# Chapter 5

# Model Reference Adaptive Control





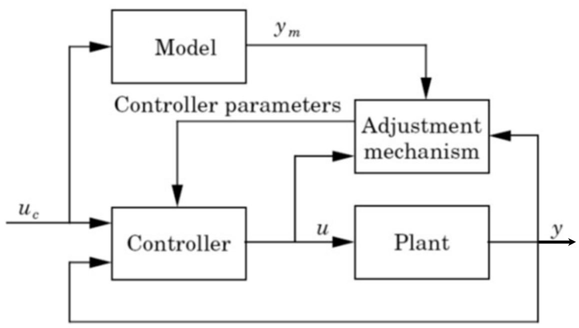
## 5.1 Adaptive Control Based on the Gradient Method

In the model reference adaptive control (MRAC) framework, the desired response $y_m(t)$ to command signal $u_c(t)$ is specified by the reference model.

The design problem is how to determine the parameter adjustment mechanism of the controller.

Among the MRAC design methods, the gradient method is a basic method.

It is the simplest and also the most widely used method.



## ■ MIT adaptive law

Let $e(t)$ be the difference (the tracking error) between the actual output $y_p(t)$ of the system and the output $y_m(t)$ of the reference model,

$$e(t) = y_m(t) - y_p(t)$$

and

$\theta$ be the unknown or slow time-varying parameter of the controller of the plant.

The control objective is to adjust the controller parameter so that $e(\infty) = 0$.

The performance index function is

$$J(\theta) = \frac{1}{2}e^2(t)$$

3

To make $J(\theta)$ be the minimum, it is more reasonable to change the parameters along the negative gradient direction of $J(\theta)$.

Change parameters such that

$$\begin{aligned}\frac{d\theta(t)}{dt} &= -\gamma\frac{\partial J(\theta)}{\partial\theta} \\ &= -\gamma e(t)\frac{\partial e(t)}{\partial\theta}\end{aligned}$$

where $\frac{\partial e(t)}{\partial\theta}$ is the sensitivity derivative and $\gamma$ is the adjustment rate.

The parameter adjustment law is usually called the MIT adaptive control law.

4

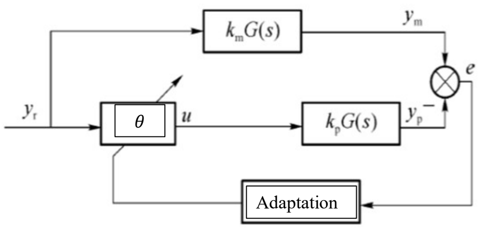
## ■ Adjustable gain MRAC based on the MIT law

The MIT adaptive control law can be used in the case of a single adjustable parameter or in the case of multiple adjustable parameters, namely, θ can be a scalar or vector.

The adjustable gain MRAC with a single adjustable parameter will be described below.

Let the plant with a feedforward gain be $k_p G(s)$

where $k_p$ is an unknown or slowly time-varying gain and $G(s)$ is a known transfer function and is stable and minimum phase.




Let the reference model be $k_m G(s)$

where $k_m$ is the known reference model gain.

The controller is designed according to the principle of matching the structure of the controlled plant and the reference model.

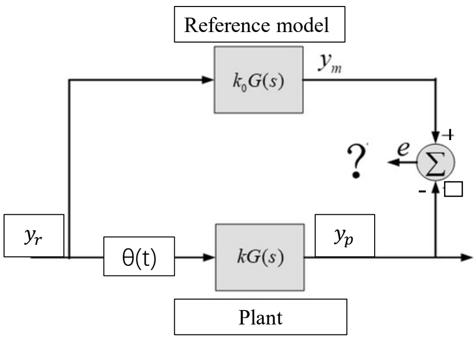
Its structure is shown above, where $\theta(t)$ is an adjustable gain.



