Chapter 2

Various Least Squares Methods

2.1 Least Squares Method

- **Method** (LS)

The least squares (LS) method was proposed by C.F. Gauss in his famous research work on star motion orbit prediction in 1795.

Later, the least squares method became the cornerstone of estimation theory.

The least squares method is widely used in system parameter estimation due to

- ✓ concise principle
- ✓ fast convergence
- ✓ easy understanding
- ✓ easy programming

♦ **Statement of problem**

In 1795, the basic principle of the least squares method proposed by Gauss is

The most probable value of an unknown value is to make the sum of the squares of the difference between the actual observed value and the calculated value be the smallest.

Karl Friedrich Gauss.

(1777-1855)

$$z(k) = y(k) + v(k)$$

$$\boxed{\text{Minimize} \sum_{k=1}^{m} (z(k) - y(k))^2}$$

where $z(k)$ is the measurement, $v(k)$ a noise and $y(k)$ the real signal.

♦ **Basic concepts of the least squares method**

Let us illustrate the LS concepts by an example. Determine the relationship between the thermistor resistance R and temperature t by experiments

$t(°C)$	t_1	t_2	$\cdots$	t_{N-1}	t_N
$R(\Omega)$	R_1	R_2	$\cdots$	R_{N-1}	R_N

$$R = a + bt$$

where a and b are the unknown parameters.

When there is no error in the measurement, only 2 measurements are required.

There is always a random error in every measurement.

$$y_i = R_i + v_i \quad \text{or} \quad y_i = a + bt_i + v_i$$

$$v_i = y_i - R_i \quad \text{or} \quad v_i = y_i - a - bt_i$$

♦ **Finding the model parameters using the least squares method**

According to the cost function of the least squares method, there is

$$J_{min} = \sum_{i=1}^{N} v_i^2 = \sum_{i=1}^{N} (R_i - (a + bt_i))^2$$

According to the method of finding the extreme value, the derivatives of the above function with respect to a and b are

$$\begin{cases} \left.\dfrac{\partial J}{\partial a}\right|_{a=\hat{a}} = -2\sum_{i=1}^{N}\left(R_i - \hat{a} - \hat{b}t_i\right) = 0 \\ \left.\dfrac{\partial J}{\partial b}\right|_{b=\hat{b}} = -2\sum_{i=1}^{N}(R_i - \hat{a} - \hat{b}t_i)t_i = 0 \end{cases}$$

$$\begin{cases} N\hat{a} + \hat{b}\sum_{i=1}^{N} t_i = \sum_{i=1}^{N} R_i \\ \hat{a}\sum_{i=1}^{N} t_i + \hat{b}\sum_{i=1}^{N} t_i^2 = \sum_{i=1}^{N} R_i t_i \end{cases} \qquad \begin{bmatrix} N & \sum_{i=1}^{N} t_i \\ \sum_{i=1}^{N} t_i & \sum_{i=1}^{N} t_i^2 \end{bmatrix} \begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N} R_i \\ \sum_{i=1}^{N} R_i t_i \end{bmatrix}$$

$$\begin{cases} \hat{a} = \dfrac{\sum_{i=1}^{N} R_i \sum_{i=1}^{N} t_i^2 - \sum_{i=1}^{N} R_i t_i \sum_{i=1}^{N} t_i}{N\sum_{i=1}^{N} t_i^2 - \left(\sum_{i=1}^{N} t_i\right)^2} \\ \hat{b} = \dfrac{N\sum_{i=1}^{N} R_i t_i - \sum_{i=1}^{N} R_i \sum_{i=1}^{N} t_i}{N\sum_{i=1}^{N} t_i^2 - \left(\sum_{i=1}^{N} t_i\right)^2} \end{cases}$$

Example: In the table below, the resistance value of the same thermistor is measured at different temperatures. Determine the mathematical model of the resistance according to the measured values. What is the resistance value at the temperature of 70°C?

Measured values of the thermistor

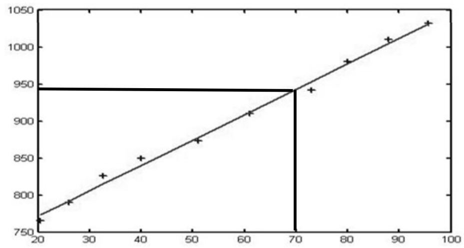
$t(°C)$	20.5	26	32.7	40	51	61	73	80	88	95.7
$R(\Omega)$	765	790	826	850	873	910	942	980	1010	1032

$R = a + bt$

$\hat{a} = 702.762$

$\hat{b} = 3.4344$

$t = 70°C$ $\quad R = 943.168\Omega$



Self Assessment Question

For the system $y(t) = Kx(t)$, there are two sets of measurement data:

$$x(1) = 1.1, \qquad y(1) = 2.0$$
$$x(2) = 3.0, \qquad y(2) = 5.9$$

Use the least squares method to obtain the estimated value of the system parameter K.

