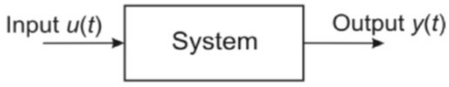
1.2 Linear Time-invariant Systems

■ **What is a linear time-invariant system**



A system is linear if it satisfies:

Superposition: If for input $u_1(t)$ the system responds with output $y_1(t)$, and
for input $u_2(t)$ the system responds with output $y_2(t)$,
then for input $u_1(t)+u_2(t)$ the system will respond with $y_1(t)+y_2(t)$,
$i.e.$, if $\mathrm{y(t)} = \mathrm{T}[u(t)]$, then

$$\mathrm{T}[u_1(t)+u_2(t)] = y_1(t)+y_2(t)$$

Homogeneity: If for input $u(t)$ the system responds with output $y(t)$,
then for input $Ku(t)$ the system will respond with output $Ky(t)$,
$i.e.$,

$$\mathrm{T}[Ku(t)] = Ky(t)$$

The systems that do not meet the above relationship are nonlinear systems.

♦ **Time-invariant systems**

The parameters of the system do not change with time, that is, regardless of the time sequence of the input signal, the shape of the output signal response is the same, only from the time of occurrence.

It is mathematically expressed as

$$\mathrm{T}[u(t)] = y(t)$$

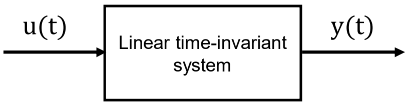
then

$$\mathrm{T}[u(t-t_0)] = y(t-t_0)$$

which shows that the sequence $u(t)$ shifts first and then transform is equivalent to it.

♦ **Linear time-invariant systems**

The system has both the linear and time-invariant characteristics, which can be expressed by a unit impulse response.

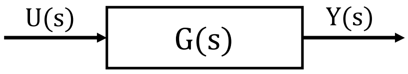


If the unit impulse response $g(\tau)$ and the system input $u(t)$ are known, the system output $y(t)$ can be calculated by

$$y(t) = \int_0^{\infty} g(\tau)u(t-\tau)\, d\tau$$

♦ **Convolution and Laplace transform**

Because it is hard to calculate the convolution,



taking Laplace transform gives

$$Y(s) = G(s)U(s)$$

■ Unit impulse response models

For a discrete identification system, it is assumed that the output of the system is observed every T moment.

Then, the system output will be observed at the following moment $t_k = kT$, for k= $1,2,\ldots,\infty$

$$y(kT) = \int_0^{\infty} g(\tau)u(kT-\tau)d\tau$$

where T is the sampling period.

In a computer control system, the system input between two sampling periods is generally constant, *i.e.,*

$$u(t) = u(kT), = u_k, \qquad kT \le t < (k+1)T$$

Substituting $u(kT)$ into $y(kT)$ yields

$$y(kT) = \int_0^{\infty} g(\tau)u(kT-\tau)d\tau = \sum_{l=1}^{\infty}\int_{\tau=(l-1)T}^{lT} g(\tau)u(kT-\tau)d\tau$$

$$= \sum_{l=1}^{\infty}\int_{\tau=(l-1)T}^{lT} g(\tau)d\tau\, u_{k-l} = \sum_{l=1}^{\infty} g_T(l)u_{k-l}$$

where

$$g_T(l) = \int_{\tau=(l-1)T}^{lT} g(\tau)d\tau$$

Omitting T in the above, the output can be expressed as

$$y(k) = \sum_{i=1}^{\infty} g(i)u(k-i),\ \mathrm{k} = 0, 1, 2, \dots.$$

It can be seen that the output $y(k)$ of the system in a sampling period T can be determined by the input $u(k-i)$ at the past time and the system model $g(\mathrm{i})$, i=1, 2, 3, ….

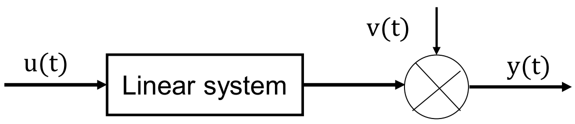
This is the unit impulse response model of the system.

■ Disturbances

♦ Effect of disturbances

If the influence of disturbance is considered, the output can be written as

$$y(k) = \sum_{i=1}^{\infty} g(i)u(k-i) + v(k)$$



Type of disturbances

— Measurement errors (noise, drift)

— Uncontrolled inputs (such as outdoor temperature change)

♦ **Quantification of disturbances**

A relatively simple method can be used to describe a disturbance

$$v(k) = \sum_{i=0}^{\infty} h(i)e(k-i)$$

$$h(0) = 1$$

where $e(t)$ is a white noise.

The white noise refers to the noise whose power spectral density is uniformly distributed in the whole frequency domain.

A random noise with the same energy at all frequencies is called a white noise.

7

Self Assessment Question

Which of the following statements is correct:

A. The unit impulse response of a linear system is the output response generated when the system input is a unit step signal.

B. The noise with a constant power spectral density of 2 is a white noise.

C. The linear system's unit impulse response function g(k) is related to the system input signal.

D. The discrete unit impulse response model is independent of the sampling period.

8

■ Transfer functions

Define q as the forward shift operator

$$qu(k) = u(k+1)$$
$$q^{i}u(k) = u(k+i)$$

Similarly, q^{-1} is the backward shift operator

$$q^{-1}u(k) = u(k-1)$$
$$q^{-i}u(k) = u(k-i)$$

Then

$$y(k) = \sum_{i=1}^{\infty} g(i)u(k-i) = \sum_{i=1}^{\infty} g(i)q^{-i}\, u(k)$$
$$= \sum_{i=1}^{\infty} g(i)q^{-i}\, u(k) = G(q^{-1})u(k)$$

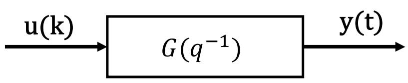
9

$G(q^{-1})$ is the transfer function of the system

$$G(q^{-1}) = \sum_{i=1}^{\infty} g(i)q^{-i}$$

Without the disturbance term,

$$y(k) = G(q^{-1})u(k)$$



If the system is stable and $g(k)$ approaches 0 with the increase of k, the above can be simplified as

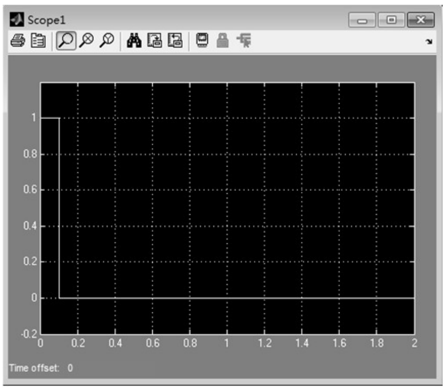
$$G(q^{-1}) = \sum_{i=1}^{n} g(i)q^{-i}$$

where g(n+1),g(n+2),... are close to 0 and can be ignored.

