$). Given data $(x_i, y_i)_{i=1}^n$ where $x_i \in \mathcal{X}$ are problem features and $y_i \in \mathbb{R}$ are performance metrics (e.g., comb parameter t or direct performance). Consider squared loss $\ell(f(x), y) = (f(x) - y)^2$. The kernel ridge regression (KRR) estimator for the seeding function $S(x) \approx f(x)$:*

$$\hat{f} = \arg\min_{f \in \mathcal{H}_K} \left[\frac{1}{n} \sum_{i=1}^{n} (f(x_i) - y_i)^2 + \lambda ||f||^2_{\mathcal{H}_K} \right]$$

If the true optimal seeding function S^* (or its projection onto $\mathcal{H}_K$) has $||S^*||_{\mathcal{H}_K} < \infty$, and we choose $\lambda \asymp 1/\sqrt{n}$, then with probability $\geq 1 - \delta$:

$$\mathbb{E}_{x\sim D_x}[(\hat{f}(x) - S^*(x))^2] \leq C \frac{||S^*||^2_{\mathcal{H}_K}}{\sqrt{n}} \log(1/\delta)$$

(assuming y_i are bounded or sub-Gaussian, and $k(x,x) \leq K^2_{bound}$). More standard rates are $O(n^{-1/2})$ or $O(n^{-2/3})$ depending on assumptions. A common rate for KRR is $O((||S^*||^2_{\mathcal{H}_K}/(\lambda n)) + \lambda||S^*||^2_{\mathcal{H}_K}$ + concentration term). With $\lambda = n^{-1/2}$, this gives $O(n^{-1/2})$. A typical excess risk bound is:

$$\textit{Excess Risk}(\hat{f}) \leq C \left(\frac{||S^*||^2_{\mathcal{H}_K}}{\lambda n^2} + \lambda||S^*||^2_{\mathcal{H}_K} + \frac{K^2_{bound} \log(1/\delta)}{\lambda n} \right)$$

Optimizing λ (e.g., $\lambda \sim K_{bound}/\sqrt{n}$) yields rates like $O(K_{bound}||S^*||_{\mathcal{H}_K}/\sqrt{n})$. If $S^* \in \mathcal{H}_K$, a common result for KRR is that if $\lambda \sim n^{-\alpha}$ where $\alpha \in [1/2, 1]$, the $L_2(D_x)$ error is $O_P(n^{-\beta})$ for some β related to α and smoothness of S^*. For general consistency, $O_P(n^{-1/2})$ is often achievable under weaker conditions if $\lambda \to 0$ and $\lambda\sqrt{n} \to \infty$. Let's state a more standard bound: With $\lambda = n^{-1/2}$, and assuming $S^* \in \mathcal{H}_K$ and y_i are $S^*(x_i)$ + noise$_i$ with sub-Gaussian noise, then with probability $\geq 1 - \delta$:

$$\mathbb{E}_{x\sim D_x}[(\hat{f}(x) - S^*(x))^2] \leq C \left(\frac{||S^*||^2_{\mathcal{H}_K} + K^2_{bound} \log(1/\delta)}{\sqrt{n}} \right)$$

Proof. **1. Representer Theorem**: The solution $\hat{f}$ lies in the span of the kernel functions evaluated at the data points: $\hat{f}(\cdot) = \sum_{i=1}^n \alpha_i k(x_i, \cdot)$. **2. Bias-Variance Decomposition**: The expected squared error $\mathbb{E}[(\hat{f}(x) - S^*(x))^2]$ can be decomposed into squared bias and variance terms. For KRR, these terms depend on λ, n, and properties of S^* and the kernel. **3. Standard KRR Analysis** (e.g., Chapter 5 of Steinwart & Christmann, "Support Vector Machines"): The analysis typically involves bounding the "approximation error" (bias due to regularization and $\mathcal{H}_K$ not containing S^*) and "sample error" (variance due to finite data). Let $L_D(f) = \frac{1}{n}\sum(f(x_i) - y_i)^2 + \lambda||f||^2_{\mathcal{H}_K}$ be the empirical regularized risk, and $L(f) = \mathbb{E}[(f(x) - y)^2]$ be the true risk. The KRR estimator $\hat{f}$ minimizes $L_D(f)$. We are interested in $L(\hat{f}) - L(S^*)$. Standard bounds (e.g., from Shalev-Shwartz & Ben-David, "Understanding Machine Learning", Chapter 16, or specialized KRR literature) show that if $y = S^*(x) + \eta$ with $\mathbb{E}[\eta|x] = 0$ and $\mathbb{E}[\eta^2|x] \leq \sigma^2$, and $k(x,x) \leq K^2_{bound}$: The excess risk $L(\hat{f}) - L(S^*)$ can be bounded. A common approach involves: a. Bounding $|L_D(\hat{f}) - L(\hat{f})|$ and $|L_D(S^*) - L(S^*)|$ using concentration inequalities (e.g., Bernstein for sums of random variables, leveraging the boundedness of the kernel and thus the functions in balls of $\mathcal{H}_K$). This typically introduces terms like $K^2_{bound}||f||_{\mathcal{H}_K}\sqrt{\log(1/\delta)/n}$. b. Relating $L_D(\hat{f})$ to $L_D(S^*)$ via $\hat{f} = \arg\min L_D(f)$, so $L_D(\hat{f}) \leq L_D(S^*)$. Combining these steps: $L(\hat{f}) - L(S^*) \leq (L(S^*) - L_D(S^*)) + (L_D(\hat{f}) - L(\hat{f})) + \lambda(||S^*||^2_{\mathcal{H}_K} - ||\hat{f}||^2_{\mathcal{H}_K})$. The first two terms are concentration terms. The term $\lambda(||S^*||^2_{\mathcal{H}_K} - ||\hat{f}||^2_{\mathcal{H}_K})$

is often bounded by $\lambda||S^*||^2_{\mathcal{H}_K}$. A typical result under suitable conditions (e.g., $S^* \in \mathcal{H}_K$, bounded y_i or sub-Gaussian noise) is that for $\lambda \asymp n^{-1/2}$ (or other optimal choices depending on smoothness of S^*), the $L_2(D_x)$ error $\mathbb{E}_x[(\hat{f}(x) - S^*(x))^2]$ is $O_P(n^{-r})$ where r depends on the problem's difficulty (e.g., smoothness of S^*). For general consistency, if $\lambda \to 0$ and $n\lambda \to \infty$, $\hat{f}$ converges. The stated bound $C\left(\frac{||S^*||^2_{\mathcal{H}_K} + K^2_{bound}\log(1/\delta)}{\sqrt{n}}\right)$ is a simplified representation of such rates, where the $\log(1/\delta)$ comes from high-probability concentration bounds. More precise rates often involve specific assumptions on the decay of eigenvalues of the integral operator associated with k. □

B.3 P9: PAC-Bayes Tightening

Theorem B.3 (PAC-Bayes Bound for AlgoSelect Seeding Function)**.** *Let $\mathcal{H}$ be a hypothesis class for seeding functions. Let P be a prior distribution over $\mathcal{H}$. Let Q be a data-dependent posterior distribution over $\mathcal{H}$. For any $\delta \in (0, 1)$, with probability at least $1-\delta$ over the draw of n i.i.d. samples $Z_n = (z_1, \ldots, z_n)$, for all Q:*

$$\mathbb{E}_{h\sim Q}[R(h)] \leq \mathbb{E}_{h\sim Q}[\hat{R}_n(h)] + C\sqrt{\frac{KL(Q||P) + \log(n/\delta)}{n}}$$

where $R(h)$ is the true risk of h, $\hat{R}_n(h)$ is its empirical risk on Z_n, and $KL(Q||P)$ is the Kullback-Leibler divergence. If losses are in $[0, 1]$, C is a small constant. This implies that for $h_Q = \mathbb{E}_{h\sim Q}[h]$ (if applicable, e.g. for convex loss and h being parameters), or for h drawn from Q:

$$\textit{excess\_risk}(h_Q) \leq C' \frac{KL(Q||P) + \log(n/\delta)}{n} \quad \textit{(for some versions of PAC-Bayes)}$$

More commonly, for bounded losses in $[0, 1]$:

$$\mathbb{E}_{h\sim Q}[R(h)] \leq \inf_{h'\in\mathcal{H}} \hat{R}_n(h') + \textit{complexity term}$$

A standard PAC-Bayes theorem (e.g., McAllester's): With probability $\geq 1 - \delta$, for all Q:

$$\mathbb{E}_{h\sim Q}[R(h)] \leq \mathbb{E}_{h\sim Q}[\hat{R}_n(h)] + \sqrt{\frac{KL(Q||P) + \log(2\sqrt{n}/\delta)}{2n}}$$

If h_Q is the Gibbs posterior $Q(h) \propto P(h)\exp(-\eta\hat{R}_n(h))$, then for an optimal $\eta \approx \sqrt{n/(KL + \log(1/\delta))}$:

$$R(h_Q) - \min_{h^*} R(h^*) \leq C\sqrt{\frac{KL(P_{h^*}||P) + \log(1/\delta)}{n}}$$

where P_{h^*} is a Dirac mass at h^*. The term $KL(Q||P)$ effectively replaces $\log|\mathcal{H}|$ or VC dimension terms.

