6.3 Generalized Minimum Variance Self-tuning Control

■ Introduction

To overcome some inherent defects of minimum variance control, especially when it is not suitable for non-minimum phase systems and the control input is not constrained.

- ✓ D. W. Clarke and P. J. Gawthrop proposed the generalized minimum variance control (GMVC) algorithm in 1975.

Basic idea: in solving the cost function of the control law, the weighted term of the control input is introduced so as to limit the excessive change of the control effect.

- ✓ The generalized minimum variance control can be applied to non-minimum phase systems as long as the weighted polynomials in the cost function are properly selected.

- ✓ The algorithm still uses the d-step prediction model described in Theorem 6.2.1, and retains the advantages of the minimum variance control algorithm that is simple and easy to understand.

■ Generalized minimum variance control law

The controlled plant: $A(q^{-1})y(k) = q^{-d}B(q^{-1})u(k) + C(q^{-1})\xi(k)$ (6.3.1)

The cost function:

$$J(k) = \mathcal{E}\left\{(P(q^{-1})y(k+d) - R(q^{-1})y_r\,(k+d))^2 + \left(Q(q^{-1})u(k)\right)^2\right\} \quad (6.3.2)$$

where $y(k+d)$ and $y_r\,(k+d)$ are the actual output and expected output of the system at time $\mathrm{k{+}d}$, $u(k)$ is the control input at time k, $P(q^{-1})$, $R(q^{-1})$ and $Q(q^{-1})$ are the weighted polynomials of the actual output, expected output and control input, respectively, which have the effect of improving the performance of the closed-loop system, softening the expected output and constraining the control input, and

$$P(q^{-1}) = 1 + p_1 q^{-1} + \ \dots + p_{n_p} q^{-n_p}$$
$$R(q^{-1}) = r_0 + r_1 q^{-1} + \ \dots + r_{n_r} q^{-n_r}$$
$$Q(q^{-1}) = q_0 + q_1 q^{-1} + \ \dots + q_{n_q} q^{-n_q}$$

The order and parameters of the above polynomials are determined according to actual needs.

The objective is to design $u(k)$ to minimize the cost function $J(k)$.

Theorem 6.3.1 (Generalized minimum variance control): For plant (6.3.1), the generalized minimum variance control law that minimizes the cost function (6.3.2) is

$$u(k)=\frac{R(q^{-1})y_r(k+d)-P(q^{-1})y^*(k+d|k)}{\frac{q_0}{b_0}Q(q^{-1})} \tag{6.3.3}$$

or

$$u(k)=\frac{C(q^{-1})R(q^{-1})y_r(k+d)-G(q^{-1})P(q^{-1})y(k)}{\frac{q_0}{b_0}C(q^{-1})Q(q^{-1})+F(q^{-1})P(q^{-1})} \tag{6.3.4}$$

Proof: Substituting (6.2.6) into (6.3.2) yields

$$J(k)=\mathcal{E}\left\{(P(q^{-1})E(q^{-1})\xi(k+d)+P(q^{-1})y^*(k+d|k)-R(q^{-1})y_r\,(k+d))^2+\left(Q(q^{-1})u(k)\right)^2\right\}$$

$\xi(k)$ is the system random interference. The linear combination of $\xi(k+j)$ $(j\geq 1)$ is independent of the current and past input/output observations $\{Y_k,U_k\}$ and the expected output sequence $\{y_r(k)\}$ of the system. So, the above equation can be written as

$$J(k)=\mathcal{E}\left\{\left(P(q^{-1})E(q^{-1})\xi(k+d)\right)^2\right\}+\mathcal{E}\{(P(q^{-1})y^*(k+d|k)-R(q^{-1})y_r\,(k+d))^2\}+\mathcal{E}\left\{\left(Q(q^{-1})u(k)\right)^2\right\}$$

Take the partial derivative of $J(k)$ with respect to $u(k)$ and make it be zero, namely

$$\frac{\partial J(k)}{\partial u(k)}=\mathcal{E}\left\{2(P(q^{-1})y^*(k+d|k)-R(q^{-1})y_r\,(k+d))\frac{\partial P(q^{-1})y^*(k+d|k)}{\partial u(k)}\right\}$$
$$+\mathcal{E}\left\{2Q(q^{-1})u(k)\frac{\partial Q(q^{-1})u(k)}{\partial u(k)}\right\}=0$$

From the expressions of equation (6.2.2) and $Q(q^{-1})$

$$\frac{\partial P(q^{-1})y^*(k+d|k)}{\partial u(k)} = b_0 \qquad \frac{\partial Q(q^{-1})u(k)}{\partial u(k)} = q_0$$

Then, the necessary condition of minimizing the performance index function $J(k)$ is

$$(P(q^{-1})y^*(k+d|k) - R(q^{-1})y_r\,(k+d))b_0 + Q(q^{-1})u(k)q_0 = 0$$

From this, the generalized minimum variance control law is

$$u(k) = \frac{R(q^{-1})y_r(k+d) - P(q^{-1})y^*(k+d|k)}{\frac{q_0}{b_0}Q(q^{-1})}$$

5

Substitute equation (6.2.2)

$$C(q^{-1})y^*(k+d|k) = G(q^{-1})y(k) + F(q^{-1})u(k)$$

into (6.3.3), and another expression of the generalized minimum variance control law is

$$u(k) = \frac{C(q^{-1})R(q^{-1})y_r(k+d) - G(q^{-1})P(q^{-1})y(k)}{\frac{q_0}{b_0}C(q^{-1})Q(q^{-1}) + F(q^{-1})P(q^{-1})}$$

So, (6.3.4) is established.

The sufficient condition for minimizing the cost function is that the following inequality is true:

$$\frac{\partial}{\partial u(k)}\frac{\partial J}{\partial u(k)} = 2b_0^2 + 2q_0^2 > 0$$

Therefore, the control law expressed by equations (6.3.3) and (6.3.4) can ensure that the cost function (6.3.2) of the system is minimal. Therefore, Theorem 6.3.1 is proved.