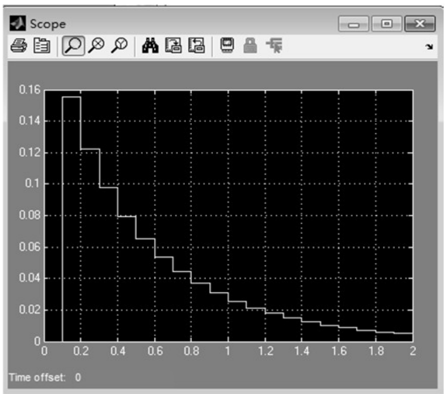
10

♦ How to obtain the value of $g(k)$ by designing experiments, where $k = 1, 2, \dots, n$.

Method 1: Impulse response method



Impulse input

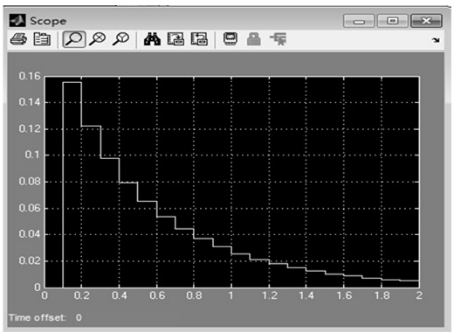


System output response

11

Observe the output of the system. There is an impulse input at time t=0. Because of this impulse input, the system has an output at time $k = 1, 2, \dots$.

If the amplitude of an impulse input is 1, then $g(k) = \mathrm{y(k)}, k = 1, 2, \dots$.



The limitations of the impulse response experiment:

— the impulse duration is only one sampling period, which makes the input energy received by the system not large enough

— the response amplitude of the system is limited

— inaccurate measurements may cause errors

Can the step response curve of the system be used to determine the value of $g(k)$?

12

Method 2: Step response method

To use the step response experiment, tanking an increment operation on $u(k)$ gives

$$\Delta u(k) = u(k) - u(k-1) = (1 - q^{-1})u(k)$$

So,
$$u(k) = \frac{\Delta u(k)}{(1 - q^{-1})}$$

The transfer function of a system can be written in an incremental form

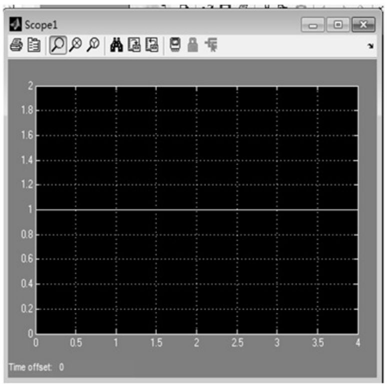
$$y(k) = \frac{G(q^{-1})}{1 - q^{-1}} \Delta u(k)$$

Let
$$\tilde{G}(q^{-1}) = \frac{G(q^{-1})}{1 - q^{-1}} \quad \Longrightarrow \quad \tilde{G}(q^{-1}) = \sum_{k=1}^{\infty} \tilde{g}(k) q^{-k}$$

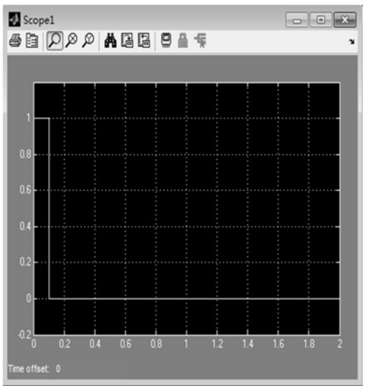
Then
$$y(k) = \tilde{G}(q^{-1}) \Delta u(k)$$

Apply a unit step input to the system

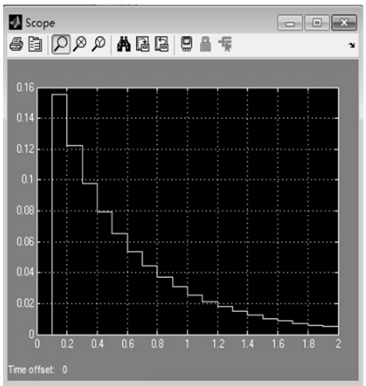
Output of the system is



u(t)



Δu(t)



y(t)

By observing the output of the system at $t=1, 2, 3, \ldots$, it can be derived that

$$\tilde{g}(k) = y(k)$$

Then, the transfer function $G(q^{-1})$ can be obtained below.

$$G(q^{-1}) = (1 - q^{-1})\tilde{G}(q^{-1})$$

Note:

For a system with disturbance, the disturbance term can be expressed as

$$v(k) = H(q^{-1})e(k)$$

where

$$H(q^{-1}) = \sum_{i=1}^{\infty} h(i)q^{-i}$$

The system can be written as

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k)$$

Self Assessment Question

If the input-output relationship of the system is described as

$$y(k) = 0.5\Delta y(k-1) + \Delta u(k-2) + 2\Delta u(k-1)$$

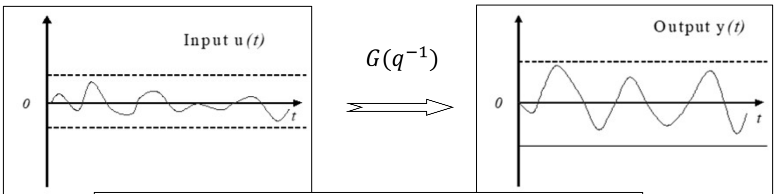
Determine the transfer function $G(q^{-1})$ of the system.