■ Parameter estimation method

In the previous sections, we introduced various models (e.g., the ARX model and ARMAX model) of linear time-invariant systems:

$$y(k) = W_y(q^{-1},\theta)y(k) + W_u(q^{-1},\theta)u(k) + e(k)$$

Assuming that the model structure is known, the model group conforming to this structure can be defined as follows:

$$M^* = \{M(\theta)|\theta \in D_m\}$$

Given θ in this case, the one-step prediction of the model can be calculated by

$$M(\theta): \quad \hat{y}(k|\theta) = W_y(q^{-1},\theta)y(k) + W_u(q^{-1},\theta)u(k)$$

Consider a system:

$$y(k) = G(q^{-1},\theta)u(k) + H(q^{-1},\theta)e(k)$$

where $W_y(q^{-1},\theta)$, $W_u(q^{-1},\theta)$, $G(q^{-1},\theta)$ and $H(q^{-1},\theta)$ have the following relationship:

$$W_y(q^{-1},\theta) = [1 - H^{-1}(q^{-1},\theta)]$$

$$W_u(q^{-1},\theta) = H^{-1}(q^{-1},\theta)G(q^{-1},\theta)$$

For an unknown system, the input and output sequence of the system can be obtained through observation and experiments

$$Z^N = [y(1), u(1), y(2), u(2), \cdots, y(N), u(N)]$$

How to estimate the parameter vector θ from these historical sequences is the research of parameter estimation methods.

■ How to evaluate a mathematical model

For a mathematical model, how to evaluate its quality requires the introduction of prediction error:

$$\varepsilon(k, \theta^*) = y(k) - \hat{y}(k|\theta^*)$$

If the sequence Z^N is known, the prediction error of each step can be calculated.

For a good mathematical model, its prediction will be more accurate. So the prediction error is relatively small.

This is an important criterion for evaluating the quality of the model.

♦ Minimize prediction error

Because the prediction error is a sequence, there are many methods to quantify this error. The following methods are used to quantify the prediction error:

First, define a filter $L(q^{-1})$ such that

$$\varepsilon_F(k, \theta) = L(q^{-1})\varepsilon(k, \theta), \qquad 1 \le t \le N$$

Then, define a cost function as

$$J(\theta, Z^N) = \frac{1}{N}\sum_{k=1}^{N} l(\varepsilon_F(k, \theta))$$

where $l(\varepsilon_F(k, \theta))$ is a scalar function

Given the historical data sequence Z^N, how to find the estimation of the vector θ ?

Minimizing $J(\theta, Z^N)$ results in

$$\hat{\theta}_N = \hat{\theta}_N(Z^N) = \arg\min_{\theta \in D_M} J(\theta, Z^N)$$

After defining the error quantification, various parameter estimation methods can be used for parameter estimation.

The function of $L(q^{-1})$ is to filter the high-frequency error in the system.

For $l(\varepsilon_F(k,\theta))$, the direct method is the square function

$$l(\varepsilon_F) = \frac{1}{2}\varepsilon_F^2$$

■ Least squares estimation

Given a system and its structure, how to determine its parameter vector θ by the least squares method.

The one-step prediction of a system is

$$\hat{y}(k|\theta) = \phi^T(k)\theta$$

For ARX model,

$$\hat{y}(k|\theta) = [1 - A(q^{-1})]y(k) + B(q^{-1})u(k)$$

The following correspondence can be obtained

$$\theta = [a_1 \quad a_2 \quad \cdots \quad a_{n_a} \quad b_1 \quad \cdots \quad b_{n_b}]^T$$

$$\phi(k) = [-y(k-1) \quad \cdots \quad -y(k-n_a) \quad u(k-1) \quad \cdots \quad u(k-n_b)]^T$$

The calculation of the prediction error is

$$\varepsilon(k,\theta) = y(k) - \phi^T(k)\theta$$

For the prediction error, select

$$L(q^{-1}) = 1 \qquad l(\varepsilon_F) = \frac{1}{2}\varepsilon_F^2 = \frac{1}{2}\varepsilon^2$$

So, the cost function is

$$J(\theta, Z^N) = \frac{1}{N}\sum_{k=1}^{N}\frac{1}{2}[y(k) - \varphi^T(k)\theta]^{\,2}$$