- ✓ Eq (6.3.3) uses the optimal output prediction $y^*(k+d|k)$ as feedback to form a control function, which is called implicit control law.
- ✓ Eq (6.3.4) directly uses the output of the controlled plant as feedback to form a control function, which is called explicit control law.

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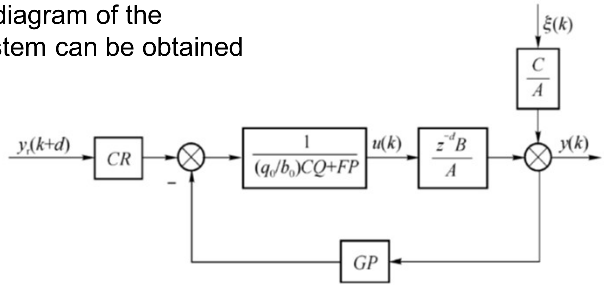
♦ **Closed-loop system analysis**

For the controlled plant (6.3.1), the block diagram of the generalized minimum variance control system can be obtained from equation (6.3.4).



From the diagram, it is easy to obtain the closed-loop system output as

$$y(k)=\frac{\dfrac{C(q^{-1})R(q^{-1})}{\frac{q_0}{b_0}C(q^{-1})Q(q^{-1})+F(q^{-1})P(q^{-1})}\dfrac{q^{-d}B(q^{-1})}{A(q^{-1})}y_r(k+d)+\dfrac{C(q^{-1})}{A(q^{-1})}\xi(k)}{1+\dfrac{1}{\frac{q_0}{b_0}C(q^{-1})Q(q^{-1})+F(q^{-1})P(q^{-1})}\dfrac{q^{-d}B(q^{-1})}{A(q^{-1})}G(q^{-1})P(q^{-1})}$$

$$y(k)=\frac{C(q^{-1})R(q^{-1})B(q^{-1})y_r(k)+C(q^{-1})\left(\frac{q_0}{b_0}C(q^{-1})Q(q^{-1})+F(q^{-1})P(q^{-1})\right)\xi(k)}{\frac{q_0}{b_0}A(q^{-1})C(q^{-1})Q(q^{-1})+A(q^{-1})F(q^{-1})P(q^{-1})+q^{-d}B(q^{-1})G(q^{-1})P(q^{-1})}$$

Substitute eq (6.2.3) $(i.e.,\ q^{-d}G=C-AE, F=BE)$ into the above and simplify it. Then,

$$y(k)=\frac{C(q^{-1})R(q^{-1})B(q^{-1})y_r(k)+C(q^{-1})\left(\frac{q_0}{b_0}C(q^{-1})Q(q^{-1})+F(q^{-1})P(q^{-1})\right)\xi(k)}{C(q^{-1})\left(\frac{q_0}{b_0}A(q^{-1})Q(q^{-1})+B(q^{-1})P(q^{-1})\right)}$$

Then the characteristic equation of the closed-loop system is

$$\frac{q_0}{b_0}A(q^{-1})Q(q^{-1})+B(q^{-1})\ P(q^{-1})=0$$

✓ It can be seen from the above that as long as $Q(q^{-1})$ (including q_0) and $P(q^{-1})$ are properly selected, the stability and good closed-loop dynamic characteristics of the system can be guaranteed.

♦ **Discussions**

When the controlled plant is a non-minimum phase system, that is, when $B(q^{-1})$ is an unstable polynomial, the weighted polynomials $Q(q^{-1})$ and $P(q^{-1})$ can still be properly selected to make the system characteristic roots within the stable area, so that the generalized minimum variance control can be used to control the non-minimum phase system.

According to eq (6.3.4), the selection of weighted polynomials, especially q_0, also has a certain constraint on the control input $\mathrm{u(k)}$.

If $Q(q^{-1}) = 0$, the cost function (6.3.2) does not contain constraints on the control action $u(k)$.

- ✓ At this time, the generalized minimum variance control degenerates to the minimum variance control, and the characteristic equation degenerates to $B(q^{-1})P(q^{-1}) = 0$.
- ✓ Also, the zeros of polynomial $B(q^{-1})$ is the poles of the closed-loop system.
- ✓ When the controlled plant is a non-minimum phase system, the closed-loop system is unstable.

When designing the control system, in general, the weighted polynomials $P(q^{-1}) = 1$, $R(q^{-1}) = 1$, $Q(q^{-1}) = q_0$ can be used.

- ✓ At this time, the selection of q_0 needs to be balanced in terms of stability and performance.
- ✓ If q_0 is too small, it almost loses the constraint on the control effect, and it is difficult to guarantee the closed-loop stability of the non-minimum phase system, which is close to the minimum variance control.
- ✓ If q_0 is too large, it is easy to meet the stability of the closed-loop system. But, because of the strong constraint on the control action, the system almost runs in the open-loop state, and the purpose of minimum variance control is not achieved at all.
- ✓ When it is difficult to adjust the system by changing q_0 alone, the order of the weighted polynomial $Q(q^{-1})$ can be appropriately increased.

♦ **Algorithm 6.3.1** (Generalized minimum variance control, GMVC)

Given the structure and parameters of the controlled plant.

Step 1: Set the initial data and choose the weighted polynomials $P(q^{-1})$, $R(q^{-1})$ and $\mathrm{Q}(q^{-1})$.

Step 2: Solve Diophantine equation (6.1.1) and obtain the coefficients of polynomials $E(q^{-1})$, $F(q^{-1})$ and $\mathrm{G}(q^{-1})$.

Step 3: Sample the current actual output $y(k)$ and the expected output $y_r(k+d)$.

Step 4: Use equation (6.3.4) to calculate and implement $\mathrm{u}(k)$.

Step 5: Return to Step 3 (k → k+1) and continue the loop.