■ System stability

If the transfer function of the system meets the following conditions

$$G(q^{-1}) = \sum_{k=1}^{\infty} g(k)q^{-k}, \qquad \sum_{k=1}^{\infty} |g(k)| < \infty$$

the system is BIBO stable (also known as bounded input bounded output stability), that is, when the amplitude of input $u(t)$ is within a certain boundary $|u(t)| < C$, then the output $y(t)$ is also within a certain boundary.



$$|y(k)| < \infty \quad \text{if } |u(k)| < \infty, \quad \text{for } k = 0, 1, 2, \ldots.$$

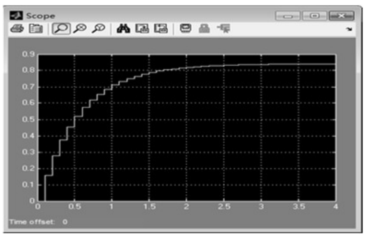
17

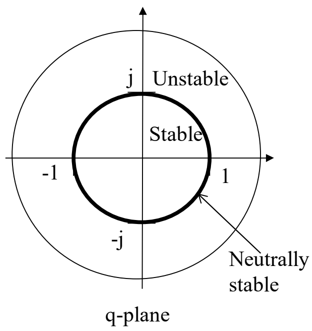
♦ System stability conditions

Stable: all poles of $G(q^{-1})$ are within the unit circle in the q plane.

Unstable: at least one pole of $G(q^{-1})$ is outside the unit circle.

Neutrally stable: $G(q^{-1})$ has at least one pole on the unit circle, and the other poles are within the unit circle.





18

Self Assessment Question

Determine the stability of the following system:

$$G(q^{-1}) = \frac{q^{-1} - 1.2}{q^{-2} + 0.1q^{-1} - 0.06}$$

A. Stable
B. Neutrally stable
C. Unstable
D. Unknown

1.3 Linear Time-invariant System Models

- **Common mathematical models for system identification**
  - **Parametric models**

1) Continuous input-output models

The typical form of continuous input-output models

$$\frac{d^{n_a}y(t)}{dt^{n_a}} + a_1\frac{d^{n_a-1}y(t)}{dt^{n_a-1}} + \cdots + a_{n_a-1}\frac{dy(t)}{dt} + a_{n_a}y(t)$$
$$= b_1\frac{d^{n_b-1}u(t)}{dt^{n_b-1}} + \cdots + b_{n_b-1}\frac{du(t)}{dt} + b_{n_b}u(t) \qquad n_a \geq n_b$$

The transfer function is a common expression of the system input-output relationship:

$$\xrightarrow{\text{Laplace Transform}} \quad G(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^{n_b-1} + \cdots + b_{n_b-1}s + b_{n_b}}{s^{n_a} + a_1 s^{n_a-1} + \cdots + a_{n_a-1}s + a_{n_a}} = \frac{B(s)}{A(s)}$$

2) Discrete input-output models

The input-output model of a discrete system can be in the form of a difference equation:

$$y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a) = b_1 u(k-1) + \cdots + b_{n_b} u(k-n_b)$$

q –Transform
(or z –Transform)

$$G(q^{-1}) = \frac{q\{y(k)\}}{q\{u(k)\}} = \frac{b_1 q^{-1} + b_2 q^{-2} + \cdots + b_{n_b} q^{-n_b}}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}}$$

Discrete impulse transfer function

The above can be written in a more generic form of

$$\boxed{A(q^{-1}) y(k) = B(q^{-1}) u(k)}$$

where $A(q^{-1}) = 1 + a_1 q^{-1} + a_2 q^{-2} \cdots + a_{n_a} q^{-n_a}$

$B(q^{-1}) = b_1 q^{-1} + b_2 q^{-2} + \cdots + b_{n_b} q^{-n_b}$

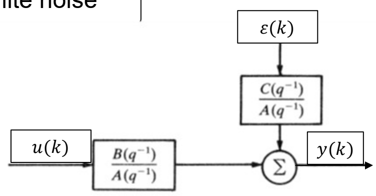
If the system is affected by a noise, then

$\varepsilon(k)$ white noise

$$\boxed{A(q^{-1}) y(k) = B(q^{-1}) u(k) + D(q^{-1}) \varepsilon(k)}$$

where

$$D(q^{-1}) = 1 + d_1 q^{-1} + d_2 q^{-2} + \cdots + d_{n_d} q^{-n_d}$$



According to the above different noise forms, the models can be divided into the following time series models:

(1) Auto-regressive moving average model with control (ARMAX or CARMA)

$$\boxed{A(q^{-1}) y(k) = B(q^{-1}) u(k) + D(q^{-1}) \varepsilon(k)}$$

(2) Auto-regressive model with control (ARX or CAR)

$$\boxed{A(q^{-1}) y(k) = B(q^{-1}) u(k) + \varepsilon(k)}$$

(3) Auto-regressive moving average (ARMA) model

$$\boxed{A(q^{-1})y(k) = D(q^{-1})\varepsilon(k)}$$

(4) Auto-regressive (AR) models

$$\boxed{A(q^{-1})y(k) = \varepsilon(k)}$$

(5) Moving average (MA) model

$$\boxed{y(k) = D(q^{-1})\varepsilon(k)}$$

(6) Box-Jenkins model (BJ model for short)

$$\boxed{y(k) = \frac{B(z^{-1})}{F(z^{-1})}u(k) + \frac{C(z^{-1})}{D(z^{-1})}\varepsilon(k)}$$

where

$$\begin{cases} B(z^{-1}) = b_1 + b_2 z^{-2} \cdots + b_{n_b} z^{-n_b} \\ F(z^{-1}) = 1 + f_1 z^{-1} + f_2 z^{-2} \cdots + f_{n_f} z^{-n_f} \\ C(z^{-1}) = 1 + c_1 z^{-1} + c_2 z^{-2} \cdots + c_{n_c} z^{-n_c} \\ D(z^{-1}) = 1 + d_1 z^{-1} + d_2 z^{-2} \cdots + d_{n_d} z^{-n_d} \end{cases}$$

n_b, n_f, n_c, n_d are the orders of the corresponding polynomials.

Self Assessment Question

What type of models is the following?

$$\boxed{y(k) = \varepsilon(k) + 0.5\varepsilon(k-1) + 0.2u(k) + 0.8u(k-1)}$$

A. ARMAX model

B. CMA model

C. ARMA model

D. ARX model

♦ **Non-parametric models**

The non-parametric model refers to the response obtained directly or indirectly from the experimental process of a system, such as the step response, impulse response, frequency response, etc. of the system.

Features:

— The non-parametric model cannot be expressed as a finite parameter model of the plant.

— It does not need to select the model structure.

— It does not need to take into account the model parameters.

— It is suitable for describing any complex system.