Determine θ by minimizing $J(\theta, Z^N)$, *i.e.*,

$$\hat{\theta}_N^{LS} = \arg\min J\,(\theta, Z^N)$$

To find the extreme value of $J(\theta, Z^N)$, we can set its derivative to 0

$$0 = \frac{d}{d\theta}J(\theta, Z^N) = -\frac{1}{N}\sum_{k=1}^{N}\varphi(k)(y(k) - \varphi^T(k)\theta)$$

which leads to

$$\sum_{k=1}^{N}\phi(k)y(k) = \sum_{k=1}^{N}\phi(k)\phi^T(k)\theta$$

Figure out the estimated value of θ by

$$\hat{\theta}_N = \left[\sum_{k=1}^{N}\phi(k)\phi^T(k)\right]^{-1}\sum_{k=1}^{N}\phi(k)y(k)$$

Let

$$R(N)=\sum_{k=1}^{N}\phi(k)\phi^T(k) \qquad f(N)=\sum_{k=1}^{N}\phi(k)y(k)$$

where $R(N)$ is a d×d matrix, $f(N)$ is a column vector of the d-th row.

The estimated value of parameter θ is

$$\hat{\theta}_N=R^{-1}(N)f(N)$$

In addition,

$$\frac{d^2}{d\theta^2}J(\theta,Z^N)=\frac{1}{N}\sum_{k=1}^{N}\varphi(k)\varphi^T(k)>0$$

Therefore, $\hat{\theta}_N$ lets $J(\theta,Z^N)$ take the minimum.

Features of the LS estimation

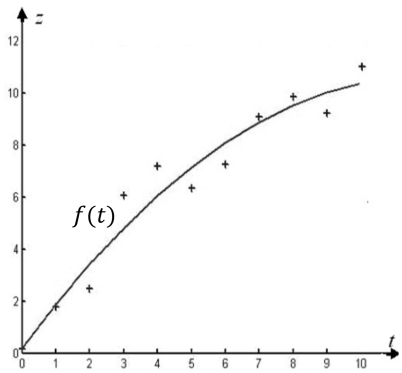
① The least squares estimation cannot satisfy every equation in the formula.

② Each equation has a deviation.

③ It minimizes the sum of the squares of the deviations of all equations

④ It takes into account the approximation of all equations

⑤ It minimizes the overall error, which is beneficial to suppress measurement errors.



■ Example

Consider the following system:

$$y(k) - 1.5y(k-1) + 0.7y(k-2) = u(k-3) + 0.5u(k-4) + \xi(k)$$

where ξ(k) is a white noise of the variance σ^2.

Assuming that the historical input/output data u(-3), u(-2), u(-1), u(0), y(-1), and y(0) are all 0.

The inverse M-sequence in section 1.6 is used as the input signal u={-1, -1,1, -1,...} and σ^2=0.

From the system difference equation, we have

$$y(1) = 1.5y(0) - 0.7y(-1) + u(-2) + 0.5u(-3) = 0$$
$$\varphi(1) = [-y(0), -y(-1), u(-2), u(-3)]^T = [0,0,0,0]^T$$
$$y(2) = 1.5y(1) - 0.7y(0) + u(-1) + 0.5u(-2) = 0$$
$$\varphi(2) = [-y(1), -y(0), u(-1), u(-2)]^T = [0,0,0,0]^T$$
$$y(3) = 1.5y(2) - 0.7y(1) + u(0) + 0.5u(-1) = 0$$
$$\varphi(3) = [-y(2), -y(1), u(0), u(-1)]^T = [0,0,0,0]^T$$

$$y(4) = 1.5y(3) - 0.7y(2) + u(1) + 0.5u(0) = -1$$
$$\varphi(4) = [-y(3), -y(2), u(1), u(0)]^T = [0,0,-1,0]^T$$
$$y(5) = 1.5y(4) - 0.7y(3) + u(2) + 0.5u(1) = -1.5 - 1 - 0.5 = -3$$
$$\varphi(5) = [-y(4), -y(3), u(2), u(1)]^T = [1,0,-1,-1]^T$$
$$y(6) = 1.5y(5) - 0.7y(4) + u(3) + 0.5u(2) = -4.5 + 0.7 + 1 - 0.5 = -3.3$$
$$\varphi(6) = [-y(5), -y(4), u(3), u(2)]^T = [3,1,1,-1]^T$$
$$y(7) = 1.5y(6) - 0.7y(5) + u(4) + 0.5u(3) = -3.35$$
$$\varphi(7) = [-y(6), -y(5), u(4), u(3)]^T = [3.3,3,-1,1]^T$$

$$Y = \begin{bmatrix} y(1) \\ y(2) \\ \vdots \\ y(7) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -3 \\ -3.3 \\ -3.35 \end{bmatrix} \qquad \Phi^T = \begin{bmatrix} \varphi^T(1) \\ \varphi^T(2) \\ \vdots \\ \varphi^T(7) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 \\ 3 & 1 & 1 & -1 \\ 3.3 & 3 & -1 & 1 \end{bmatrix}$$