♦ **Example 6.3.1:**

The controlled plant is the following open-loop unstable non-minimum phase system:

$$y(k) - 1.7y(k-1) + 0.7y(k-2) = u(k-4) + 2u(k-5) + \ \xi(k) + 0.2\,\xi(k-1)$$

where $\xi(k)$ is a white noise with a variance of 0.1.

According to Example 6.1.1, the solution of Diophantine equation of the system is

$$F(q^{-1}) = 1 + 3.9q^{-1} + 6.33q^{-2} + 8.031q^{-3} + 5.942q^{-4}$$
$$G(q^{-1}) = 3.2797 - 2.0797q^{-1}$$
$$E(q^{-1}) = 1 + 1.9q^{-1} + 2.53q^{-2} + 2.971q^{-3}$$

Take $P(q^{-1}) = 1$, $R(q^{-1}) = 1$, $Q(q^{-1}) = 2$, then the generalized minimum variance control law of the system is obtained from equation (6.3.4)

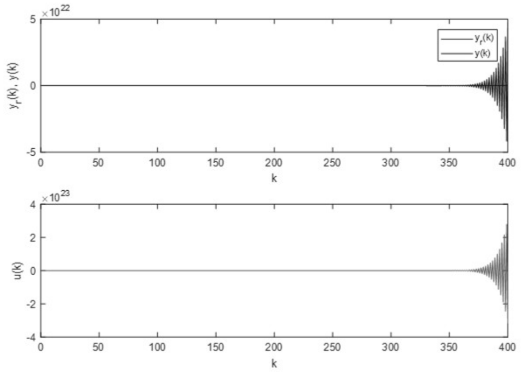
$$u(k) = \frac{(1 + 0.2q^{-1})y_r(k + 4) - (3.2797 - 2.0797q^{-1})y(k)}{5 + 4.7q^{-1} + 6.33q^{-2} + 8.031q^{-3} + 5.942q^{-4}}$$

$$u(k) = -0.94u(k - 1) - 1.266u(k - 2) - 1.6062u(k - 3) - 1.1884u(k - 4) + 0.2y_r(k + 4) + 0.04y_r(k + 3) - 0.6559y(k) + 0.4159y(k - 1)$$

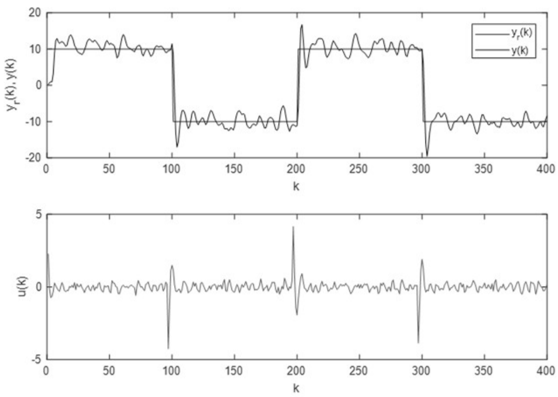
The expected output $y_r(k)$ is a square wave signal with amplitude of 10.

Take $P(q^{-1}) = 1$, $R(q^{-1}) = 1$, $Q(q^{-1}) = q_0$, where q_0 is taken as 0.5, 2 and 20, respectively.

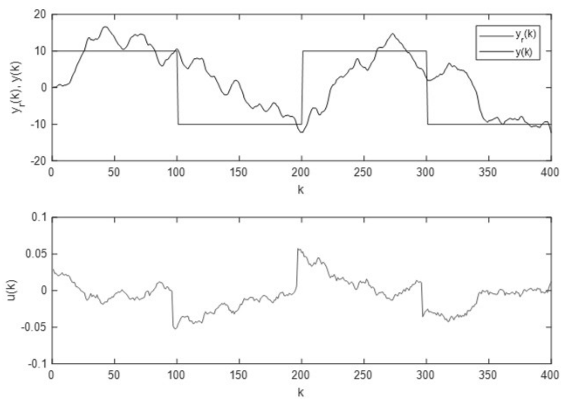
Case 1: q_0 =0.5



Case 2: q_0 =2



Case 3: q_0 =20



When q_0 is small, the generalized minimum variance control is approximately degenerated into the minimum variance control, so the non-minimum phase system cannot be controlled well.

When q_0 is too large, the constraint on the control quantity is too strong to realize the optimal control.

♦ **Simulation codes**

example631.m

■ Generalized minimum variance indirect self-tuning control

When the parameters of the controlled plant (6.3.1) are unknown, the self-tuning control algorithm can be used.

Similar to the minimum variance self-tuning control algorithm, the generalized minimum variance self-tuning control is also divided into an indirect algorithm and direct algorithm.

The indirect algorithm: First use the recursive least squares method to estimate the parameters of the plant online in real time, and then use the estimated parameters to design the generalized minimum variance control law.

♦ **Algorithm 6.3.2** (generalized minimum variance indirect self-tuning control)

Given the model order n_a, n_b, n_c and delay d.

Step 1: Set initial values $\hat{\theta}(0)$ and $P(0)$, enter the initial data, and set the weighted polynomials $P(q^{-1})$, $R(q^{-1})$ and $\mathrm{Q}(q^{-1})$.

Step 2: Sample the current actual output $y(k)$ and the expected output $y_r(k+d)$.

Step 3: Use the recursive least squares method to estimate the parameters $\hat{\theta}(k)$ of the controlled plant online in real time, namely $\hat{A}(q^{-1})$, $\hat{B}(q^{-1})$ and $\hat{C}(q^{-1})$.

Step 4: Solve Diophantine equation (6.1.1) to obtain the coefficients of polynomials $E(q^{-1})$, $F(q^{-1})$ and $\mathrm{G}(q^{-1})$.

Step 5: Uses (6.3.4) to calculate and implement $u(k)$.

Step 6: Return to Step 2 (k → k+1) and continue the loop.