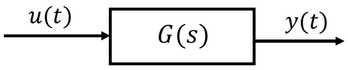
Types:

— Nonparametric models of continuous systems

— Nonparametric models of discrete systems

(1) Non-parametric models of continuous systems

Impulse response $g(t)$



Any input excitation signal can be decomposed into the sum of impulse signals.

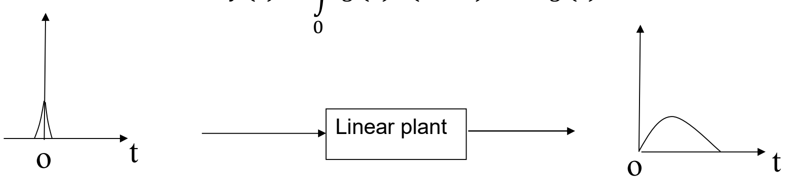
According to the principle of superposition, when all initial conditions are zero, the output impulse response of the linear time-invariant system can be calculated by

$$\boxed{y(t) = \int_0^\infty g(\tau)u(t-\tau)d\tau}$$

27

If $u(t) = \delta(t)$, *i.e.*, the input excitation signal is a unit impulse function, then

$$y(t) = \int_0^\infty g(\tau)\delta(t-\tau)d\tau = g(t)$$



Impulse response

28

Frequency response $G(j\omega)$

If the system input is an ideal unit impulse function $u(t) = \delta(t)$, then the transfer function is $G(s)$.

The unit impulse response $g(t)$ and system $G(s)$ have the following relationship

$$G(s) = L[g(t)] \qquad g(t) = L^{-1}[G(s)]$$

If s is substituted by $j\omega$, the frequency response $G(j\omega)$ is

$$G(j\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt$$

The frequency response is expressed as the magnitude-frequency characteristics and phase-frequency characteristics of the Bode plot in the rectangular coordinate system and as the Nyquist plot in the polar coordinate system.

These frequency response curves combined with Fourier transform can constitute a frequency-domain identification method.

29

(2) Non-parametric models of discrete systems

The expression form of a discrete system using a non-parametric model is called the weight form, which is defined as the system response after the system is excited by a unit impulse (delta) function when the initial condition is zero at time zero.

The weight sequence is denoted as $\{g(k)\},\ \ k = 0,1,2,\cdots$

The convolution formula representing the input-output relationship of a discrete system is

$$y(k) = \sum_{i=0}^{k} g(k-i)u(i)$$

The q-transform of the weight sequence $\{g(k)\}$ is the impulse transfer function

$$G(q^{-1}) = \sum_{i=0}^{\infty} g(i)q^{-i}$$

30

Self Assessment Question

A non-parametric model

A. cannot be expressed as a finite parameter model of the plant;

B. may not be expressed in a mathematical equation;

C. does not need to consider model parameters or structure;

D. is difficult to describe a complex system.

Which of the above is correct?

31

■ Linear time-invariant system model group

♦ Model group

Consider a linear time-invariant system

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k) \qquad (1.3.1)$$

where

$$G(q^{-1}) = \sum_{i=1}^{\infty} g(i)q^{-i} \qquad H(q^{-1}) = 1 + \sum_{i=1}^{\infty} h(i)q^{-i}$$

How to estimate the parameters in $G(q^{-1})$ and $H(q^{-1})$ is very important, which is a key part of system identification.

The determination of those parameters may not be completely analyzed from the physical principle, which requires the use of some mathematical parameter estimation methods.

The finite parameters to be determined can be expressed as a vector θ.

32

Another mathematical expression

$$y(k,\theta) = G(q^{-1},\theta)u(k) + H(q^{-1},\theta)e(k) \qquad (1.3.2)$$

Vector θ contains the parameters to be determined. Its value range is a subset of R^d, where d is the dimension of θ, *i.e.,*

$$\theta \in D_m \subset R^d$$

Note that (1.3.2) is not a simple mathematical model, but a group of models.

♦ **One-step prediction**

For the one-step prediction of $y(k)$, following (1.3.2) leads to

$$\hat{y}(k|\theta) = H^{-1}(q^{-1},\theta)G(q^{-1})u(k) + [1 - H^{-1}(q^{-1},\theta)]y(k) \qquad (1.3.3)$$

Unlike (1.3.2), which is a probability model group, (1.3.3) is a prediction model group.

(1.3.3) contains information about the model structure, and several mathematical forms will be used to describe the model structure of (1.3.3).

♦ **ARX model**

ARX model is probably the simplest mathematical expression to describe the relationship between the system input and output

$$\begin{aligned} y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a) \\ = b_1 u(k-1) + \cdots + b_{n_b} u(k-n_b) + e(k) \end{aligned} \qquad (1.3.4)$$

The expression (1.3.4) is called the ARX model, where AR (AutoRegression) refers to the autoregressive part $A(q^{-1})y(k)$ and X refers to the additional input part $B(q^{-1})u(t)$.

In the ARX model, the parameters to be determined are

$$\theta = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n_a} & b_1 & b_2 & \cdots & b_{n_b} \end{bmatrix}^T$$

Let

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}$$

$$B(q^{-1}) = b_1 q^{-1} + b_2 q^{-1} + \cdots + b_{n_b} q^{-n_b}$$

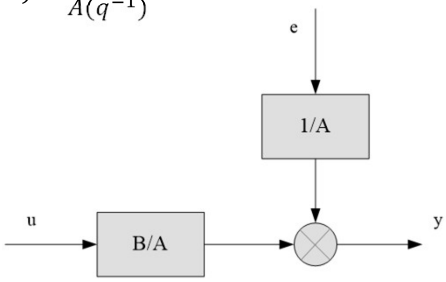
Then (1.3.4) can be rewritten as

$$A(q^{-1})y(k) = B(q^{-1})u(k) + e(k) \qquad (1.3.5)$$

Comparing (1.3.5) and (1.3.1), there can be the following correspondence:

$$G(q^{-1},\theta) = \frac{B(q^{-1})}{A(q^{-1})} \qquad H(q^{-1},\theta) = \frac{1}{A(q^{-1})}$$

The block diagram of ARX model:



Eq (1.3.5) can be expressed as

$$y(k) = B(q^{-1})u(k) + [1 - A(q^{-1})]y(k) + e(k)$$

The one-step prediction of output $y(k)$ can be written as

$$\hat{y}(k|\theta) = B(q^{-1})u(k) + [1 - A(q^{-1})]y(k) \qquad (1.3.6)$$

Because the random term $e(k)$ is not important or difficult to predict, the deterministic model (1.3.6) is used for further derivation.