The tracking error

$$e(t) = y_m(t) - y_p(t) = k_0 G(p) y_r(t) - kG(p)u(t)$$

where the differential operator is $p = \dfrac{d}{dt}$




The controlled plant: $y_p(t) = kG(p)u(t)$

The desired response: $y_m(t) = k_0 G(p) y_r(t)$

The controller: $u(t) = \theta(t) y_r(t)$

The sensitivity derivative:

$$\frac{\partial e(t)}{\partial \theta} = -kG(p) y_r(t) = -\frac{k}{k_0} y_m(t)$$

The MIT adaptive control rule

$$\frac{d\theta(t)}{dt} = -\gamma_0 e(t)\frac{\partial e(t)}{\partial \theta} = \gamma_0 \frac{k}{k_0} e(t) y_m(t) = \gamma e(t) y_m(t) \qquad (5.1.1)$$

where the adaptive gain $\gamma$

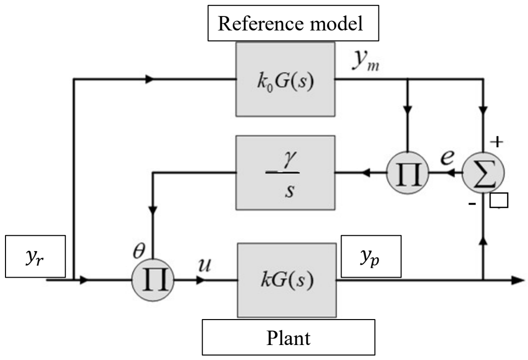
$$\gamma = \gamma_0 \frac{k}{k_0}$$

The controller

$$u(t) = \theta(t) y_r(t)$$

where

$$\theta(t) = \gamma \int e(t) y_m(t) dt$$






## ■ Algorithm 5.1.1 (adjustable gain MIT-MRAC)

Given $G(s)$.

**Step 1:** Select the reference model $k_m G(s)$.

**Step 2:** Select reference input signal $y_r(t)$ and adaptive gain γ.

**Step 3:** Sample the current reference model output $y_m(t)$.
and the actual system output $y_p(t)$.

**Step 4:** Calculate $u(t)$using the following

$$u(t) = \theta(t) y_r(t)$$

$$\theta(t) = \gamma \int e(t) y_m(t) dt \qquad (5.1.2)$$

**Step 5:** $t \rightarrow t+h$, return to Step 3 and continue the loop.

## ■ Simulation example 5.1.1

Let the stable controlled plant be $k_pG(s) = \dfrac{k_p}{s^2 + a_1 s + a_0}$

where $k_p$ is unknown ($k_p = 1$ for simulation) and let $a_1 = a_0 = 1$.

The reference model:

$$k_mG(s) = \frac{k_m}{s^2 + a_1 s + a_0}$$ where $a_1 = a_0 = 1$.

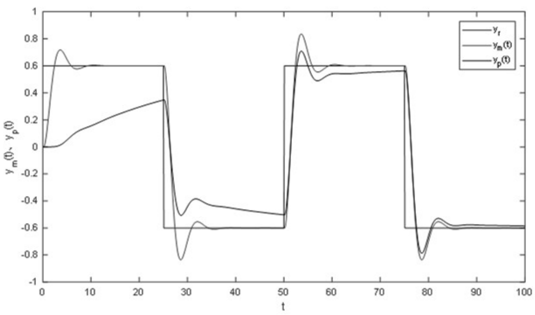
Let the adaptive gain $\gamma = 0.1$, the reference input $y_r(t)$ is a square wave signal, and the signal amplitude $r$ is taken as 0.6, 1.2 and 3.2, respectively.

The adjustable gain MIT-MRAC algorithm is adopted, and the simulation results are shown on the next slide.