♦ **Example 6.3.2:**

Let the controlled plant be an open-loop unstable non-minimum phase system:

$$y(k) - 1.7y(k-1) + 0.7y(k-2) = u(k-4) + 2u(k-5) + \xi(k) + 0.2\xi(k-1)$$

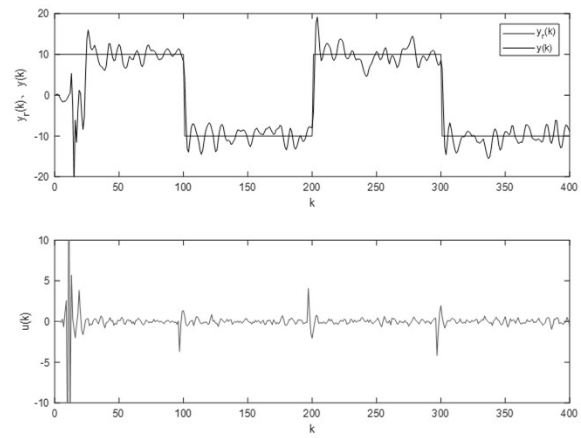
where $\xi(k)$ is a white noise with a variance of 0.1.

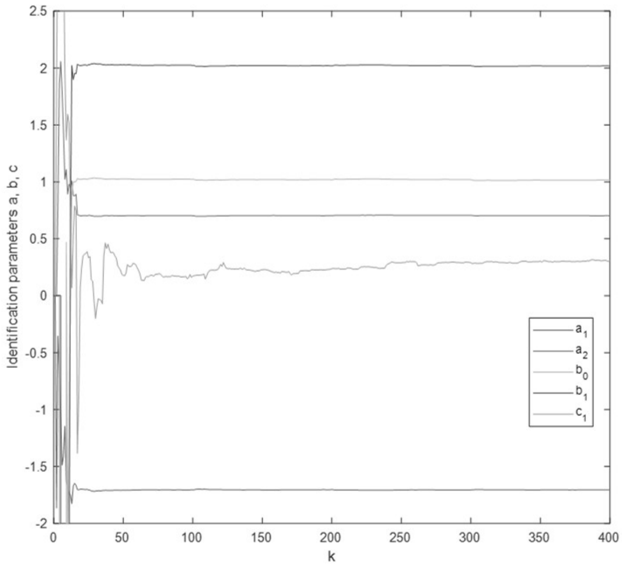
Take the initial values $\hat{\theta}(0) = 0.001$, and $P(0) = 10^6 I$.

Set the weighted polynomial $P(q^{-1}) = 1$, $R(q^{-1}) = 1$, $Q(q^{-1}) = 2$.

The expected output $y_r(k)$ is a square wave signal with amplitude of 10.

The indirect algorithm of generalized minimum variance self-tuning control is adopted, and its results are shown on the right.





Note: You can't take it here $\hat{\theta}(0) = 0$, because at the moment of $k = 1, f_0(1) = \hat{b}_0(1) = 0$, when the control input is calculated using equation (6.3.4), the phenomenon of dividing by zero occurs.

♦ **Simulation codes**

example632.m

6.4 Pole Assignment Self-tuning Control

■ Introduction

As we all know, for linear time-invariant systems, not only the stability of the system depends on the distribution of poles, but also the dynamic performance of the system, such as rise time, overshoot, oscillation number, etc., is closely related to the position of poles to a large extent.

Therefore, the designer selects a feedback control law, so long as the closed-loop poles are moved to the desired position.

The closed-loop system performance can meet the predetermined performance requirements, which is the pole assignment (or pole placement) design method.

♦ **Features of pole assignment control**

- ✓ Unlike the minimum variance control and the generalized minimum variance control, this method does not need to cancel the zeros of the controlled plant.
- ✓ There is no instability problem when controlling the non-minimum phase system, and it also avoids the difficulty of carefully trying to control the weighted parameters.
- ✓ In addition, because the desired pole position is based on the performance requirements of the transient response, it also has the advantages of intuitive engineering concept and easy to consider various engineering constraints.
- ✓ Generally speaking, the pole assignment design method is not the control in the "optimal sense", and the algorithm is relatively complex.

■ Pole assignment control

The system adopts the following mathematical model

$$\begin{aligned} A(q^{-1})y(k) &= q^{-d}B(q^{-1})u(k) + C(q^{-1})\xi(k) \\ &= q^{-d}B(q^{-1})u(k) + v(k) \end{aligned} \qquad (6.4.1)$$

where $y(k)$ and $u(k)$ represent the output and input of the system, $\xi(k)$ and $v(k)$ represent a white noise and a colored noise, respectively, $d \geq 1$ is pure delay, and

$$\begin{aligned} A(q^{-1}) &= 1 + a_1q^{-1} + a_2q^{-2} + \cdots + a_{n_a}\, q^{-n_a} \\ B(q^{-1}) &= b_0 + b_1q^{-1} + b_2q^{-2} + \cdots + b_{n_b}q^{-n_b} \\ C(q^{-1}) &= 1 + c_1q^{-1} + c_2q^{-2} + \cdots + c_{n_c}q^{-n_c} \end{aligned} \qquad v(k) = C(q^{-1})\xi(k)$$

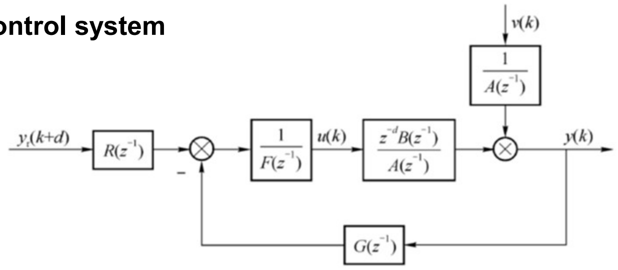
and $A(q^{-1})$ and $B(q^{-1})$ are mutually prime, that is, there is no common factor.

The pole assignment control (PAC) is designed as follows:

$$F(q^{-1})u(k) = R(q^{-1})y_r(k+d) - G(q^{-1})y(k) \qquad (6.4.2)$$

where $F(q^{-1})$, $R(q^{-1})$ and $G(q^{-1})$ are undetermined polynomials, and $y_r(k)$ is the reference input.