Let

$$\phi(k) = [-y(k-1) \quad \cdots \quad -y(k-n_a) \quad u(k-1) \quad \cdots \quad u(k-n_b)]^T$$

$$\theta = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n_a} & b_1 & b_2 & \cdots & b_{n_b} \end{bmatrix}^T$$

Thus (1.3.6) can be expressed in a more concise way

$$\hat{y}(k|\theta) = \theta^T \phi(k) = \phi^T(k)\theta$$

where vector $\phi(t)$ is the historical input and output data of the system, which is known; θ is the parameter to be estimated and is unknown.

How to use mathematical tools to estimate parameter vectors θ is an important part of system identification.

37

Self Assessment Question

For the following system:

$$6y(k) + 4y(k+1) + 2y(\mathrm{k}+2) + 3u(\mathrm{k}-1) + 5u(k) + 7u(\mathrm{k}+1) - 2e(\mathrm{k}+2) = 0$$

determine the expressions of polynomials $\mathrm{A}(q^{-1})$ and $\mathrm{B}(q^{-1})$ in the ARX model of the system.

38

♦ **ARMAX model**

One disadvantage of the model described in Equation (1.3.4) is that it does not adequately describe the disturbance of the system.

To solve this problem, a more complex ARMAX model is introduced

$$y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a) = b_1 u(k-1) + \cdots + b_{n_b} u(k-n_b) + e(k) + c_1 e(k-1) + \cdots + c_{n_c} e(k-n_c) \quad (1.3.7)$$

Let $C(q^{-1}) = 1 + c_1 q^{-1} + \cdots + c_{n_c} q^{-n_c}$ and $C(q^{-1})e(k)$ is called moving average.

Then (1.3.7) can be rewritten as $A(q^{-1})y(k) = B(q^{-1})u(k) + C(q^{-1})e(k)$

So,
$$y(k) = \frac{B(q^{-1})}{A(q^{-1})} u(k) + \frac{C(q^{-1})}{A(q^{-1})} e(k) \quad (1.3.8)$$

39

Comparing (1.3.8) and (1.3.2) gives

$$G(q^{-1},\theta) = \frac{B(q^{-1})}{A(q^{-1})} \qquad H(q^{-1},\theta) = \frac{C(q^{-1})}{A(q^{-1})} \quad (1.3.9)$$

So, the parameter vector θ is

$$\theta = [a_1 \ \cdots \ a_{n_a} \ \ b_1 \ \cdots \ b_{n_b} \ \ c_1 \ \cdots \ c_{n_c}]^T$$

Bring (1.3.9) into (1.3.3) to have the one-step prediction of $y(k)$

$$\hat{y}(k|\theta) = \frac{B(q^{-1})}{C(q^{-1})} u(k) + \left(1 - \frac{A(q^{-1})}{C(q^{-1})}\right) y(k)$$

$$C(q^{-1})\hat{y}(k|\theta) = B(q^{-1})u(k) + (C(q^{-1}) - A(q^{-1}))y(k) \quad (1.3.10)$$

It can be seen from (1.3.10) that it is different from ARX model. If you want to calculate the one-step prediction of ARMAX model, you should not only know the historical value of system input $\mathrm{u}(k)$ and system output $y(k)$, but also know the historical value of the predicted $\hat{y}(k|\theta)$.

40

Reorganize (1.3.10) and add $(1 - C(q^{-1}))\hat{y}(k|\theta)$ on both sides at the same time.

$$\hat{y}(k|\theta) = B(q^{-1})u(k) + [1 - A(q^{-1})]y(k) + [C(q^{-1}) - 1][y(k) - \hat{y}(k|\theta)]$$

It can be seen from the above that historical prediction errors are taken into account

$$\varepsilon(k,\theta) = y(k) - \hat{y}(k|\theta)$$

Define a vector

$$\phi(k,\theta) = [-y(k-1) \quad \cdots \quad -y(k-n_a) \quad u(k-1) \quad \cdots \quad u(k-n_b) \quad \varepsilon(k-1,\theta) \quad \cdots \quad \varepsilon(k-n_c,\theta)]^T$$

The one-step prediction of output $y(k)$ can be expressed in a concise way

$$\hat{y}(k|\theta) = \phi^T(k,\theta)\theta$$

41

Self Assessment Question

For an ARMAX model, let

$$\phi(k,\theta) = [-y(k-1) \quad \cdots \quad -y(k-n_a) \quad u(k-1) \quad \cdots \quad u(k-n_b) \quad \varepsilon(k-1,\theta) \quad \cdots \quad \varepsilon(k-n_c,\theta)]$$

$$\theta = [a_1 \quad \cdots \quad a_{n_a} \quad b_1 \quad \cdots \quad b_{n_b} \quad c_1 \quad \cdots \quad c_{n_c}]^T$$

which of the following one-step predictions of $y(k)$ is correct:

A. $\hat{y}(k|\theta) = \theta\phi(k,\theta)$

B. $\hat{y}(k|\theta) = \phi(k,\theta)\theta$

C. $\hat{y}(k|\theta) = \theta^T\phi^T(k,\theta)$

D. $\hat{y}(k|\theta) = \phi^T(k,\theta)\theta$

42

1.4 System Output Prediction

■ Applications of models

Consider a linear time-invariant system:

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k) \tag{1.4.1}$$

Applications of the above model are prediction and simulation.

♦ **Prediction**

Without considering the disturbance, based on the data before time $k-1$, the system output at time k can be predicted as

$$\hat{y}(k|k-1) = G(q^{-1})u(k)$$

♦ **Simulation**

When the system mathematical model is known, if the input sequence $u^*(k)$ $(k = 1, 2, \ldots, N)$ of the system is given, the corresponding system output can be calculated without considering the disturbance

$$y^*(k) = G(q^{-1})u^*(k), \quad k = 1, 2, \ldots, N$$

The disturbance is unknowable, but the computer's pseudo-random number $e^*(k)$ can be used to generate the disturbance signal, $i.e.$,

$$v^*(k) = H(q^{-1})e^*(k), \quad k = 1, 2, \ldots, N$$

Such simulation is only experimental and cannot fully reflect all dynamic characteristics of a real system.