Due to rank(Φ_6^T)=3, rank(Φ_7^T) =4, and the LS algorithm requirements $\Phi\Phi^T$ is a full rank matrix. So, select N ≥ 7.

If N=7 is taken here, the least squares estimate of the system parameters obtained is

$$\hat{\theta} = [\hat{a}_1 \quad \hat{a}_2 \quad \hat{b}_1 \quad \hat{b}_2] = [-1.5 \quad 0.7 \quad 1.0 \quad 0.5]$$

which is the same as the true value of the system parameter.

Select the inverse M-sequence in simulation in section 1.6 as the input signal u(k), and set σ^2=1, the simulation results of parameter estimation using the LS algorithm are shown below.

Parameter				
True value	-1.5	0.7	1.0	0.5
	-1.5	0.7	1.0	0.5
	-1.5049	0.7162	1.0227	0.5629

As can be seen from the above, due to relatively large system interference, the accuracy of parameter estimation results has been greatly affected, but the parameter estimation value is still relatively close to the true value.

♦ **Simulation codes**

example211.m

Self Assessment Question

Consider the following system:

$$y(k+2) - 1.2y(k+1) + 0.5y(k) = u(k) - 0.7u(k-1) + 1.5u(k-2) + \xi(k+2)$$

where ξ(k) is a white noise of the variance 1 and |ξ(k)|≤0.25.

saq211.m

■ Simulation example of parameter estimation

Consider a first-order system

$$G(s) = \frac{8}{s+5}$$

Under the Simulink of Matlab, assume that the above system is unknown, that is, its transfer function is not known.

But it is known that it is a first-order ARX system, that is, n_a=n_b=1, and the sampling period of the system is 0.1s.

Now, by simulation, determine the parameters of the discrete transfer function of this system.

From $n_a = n_b = 1$, the discrete transfer function of the system is

$$(1 + a_1 q^{-1})y(k) = b_1 q^{-1} u(k)$$

Conversion form

$$y(k) = -a_1 y(k-1) + b_1 u(k-1)$$

Let

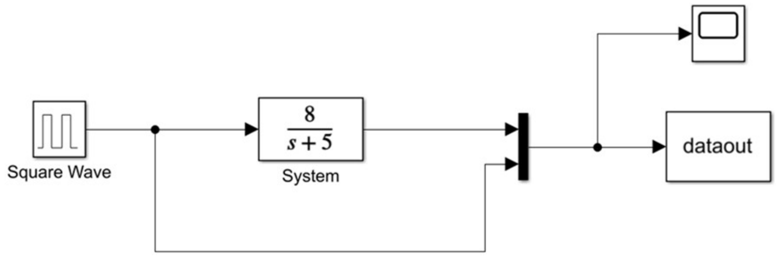
$$\theta = [a_1 \quad b_1]^T$$

$$\phi(k) = [-y(k-1) \quad u(k-1)]^T$$

$$y(k) = \phi^T(k)\theta$$

The transfer function of the unknown system can be obtained by using the least squares method to obtain the parameters of a_1 and b_1.

Design a simulation experiment in Simulink, as shown below



The input and output data are stored in the array ‘dataout’.

♦ **Simulation codes**

example212s.slx
example212.m

Self Assessment Question

Following the previous example, identify the 2nd order system below:

$$G(s) = \frac{4s+5}{s^2+2s+3}$$

saq212s.slx
saq212.m

2.2 Weighted Least Squares Method

■ **Principle and algorithm of the weighted least squares method**

One of the reasons for the low accuracy of the general least squares estimation is that the measurement data are treated equally.

It is difficult to obtain all measurement data under the same conditions.

Some measurements have high confidence and others have low confidence.

Measured values with different confidence levels can be treated separately by weighting.

If the confidence is high, the weight will be larger; if the confidence is low, the weight will be smaller.