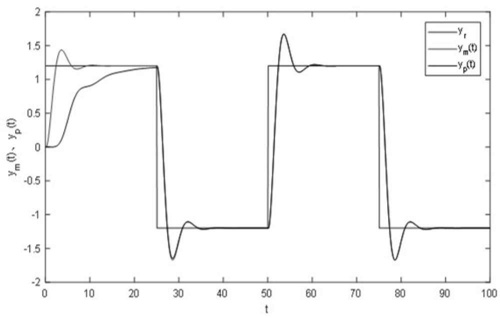


### ♦ Simulation results

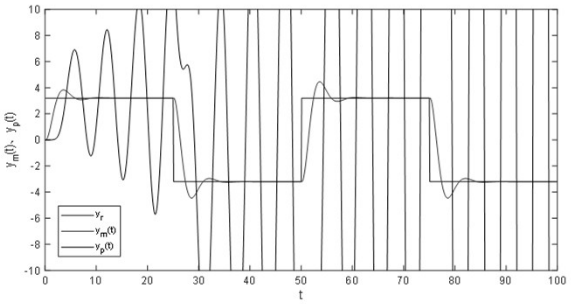
- When the reference input signal amplitude $r = 0.6$, the output response of the closed-loop system is particularly slow;
- When $r = 1.2$, the output response performance is quite good;
- When $r = 3.2$, the system becomes unstable.




$r = 0.6$




$r = 1.2$




$r = 3.2$

It can be seen that the convergence rate and stability of the system are closely related to the amplitude of the reference input signal.

**SAQ:** When the reference input signal takes a fixed amplitude value ( $r = 1.2$) and the adaptive gain $\gamma$ are taken to be 0.01, 0.1 and 0.7, respectively, a similar phenomenon will occur.

It can be seen that if the reference input signal amplitude or adaptive gain is too large, it will easily lead to system instability.



♦ **Simulation codes**

**example511.m**

**Self Assessment Question**

When the reference input signal takes a fixed amplitude value ( $r = 1.2$) and the adaptive gain $\gamma$ are taken to be 0.01, 0.1 and 0.7, respectively, what will occur?

## ■ Stability analysis

From eq (5.1.1) , if the reference input signal amplitude $r$ or adaptive gain $\gamma$ is too large, the adjustable gain change rate $\theta(t)$ will be too large, resulting in the divergence of the closed-loop system.

This conclusion can also be verified from the following theoretical analysis.

Take the derivative on both sides of eq (5.1.2) and obtain from $y_r(t) = r$ that

$$\dot{u}(t) = \dot{\theta}(t)r$$

The plant is

$$y_p(t) = k_p G(p)u(t)$$

From the transfer function $G(p)$ given in simulation example 5.1.1,

$$\ddot{y}_p(t) + a_1\dot{y}_p(t) + a_0 y_p(t) = k_p u(t)$$



Taking derivative on the previous equation gives

$$\dddot{y}_p(t) + a_1\ddot{y}_p(t) + a_0\dot{y}_p(t) = k_p\dot{u}(t)$$

$$\dddot{y}_p(t) + a_1\ddot{y}_p(t) + a_0\dot{y}_p(t) = k_p\gamma e(t)y_m(t)r = k_p\gamma\left(y_m(t) - y_p(t)\right)y_m(t)r$$

$$\dddot{y}_p(t) + a_1\ddot{y}_p(t) + a_0\dot{y}_p(t) + k_p\gamma y_m(t)r y_p(t) = k_p\gamma y_m^2(t)r \qquad (5.1.3)$$

**Note:** $y_m(t)$ is a known function of time t.

The above is a time-varying linear differential equation.

It is not easy to analyze its characteristics.

Since the reference input signal is constant, the reference model output $y_m(t)$ will also tend to be constant, *i.e.,*

$$y_m(\infty) = \lim_{s\to 0} s y_m(s) = \lim_{s\to 0}\frac{k_m r}{s^2 + s + 1} = k_m r$$

Substituting $y_m(t)$ in equation (5.1.3) with $k_m r$ leads to

$$\dddot{y}_p(t) + a_1\ddot{y}_p(t) + a_0\dot{y}_p(t) + k_p\gamma k_m r^2 y_p(t) = k_p\gamma k_m^2 r^3$$

Then, the characteristic equation of the closed-loop system is

$$s^3 + a_1 s^2 + a_0 s + k_p\gamma k_m r^2 = 0$$

According to Hurwitz stability criterion, if

$$a_1 a_0 > k_p\gamma k_m r^2 \quad (5.1.4)$$

the MRAC closed-loop system is stable.