If the system is properly modelled, the key characteristics of a real system can still be reflected in the simulation, which is also the reason why simulation plays a very important role in engineering.

Simulation technology has important applications in many industrial fields, such as flight simulator, spacecraft simulator, thermal power plant simulator, etc.

Those simulators may use more complex expressions than (1.4.1) to describe the system, but fundamentally, their most basic principles are the same.

■ Estimation of disturbances

The real system output should be affected by a disturbance, such as (1.4.1).

If the disturbance is not considered, there must be a prediction error:

$$\begin{aligned} y(k) - \hat{y}(k|k-1) &= G(q^{-1})u(k) + H(q^{-1})e(k) - G(q^{-1})u(k) \\ &= H(q^{-1})e(k) \end{aligned}$$

To reduce the influence of a disturbance, the disturbance term $v(k)$ must also be considered for prediction.

When predicting $y(k)$, the disturbance $v(k)$ at time k is not measurable because it has not yet occurred, but the disturbance term $v(k-i)$ at time $k-i$ is known, $i = 1,\ 2,\ 3,\ \ldots$.

$$v(k-i) = y(k-i) - G(q^{-1})u(k-i), \qquad i = 1,\ 2,\ 3,\ \ldots$$

Although the disturbance $v(k)$ at time k is not available, it can be estimated by the historical value $v(k-i)$

Reversibility of the disturbance model: the mathematical model of a given disturbance

$$v(k) = H(q^{-1})e(k)$$

Then $e(k)$ can be back-calculated by $v(k)$, which conforms to the following calculation:

$$e(k) = H^{-1}(q^{-1})v(k)$$

where

$$H^{-1}(q^{-1}) = \frac{1}{H(q^{-1})}$$

47

Because $H(q^{-1})$ is a polynomial with $h(0) = 1$

$$v(k) = \sum_{i=0}^{\infty} h(i)e(k-i) = e(k) + \sum_{i=1}^{\infty} h(i)e(k-i)$$

$e(k)$ is unknown because the disturbance at time k has not yet occurred.

Let

$$m(k-1) = \sum_{i=1}^{\infty} h(i)e(k-i)$$

Based on the data at time $k-1$, get the estimated value $\hat{v}(k)$ closest to $v(k)$, that is, to meet the following condition:

$$\min E\,[(v(k) - \hat{v}(k))^2]$$

After calculation, it can be obtained that $\hat{v}(k) = m(k-1)$.

48

Therefore, it is considered that $m(k-1)$ is the estimated value of $v(k)$ based on the data at time $k-1$, which is represented by

$$\hat{v}(k|k-1) = m(k-1) = \sum_{i=1}^{\infty} h(i)e(k-i)$$

$e(k-i)$ is not measurable, but $v(k-i)$ is known.

Using $v(k) = H(q^{-1})e(k)$ gives

$$\begin{aligned}\hat{v}(k|k-1) &= \sum_{i=1}^{\infty} h(i)q^{-i}e(k) = [H(q^{-1}) - 1]e(k) \\ &= \frac{H(q^{-1}) - 1}{H(q^{-1})}v(k) = [1 - H^{-1}(q^{-1})]v(k)\end{aligned}$$

In this way, based on knowing the historical values of $v(k)$ at time $k-1$, the estimated value $\hat{v}(k|k-1)$ of $\mathrm{v}(k)$ at time k can be calculated.

Self Assessment Question

Prove the estimated value of noise $\hat{v}(k) = \sum_{i=1}^{\infty} h(i)e(k-i)$

satisfies the following condition: $\min E\,[(v(k) - \hat{v}(k))^2]$

where

$$v(k) = \sum_{i=0}^{\infty} h(i)e(k-i)$$

and $e(t)$ is a white noise.

■ Prediction of outputs

The estimation of $y(k)$ at time k can be calculated using the data before time $k-1$.

$$\begin{aligned}\hat{y}(k|k-1) &= G(q^{-1})u(k) + \hat{v}(k|k-1)\\ &= G(q^{-1})u(k) + [1 - H^{-1}(q^{-1})]v(k)\\ &= G(q^{-1})u(k) + [1 - H^{-1}(q^{-1})][(y(k) - G(q^{-1})u(k)]\end{aligned}$$

The above can also be expressed in the following form:

$$\hat{y}(k|k-1) = H^{-1}(q^{-1})G(q^{-1})u(k) + [1 - H^{-1}(q^{-1})]y(k)$$

$$H(q^{-1})\hat{y}(k|k-1) = G(q^{-1})u(k) + [H(q^{-1}) - 1]y(k)$$

51

Define $\{l(i)\}$ as $$\frac{G(q^{-1})}{H(q^{-1})} = \sum_{i=1}^{\infty} l(i)q^{-i}$$

Similarly, define $\tilde{h}(k)$ as $$H^{-1}(q^{-1}) = \sum_{i=0}^{\infty} \tilde{h}(i)\, q^{-i}$$

Then $$1 - H^{-1}(q^{-1}) = -\sum_{i=1}^{\infty} \tilde{h}(i)\, q^{-i}$$

The output prediction can be rewritten as

$$\hat{y}(k|k-1) = \sum_{i=1}^{\infty} l(i)u(k-i) - \sum_{i=1}^{\infty} \tilde{h}(i)y(k-i)$$

It can be seen from the above that the one-step prediction of $y(k)$ is not a probability model, but a deterministic value.

52

♦ **Initial value problem**

The above calculations assume that the historical data before the time $k-1$ is known, but in a practical system, only the data within the range of [0, $k-1$] are real.

The output prediction can be rewritten as an approximate form:

$$\hat{y}(k|k-1) \approx \sum_{i=1}^{k} l(i)u(k-i) - \sum_{i=1}^{k} \tilde{h}(i)y(k-i)$$

♦ **Prediction error**

The prediction error can be calculated as

$$\begin{aligned} y(k) - \hat{y}(k|k-1) &= y(k) - H^{-1}(q^{-1})G(q^{-1})u(k) - [1 - H^{-1}(q^{-1})]y(k) \\ &= -H^{-1}(q^{-1})G(q^{-1})u(k) + H^{-1}(q^{-1})y(k) \\ &= H^{-1}(q^{-1})v(k) \\ &= e(k) \end{aligned}$$

Because $e(k)$ is a random quantity at time k, it is impossible to estimate it from the historical data at time $k-1$.