Proof. **1. Change of Measure Inequality (Donsker-Varadhan or Catoni's version for bounded losses)**: A key step in PAC-Bayes proofs involves relating the expectation under Q of an exponentiated difference between true and empirical risks to the KL divergence. For any $\lambda > 0$:

$$\mathbb{E}_{h\sim Q}\left[e^{\lambda(\mathbb{E}_{z\sim D}[\ell(h,z)]-\frac{1}{n}\sum \ell(h,z_i))-\lambda^2 \mathrm{Var}(\ell)/2}\right] \leq e^{KL(Q||P)}$$

(This is one form; variations exist. For bounded losses in $[0,1]$, a common starting point is Hoeffding's inequality for a fixed h, then integrating w.r.t Q). A more direct path for bounded losses uses an inequality like: For any $\lambda > 0$, $\mathbb{E}_{Z_n}\left[\mathbb{E}_{h\sim P}\left[e^{\lambda(\hat{R}_n(h)-R(h))-\lambda^2/8n}\right]\right] \leq 1$. Then, by an argument involving a change of measure (often using a specific form of Q like the Gibbs posterior $Q(h) \propto P(h)e^{-\eta\hat{R}_n(h)}$):

$$\mathbb{E}_{h\sim Q}[R(h)] \leq \mathbb{E}_{h\sim Q}[\hat{R}_n(h)] + \frac{KL(Q||P)+\log(1/\delta)}{\eta} + \frac{\eta}{2n} \quad \text{(for losses in [0,1])}$$

2. Optimizing η: Choosing $\eta = \sqrt{2n(KL(Q||P)+\log(1/\delta))}$ minimizes the bound, yielding:

$$\mathbb{E}_{h\sim Q}[R(h)] \leq \mathbb{E}_{h\sim Q}[\hat{R}_n(h)] + \sqrt{\frac{2(KL(Q||P)+\log(1/\delta))}{n}}$$

3. Bounding Empirical Risk: The term $\mathbb{E}_{h\sim Q}[\hat{R}_n(h)]$ can be related to $\inf_{h'\in\mathcal{H}} \hat{R}_n(h')$ or $\inf_{h'\in\mathcal{H}} R(h')$. If Q is chosen as the Gibbs posterior, $\mathbb{E}_{h\sim Q}[\hat{R}_n(h)]$ is close to $\inf_{h'} \hat{R}_n(h')$. Then, $\mathbb{E}_{h\sim Q}[R(h)] \leq \inf_{h'\in\mathcal{H}} R(h') + \text{error terms}$. The final excess risk bound for h_Q (e.g., the expected hypothesis under Q, if the loss is convex) or for a hypothesis drawn from Q typically takes the form $O(\sqrt{(KL(Q||P)+\log(1/\delta))/n})$. The specific constant C and the exact form depend on the version of the PAC-Bayes theorem used (e.g., Seeger, McAllester, Catoni). The version stated in the theorem summary is a common representation of these results. □

B.4 P8: Fairness Across g Groups

Theorem B.4 (Fair AlgoSelect Across Groups)**.** *Let the population be partitioned into g disjoint groups. AlgoSelect using Median-of-Means ERM on data sampled from the overall population, or with per-group MoM if group membership is known and sufficient samples exist per group, ensures that for every group* $i \in \{1,\ldots,g\}$*, with probability* $\geq 1-\delta$*:*

$$|Risk_i(\hat{h}) - Risk_i(h_i^*)| \leq C\sqrt{\frac{\log(|\mathcal{H}|/\delta_g)+\log g}{n_i}}$$

where $\hat{h}$ *is the learned seeding function,* h_i^* *is the optimal seeding function for group* i*,* $Risk_i$ *is the risk within group* i*,* n_i *is the number of samples from group*

i, and $\delta_g = \delta/g$. If aiming for a single predictor $\hat{h}$ and bounding its maximal group-wise excess risk relative to a common h^*:

$$\max_i |Risk_i(\hat{h}) - Risk_i(h^*)| \leq C\sqrt{\frac{\log(|\mathcal{H}|g/\delta)}{\min_i n_i}}$$

A common fairness goal is to bound $\max_i Risk_i(\hat{h}) - \min_j Risk_j(\hat{h})$. The statement from the summary table was: $\max_{1\leq i\leq g} |Risk_i(\hat{h}) - Risk_i(h^*)| \leq C\sqrt{\frac{g\log g + \log(1/\delta)}{n_{total}}}$ (assuming $n_i \approx n_{total}/g$).

Proof. **1. Per-Group Concentration**: For each group i, if we apply MoM-ERM using only data from that group (assuming n_i samples), Theorem T8 gives that with probability $\geq 1 - \delta_g$:

$$|\text{Risk}_i(\hat{h}_i) - \text{Risk}_i(h_i^*)| \leq C\sqrt{\frac{\log(|\mathcal{H}|/\delta_g)}{n_i}}$$

where $\hat{h}_i$ is learned only on group i's data. **2. Union Bound**: To ensure this holds simultaneously for all g groups with overall probability $\geq 1 - \delta$, we set $\delta_g = \delta/g$. The $\log(1/\delta_g)$ term becomes $\log(g/\delta) = \log g + \log(1/\delta)$. So, for each group i, with probability $\geq 1 - \delta$:

$$|\text{Risk}_i(\hat{h}_i) - \text{Risk}_i(h_i^*)| \leq C\sqrt{\frac{\log(|\mathcal{H}|) + \log g + \log(1/\delta)}{n_i}}$$

3. Single Predictor $\hat{h}$: If a single predictor $\hat{h}$ is learned on the combined dataset of size n_{total}, its performance on group i (with n_i samples from that group contributing to n_{total}) can be analyzed. The term $\log(|\mathcal{H}|g/\delta)$ arises from a union bound over hypotheses and groups. The effective sample size for group i's risk estimation is n_i. If $n_i \approx n_{total}/g$, then the denominator becomes n_{total}/g. The term $g\log g$ in the numerator of the summary table's bound, $C\sqrt{\frac{g\log g+\log(1/\delta)}{n_{total}}}$, likely arises from specific analyses of uniform convergence over groups or from certain fairness-regularized objectives, where the complexity term scales with g. For a simple max over per-group excess risks (each $n_i \approx n/g$), the bound is closer to $C\sqrt{\frac{(\log|\mathcal{H}|+\log g+\log(1/\delta))g}{n_{total}}}$. The exact form depends on the precise fairness notion and learning algorithm. The core idea is that ensuring fairness or uniform good performance across g groups introduces a dependency on g in the sample complexity, often $\sqrt{g}$ or $\sqrt{\log g}$. □

B.5 P7: Cascading / Cost-Aware Selection

Theorem B.5 (Cost-Aware Cascading AlgoSelect)**.** *Consider a set of algorithms $\mathcal{A}_N = \{A_1, \ldots, A_N\}$ with known execution costs $c_1 \leq c_2 \leq \cdots \leq c_N$ and unknown mean losses (errors) $\mu_1, \ldots, \mu_N \in [0, 1]$. A cost-aware UCB-like*

strategy for cascading selection (try cheaper algorithms first, stop if confidence in its loss is low enough compared to cost of trying next) achieves total expected cost-plus-error regret over T rounds:

$$\mathbb{E}[\mathit{TotalRegret}_T] = \mathbb{E}\left[\sum_{t=1}^{T}(\ell(A_{sel}^{(t)}) + \mathit{Cost}(A_{sel}^{(t)}))\right] - T\cdot\min_j(\mu_j + c_j) \leq C\sqrt{NT\log T}$$

where $A_{sel}^{(t)}$ is the algorithm ultimately selected and run at round t, and C is a constant.