Therefore, when adaptive gain $\gamma$ or the input signal amplitude $r$ is too large, eq (5.1.4) will not hold, resulting in system instability.



## 5.2 Adaptive Control Based on Lyapunov Stability Theory

To overcome the defect that the MRAC designed by the local parameter optimization method in Section 5.1 cannot guarantee stability, P. C. Parks proposed the MRAC adaptive control law derived by Lyapunov method in 1966 to ensure the global asymptotic stability of the system.

Thus, the MRAC design entered the stage of adopting the design criteria of stability theory, and brought new vitality to the adaptive control technology, enabling it to develop and apply rapidly.

### ■ Lyapunov-MRAC with adjustable gain

The adaptive control scheme is similar to the adjustable gain MIT-MRAC designed by the gradient method, and the error between the actual output of the system and the output of the reference model tends to zero by adjusting the gain of the adjustable system.

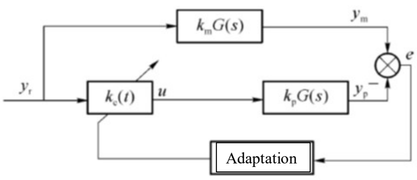
But the difference is that the system designed by this method can ensure the stability of the closed-loop system.

**♦ Plant and reference models**

Let the transfer functions of the actual plant and the reference model be

$$G_p(s) = k_p \frac{N(s)}{D(s)} = k_p \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

$$G_m(s) = k_m \frac{N(s)}{D(s)} = k_m \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$




where $k_p$ is a gain, unknown or slowly time-varying, $k_m$, $n$, $a_i$, $b_i$ $(i = 0, 1, \ldots, n-1)$ are known.

The controller gain $k_c(t)$ is used to compensate for the plant parameter $k_p$.

The design task of the control system is to find the regulation law of the adjustable gain $k_c(t)$ according to the Lyapunov stability theory, so that the difference $e(t)$ between the reference model output and the actual output of the system tends to zero.



The output error

$$e(t) = y_m(t) - y_p(t) = k(t)\frac{N(p)}{D(p)}\, y_r$$

where

$$k(t) = k_m - k_c(t)k_p$$

Convert the output error equation into an observable canonical form in state space

$$\begin{cases} \dot{x}(t) = Ax(t) + k(t)by_r \\ e(t) = cx(t) \end{cases} \qquad (5.2.1)$$

where $A$ is a matrix, $b$ and $c$ are vectors.

**Lemma:** If the homogeneous system $\dot{x}(t) = Ax(t)$ in the above equation is asymptotically stable, then there are positive definite matrices $P$ and $Q$ so that

$$A^T P + PA = -Q$$

♦ **Lyapunov method**

Select a Lyapunov function as

$$V(t) = \gamma' x^T(t)Px(t) + k^2(t), \qquad \gamma' > 0$$

The derivative of $V(t)$ is

$$\dot{V}(t) = \frac{dV(t)}{dt} = \gamma' \left(\frac{dx^T(t)}{dt} Px(t) + x^T(t)P\frac{dx(t)}{dt}\right) + 2k(t)\frac{dk(t)}{dt}$$
$$= \gamma' x^T(t)A^T Px(t) + \gamma' x^T(t)PAx(t) + 2k(t)\gamma' y_r b^T Px(t) + 2k(t)\frac{dk(t)}{dt}$$
$$= -\gamma' x^T(t)Qx(t) + 2k(t)\left(\dot{k}(t) + \gamma' y_r b^T Px(t)\right)$$

To make $\dot{V}(t) < 0$, take the parameter adjustment law as

$$\dot{k}(t) = -\gamma' y_r b^T Px(t)$$



♦ **Adaptive gain law**

Since $k_p$ is unknown or slowly time-varying, it can be approximated as a constant and the adaptive gain law is

$$\dot{k}_c(t) = -\frac{\dot{k}(t)}{k_p} = \frac{\gamma'}{k_p} y_r b^T Px(t) \qquad (5.2.2)$$

The obtained adaptive law depends on state variables, that is, all state variables are required to be measurable.