♦ **The structure of a pole assignment control system**



The output of the closed-loop system is

$$y(k) = \frac{\frac{R(q^{-1})}{F(q^{-1})}\frac{q^{-d}B(q^{-1})}{A(q^{-1})}}{1+\frac{q^{-d}B(q^{-1})}{A(q^{-1})}\frac{G(q^{-1})}{F(q^{-1})}} y_r(k+d) + \frac{\frac{1}{A(q^{-1})}}{1+\frac{q^{-d}B(q^{-1})}{A(q^{-1})}\frac{G(q^{-1})}{F(q^{-1})}} v(k)$$

$$= \frac{q^{-d}R(q^{-1})B(q^{-1})y_r(k+d) + F(q^{-1})v(k)}{A(q^{-1})F(q^{-1}) + q^{-d}B(q^{-1})G(q^{-1})} \qquad (6.4.3)$$

The closed-loop characteristic equation is Diophantine equation in the following form:

$$A(q^{-1})F(q^{-1}) + q^{-d}B(q^{-1})G(q^{-1}) = T(q^{-1}) \qquad (6.4.4)$$

♦ **Desired closed-loop transfer function**

The design task of pole assignment control is to determine the expected closed-loop characteristic polynomial $T(q^{-1})$ according to the inherent characteristics and control requirements of the system, and then determine $F(q^{-1})$ and $G(q^{-1})$ through equation (6.4.4), and finally calculate the control input from equation (6.4.2).

Set the expected input and output expression as

$$A_m(q^{-1})y_m(k) = q^{-d}B_m(q^{-1})y_r(k+d) \qquad (6.4.5)$$

where $A_m(q^{-1})$ and $B_m(q^{-1})$ are the denominator and the numerator polynomials of the expected transfer function, and the two polynomials are mutually prime.

To obtain the desired input and output response, it is obtained from equation (6.4.3) and equation (6.4.5)

$$\frac{q^{-d}R(q^{-1})B(q^{-1})}{A(q^{-1})F(q^{-1}) + q^{-d}B(q^{-1})G(q^{-1})} = \frac{q^{-d}B_m(q^{-1})}{A_m(q^{-1})} \qquad (6.4.6)$$

♦ **Coping with zeros of the plant**

The designed controller may cancel some zeros of the original controlled plant, but in practical engineering, the poles of the controller can only cancel with the zeros of the stable controlled plant, while the unstable zeros of the plant are not expected to cancel with the poles of the controller.

For this purpose, the polynomial $\mathrm{B}(q^{-1})$ is decomposed into

$$B(q^{-1}) = B^{+}(q^{-1})B^{-}(q^{-1}) \qquad (6.4.7)$$

where $B^{+}(q^{-1})$ is the first polynomial composed of stable and well-damped zeros, which can be eliminated from the poles of the controller;
$B^{-}(q^{-1})$ is a polynomial composed of unstable zeros.

When $B^{+}(q^{-1}) = 1$, it means that no zeros in $\mathrm{B}(q^{-1})$ is cancelled.

When $B^{-}(q^{-1}) = b_0$, all zeros in $\mathrm{B}(q^{-1})$ can be cancelled.

Then eq (6.4.6) can be rewritten as

$$\frac{R(q^{-1})B^{+}(q^{-1})B^{-}(q^{-1})}{A(q^{-1})F(q^{-1}) + q^{-d}B^{+}(q^{-1})B^{-}(q^{-1})G(q^{-1})} = \frac{B_m(q^{-1})}{A_m(q^{-1})} \qquad (6.4.8)$$

As $\mathrm{A}(q^{-1})$ and $\mathrm{B}(q^{-1})$ are mutually prime, according to eq (6.4.8), if $B^{+}(q^{-1})$ is cancelled, $F(q^{-1})$ should be divided by $B^{+}(q^{-1})$, that is

$$F(q^{-1}) = F_1(q^{-1})B^{+}(q^{-1}) \qquad (6.4.8a)$$

Then, eq (6.4.8) becomes

$$\frac{R(q^{-1})B^{-}(q^{-1})}{A(q^{-1})F_1(q^{-1}) + q^{-d}B^{-}(q^{-1})G(q^{-1})} = \frac{B_m(q^{-1})}{A_m(q^{-1})} \qquad (6.4.9)$$

Since the unstable zeros of $B(q^{-1})$ of the plant is not cancelled, $B^-(q^{-1})$ cannot be a factor of $A(q^{-1})F_1(q^{-1}) + q^{-d}B^-(q^{-1})G(q^{-1})$, and it can be seen from the polynomials in eq (6.4.9) that the reserved zeros of $B^-(q^{-1})$ should be reflected by $B_m(q^{-1})$, that is

$$B_m(q^{-1}) = B_m'(q^{-1})B^-(q^{-1}) \tag{6.4.10}$$

The closed-loop system output of eqs (6.4.3), (6.4.8) and (6.4.10) is

$$y(k) = \frac{B_m'(q^{-1})B^-(q^{-1})}{A_m(q^{-1})}y_r(k) + \frac{F(q^{-1})}{A(q^{-1})F(q^{-1}) + q^{-d}B(q^{-1})G(q^{-1})}v(k)$$

It can be seen from the above that to make the steady-state output of the closed-loop system have no deviation.

$$B_m'(1) = \frac{A_m(1)}{B^-(1)}$$

◆ **Determine polynomial $R(q^{-1})$**

So, eq (6.4.9) can be changed to

$$\frac{R(q^{-1})}{A(q^{-1})F_1(q^{-1}) + q^{-d}B^-(q^{-1})G(q^{-1})} = \frac{B_m'(q^{-1})}{A_m(q^{-1})} \tag{6.4.11}$$

According to eq (6.4.11), $A_m(q^{-1})$ is a factor of $A(q^{-1})F_1(q^{-1}) + q^{-d}B^-(q^{-1})G(q^{-1})$, then

$$A(q^{-1})F_1(q^{-1}) + q^{-d}B^-(q^{-1})G(q^{-1}) = A_0(q^{-1})\, A_m(q^{-1}) \tag{6.4.12}$$

$$R(q^{-1}) = B_m'(q^{-1})A_0(q^{-1}) \tag{6.4.13}$$

where $A_0(q^{-1})$ is the introduced stable polynomial.