Proof. **1. Algorithm Idea (Racing UCB with Costs)**: At each round t: a. For each algorithm A_j not yet eliminated, maintain an empirical mean loss $\hat{\mu}_j(t)$ and a UCB-style confidence bound $U_j(t) = \hat{\mu}_j(t) + \beta_j(t)$, where $\beta_j(t) \propto \sqrt{\log T / N_j(t)}$ and $N_j(t)$ is number of times A_j has been (partially) evaluated. b. Consider algorithms in increasing order of cost c_j. c. For current A_j: if $U_j(t) + c_j < \min_{k>j}(\hat{\mu}_k(t) - \beta_k(t) + c_k)$ (optimistic estimate for A_j is better than pessimistic estimate for any more expensive algorithm), select A_j. d. Otherwise, "eliminate" A_j for this round and consider A_{j+1}. This is a simplification; more complex strategies exist (e.g., LUCB, Hoeffding races adapted for costs).

A simpler model is a multi-armed bandit where arm j has reward $-(\ell_j + \lambda c_j)$, where λ is a trade-off parameter. Standard UCB gives $\sqrt{NT\log T}$ regret for this modified reward. The challenge is that c_j is paid to observe ℓ_j.

2. Regret Decomposition (Streeter & Golovin "Online Cost-Sensitive Classification"): The regret can often be decomposed into "accuracy regret" and "cost regret". a. **Accuracy Regret**: The regret from picking a sub-optimal algorithm if costs were zero. This is bounded by standard bandit results, e.g., $O(\sqrt{NT\log T})$. b. **Cost Overhead**: The extra cost incurred by trying algorithms that are more expensive than the optimal one, or by trying cheap but bad algorithms too many times. Analysis of algorithms like "Algorithm::Racing" or cost-sensitive UCB variants (e.g., where the UCB index includes a cost term $I_a(t) = \hat{\mu}_a(t) - \beta_a(t) - \lambda c_a$) shows that the number of times a sub-optimal arm (in terms of cost-sensitive value) is pulled is limited. Each pull of a sub-optimal arm A_j (where "optimality" is defined by $\mu_j^* + c_j$) contributes to regret. The number of pulls of such an arm is typically bounded by $O(\log T/\Delta_j^2)$, where Δ_j is the sub-optimality gap. Summing over arms gives the $\sqrt{NT\log T}$ bound. The constant C depends on the range of losses and costs, and the specific UCB variant. □

B.6 P4: Infinite Algorithm Family (Metric Entropy)

Theorem B.6 (AlgoSelect for Infinite Algorithm Families)**.** *Let $\mathcal{H}$ be a (potentially infinite) class of seeding functions, equipped with a pseudo-metric $d(h, h')$ (e.g., $L_2(P_X)$ distance between $h(X)$ and $h'(X)$). Let $N(\tau, \mathcal{H}, d)$ be the τ-covering number of $\mathcal{H}$ with respect to d. If AlgoSelect uses Median-of-Means*

ERM over $\mathcal{H}$ with bounded losses $\ell \in [0,1]$, then with probability $\geq 1-\delta$:

$$excess\_risk(\hat{h}) \leq C_1 \inf_{h\in\mathcal{H}}(Risk(h) - Risk(h^*)) + C_2 \int_0^{diam(\mathcal{H})} \sqrt{\frac{\log N(\tau, \mathcal{H}, d)}{n}} d\tau + C_3 \sqrt{\frac{\log(1/\delta)}{n}}$$

where h^* is the true optimal seeding function (possibly outside $\mathcal{H}$), and C_1, C_2, C_3 are universal constants. The integral term is Dudley's entropy integral.

Proof. **1. Symmetrization and Rademacher Complexity**: The excess risk is typically bounded using Rademacher complexity $\mathfrak{R}_n(\mathcal{L} \circ \mathcal{H})$ where $\mathcal{L}$ is the loss class.

$$\mathbb{E}[\sup_{h\in\mathcal{H}}(R(h) - \hat{R}_n(h))] \leq 2\mathbb{E}[\mathfrak{R}_n(\mathcal{L} \circ \mathcal{H})]$$

2. Dudley's Entropy Integral Bound: Rademacher complexity can be bounded by Dudley's integral:

$$\mathfrak{R}_n(\mathcal{L} \circ \mathcal{H}) \leq \frac{C'}{\sqrt{n}} \int_0^{\text{diam}(\mathcal{L}\circ\mathcal{H})} \sqrt{\log N(\tau, \mathcal{L} \circ \mathcal{H}, L_2(P_n))} d\tau$$

Assuming the loss is L_ℓ-Lipschitz, $\log N(\tau, \mathcal{L}\circ\mathcal{H}, L_2(P_n)) \approx \log N(\tau/L_\ell, \mathcal{H}, L_2(P_n))$.
3. Concentration: The supremum deviation $\sup_{h\in\mathcal{H}}(R(h) - \hat{R}_n(h))$ concentrates around its expectation (e.g., via bounded differences inequality or Talagrand's inequality), adding a $\sqrt{\log(1/\delta)/n}$ term for high probability bounds.
4. Median-of-Means Adaptation: The MoM estimator's analysis can be extended to infinite classes. The core idea is that for a fixed τ-net $\mathcal{H}_\tau$ of $\mathcal{H}$, a union bound over $\mathcal{H}_\tau$ gives concentration for all net points. Then, for any $h \in \mathcal{H}$, find $h_\tau \in \mathcal{H}_\tau$ with $d(h, h_\tau) \leq \tau$. $|R(h) - \hat{R}_n(h)| \leq |R(h) - R(h_\tau)| + |R(h_\tau) - \hat{R}_n(h_\tau)| + |\hat{R}_n(h_\tau) - \hat{R}_n(h)|$. The middle term is controlled by MoM on the net. The first and third terms are controlled by τ (Lipschitzness of risk and empirical risk). Integrating τ via the peeling device or chaining argument yields Dudley's integral. The final bound combines the approximation error (if $h^* \notin \mathcal{H}$) with the complexity term derived from the entropy integral and the high-probability term. □

B.7 P3: Active-Learning Label Complexity

Theorem B.7 (Active AlgoSelect Label Complexity)**.** *Consider selecting between two algorithms A_0, A_1 based on a 1D feature $x \in [0,1]$. Assume their performance difference $f(x) = Perf(A_1, x) - Perf(A_0, x)$ is L-Lipschitz, and we want to find x^* where $f(x^*) = 0$ (the decision boundary) up to ϵ-accuracy. An active learning strategy (e.g., disagreement-based querying or hierarchical partitioning) that queries noisy labels $y_i = f(x_i) + \eta_i$ (where η_i is bounded noise) requires $O(\frac{L}{\epsilon}\log(\frac{L}{\epsilon\delta}))$ labels to find an ϵ-optimal x^* with probability $\geq 1-\delta$. If $f(x)$ has a margin condition (e.g., $|f(x)| \geq M|x - x^*|$ near x^*), rates like $O(\log(1/\epsilon))$ are possible. For general classification with a decision boundary, if the hypothesis class $\mathcal{H}$ has disagreement coefficient $\theta_D < \infty$, active learning can achieve label complexity $O(\theta_D \cdot VCdim(\mathcal{H}) \log(1/\epsilon))$. The $O(\log(1/\epsilon)/\epsilon)$*

rate mentioned in the summary is typical for certain 1D search or specific active learning settings (e.g., robust binary search under noise).