So, the application of this adaptive law is greatly limited.

## ■ Lemma (Kalman-Yakubovich lemma)

**Lemma:** Assume the system below is completely controllable and completely observable.

$$\begin{cases} \dot{x}(t) = Ax(t) + bv(t) \\ e(t) = cx(t) \end{cases}$$

Then, the transfer function $G(s) = c(sI - A)^{-1}b$ is a strictly positive real (SPR) function if and only if there are positive definite matrices $P$ and $Q$ such that

$$A^T P + PA = -Q, \quad b^T P = c \qquad (5.2.3)$$

If the transfer function $G(s)$ of the system ($G(s) = N(s)/D(s) = c(sI - A)^{-1}b$) is a strictly positive real function, the following adjustable gain Lyapunov-MRAC adaptive law can be derived from Equation (5.2.1) and Equation (5.2.2):

$$\dot{k}_c(t) = -\frac{\dot{k}(t)}{k_p} = \gamma e(t) y_r(t) \qquad (5.2.4)$$

where $\gamma$=$\gamma'$/ $k_p$ is an adaptive gain, $\gamma > 0$.

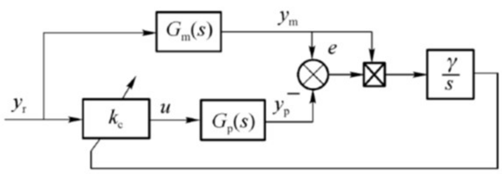


♦ **Comparison between gradient-MRAC and Lyapunov-MRAC**

The adaptive law (5.2.4) is very similar to the adaptive law (5.1.2) obtained by the gradient method, and their structures are shown below.

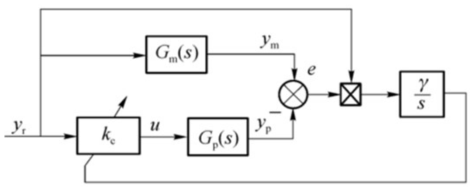
Gradient-MRAC

$$k_c(t) = \gamma \int e(t) y_m(t) dt$$




Lyapunov-MRAC

$$k_c(t) = \gamma \int e(t) y_r(t) dt$$




From the above figures, the control law of the system is

$$u(t) = k_c(t) y_r(t) \qquad (5.2.5)$$

## ■ Algorithm 5.2.1 (adjustable gain Lyapunov-MRAC)

Given $N(s)$ and $D(s)$.

**Step 1:** Select the reference model, namely $G_m(s)$.

**Step 2:** Select input signal $y_r(t)$ and adaptive gain $\gamma$.

**Step 3:** Sample the current reference model output $y_m(t)$ and the actual system output $y_p(t)$.

**Step 4:** Use formula (5.2.4) and formula (5.2.5) to calculate $u(t)$.

**Step 5:** t → t+h, return to Step 3 and continue the loop.



## ■ Simulation example 5.2.1

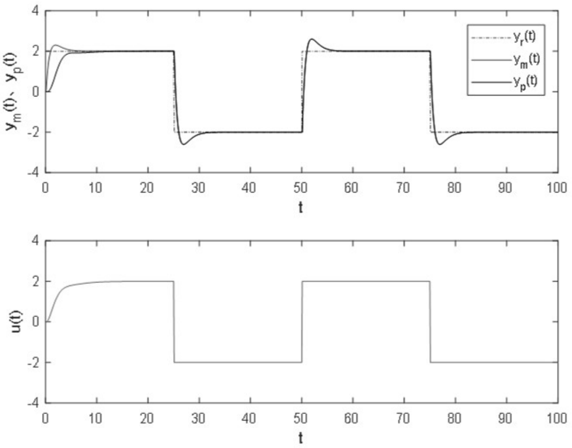
Consider the controlled object model: $$G_p(s) = \frac{k_p(2s+1)}{s^2+2s+1}$$

Select the reference model as

$$G_m(s) = \frac{k_m(2s+1)}{s^2+2s+1}, \qquad k_m = 1$$

Clearly, it can be verified that $G_p(s)$ and $G_m(s)$ are strictly positive real functions.