53

Self Assessment Question

For the following system

$$y(k) = G(q^{-1})u(k) + H(q^{-1})e(k)$$

Under what conditions are the following two output predictions equivalent?

1) $\hat{y}_1(k|k-1) = G(q^{-1})u(k)$

2) $\hat{y}_2(k|k-1) = G(q^{-1})u(k) + \hat{v}(k|k-1)$

where

$$\hat{v}(k|k-1) = \sum_{i=1}^{\infty} h(i)e(k-i)$$

54

1.5 Description and Analysis of Random Signals

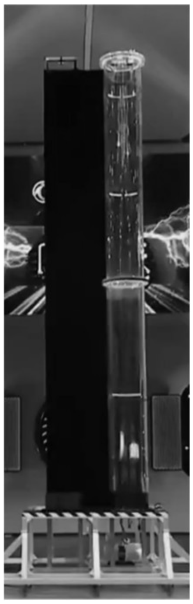
■ Physical concept of stochastic processes

♦ Deterministic processes

Examples of deterministic process:

- free falling motion in vacuum.
- calculating what your savings account balance will be in a month (add up your deposits and the prevailing interest rate).
- the relationship between a circumference and radius of a circle, or the area and radius of a circle.

Features: knowing the past and current status can accurately predict the future states. It will always produce the same output from a given starting condition or initial state.



55

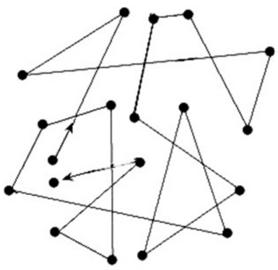
♦ Stochastic processes

Examples of stochastic processes:

- flip a coin to judge the front and back
- the undulation of the sea in wind and waves
- the output value of a force sensor at zero input
- zero drift of an amplifier
- Brownian motion

Toss a coin

Ups and downs of the waves



Schematic diagram of Brownian motion

Features: It is impossible to predict the future state exactly.

56

■ Definition of a stochastic process

Description 1:

- Let e be a random experiment and $S = \{e\}$ be its sample space.
- For each e, a real-valued function $X(e, t)$ with a parameter of t can always be determined according to certain rules, corresponding to it.
- When e is taken through S, a family of ordinary functions of t defined on set T is obtained.
- The functions of parameter t in this family are called stochastic processes, and each function in this family is called the sample function of this stochastic process.
- T is the variation range of parameter t, called the parameter set, *i.e.*, $t \in T$.

Example: Using the experiment of tossing a coin, define

$$X(e,t) = \begin{cases} \cos(\pi t), & e = H, \\ t, & e = T, \end{cases} \quad t \in (-\infty, +\infty)$$

where

$$P(H) = P(T) = 0.5$$

The sample function family is $\{\cos(\pi t), t\}$

57

Description 2:

Let e be a random experiment and $S = \{e\}$ be its sample space.
A stochastic process is defined as a function family of random variables that depend on the parameter t on T, and is recorded as:

$$\{X(e,t), t \in T\}$$

Note:

(1) The value of each instant is random. It is a random variable. The concept of the one-dimensional random variable of a stochastic process: $P(v, t)$

(2) From the perspective of a time coordinate, the concept of the two-dimensional random variable of a stochastic process is introduced: $P(v_1, v_2; t_1, t_2)$

(3) To describe the stochastic process more accurately, we can also introduce the concept of the high-dimensional random variable of a stochastic process.

58

Example: There is a random error when measuring the distance of the moving target, so that the measurement error at time t is a random variable at time t. When the target moves according to a certain rule with time, the measurement error also changes with time t, forming a stochastic process.

Classification of stochastic processes

— According to whether the state at any time is continuous random variable or discrete random variable, it can be divided into continuous stochastic process and discrete stochastic process (sequence).

— According to whether the parameter is continuous or discrete, it can be divided into continuous parameter stochastic process and discrete parameter stochastic process (sequence). $X(e,t), t \in T$ $T = \{0,1,2,\cdots\}$ $T = \{0,\pm 1,\pm 2,\cdots\}$

1) When the time set T is a finite or infinite interval, it is called a continuous parameter stochastic process;

2) If T is a discrete set, it is called a stochastic sequence.

Mathematical description of stochastic processes

Statistical description of stochastic processes

One-dimensional distribution function: given a stochastic process, for each fixed t, the distribution function $\{F(x,t), t \in T\}$ of random variable $\{x(t), t \in T\}$ is generally related to t, which is recorded as

$$F(x,t) \equiv P\{X(t) \le x\}, x \in R$$

$F(x,t)$ is called a family of one-dimensional distribution functions of a stochastic process.

n-dimensional distribution function: for any moment, introduce n-dimensional random variables $\{x_1(t_1), x_2(t_2), \cdots, x_n(t_n)\}$, $x_i(t) \in R, i = 1,2,\cdots,n$, and the distribution function $\{F(x_1,x_2,\cdots,x_n; t_1,t_2,\cdots,t_n), t_1,t_2,\cdots t_n \in T\}$ is calculated as

$$F(x_1,x_2,\cdots,x_n; t_1,t_2,\cdots,t_n) \equiv P\{x(t_1) \le x_1, x(t_2) \le x_2, \cdots, x(t_n) \le x_n\}, \quad \text{for } t_1,t_2,\cdots t_n \in T$$

which is called a family of n-dimensional distribution functions of a stochastic process.

Self Assessment Question

Which of the statements below are true?

A. The output of a deterministic process can accurately be predicted from a given starting condition.

B. A stochastic process is defined as a function family of random variables that depend on the parameter t.

C. If the process state at any time is continuous random variable, it is a continuous stochastic sequence.

D. The distribution function $F(x_1, x_3; t_1, t_3)$ is called a family of 3-dimensional distribution functions of stochastic processes.

61

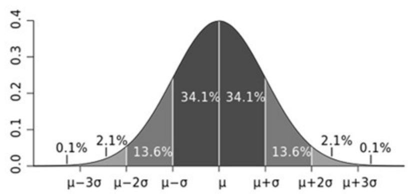
■ Numerical characteristics of stochastic processes

♦ **Mean function**

The mean of a probability distribution is the long-run arithmetic average value of a random variable having that distribution.