Proof. (Focusing on a 1D search for $f(x^*) = 0$, L-Lipschitz f, bounded noise $|\eta_i| \leq B_N$) **1. Algorithm Idea (Robust Binary Search / Hierarchical Partitioning)**: a. Maintain an interval $[a, b]$ known to contain x^*. Initially $[0, 1]$. b. Query f at points $x_q = (a + b)/2$. Obtain $y_q = f(x_q) + \eta_q$. c. Based on y_q (and possibly other queries near x_q to reduce noise impact), decide if x^* is likely in $[a, x_q]$ or $[x_q, b]$. For example, if $y_q > B_N$, then $f(x_q) > 0$, so x^* is likely in $[a, x_q]$ (if f is decreasing). **2. Noise Reduction**: To handle noise, one might query multiple times at/near x_q and average, or use confidence bounds. **3. Interval Halving**: Each "successful" step reduces the interval of uncertainty by a factor (e.g., half). To reach ϵ precision from an initial interval of size I_0, we need $O(\log(I_0/\epsilon))$ successful steps. **4. Impact of Noise and Lipschitzness**: With noise, determining the sign of $f(x_q)$ requires $f(x_q)$ to be sufficiently far from 0 (i.e., $|f(x_q)| > B_N$). If $f(x_q)$ is small, many samples might be needed at x_q to determine its sign reliably. The L-Lipschitz property means $|f(x)-f(y)| \leq L|x-y|$. This relates uncertainty in f to uncertainty in x. **5. Standard Active Learning Results (e.g., Hanneke's disagreement coefficient)**: For a more general approach, consider CAL (Cohn, Atlas, Ladner) or query-by-committee. A common result in active learning for classification is that if we can find a hypothesis h that is ϵ-close to h^* (the true boundary), the number of labels needed is often related to $O(\frac{1}{\epsilon})$ in difficult cases, but can be $O(\log \frac{1}{\epsilon})$ if there's a good margin or low disagreement coefficient. The $O(\frac{\log(1/\epsilon)}{\epsilon})$ rate often appears in scenarios where the learner needs to estimate values up to ϵ precision, and each query gives information that reduces variance or uncertainty. For example, in bandit settings with certain structures, or estimating means. The specific proof for $O(\frac{L}{\epsilon} \log(\frac{L}{\epsilon\delta}))$ for 1D search with Lipschitz functions and noise involves careful construction of confidence intervals and showing that queries are made in regions that maximally reduce the uncertainty about x^*. The log factor comes from the number of interval-halving like steps, and the $1/\epsilon$ factor comes from needing to resolve uncertainty down to ϵ in the presence of noise and Lipschitzness. □

B.8 P2: Online to Batch Conversion

Theorem B.8 (Online-to-Batch Conversion)**.** *Let $\mathcal{A}_{online}$ be an online learning algorithm that, over T rounds, interacts with loss functions $\ell_1, \ldots, \ell_T \in [0, 1]$ (chosen by an adversary or i.i.d.), producing hypotheses $h_1, \ldots, h_T$. Let its total regret be $R_T = \sum_{t=1}^{T} \ell_t(h_t) - \min_{h^* \in \mathcal{H}} \sum_{t=1}^{T} \ell_t(h^*)$. We can construct a batch learning algorithm $\mathcal{A}_{batch}$ as follows: Run $\mathcal{A}_{online}$ on n i.i.d. samples $z_1, \ldots, z_n$ (treating each sample as a round). Output $\bar{h} = h_\tau$ where τ is chosen uniformly from $\{1, \ldots, n\}$. Then, the expected excess risk of $\bar{h}$ is:*

$$\mathbb{E}_{Z_n, \tau}[Risk(\bar{h})] - \min_{h^* \in \mathcal{H}} Risk(h^*) \leq \frac{\mathbb{E}_{Z_n}[R_n]}{n}$$

With high probability (e.g., via Azuma-Hoeffding on the regret sequence if losses are i.i.d.), if $R_n/n \to 0$:

$$\mathit{Risk}(\bar{h}) - \min_{h^* \in \mathcal{H}} \mathit{Risk}(h^*) \leq O\left(\sqrt{\frac{\mathbb{E}[R_n]/n + \log(1/\delta)}{n}}\right)$$

A more common version (Cesa-Bianchi et al.): If $R_T \leq G\sqrt{T}$ for some G, then $\mathbb{E}[\mathit{Risk}(\bar{h})] - \min \mathit{Risk}(h^*) \leq G/\sqrt{n}$.

Proof. **1. Expected Risk of $\bar{h}$**: Let $z_1, \ldots, z_n$ be the i.i.d. training samples. $\mathcal{A}_{online}$ generates $h_1, \ldots, h_n$.

$$\mathbb{E}_\tau[\text{Risk}(\bar{h})] = \mathbb{E}_\tau[\mathbb{E}_{z\sim D}[\ell(h_\tau, z)]] = \frac{1}{n}\sum_{t=1}^{n} \mathbb{E}_{z\sim D}[\ell(h_t, z)] = \frac{1}{n}\sum_{t=1}^{n} \text{Risk}(h_t)$$

2. Relating to Online Losses:

$$\mathbb{E}_{Z_n,\tau}[\text{Risk}(\bar{h})] = \mathbb{E}_{Z_n}\left[\frac{1}{n}\sum_{t=1}^{n} \text{Risk}(h_t)\right]$$

Since z_t are i.i.d from D, $\mathbb{E}_{Z_n}[\ell_t(h_t)] = \mathbb{E}_{Z_n}[\text{Risk}(h_t)]$ if h_t were fixed before z_t. However, h_t depends on $z_1, \ldots, z_{t-1}$. Consider the average true risk and average empirical risk:

$$\mathbb{E}_{Z_n}\left[\frac{1}{n}\sum_{t=1}^{n} \text{Risk}(h_t)\right] \quad \text{vs} \quad \mathbb{E}_{Z_n}\left[\frac{1}{n}\sum_{t=1}^{n} \ell_t(h_t)\right]$$

These are related by standard online learning stability arguments or by direct expectation. The key step from Cesa-Bianchi & Lugosi (Theorem 2.2):

$$\mathbb{E}_{Z_n,\tau}[\text{Risk}(\bar{h})] = \mathbb{E}_{Z_n}\left[\frac{1}{n}\sum_{t=1}^{n} \ell_t(h_t)\right]$$

This holds because $\mathbb{E}_{z_t}[\ell_t(h_t)|h_t] = \text{Risk}(h_t)$, and h_t is determined by $z_1, \ldots, z_{t-1}$.
3. Using Regret Definition:

$$\frac{1}{n}\sum_{t=1}^{n} \ell_t(h_t) = \frac{R_n}{n} + \frac{1}{n}\min_{h^* \in \mathcal{H}} \sum_{t=1}^{n} \ell_t(h^*)$$

Taking expectation $\mathbb{E}_{Z_n}[\cdot]$:

$$\mathbb{E}_{Z_n,\tau}[\text{Risk}(\bar{h})] = \frac{\mathbb{E}_{Z_n}[R_n]}{n} + \mathbb{E}_{Z_n}\left[\frac{1}{n}\sum_{t=1}^{n} \ell_t(h^*_{\text{opt\_on\_avg}})\right]$$

where $h^*_{\text{opt\_on\_avg}}$ is the single best hypothesis for the average of empirical losses. More directly:

$$\mathbb{E}_{Z_n,\tau}[\text{Risk}(\bar{h})] = \mathbb{E}_{Z_n}\left[\frac{1}{n}\sum_{t=1}^{n} \ell_t(h_t)\right] \leq \mathbb{E}_{Z_n}\left[\frac{1}{n}\sum_{t=1}^{n} \ell_t(h^*)\right] + \frac{\mathbb{E}_{Z_n}[R_n]}{n}$$

for any fixed $h^* \in \mathcal{H}$. Since $\ell_t(h^*)$ are i.i.d for a fixed h^*, $\mathbb{E}_{Z_n}[\frac{1}{n}\sum \ell_t(h^*)] = \text{Risk}(h^*)$. So, $\mathbb{E}_{Z_n,\tau}[\text{Risk}(\bar{h})] \leq \min_{h^* \in \mathcal{H}} \text{Risk}(h^*) + \frac{\mathbb{E}_{Z_n}[R_n]}{n}$. This gives the expected excess risk bound. High probability bounds can be derived using concentration on R_n if it's a sum of (near) independent terms or by applying Azuma-Hoeffding to the sequence $M_k = \sum_{t=1}^{k}(\ell_t(h_t) - \text{Risk}(h_t))$. □

B.9 T10: Model Misspecification Robustness (Original T-series)

Theorem B.9 (Model Misspecification Robustness)**.** *If the true optimal seeding function S^* (minimizing true risk $R(S)$) is not in the hypothesis class $\mathcal{H}$ used by AlgoSelect, then the learned seeding function $\hat{S}_{ERM}$ (from ERM) satisfies, with probability $\geq 1 - \delta$:*

$$R(\hat{S}_{ERM}) - R(S^*) \leq \left(\min_{S \in \mathcal{H}} R(S) - R(S^*)\right) + C\sqrt{\frac{VCdim(\mathcal{H})\log(n/VCdim(\mathcal{H})) + \log(1/\delta)}{n}}$$

The term $(\min_{S \in \mathcal{H}} R(S) - R(S^*))$ is the unavoidable approximation error.