From adaptive gain $\gamma = 0.1$, the reference input $y_r(t)$ is a square wave signal with amplitude $r = 2$, and the adjustable gain Lyapunov-MRAC algorithm is adopted.



♦ **Simulation code**

**example521.m**

**Self Assessment Question**

In example 5.2.1, try different adaptive gain $\gamma$ (e.g.,0.01, 0.1, 1, 2).



## ■ MRAC when system state variables are measurable

♦ **System models**

When the control system is described by a state space model and the states are completely observable, the adaptive control law can be formed by the system state variables.

Assuming that the state variables are completely observable, let its state-space equation of the plant be

$$\dot{x}_p(t) = A_p x_p(t) + B_p u(t)$$

where $x_p(t)$ is the n-dimensional state vector,
$u(t)$ is the m-dimensional control vector,
$A_p$ and $B_p$ are $n \times n$ and $n \times m$ matrices, respectively.

Let the reference model be

$$\dot{x}_m(t) = A_m x_m(t) + B_m u_m(t)$$

where $x_m(t)$ is the state vector of the n-dimensional reference model,
$u_m(t)$ is the m-dimension reference input vector, and
$A_m$ and $B_m$ are $n \times n$ and $n \times m$ constant matrices, respectively.

**Note:** When the control system is described by a transfer function, the input and output variables of the system can be used to form an adaptive control law.

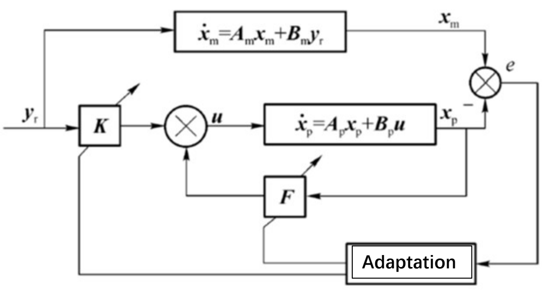


♦ **Controllers**

Since the plant parameter matrices $A_p$ and $B_p$ are generally unknown and cannot be adjusted directly to change the dynamic characteristics of the plant, the state feedback controller $F$ and feedforward controller $K$ can be used to form an adjustable system, namely,

$$u(t) = K(t)y_r(t) + F(t)x_p(t) \tag{5.2.6}$$

where $K(t)$ and $F(t)$ are $m \times m$ and $m \times n$ gain matrices, respectively.



Substituting the controller into the plant leads to

$$\dot{x}_p(t) = \left(A_p + B_pF(t)\right)x_p(t) + B_pK(t)y_r(t)$$

Then the state error equation is

$$\dot{e}(t) = \dot{x}_m(t) - \dot{x}_p(t) = A_m e(t) + \left(A_m - A_p - B_pF(t)\right)x_p(t) + \left(B_m - B_pK(t)\right)y_r(t)$$

The design task of the control system is to seek the adaptive law of adjusting $K$ and $F$ and to achieve state convergence

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left(x_m(t) - x_p(t)\right) = 0$$

and/or parameter convergence (the reference model parameters match with adjustable system parameters)

$$\lim_{t\to\infty}\left(A_p + B_pF(t)\right) = A_m$$

$$\lim_{t\to\infty} B_pK(t) = B_m$$



When $F(t) = F_0$ and $K(t) = K_0$ , the reference model and the adjustable system are completely matched, *i.e.*,

$$A_p + B_pF_0 = A_m$$
$$B_pK_0 = B_m$$

Using the above, the state error equation becomes

$$\begin{aligned}\dot{e}(t) &= A_m e(t) + B_p(F_0 - F(t))x_p(t) + B_p(K_0 - K(t))y_r(t)\\ &= A_m e(t) + B_mK_0^{-1}\tilde{F}(t)x_p(t) + B_mK_0^{-1}\tilde{K}(t)y_r(t)\end{aligned}$$

where $\tilde{F}(t) = F_0 - F(t),\ \tilde{K}(t) = K_0 - K(t)$.