If the measured values at different times have different weights, the cost function of the estimation is defined as

$$J(\theta, Z^N) = \sum_{k=1}^{N} \beta(N,k)[y(k) - \phi^T(k)\theta]^2$$

where $\beta(N,k)$ is the weighting coefficient.

The estimated value of θ is

$$\hat{\theta}_N = \left[\sum_{k=1}^{N} \beta(N,k)\phi(k)\phi^T(k)\right]^{-1} \sum_{k=1}^{N} \beta(N,k)\phi(k)y(k)$$

♦ **Principle and algorithm of the weighted least squares method**

When $\beta(N,k)$=1, the weighted least squares algorithm becomes the general least squares algorithm.

When $\beta(N,k) = a\lambda^{\mathrm{N}-k}(a > 0,\ 0 < \lambda < 1)$, the weighted least squares algorithm is also called the fading memory least squares algorithm.

The weighted least squares method is only used to estimate the influence of the measurement errors on parameter estimates in advance.

Self Assessment Question

Prove the estimated value of the parameter θ using the weighted least squares is

$$\hat{\theta}_N = \left[\sum_{k=1}^{N} \beta(N,k)\phi(k)\phi^T(k)\right]^{-1} \sum_{k=1}^{N} \beta(N,k)\phi(k)y(k)$$

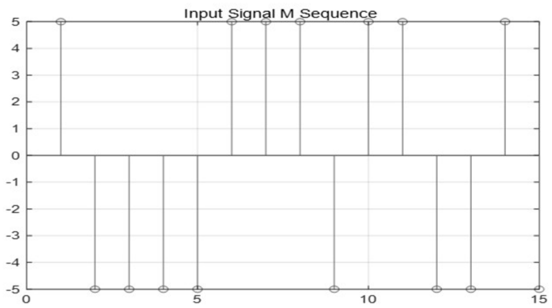
Example: Consider a plant

$$y(k) + 1.5y(k-1) + 0.7y(k-2) = u(k-1) + 0.5u(k-2) + V(k)$$

where $V(k)$ is a white noise that obeys the normal distribution $N(0,1)$. The input signal adopts a 4th-order M-sequence with an amplitude of 5.

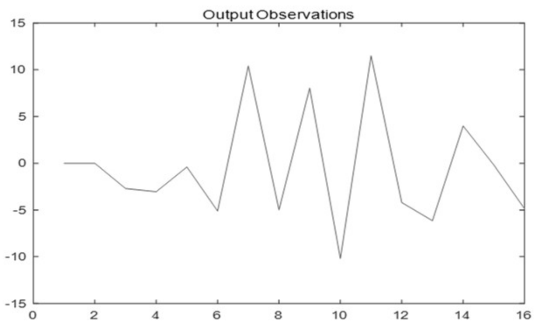
Select the following identification model for general least squares parameter identification.

$$y(k) + a_1y(k-1) + a_2y(k-2) = b_1u(k-1) + b_2u(k-2) + V(k)$$



The 4th-order M-sequence

The output signal



Solution: Since the input signal is a 4-order M-sequence, the cycle length of the sequence is $L = 2^4 - 1 = 15$. Therefore, assuming that the value of the input signal is an M-sequence from $k = 1$ to $k = 16$, then we can get

$$Y = \begin{bmatrix} y(3) \\ y(4) \\ \vdots \\ y(16) \end{bmatrix} \qquad \Phi^T = \begin{bmatrix} \varphi^T(3) \\ \varphi^T(4) \\ \vdots \\ \varphi^T(16) \end{bmatrix} = \begin{bmatrix} -y(2) & -y(1) & u(2) & u(1) \\ -y(3) & -y(2) & u(3) & u(2) \\ \vdots & \vdots & \vdots & \vdots \\ -y(15) & -y(14) & u(15) & u(14) \end{bmatrix} \qquad \hat{\theta} = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix}$$

$$\beta(N,k) = 0.95^{N-k}, \text{for } \mathrm{k} = 3, 4, \ldots, 16$$

Identification results of the weighted least squares algorithm

Parameter	a_1	a_2	b_1	b_2
Actual value	1.5	0.7	1.0	0.5
Estimated value	1.4964	0.6985	0.9661	0.4845

♦ **Simulation codes**

example221.m

Exercise 2.1

Identify the 2nd order system below using the weighted least squares method.

$$Y(s) = \frac{2s + 3}{s^2 + 4s + 5} U(s)$$

where $\mathrm{Y}(s)$ and $\mathrm{U}(s)$ are the output and input of the system. Let the weighting factor be $\beta(N,k) = 0.9^{N-k}, \text{for } \mathrm{k} = 1,2, \ldots.$