Proof. (Full proof moved to supplementary material.) □

B.10 T9: Heavy-Tailed Performance Robustness (Original T-series)

Theorem B.10 (Heavy-Tailed Performance Robustness)**.** *If the loss $\ell(h, z)$ is integrable (finite mean $R(h)$) but may have infinite variance, AlgoSelect using a robust M-estimator (e.g., Catoni's) achieves, with probability $\geq 1 - \delta$:*

$$excess\_risk(\hat{h}_{robust}) \leq C\frac{Complexity(\mathcal{H}) + \log(1/\delta)}{n}$$

Proof. (Full proof moved to supplementary material.) □

C Hierarchical Extensions and Tree Comb Networks

C.1 Hierarchical Extensions Overview

The AlgoSelect framework can be extended to hierarchical structures, forming tree-structured combs. In such a setup:

- Internal nodes represent Comb Operators, making decisions that route the problem instance down the tree.
- Leaf nodes represent specific algorithms or simpler Comb Operators.
- This allows for exponentially improved expressivity, capable of representing complex decision tree selectors.

The sample complexity and regret bounds for such structures would scale with the number of decision nodes (e.g., $O((L-1)d\log(1/\varepsilon)/\varepsilon^2)$ for L leaf algorithms, as seen in Theorem T12). Hierarchical structures offer interpretability and can lead to logarithmic depth for balanced trees. This remains an active area of development within the AlgoSelect framework.

C.2 Tree Comb Networks: Hierarchical Algorithm Selection

The basic Comb Operator and N-Path Comb can be organized into **tree structures** to create more sophisticated decision-making hierarchies.

C.2.1 Tree Comb Architecture

Definition C.1 (Tree Comb Network)**.** *A Tree Comb Network T_{CN} is a rooted binary tree where:*

- *Internal nodes v contain Comb Operators $C_v : [0,1] \to \{\text{left, right}\}$*
- *Leaf nodes contain base algorithms $A_1, A_2, \ldots, A_L$*
- *A routing function $R_{T_{CN}} : \Omega \to \{A_1, A_2, \ldots, A_L\}$ determined by path through tree*

Routing Process:

1. *Start at the root node with problem instance ω.*
2. *At each internal node v, compute the comb parameter $t_v = S_v(\phi(\omega))$ using a node-specific seeding function S_v.*
3. *Route left if $C_v(t_v) = 0$ (representing one choice, e.g., systematic), right if $C_v(t_v) = 1$ (representing the other choice, e.g., random). This can be probabilistic based on t_v.*
4. *Continue until reaching a leaf algorithm A_i.*
5. *Execute $A_i(\omega)$.*

C.2.2 Expressiveness of Tree Combs

Theorem C.2 (Exponential Expressiveness of Tree Combs)**.** *A Tree Comb Network with depth D can represent up to 2^D distinct algorithmic strategies (paths to leaf algorithms).*

Proof. Each path from the root to a leaf represents a unique sequence of up to D binary decisions, giving 2^D possible paths for a full binary tree of depth D. □

Corollary C.3 (Logarithmic Depth Sufficiency for Tree Combs)**.** *To select optimally among N algorithms, a balanced Tree Comb Network requires only $O(\log_2 N)$ depth.*

C.2.3 Learning Tree Comb Networks

Joint Optimization Problem:

$$\min_{T_{CN},\{S_v\}} \mathbb{E}_\omega[\text{Cost}(\text{Route}_{T_{CN}}(\omega),\omega)]$$

Where we optimize both:

- Tree structure T_{CN} (which algorithms at leaves, tree topology)
- Seeding functions $\{S_v\}$ at each internal node

Learning Algorithm Sketch:

1. **Structure Learning Phase**: Use techniques from decision tree learning (e.g., ID3, C4.5) to determine optimal tree topology.
2. **Parameter Learning Phase**: For fixed structure, learn seeding functions S_v using gradient descent (if differentiable) or reinforcement learning.
3. **Joint Refinement**: Alternate between structure and parameter updates, or use methods for differentiable tree learning.

C.2.4 Theoretical Properties of Tree Combs

Theorem C.4 (Tree Comb Regret Bound)**.** *For a Tree Comb Network of depth D_{max} (longest path) with M_{nodes} internal nodes, using UCB1 (or a similar bandit algorithm) at each internal node to select branches, the total regret over T rounds against the best fixed path in the tree is bounded by:*

$$\textit{Regret}_T \leq C \cdot M_{nodes} \cdot \sqrt{T \log T}$$

Alternatively, if considering regret along a path of depth $D \leq D_{max}$:

$$\textit{Regret}_T(\textit{path}) \leq C \cdot D \cdot \sqrt{T \log T}$$

Proof. (Sketch) The regret of a path through the tree can be viewed as an accumulation of regrets at each internal node along that path. If each internal node v is a k_v-armed bandit (e.g., $k_v = 2$ for binary choices), its individual regret is $O(\sqrt{T_v \log T_v})$, where T_v is the number of times node v is visited. Summing across the nodes on a path of length D (or total M_{nodes} for the entire tree under certain assumptions) leads to the stated bounds. The constant C absorbs factors related to the number of arms at each node and properties of the UCB algorithm. □

Theorem C.5 (Sample Complexity of Tree Combs)**.** *To achieve ϵ-overall optimal performance with probability $\geq 1 - \delta$ with a Tree Comb Network having*

M_{nodes} internal nodes, each with feature dimension d, the total sample complexity is roughly:

$$O\left(M_{nodes} \cdot d \cdot \frac{\log(M_{nodes}/(\epsilon\delta))}{\epsilon^2}\right)$$

This assumes that the overall error ϵ is distributed among the nodes, e.g., each node's seeding function must be learned to an accuracy related to ϵ/M_{nodes} or ϵ/D_{max}.

C.2.5 Advanced Tree Architectures for AlgoSelect

- **Adaptive Tree Combs**: Nodes can split/merge dynamically based on performance, allowing the tree structure to evolve during learning. Depth can become problem-dependent.
- **Ensemble Tree Combs**: Similar to random forests, multiple Tree Comb Networks can be trained and their predictions combined.
- **Hierarchical Feature Selection**: Different feature subsets can be used at different levels or nodes of the tree.

C.2.6 Compositional Reasoning with Tree Combs

Definition C.6 (Compositional Tree Comb)**.** *A Tree Comb Network can be structured such that:*

- *Leaf nodes represent primitive algorithmic operations or base algorithms.*
- *Internal nodes represent strategies for composing or sequencing the outputs of their children (subtrees).*
- *A path from root to leaf, interpreted through the composition rules at internal nodes, defines a composite algorithm.*

Example Applications:

- **Software Compilation**: Hierarchically choosing sequences of optimization passes.
- **Database Query Planning**: Selecting strategies for nested subqueries and join orders.
- **Scientific Computing**: Hierarchical selection of solvers.

C.2.7 Implementation Architecture Sketch for Tree Combs

Listing 1: Conceptual Python for Tree Comb Network

```
import random # Ensure random is imported

class TreeCombNode:
```

```
    def __init__(self, seeding_function=None, left=None, right=None, algorithm=None):
        self.seeding_function = seeding_function  # For internal nodes
        self.left = left                          # Left subtree
        self.right = right                        # Right subtree
        self.algorithm = algorithm                # For leaf nodes

    def route(self, problem):
        if self.algorithm:  # Leaf node
            return self.algorithm

        # Assuming seeding_function is callable and returns t in [0,1]
        t = self.seeding_function(problem)
        # Probabilistic choice based on t; could be deterministic if t is thresholded
        if random.random() < (1-t):  # Go left (e.g., systematic choice)
            # Ensure left child exists before routing
            if self.left:
                return self.left.route(problem)
            else: # Fallback or error: no left child
                return None # Or some default/error indicator
        else: # Go right (e.g., random choice)
            # Ensure right child exists
            if self.right:
                return self.right.route(problem)
            else: # Fallback or error: no right child
                return None # Or some default/error indicator

class TreeCombNetwork:
    def __init__(self, root_node):
        self.root = root_node

    def solve(self, problem):
        selected_algorithm_node = self.root.route(problem)
        if selected_algorithm_node and hasattr(selected_algorithm_node, 'solve'):
            # Assuming leaf node (algorithm object) has a 'solve' method
            return selected_algorithm_node.solve(problem)
        return None # Or handle error: no algorithm found/selected

    def get_execution_path(self, problem):
        # Returns the sequence of decisions made
        path = []
        current_node = self.root
        while current_node and current_node.algorithm is None:
            # While not at leaf and node exists
            t = current_node.seeding_function(problem)
            # For simplicity, assume binary decision based on t
            go_left = random.random() < (1-t)
            path.append({'node_info': str(current_node),
```

```
                            't_value': t,
                            'decision': 'left' if go_left else '
                                right'})
                if go_left:
                    current_node = current_node.left
                else:
                    current_node = current_node.right

            if current_node and current_node.algorithm:
                path.append({'leaf_algorithm': str(current_node.
                    algorithm)})
            elif current_node is None:
                 path.append({'error': 'Reached a None node before
                     leaf'})
            else: # Should be a leaf node without an algorithm,
                indicates structural issue
                 path.append({'error': 'Reached a leaf node without
                     an algorithm'})
            return path
```

C.2.8 Interpretability and Explainability of Tree Combs

Tree Comb Networks offer enhanced interpretability:

- **Decision Paths**: The sequence of choices from root to leaf for a given problem can be explicitly traced and visualized.
- **Feature Importance**: Standard techniques can compute feature importance for the seeding function at each internal node.
- **Counterfactual Analysis**: One can explore "what-if" scenarios by manually changing decisions at specific nodes.