Construct a Lyapunov function

$$V(t) = e^T(t)Pe(t) + tr\left(\tilde{F}^T(t)P_F^{-1}\,\tilde{F}(t)\right) + tr\left(\tilde{K}^T(t)P_K^{-1}\,\tilde{K}(t)\right)$$

where $P, P_F$ and $P_K$ are $n \times n, n \times n$ and $m \times m$ symmetric positive definite matrices, respectively, and $tr$ is the mathematical symbol of trace.

Taking the derivative on both sides of the Lyapunov function according to the property of the matrix trace results in

$$\dot{V}(t) = \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) + tr\left(\dot{\tilde{F}}^T(t)P_F^{-1}\tilde{F}(t) + \tilde{F}^T(t)P_F^{-1}\dot{\tilde{F}}(t)\right)$$
$$+tr\left(\dot{\tilde{K}}^T(t)P_K^{-1}\tilde{K}(t) + \tilde{K}^T(t)P_K^{-1}\dot{\tilde{K}}(t)\right)$$
$$= e^T(t)\left(A_m^T(t)P + PA_m\right)e(t) + 2tr\left(\dot{\tilde{F}}^T(t)P_F^{-1}\tilde{F}(t) + x_p(t)e^T(t)PB_mK_0^{-1}\tilde{F}(t)\right)$$
$$+2tr\left(\dot{\tilde{K}}^T(t)P_K^{-1}\tilde{K}(t) + y_r(t)e^T(t)PB_mK_0^{-1}\tilde{K}(t)\right) \qquad Note\text{: } ab = tr(ba) \qquad (5.2.7)$$

Since $A_m$ is a stable matrix, it is known that

$$A_m^TP + PA_m = -Q, \quad Q^T = Q > 0$$

The first item of eq (5.2.7) is negative. In order to ensure that $\dot{V}(t)$ is negative, the last two items on the right of (5.2.7) can be set to 0, respectively

$$\dot{\tilde{F}}(t) = -P_FK_0^{-T}B_m^TPe(t)x_p^T(t) \qquad \dot{\tilde{K}}(t) = -P_KK_0^{-T}B_m^TPe(t)y_r^T(t)$$

31

The above equations can be further expressed as

$$\dot{F}(t) = P_FK_0^{-T}B_m^TPe(t)x_p^T(t) \qquad \dot{K}(t) = P_KK_0^{-T}B_m^TPe(t)y_r^T(t)$$

Since the plant parameters $A_p$ and $B_p$ are unknown, it is difficult to determine $F_0$ and $K_0$, but considering the randomness of the values of $P_F$ and $P_K$, the above adaptive law can be rewritten as

$$\dot{F}(t) = R_1B_m^TPe(t)x_p^T(t) \qquad \dot{K}(t) = R_2B_m^TPe(t)y_r^T(t) \qquad (5.2.8)$$

where $R_1$ and $R_2$ are $m \times m$ matrices, which can be determined by test.

The adaptive law (5.2.8) can ensure the global asymptotic stability of the closed-loop system, that is, it satisfies the state convergence:

$$\lim_{t\to\infty} e(t) = 0 \qquad \text{because } \dot{V}(t) = -e^T(t)Qe(t).$$

32

## ■ Algorithm 5.2.2 (Lyapunov-MRAC based on state variables)

Given the plant structure, namely the dimension of $A_p$ and $B_p$.