If the random variable is denoted by x, then it is also known as the expected value of x (denoted $E[X(t)]$).

$$\mu_X(t) \equiv E[x(t)] = \int_{-\infty}^{+\infty} x(t)p(x(t), t)dx(t)$$



For a discrete-time case, the mean is given by

$$\mu_X(k) \equiv E[x(t)]\Big|_{t=k}$$

It is the average of the function values at time t, called "set (combined) average" or "statistical average“

Note: set average ≠ time average

62

♦ **Mean square function**

The mean square is normally defined as the arithmetic mean of the squares of a random variable.

For the continuous time,

$$\psi_x^2(t) \equiv E[x^2(t)] = \int_{-\infty}^{+\infty} x^2(t) p(x(t), t) dx(t)$$

For the discrete time case,

$$\psi_x^2(k) \equiv E[x^2(t)] \Big|_{t = k}$$

♦ **Variance function**

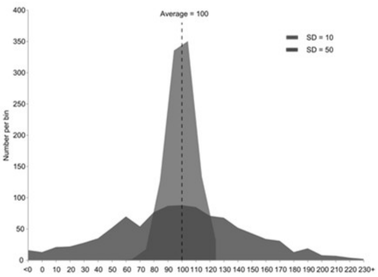
The variance is the squared deviation from the mean of a random variable. The variance is also often defined as the square of the standard deviation.

For the continuous time,

$$\sigma_x^2(t) \equiv E\{[x(t) - \mu_x(t)]^2\} = \int_{-\infty}^{+\infty} [x(t) - \mu_X(t)]^2 p(x(t), t) dx(t)$$

For the discrete time,

$$\sigma_x^2(k) \equiv E\{[x(t) - \mu_x(t)]^2\} \Big|_{t = k}$$



♦ **Autocorrelation**

Autocorrelation, sometimes known as the serial correlation in the discrete time case, is the correlation of a signal with a delayed copy of itself as a function of delay.

For the continuous time,

$$R_x(t_1, t_2) \equiv E[x(t_1)x(t_2)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t_1)x(t_2)p(x(t_1), x(t_2), t_1, t_2)dx(t_1)dx(t_2)$$

For the discrete time,

$$R_x(k_1, k_2) \equiv E[x(t_1)x(t_2)]\Big|_{t_1 = k_1, t_2 = k_2}$$

65

♦ **Covariance function**

The covariance function describes how much two random variables change together (their covariance) with varying spatial or temporal separation.

For a stochastic process $x(t)$ on a domain D, a covariance function C(t_1,t_2) gives the covariance of the values of the stochastic process $x(t)$ at the two locations t_1 and t_2 :

For the continuous time,

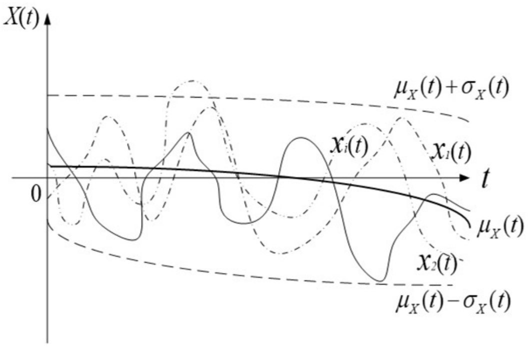
$$\begin{aligned} C_x(t_1, t_2) &\equiv E\big[\big(x(t_1) - \mu_x(t_1)\big)\big(x(t_2) - \mu_x(t_2)\big)\big] \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x(t_1) - \mu_X(t_1))(x(t_2) - \mu_X(t_1))p(x(t_1), x(t_2), t_1, t_2)dx(t_1)dx(t_2) \end{aligned}$$

For the discrete time,

$$C_x(k_1, k_2) \equiv E\big[\big(x(t_1) - \mu_x(t_1)\big)\big(x(t_2) - \mu_x(t_2)\big)\big]\Big|_{t_1 = k_1, t_2 = k_2}$$

66

♦ **Schematic diagram of digital characteristics of a stochastic process**



♦ **Relationship between digital features**

$$\Psi_x^2(k) = R_x(k,k)$$
$$\sigma_x^2(k) = R_x(k,k) - \mu_x^2(k)$$
$$C_x(k_1,k_2) = R_x(k_1,k_2) - \mu_x(k_1)\mu_x(k_2)$$

Note: The most important quantities are "mean function" and "autocorrelation function".

67

♦ **Stationary stochastic processes**

A stationary stochastic process $x(t)$ is defined as: the distribution of the random variables of the stochastic process is the same for any value of the variable parameter t.

The mean of a discrete stationary stochastic process

$$\mu_x = E[x(k)] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} x(k)$$

The variance of a discrete stationary stochastic process

$$\sigma_x^2 = E[(x(k)-\mu_x)^2] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} (x(k)-\mu_x)^2$$

68

♦ **Spectral density**

For a stationary stochastic process $x(t)$, the spectral density is defined as

$$S_x(\omega) = \frac{1}{2}\int_{-\infty}^{+\infty} R_x(\tau)e^{j\tau\omega}d\tau, \quad -\infty < \omega < \infty$$

where

$$R_x(\tau) \equiv E[x(t)x(t+\tau)]$$

For a stationary stochastic sequence $x(k)$, the spectral density is defined as

$$S_x(\omega) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} R_x(n)e^{jn\omega} \qquad -\pi < \omega < \pi$$

where

$$R_x(n) \equiv E[x(k)x(k+n)]$$

Self Assessment Question

Consider a stochastic process

$$x(t) = y\cos(\omega t) + z\sin(\omega t)$$

where y and z are independent random variables, ω is a constant, $\mu_y = \mu_z = 0$ and $\sigma_y^2 = \sigma_z^2 = \sigma^2$.

Calculate $\mu_x(t)$ and $R_x(t_1, t_2)$.

Exercise 1.2

Consider the "state-space description"

$$x(t + 1) = fx(t) + w(t)$$
$$y(t) = hx(t) + v(t)$$

where $x, f, h, w \; and \; v$ are scalars. $\{w(t)\}$ and $\{v(t)\}$ are mutually independent white Gaussian noises with variances R_1 and R_2, respectively. Show that $y(t)$ can be represented as the ARMA form:

$$y(\mathrm{t}) + a_1 y(t - 1) \ldots + a_n y(t - n) = e(t) + c_1 e(t - 1) + \cdots + c_n e(t - n)$$

Determine n, a_i, c_i, and the variance of $e(t)$ in terms of f, h, R_1 and R_2. What is the relationship between $e(t), w(t)$ and $v(t)$?