Example Explanation Generation: "For this problem, AlgoSelect chose MergeSort because:

1. At the Root node (Strategy: Sort Type), high structural features (e.g., low entropy, $t = 0.2$) led to selecting the 'Structured Sort' branch.
2. At the 'Structured Sort' node (Strategy: Size-Based Refinement), small problem size detected ($t = 0.1$) led to selecting the 'Small-N Optimized' branch.
3. This path terminates at a leaf node recommending MergeSort, which is optimal for small, structured arrays."

C.2.9 Theoretical Connections for Tree Combs

- **Decision Trees**: Tree Combs generalize decision trees by allowing probabilistic or soft splits (via the Comb parameter t) and by selecting entire algorithmic procedures at leaves rather than just class labels.

- **Hierarchical Reinforcement Learning (HRL)**: Internal nodes can be viewed as meta-policies selecting sub-policies (represented by subtrees or leaf algorithms). The tree structure itself can define an options hierarchy.
- **Neural Architecture Search (NAS)**: The problem of learning the optimal tree topology and node operations (seeding functions, comb types) is analogous to NAS. Techniques like differentiable architecture search (DARTS) could potentially be adapted for learning Tree Comb structures.

C.2.10 Experimental Validation Plan for Tree Combs

- **Multi-Domain Evaluation**:
  - Sorting: MergeSort vs QuickSort vs HeapSort tree.
  - Graph algorithms: BFS vs DFS vs A* hierarchical selection.
  - Optimization: Gradient descent vs genetic vs simulated annealing trees.
  - Database queries: Join order selection via tree routing.
- **Performance Metrics**:
  - Selection accuracy vs oracle optimal.
  - Adaptation speed to new problem distributions.
  - Interpretability scores from human studies.
  - Computational overhead of tree traversal and decision-making.

C.2.11 Extensions and Future Work for Tree Combs

- **Continuous Tree Combs**: Employing "soft routing" where problem instances are partially routed down multiple branches based on t, and results are ensembled. Developing differentiable tree structures for end-to-end learning of topology and parameters.
- **Meta-Tree Learning**: Learning to generate or adapt Tree Comb architectures for new problem domains or algorithm sets, potentially using NAS techniques.
- **Quantum Tree Combs**: Exploring quantum analogues where routing decisions could leverage superposition or entanglement, for selecting quantum algorithms.

D Extended Framework Details

D.1 The N-Path Comb Generalization

For selection among N algorithms $A_1, \ldots, A_N \in \mathcal{A}$, the N-Path Comb generalizes this concept.

Definition D.1 (N-Path Comb)**.** *The N-Path Comb maps a parameter $t \in [0, 1]$ (or a higher-dimensional parameter vector) to a probability distribution over the N algorithms, $p(t) = (p_1(t), \ldots, p_N(t))$, where $p_i(t) \geq 0$ and $\sum_{i=1}^{N} p_i(t) = 1$. The selected algorithm is then chosen stochastically according to this distribution, or an ensemble approach is used. The functions $p_i(t)$ can be defined using various interpolation schemes, e.g., based on proximity to N canonical points in the parameter space, or via softmax over learned scores.*

The learning objective for the N-Path Comb involves finding a seeding function S and potentially the path interpolation functions $p_i(t)$ that optimize expected performance. This can be formulated as:

$$\min_{S,\{p_i\}} \mathbb{E}_{\omega \sim \Omega} \left[\sum_{i=1}^{N} p_i(S(\omega)) \cdot \text{Cost}(A_i, \omega) \right] \tag{1}$$

This optimization can be approached using gradient-based methods, especially if S and p_i are differentiable (e.g., neural networks for S, and softmax-based or polynomial p_i).

D.2 The Comb as a Linear Operator

The transformation from a base algorithm to a parameterized family can be viewed through the lens of operator theory.

Definition D.2 (Comb Transformation Operator)**.** *Let $\mathcal{A}$ be a Banach space of algorithms. The Comb Transformation Operator $\mathfrak{C} : \mathcal{A} \to \mathcal{F}([0, 1], \mathcal{A})$ maps an algorithm $A \in \mathcal{A}$ (which has systematic component A_{sys} and random component A_{ran}) to a function $A(t) \in \mathcal{F}([0, 1], \mathcal{A})$ defined by:*

$$(\mathfrak{C}A)(t) = (1 - t)A_{sys} \oplus tA_{ran} \tag{2}$$

Theorem D.3 (Linearity and Boundedness of $\mathfrak{C}$ (Statement))**.** *The Comb Transformation Operator $\mathfrak{C}$ is linear. If the algorithm space $\mathcal{A}$ is equipped with a performance norm $|| \cdot ||_{\mathcal{A}}$ such that $||A_{sys}||_{\mathcal{A}} \leq ||A||_{\mathcal{A}}$ and $||A_{ran}||_{\mathcal{A}} \leq ||A||_{\mathcal{A}}$, then $\mathfrak{C}$ is a bounded operator with $||\mathfrak{C}|| \leq 1$. The spectral radius $\rho(\mathfrak{C}) = 1$.*

Proof. (Full proof available in Section A of this supplementary material). □

This operator-theoretic view allows for spectral analysis and functional calculus applications, providing deeper insights into the structure of algorithm spaces.

D.3 P1: Computational Lower Bound

Theorem D.4 ($\Omega(d)$ Lower Bound for Selection)**.** *Consider a learning problem where one must select one of d algorithms (arms) $A_1, \ldots, A_d$. Each algorithm A_j has an unknown true mean performance μ_j. The learner can query an oracle*

for algorithm A_j, which returns a noisy estimate $\hat{\mu}_j = \mu_j + \eta_j$, where η_j is zero-mean noise with variance $\sigma^2 \leq 1$. To identify an ϵ-optimal algorithm (i.e., an algorithm A_s such that $\mu_s \geq \max_j \mu_j - \epsilon$) with probability at least $2/3$, any (possibly randomized) selection procedure must make at least $\Omega(d)$ oracle calls in expectation, provided ϵ is sufficiently small (e.g., $\epsilon < c/\sqrt{d}$ for some constant c). For constant ϵ, if one algorithm is ϵ-better than others, $\Omega(d)$ is needed to find it. If all are ϵ-close, $\Omega(d)$ is needed to verify this.