When the disturbance property is known, let $A_0(q^{-1}) = C(q^{-1})$.

According to eq (6.4.13), if $B_m'(q^{-1})$ and $A_0(q^{-1})$ are selected, the polynomial $R(q^{-1})$ can be determined.

The next work is to use eq (6.4.12) to solve polynomial $F_1(q^{-1})$ and $G(q^{-1})$.

♦ **Determine polynomials $F_1(q^{-1})$ and $G(q^{-1})$**

To ensure that Diophantine equation below has a unique solution,

$$A(q^{-1})F_1(q^{-1}) + q^{-d}B^-(q^{-1})G(q^{-1}) = A_0(q^{-1})\, A_m(q^{-1})$$

the order of $A(q^{-1})F_1(q^{-1})$ and $q^{-d}B^-(q^{-1})G(q^{-1})$ is the same, and the right order of the equation is less than the left order, that is, the order of each polynomial is required to meet the following relationship:

$$\begin{aligned} &\deg F_1 = \deg B^- + d - 1 \\ &\deg G = \deg A \; - 1 \\ &(\deg A_m + \deg A_0) \le (\deg A \; + \deg B^- \; + d - 1) \qquad \text{(6.4.13a)} \end{aligned}$$

The solution of Diophantine equation is essentially the same as that of linear equations.

♦ **Generic case**

Without losing generality, set the key PAC equation of to be

$$A(q^{-1})F(q^{-1}) + q^{-d}B(q^{-1})G(q^{-1}) = T(q^{-1}) \qquad \text{(6.4.14)}$$

where $A(q^{-1})$, $B(q^{-1})$ and $T(q^{-1})$ are known polynomials with orders of n_a, n_b and $n_t = n_a + n_b - 1$.

$F(q^{-1})$ and $G(q^{-1})$ are polynomials to be solved with the orders of $n_f = n_b + d - 1$ and $n_g = n_a - 1$, respectively.

$$\begin{bmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & \vdots & b_1 & b_0 & \cdots & 0 \\ \vdots & a_1 & \cdots & 0 & b_2 & b_1 & \cdots & 0 \\ & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n_a} & & & a_1 & b_{n_b} & & & b_1 \\ 0 & a_{n_a} & & & 0 & b_{n_b} & & b_2 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n_a} & 0 & \cdots & 0 & b_{n_b} \\ & \scriptstyle n_f+1=n_b\text{列} & & & & \scriptstyle n_g+1=n_a\text{列} & & \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n_f} \\ g_0 \\ g_1 \\ \vdots \\ g_{n_g} \end{bmatrix} = \begin{bmatrix} t_0 \\ t_1 \\ \vdots \\ t_{n_t} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

♦ **Discussions**

- ✓ The matrix on the left of the previous equation is called the Sylvester matrix.
- ✓ If the polynomials $A(q^{-1})$ and $B(q^{-1})$ are mutually prime, then the Sylvester matrix is nonsingular, that is, there is a unique solution to equation (6.4.14).
- ✓ There are many methods to solve equation (6.4.14), such as Gaussian elimination method.
- ✓ In MATLAB, it can be solved directly by "left division" operation.
- ✓ In addition, to ensure that the control law (6.4.2) is causal, that is, the controller is physically realizable, the necessary conditions are:

$$\deg A_0 \geq (2\deg A - \deg A_m - \deg B^+ - 1) \qquad (6.4.15)$$
$$(\deg A_m - \deg B_m) \geq (\deg A - \deg B)$$

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♦ **Algorithm 6.4.1** (pole assignment control, PAC)

Given polynomials $A(q^{-1})$, $B(q^{-1})$ and d.
Performance requirements: polynomials $A_m(q^{-1})$, $B_m(q^{-1})$ and $A_0(q^{-1})$.
Compatibility conditions: $B_m(q^{-1}) = B'_m(q^{-1})B^-(q^{-1})$ (that is, B^- can divide all B_m), $degA_m - degBm \geq degA - degB$ and $degA_0 \geq 2degA - degAm - degB^+ - 1$

Step 1: Factorize polynomial $B(q^{-1}) = B^+(q^{-1})B^-(q^{-1})$.

Step 2: According to the compatibility conditions and system performance requirements, determine the polynomials $A_m(q^{-1})$ and $B_m(q^{-1})$, where $B'_m(1) = A_m(1)/B^-(1)$ to ensure that the system's steady-state output is error-free.

Step 3: Consider the compatibility conditions to determine stable polynomial $A_0(q^{-1})$ (if polynomial $C(q^{-1})$ is known, $A_0(q^{-1}) = C(q^{-1})$ can be selected). In addition, the order of $A_0(q^{-1})$ can also be simply taken as $degA_0 \geq 2degA - degAm - degB^+ - 1$.

Step 4: Solve $F_1(q^{-1})$ and $G(q^{-1})$ in Diophantine equation (6.4.12), $F(q^{-1})$ by equation (6.4.8a), and $R(q^{-1})$ by equation (6.4.13).

Step 5: Input the initial data.

Step 6: Sample the current actual output $y(k)$ and reference input $y_r(k+d)$.

Step 7: Calculate and implement the control quantity $u(k)$ according to formula (6.4.2).

Step 8: Return to Step 6 (k → k+1) and continue the loop.

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♦ **Example 6.4.1**

Consider the controlled plant $G(s) = \frac{1}{s(s+1)} e^{-s}$

The expected closed-loop system is required to be natural frequency $\omega = 1$ rad/s and damping coefficient $\xi = 0.7$, and the steady-state output has no error.