**Step 1:** Select the reference model, *i.e.*, $A_m$ and $B_m$.

**Step 2:** Select input signal $y_r(t)$ and parameters $P$, $R_1$ and $R_2$.

**Step 3:** Sample the current reference model state $x_m(t)$ and the actual system state $x_p(\mathrm{t})$.

**Step 4:** Use (5.2.6) and (5.2.8) to calculate $u(t)$.

**Step 5:** t → t+h, return to Step 3 and continue the loop.

33

## ■ Simulation example 5.2.2

Consider that the state equation of the controlled plant is

$$\dot{x}_p(t) = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} x_p(t) + \begin{bmatrix} 0 \\ 8 \end{bmatrix} u(t)$$

Select the reference model and matrix $P$ as

$$\dot{x}_m(t) = \begin{bmatrix} 0 & 1 \\ -10 & 5 \end{bmatrix} x_m(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} y_r(t) \qquad\qquad P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

Take the matrices $R_1 = R_2 = 1$, which can be verified that $P$ meets

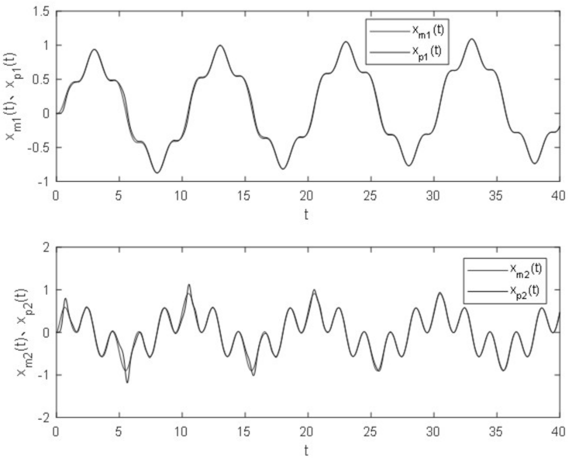
$$A_m^T P + P A_m = -Q, \quad Q^T = Q > 0$$

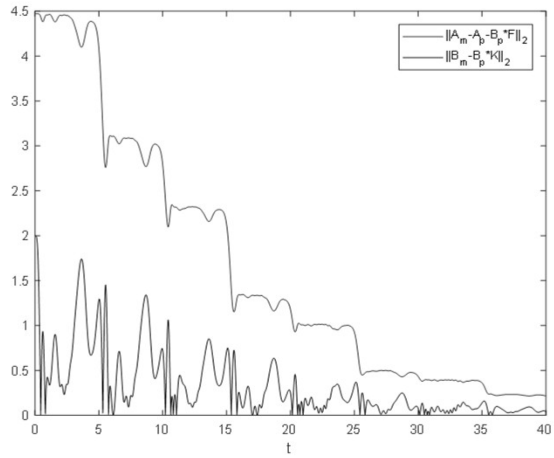
The reference input $y_r(t) = \sin(0.01\pi t) + 4\sin(0.2\pi t) + \sin(\pi t)$.

Lyapunov-MRAC algorithm based on state variables is adopted.

34

♦ ***Simulation results***




It can be seen that the adjustable system states can track the reference model states well.

It can be seen that the system parameters also converge to the reference model parameters well.

35

♦ **Simulation code**

## example522.m

### Self Assessment Question

In example 5.2.2, change the reference input to $y_r(t) = 4e^{0.01t}$.

36

## Exercise 5.1

Consider a time-varying plant

$$\dot{x}_p(t) = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix} x_p(t) + \begin{bmatrix} 0 \\ c \end{bmatrix} u(t)$$

where $a = -6\big(1 + 0.1\sin(0.1\pi\mathrm{t})\big)$, $b = -7\big(1 - 0.1\cos(0.2\pi\mathrm{t})\big)$, $c = 8\big(1 + 0.1\sin(0.3\pi\mathrm{t})\big)$.

Assume the reference model is

$$\dot{x}_m(t) = \begin{bmatrix} 0 & 1 \\ -10 & 5 \end{bmatrix} x_m(t) + \begin{bmatrix} 0 \\ 2 \end{bmatrix} y_r(t)$$

and the reference input $y_r(t) = 4\sin(\pi t)$.

Using the Lyapunov-MRAC method, design an adaptive controller and make simulation.