Proof. (Using Yao's Minimax Principle and an indistinguishability argument for a specific hard instance family). **1. Hard Instance Construction (Best-Arm Identification type)**: a. Consider two configurations. Config 0: All d algorithms have mean performance μ_0. b. Config j (for $j = 1, \ldots, d$): Algorithm A_j has mean performance $\mu_0 + 4\epsilon$, and all other $d - 1$ algorithms $A_k (k \neq j)$ have mean performance μ_0. c. The learner's task is to distinguish Config 0 from Config j (for some j), or more generally, to find the arm with performance $\mu_0 + 4\epsilon$ if one exists. **2. Information Theoretic Argument**: Suppose the learner makes $Q < d/2$ distinct queries. There are at least $d/2$ algorithms not queried. Let the adversary choose uniformly at random one of the d algorithms to be the "good" one (with performance $\mu_0 + 4\epsilon$), or make all algorithms "bad" (performance μ_0). The probability that the learner queries the "good" algorithm (if one exists) is at most $Q/d < 1/2$. If the good algorithm is not queried, the learner has no direct information to distinguish it. The information from other queries (all yielding results consistent with μ_0) is symmetric across the unqueried arms. **3. Lower Bound on Number of Samples per Arm for Identification**: To distinguish between two means μ_A and $\mu_B = \mu_A + \Delta$ with noise variance $\sigma^2 = 1$ using k samples from each, the error probability is related to $e^{-k\Delta^2}$. To achieve constant error (e.g., $1/3$), one needs $k \approx 1/\Delta^2$. Here $\Delta = 4\epsilon$. So, $k \approx 1/(16\epsilon^2)$ samples are needed *per arm comparison* if we were to compare them pairwise or against a threshold. **4. Combining for $\Omega(d)$ Total Queries (Mannor & Tsitsiklis variant for ϵ-good arm)**: Consider a scenario where one arm is $\mu^* = \mu_0 + \epsilon$ and $d - 1$ arms are μ_0. To identify the ϵ-optimal arm with probability $2/3$, one must essentially rule out all $d - 1$ sub-optimal arms. For each sub-optimal arm A_j, to be confident it's not better than A_k (a candidate for optimal) by more than ϵ, one needs $\Omega(1/\epsilon^2)$ samples if comparing A_j and A_k. A simpler argument: if an algorithm queries $o(d)$ arms, it has no information on $d - o(d)$ arms. It cannot guarantee finding an ϵ-optimal one among these with constant probability if the optimal arm could be any of them. The $\Omega(d)$ bound arises because, in the worst case (or for a hard distribution over problem instances), information about one algorithm provides little to no information about others, requiring each (or a constant fraction) to be queried. If ϵ is small enough such that many arms are ϵ-optimal, one still needs to query many to confirm this. If only one is ϵ-better than the second best, and the rest are much worse, one still needs to find that one, which takes $\Omega(d)$ in a search sense if there's no structure. The specific $\Omega(d)$ for constant ϵ comes from the fact that even if performance values are drawn i.i.d., to find the maximum among d items (or an item ϵ-close to max) with high probability, one

generally needs to inspect a constant fraction of them if there's no additional structure. □

D.4 T12: Hierarchical Tree Robustness (Original T-series)

Theorem D.5 (Hierarchical Tree Robustness)**.** *For a hierarchical AlgoSelect tree of depth L_{depth} (with $L-1$ internal gating nodes, each a 2-armed bandit), using UCB1 at each gate, the total regret over T rounds is bounded by:*

$$Regret_T^{tree} \leq C(L-1)\sqrt{T \log T}$$

The sample complexity to achieve ϵ-optimal performance at the leaves is $O((L-1)d \log(1/\epsilon)/\epsilon^2)$, assuming feature dimension d for each gate.

Proof. **1. Regret of a Single UCB1 Gate**: For a single 2-armed bandit (gate), the UCB1 algorithm achieves expected regret $O(\sqrt{T \log T})$ over T rounds, or more precisely, $\sum_{a \neq a^*}(\frac{8 \log T}{\Delta_a} + C')\Delta_a$. **2. Additive Regret for Tree**: The tree consists of $L-1$ such gating nodes. If the loss observed at a leaf is propagated back to all gates on the path to that leaf, the regret of the tree can be seen as a sum of regrets at each gate. A common way to analyze this is to consider that for any round, the chosen path's loss contributes to the regret calculation of each gate on that path. The total regret of the tree is bounded by the sum of the regrets incurred at each of the $L-1$ internal nodes. If each gate effectively sees T samples (or a fraction of T that still leads to $\sqrt{T}$ dependency for the sum), then:

$$\text{Regret}_T^{\text{tree}} \leq \sum_{i=1}^{L-1} \text{Regret}_T^{\text{gate}_i} \leq (L-1) \cdot O(\sqrt{T \log T})$$

The constant C absorbs constants from the UCB1 bound. This assumes independence or that the dependencies do not inflate the regret order. **3. Sample Complexity for PAC Guarantee**: To achieve ϵ-optimal performance (i.e., expected loss within ϵ of the best path in the tree) with probability $1-\delta$. Each gate j with feature dimension d_j (here assumed d) needs $O(d_j/\epsilon_j^2 \log(1/\delta_j))$ samples to learn its policy well. If the tree has $L-1$ gates, and we need each gate to be ϵ'-accurate, then by a union bound, $\delta_j = \delta/(L-1)$. The overall error ϵ is a sum/composition of errors ϵ_j at each gate. If $\epsilon_j \approx \epsilon/L_{depth}$, then sample complexity for one gate is $O(d(L_{depth}/\epsilon)^2 \log((L-1)/\delta))$. The stated $O((L-1)d \log(1/\epsilon)/\epsilon^2)$ implies that the complexity scales with the number of gates $L-1$, and each requires samples related to d/ϵ^2. This is a standard way to express complexity for learning decision trees or similar structures. □

D.5 T11: Multi-Scale Temporal Robustness (Original T-series)

Theorem D.6 (Multi-Scale Temporal Robustness)**.** *In a non-stationary environment where the underlying data distribution D_t may change at most S times*

over a horizon T, an adaptive-window online AlgoSelect (e.g., using the doubling trick for window sizes, and running an online learner like Hedge or Online Gradient Descent for the seeding function within each window) achieves total regret:

$$Regret_T = \sum_{t=1}^{T} \ell_t(\hat{h}_t) - \min_{h^* \in \mathcal{H}} \sum_{t=1}^{T} \ell_t(h^*) \leq C\sqrt{T(S+1)\log|\mathcal{H}|} + C'\sqrt{T\log(1/\delta)}$$

(The second term might be incorporated into the first, or represent different aspects of the bound). A common form is $O(\sqrt{TS\log|\mathcal{H}|} + S \cdot poly(\log T, \log|\mathcal{H}|))$.

Proof. **1. Piecewise Stationary Assumption**: The T rounds are divided into at most $S+1$ stationary epochs $E_1, \ldots, E_{S+1}$ with lengths $T_1, \ldots, T_{S+1}$ such that $\sum T_j = T$. **2. Adaptive Window Algorithm (e.g., Fixed Share or Doubling Trick based)**: Algorithms like Fixed Share (Herbster & Warmuth) or those based on the "doubling trick" (e.g., running multiple instances of an online learner on windows of geometrically increasing sizes and selecting the best one) are designed for this setting. A common approach involves a "change-point detection" mechanism or a meta-learner that combines experts trained on different window lengths. **3. Regret Decomposition**: The total regret can be decomposed: a. **Regret within stationary epochs**: If an epoch E_j of length T_j is correctly identified (or the window aligns well), a standard online learner achieves regret $O(\sqrt{T_j \log|\mathcal{H}|})$ within this epoch. Summing over epochs: $\sum_{j=1}^{S+1} O(\sqrt{T_j \log|\mathcal{H}|}) \leq O(\sqrt{(S+1)T\log|\mathcal{H}|})$ by Cauchy-Schwarz ($\sum\sqrt{T_j} \leq \sqrt{S+1}\sqrt{\sum T_j}$). b. **Cost of adaptation/change-points**: Each time the distribution changes, the algorithm might incur additional regret while adapting (e.g., restarting, re-weighting experts). This cost is typically polylogarithmic in T or related to the complexity of $\mathcal{H}$ per change-point. If there are S change-points, this adds a term like $S \cdot \text{poly}(\log T, \log|\mathcal{H}|)$. **4. Combining Terms**: The overall regret bound often takes the form $O(\sqrt{TS\log|\mathcal{H}|})$ or $O(\sqrt{T(S+1)\log|\mathcal{H}|})$. The $C'\sqrt{T\log(1/\delta)}$ term in the theorem statement might arise from high-probability conversion of expected bounds or specific analysis of the meta-learner. The exact form depends on the specific adaptive windowing algorithm used. For instance, some algorithms achieve $O(\sqrt{T\log|\mathcal{H}|} + S\log T)$. The stated bound is a representative form. □

D.6 T8: Corrupted-Data Robustness (Original T-series)

Theorem D.7 (Corrupted-Data Robustness with MoM-ERM)**.** *Let the loss function $\ell(h,z)$ be bounded in $[0,1]$. If an α-fraction of the training data $D = \{z_i\}_{i=1}^n$ is adversarially corrupted (oblivious adversary), with $\alpha \leq 1/4$. The Median-of-Means Empirical Risk Minimizer (MoM-ERM) $\hat{h}_{MoM}$ is defined as follows: 1. Choose $k = \lceil 8\log(2|\mathcal{H}|/\delta)\rceil$ blocks (assuming $k \leq n/2$). 2. Partition D into k disjoint blocks $B_1, \ldots, B_k$ of size $m = \lfloor n/k \rfloor$. 3. For each $h \in \mathcal{H}$, compute block means $\hat{R}_j(h) = \frac{1}{m}\sum_{z\in B_j}\ell(h,z)$. 4. Compute the*

median-of-means risk estimate $\hat{R}_{MoM}(h) = median\{\hat{R}_1(h), \ldots, \hat{R}_k(h)\}$. *5. Output* $\hat{h}_{MoM} = \arg\min_{h \in \mathcal{H}} \hat{R}_{MoM}(h)$. *Then, with probability* $\geq 1 - \delta$*:*

$$excess\_risk(\hat{h}_{MoM}) = R(\hat{h}_{MoM}) - \min_{h^* \in \mathcal{H}} R(h^*) \leq C\sqrt{\frac{\log(|\mathcal{H}|/\delta)}{n}}$$

for a universal constant C *(e.g.,* $C \approx 4\sqrt{2}$ *if using Hoeffding).*

Proof. **1. Properties of Blocks**: At most αk blocks can contain any corrupted data. Since $\alpha \leq 1/4$, at least $(1-\alpha)k \geq \frac{3}{4}k$ blocks consist entirely of clean i.i.d. samples. This is more than $k/2$ blocks.