The zero-order holder is adopted, and the sampling period $T_s = 0.5s$ is taken to discretize the continuous system

$$G(q^{-1}) = \frac{q^{-3}(0.1065 + 0.0902q^{-1})}{1 - 1.6065q^{-1} + 0.6065q^{-2}}$$

Its poles are $p_1 = 1$ and $p_2 = 0.6065$ and its zero is $z_1 = -0.8467$.

The desired closed-loop characteristic polynomial is

$$T(s) = s^2 + 2\xi\omega s + \omega^2 = s^2 + 1.4s + 1$$

Similarly, after discretization $A_m(q^{-1}) = 1 - 1.3205q^{-1} + 0.4966q^{-2}$

The poles are $p_{m1} = 0.6602 + j0.2463$ and $p_{m2} = 0.6602 - j0.2463$.

Consider the case that the system zero is not cancelled

Step 1: The zero of the discretized system is $z_1 = -0.8467$, which is close to the unit circle and belongs to the case of poor damping. Therefore, this zero is not allowed to cancel with the controller pole, *i.e.*,

$$B^+(q^{-1}) = 1, \qquad B^-(q^{-1}) = 0.1065 + 0.0902q^{-1}$$

Step 2: To make the steady-state output error free, take $B'_m(q^{-1})$ as

$$B'_m(q^{-1}) = \frac{A_m(1)}{B^-(1)} = 0.8951, \qquad B_m(q^{-1}) = B'_m(q^{-1})B^-(q^{-1}) = 0.0954 + 0.0807q^{-1}$$

Step 3: Obtained from eq (6.4.13a)

$$degA_0 \le degA + degB^- + d - 1 - degA_m = 2 + 1 + 3 - 1 - 2 = 3$$

According to eq (6.4.15)

$$degA_0 \ge 2degA - degA_m - degB^+ - 1 = 2 \times 2 - 2 - 0 - 1 = 1$$

Here, take $\deg(A_0(q^{-1})) = 1$, consider its rapidity, and select $A_0(q^{-1}) = 1+0.5q^{-1}$

Step 4: Use eq (6.4.14), namely the MATLAB function "diophantine.m" (readers can also calculate manually)

$$G(q^{-1}) = 2.4099 - 1.3407q^{-1}$$

$$F_1(q^{-1}) = 1 + 0.4804q^{-1} + 0.4052q^{-2} + 0.1994q^{-3}$$

According to eqs (6.4.8a) and (6.4.13),

$$F(q^{-1}) = F_1(q^{-1})B^+(q^{-1}) = F_1(q^{-1}) = 1 + 0.4804q^{-1} + 0.4052q^{-2} + 0.1994q^{-3}$$

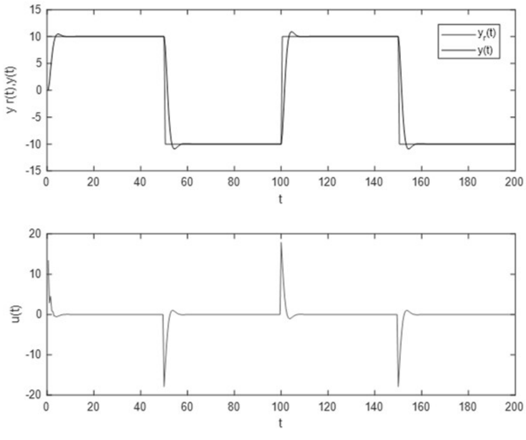
$$R(q^{-1}) = B'_m(q^{-1})A_0(q^{-1}) = 0.8951 + 0.4476q^{-1}$$

Step 5: The control law of eq (6.4.2) is

$$u(k) = (0.8951 + 0.4476q^{-1})y_r(k+3) - (2.4099 - 1.3407q^{-1})y(k)$$
$$-(0.4804q^{-1} + 0.4052q^{-2} + 0.1994q^{-3})u(k)$$

♦ **Results**

Take the expected output $y_r(k)$ as a square wave signal with amplitude of 10



♦ **Simulation codes**

example641.m

■ Indirect self-tuning control of pole assignment

The pole assignment control method is to design the controller polynomials $F(q^{-1}), G(q^{-1})$ and $R(q^{-1})$ under the assumption that the parameters of the controlled plant are known, so that the poles of the closed-loop system are configured according to the desired dynamic response.

When the parameters of the controlled plant are unknown, self-tuning control is required.

There are also two kinds of self-tuning control structures in the pole assignment design method, namely indirect self-tuning control and direct self-tuning control.

If the controlled plant is set as eq (6.4.1), it can be expressed as

$$y(k) = \varphi^T(k)\theta + \xi(k)$$

where

$$\theta = \begin{bmatrix} a_1 & \cdots & a_{n_a} & b_0 & \cdots & b_{n_b} & c_1 & \cdots & c_{n_c} \end{bmatrix}^T$$

$$\varphi(k) = [-y(k-1), \cdots, y(k-n_a), u(k-d), \cdots, u(k-d-n_b), \xi(k-1), \cdots, \xi(k-n_c)]^T$$

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Then the parameter estimation can adopt the recursive augmented least squares method with a forgetting factor, namely

$$\hat{\theta}(k) = \hat{\theta}(k-1) + K(k)\left(y(k) - \hat{\varphi}^T(k-d)\hat{\theta}(k-1)\right)$$

$$K(k) = \frac{P(k-1)\hat{\varphi}(k-d)}{\lambda + \hat{\varphi}^T(k-d)P(k-1)\hat{\varphi}(k-d)}$$

$$P(k) = \frac{1}{\lambda}\left(I - K(k)\hat{\varphi}^T(k-d)\right)P(k-1)$$

where

$$\hat{\theta} = \begin{bmatrix} \hat{a}_1 & \cdots & \hat{a}_{n_a} & \hat{b}_0 & \cdots & \hat{b}_{n_b} & \hat{c}_1 & \cdots & \hat{c}_{n_c} \end{bmatrix}^T$$

$$\hat{\varphi}(k) = \left[-y(k-1), \cdots, y(k-n_a), u(k-d), \cdots, u(k-d-n_b), \hat{\xi}(k-1), \cdots, \hat{\xi}(k-n_c)\right]^T$$

$$\hat{\xi}(k) = y(k) - \hat{y}(k) = y(k) - \hat{\varphi}^T(k)\hat{\theta}$$

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♦ **Algorithm 6.4.2** (pole assignment indirect self-tuning control)

Given model structure n_a, n_b, n_c and d and expected closed-loop characteristic polynomial $A_m(q^{-1})$.