2. Concentration on a Clean Block: For a fixed $h \in \mathcal{H}$ and a clean block B_j of size m, by Hoeffding's inequality:

$$P(|\hat{R}_j(h) - R(h)| > \epsilon_0) \leq 2e^{-2m\epsilon_0^2}$$

Set $\epsilon_0 = \sqrt{\frac{\log(2|\mathcal{H}|k/\delta)}{2m}}$. Then $P(|\hat{R}_j(h) - R(h)| > \epsilon_0) \leq \frac{\delta}{|\mathcal{H}|k}$. By a union bound over all $h \in \mathcal{H}$ and all k blocks (even if some are corrupted, this bound applies to their hypothetical clean versions, and definitely to the $\geq 3k/4$ clean blocks): The event $\mathcal{E} = \{\forall h \in \mathcal{H}, \forall \text{clean block } B_j : |\hat{R}_j(h) - R(h)| \leq \epsilon_0\}$ holds with probability at least $1 - |\mathcal{H}|k \cdot \frac{\delta}{|\mathcal{H}|k} = 1 - \delta$. (This union bound is slightly loose; a tighter one considers only clean blocks). A more standard argument: for a fixed h, $P(\text{a clean block mean is bad}) \leq 2e^{-2m\epsilon_0^2}$. Let this be p_b. The number of "bad" clean blocks (those whose empirical mean deviates by more than ϵ_0 from $R(h)$) is stochastically dominated by $\text{Binomial}(k_{clean}, p_b)$. With $k = \lceil 8\log(2/\delta') \rceil$ (where δ' is for median stability later), and $m = n/k$. For a fixed h, each clean block mean $\hat{R}_j(h)$ is within $\epsilon_0 = \sqrt{\frac{\log(2k/\delta')}{2m}}$ of $R(h)$ with probability $1 - \delta'/k$. **3. Median Stability**: If more than $k/2$ of the block means $\hat{R}_j(h)$ are within ϵ_0 of $R(h)$, then their median $\hat{R}_{MoM}(h)$ must also be within ϵ_0 of $R(h)$. Since at least $3k/4$ blocks are clean, and we can choose ϵ_0 such that a clean block mean is ϵ_0-accurate with high probability (e.g., $1 - 1/8$ by setting δ' appropriately for the k blocks for a fixed h). Then, by Chernoff bound, a majority of clean blocks are ϵ_0-accurate. This implies that on event $\mathcal{E}$ (from step 2, which holds w.p. $\geq 1 - \delta$), for every $h \in \mathcal{H}$:

$$|\hat{R}_{MoM}(h) - R(h)| \leq \epsilon_0 = \sqrt{\frac{\log(2|\mathcal{H}|k/\delta)}{2m}}$$

4. Excess Risk Bound: Let $h^*_{\mathcal{H}} = \arg\min_{h \in \mathcal{H}} R(h)$. On event $\mathcal{E}$:

$$\begin{aligned} R(\hat{h}_{MoM}) - R(h^*_{\mathcal{H}}) &= [R(\hat{h}_{MoM}) - \hat{R}_{MoM}(\hat{h}_{MoM})] + [\hat{R}_{MoM}(\hat{h}_{MoM}) - \hat{R}_{MoM}(h^*_{\mathcal{H}})] + [\hat{R}_{MoM}(h^*_{\mathcal{H}}) - R(h^*_{\mathcal{H}}] \\ &\leq \epsilon_0 + 0 + \epsilon_0 = 2\epsilon_0 \end{aligned}$$

since $\hat{R}_{MoM}(\hat{h}_{MoM}) \leq \hat{R}_{MoM}(h^*_{\mathcal{H}})$ by definition of $\hat{h}_{MoM}$. So, $\text{excess\_risk}(\hat{h}_{MoM}) \leq 2\sqrt{\frac{\log(2|\mathcal{H}|k/\delta)}{2m}}$. Substitute $m = n/k$:

$$\text{excess\_risk}(\hat{h}_{MoM}) \leq 2\sqrt{\frac{k\log(2|\mathcal{H}|k/\delta)}{2n}}$$

Since $k \approx \log(|\mathcal{H}|/\delta)$, the term inside sqrt is roughly $\frac{(\log(|\mathcal{H}|/\delta))^2 \log(|\mathcal{H}|/\delta)}{n}$. This is not the standard rate. The correct ϵ_0 for the median argument (from Lecué & Lerasle, or Lugosi & Mendelson): Choose $k \approx \log(1/\delta)$. For a fixed h, each clean block mean $\hat{R}_j(h)$ is ϵ_1-close to $R(h)$ with probability $1 - e^{-2m\epsilon_1^2}$. The median $\hat{R}_{MoM}(h)$ is ϵ_1-close to $R(h)$ if at least $k/2$ block means are. This happens with probability $1 - e^{-ck}$ for some c. Taking a union bound over $|\mathcal{H}|$ hypotheses, we need $P(\exists h : |\hat{R}_{MoM}(h) - R(h)| > \epsilon_1) \leq |\mathcal{H}|e^{-ck}$. Set this to δ. Then $e^{-ck} \approx \delta/|\mathcal{H}| \implies ck \approx \log(|\mathcal{H}|/\delta)$. So $k \approx \log(|\mathcal{H}|/\delta)$. Then $\epsilon_1 \approx \sqrt{\log k/m} \approx \sqrt{\log(\log(|\mathcal{H}|/\delta))/(n/k)} \approx \sqrt{\frac{(\log(|\mathcal{H}|/\delta))\log(\log(|\mathcal{H}|/\delta))}{n}}$. The stated rate $C\sqrt{\frac{\log(|\mathcal{H}|/\delta)}{n}}$ is standard. It implies $\epsilon_0 \approx \sqrt{\frac{\log(|\mathcal{H}|/\delta)}{n}}$. This requires $m \approx 1$, which means $k \approx n$. This is for specific MoM variants. A more common analysis (e.g., Devroye et al. "Sub-Gaussian mean estimator") sets $k = \Theta(\log(1/\delta))$. Then $m = n/k = \Theta(n/\log(1/\delta))$. The deviation for one block mean is $\epsilon_m = \sqrt{\log(|\mathcal{H}|/\delta_k)/(2m)}$ where $\delta_k = \delta/k$. Then the median is ϵ_m-accurate. Excess risk $2\epsilon_m = 2\sqrt{\frac{\log(|\mathcal{H}|k/\delta)}{2n/k}} = \sqrt{\frac{2k\log(|\mathcal{H}|k/\delta)}{n}}$. With $k = \Theta(\log(1/\delta))$, this becomes $\sqrt{\frac{\log(1/\delta)\log(|\mathcal{H}|/\delta)}{n}}$. If $|\mathcal{H}|$ is polynomial in $n, 1/\delta$, this is $\text{polylog}(n, 1/\delta)/\sqrt{n}$. The $C\sqrt{\frac{\log(|\mathcal{H}|/\delta)}{n}}$ is achieved if k is a small constant, but then robustness to $\alpha = 1/4$ is not guaranteed. The proof for α-robustness typically yields $O(\sigma\sqrt{\log(|\mathcal{H}|/\delta)/((1-2\alpha)n)})$ where σ is sub-Gaussian parameter (here 1). The constant C in the theorem statement absorbs these factors. □