Step 1: Set initial value $\hat{\theta}(0), P(0)$ and forgetting factor λ, and input initial data.

Step 2: Consider the compatibility conditions to determine stable polynomial $A_0(q^{-1})$ (if polynomial $C(q^{-1})$ is known, $A_0(q^{-1}) = C(q^{-1})$ can be selected). In addition, the order of $A_0(q^{-1})$ can be simply taken as $degA_0 \geq 2degA - degAm - degB^+ - 1$.

Step 3: Sample the current actual output $y(k)$ and reference input $y_r(k+d)$.

Step 4: Use the recursive least squares method to estimate the parameters $\hat{\theta}(k)$ of the controlled plant online, i.e., $\hat{A}(q^{-1}), \hat{B}(q^{-1})$ and $\hat{C}(q^{-1})$.

Step 5: Factorize polynomial $B(q^{-1}) = B^+(q^{-1})B^-(q^{-1})$ and determine $B'_m(1) = A_m(1) / B^-(1)$ to ensure that the steady-state output of the system has no error.

Step 6: Solve $F_1(q^{-1})$ and $G(q^{-1})$ in Diophantine equation (6.4.12), $F(q^{-1})$ by equation (6.4.8a), and $R(q^{-1})$ by equation (6.4.13).

Step 7: Calculate and implement the control input $u(k)$ according to formula (6.4.2).

Step 8: Returns to Step 3 (k → k+1) and continue the loop.

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♦ **Example 6.4.2**

Consider the controlled plant as follows: (colored noise is added on the basis of discrete plant in simulation example 6.4.1)

$$y(k) - 1.6065y(k-1) + 0.6065y(k-2) = 0.1065u(k-3) + 0.0902u(k-4) + \xi(k) + 0.5\xi(k-1)$$

where $\xi(k)$ is a white noise with a variance of 0.01.

The expected transfer function denominator polynomial is

$$A_m(q^{-1}) = 1 - 1.3205q^{-1} + 0.4966q^{-2}$$

Take the initial values $\hat{\theta}(0) = 0.001$ and $P(0) = 10^6 I$, forgetting factor $\lambda = 1$;
The expected output $y_r(k)$ is a square wave signal with amplitude of 10

From $degA_0 = 2degA - degAm - degB^+ - 1 = 2*2 - 2 - 0 - 1 = 1$, take $A_0(q^{-1}) = 1 + 0.5q^{-1}$

When the parameters of the controlled plant are unknown, it can be seen from its model structure that it has only one zero, but its value is unknown and needs to be obtained through parameter identification.

Ensure that the zero of the system will not be cancelled during the system operation.

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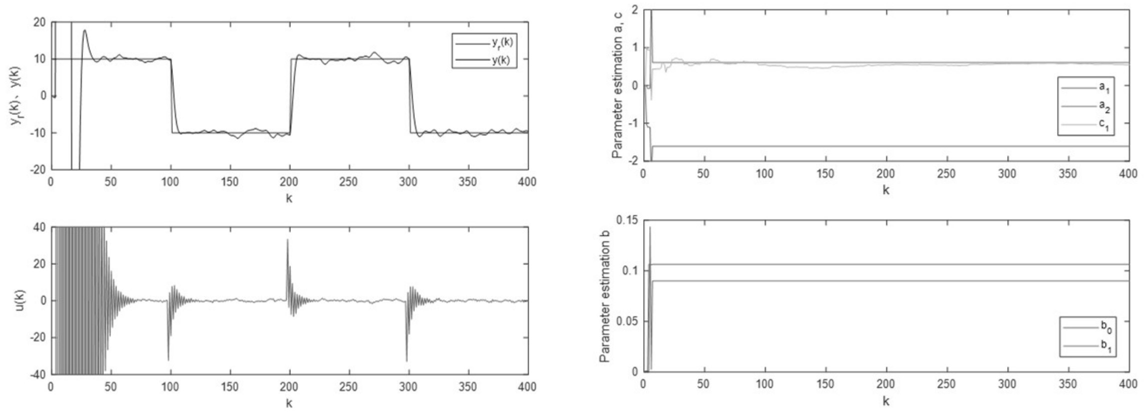
♦ **Results**



♦ **Simulation codes**

example642.m

Exercise 6.2

Consider a time-varying plant

$$y(k) + a_1 y(k-1) + a_0 y(k-2) = b_1 u(k-1) + b_0 u(k-2) + \xi(k) + 0.2\xi(k-1)$$

where $a_1 = -1.7\big(1 + 0.1\sin(0.004\pi k)\big), a_0 = 0.7\big(1 - 0.2\cos(0.005\pi k)\big),$
$b_1 = \big(1 + 0.1\sin(0.006\pi k)\big),\ b_0 = 0.5\big(1 - 0.05\cos(0.007\pi k)\big)$ and $\xi(k)$ is a white noise with variance of 0.01.

Assume the reference input $y_r(k)$ is a square wave signal with amplitude of 20 and period of 300 steps and the expected transfer function denominator polynomial is

$$A_m(q^{-1}) = 1 - 1.3205q^{-1} + 0.4966q^{-2}$$

Using the pole assignment self-tuning control scheme (indirect method), design self-tuning controllers and make simulations.