

THU-93/26
gr-qc/9310026

DIMENSIONAL REDUCTION in QUANTUM GRAVITY[†]

G. 't Hooft

Institute for Theoretical Physics
Utrecht University

Postbox 80 006, 3508 TA Utrecht, the Netherlands

Abstract

The requirement that physical phenomena associated with gravitational collapse should be duly reconciled with the postulates of quantum mechanics implies that at a Planckian scale our world is not 3+1 dimensional. Rather, the observable degrees of freedom can best be described as if they were Boolean variables defined on a two-dimensional lattice, evolving with time. This observation, deduced from not much more than unitarity, entropy and counting arguments, implies severe restrictions on possible models of quantum gravity. Using cellular automata as an example it is argued that this dimensional reduction implies more constraints than the freedom we have in constructing models. This is the main reason why so-far no completely consistent mathematical models of quantum black holes have been found.

[†] Essay dedicated to Abdus Salam

With the request to write a short paper in honor of Abdus Salam I am given the opportunity to contemplate some very deep questions concerning the ultimate unification that may perhaps be achieved when all aspects of quantum theory, particle theory and general relativity are combined. One of these questions is the dimensionality of space and time.

At first sight our world has three spacelike dimensions and one timelike. This was used as a starting point of all quantum field theories and indeed also all string theories as soon as they invoke the Kaluza Klein mechanism. Also when we quantize gravity perturbatively we start by postulating a Fock space in which basically free particles roam in a three plus one dimensional world. Naturally, when people discuss possible cut-off mechanisms, they think of some sort of lattice scheme either in 3+1 dimensional Minkowski space or in 4 dimensional Euclidean space. The cut-off distance scale is then suspected to be the Planck scale.

Unfortunately any such lattice scheme seems to be in conflict with local Lorentz invariance or Euclidean invariance, as the case may be, and most of all also with coordinate reparametrization invariance. It seems to be virtually impossible to recover these symmetries at large distance scales, where we want them. So the details of the cut-off are kept necessarily vague.

The most direct and obvious *physical* cut-off does not come from non-renormalizability alone, but from the formation of microscopic black holes as soon as too much energy would be accumulated into too small a region. From a physical point of view it is the black holes that should provide for a natural cut-off all by themselves.

This has been this author's main subject of research for over a decade. A mathematically consistent formulation of the black hole cut-off turns out to be extremely difficult to find, and in this short note I will explain what may well be the main reason for this difficulty: nature is much more crazy at the Planck scale than even string theorists could have imagined.

One of my starting points has been that quantum mechanics itself is not at all a mystery to me. The emergence of a Hilbert space with a Copenhagen interpretation of its inner products is a quite natural feature of any theory with the following characteristics at a local scale: the system must have *discrete* degrees of freedom at tiny ditance scales, and the laws of evolution must be *reversible* in time. With discrete degrees of freedom one can construct Hilbert space in a quite natural way by postulating that any state of the physical degrees of freedom corresponds to an element of a basis of this Hilbert space[1]. Reversibility in time is required if we wish to see a quantum superposition principle: the norm of all states is then preserved, even if they are quantum superpositions of these basis elements.

Needless to say, one might suspect that some or all of the quantum mechanical postulates could break down at the Planck scale[2]. But then one might as well throw away anything we know about physics, and that is not the route I want to follow. I have never seen convincing models where ordinary quantum mechanics breaks down at a microscopical level but is somehow recovered at the atomic scale. Therefore I prefer not to speculate that quantum mechanics breaks down at the Planck scale, but instead to suspect that quantum mechanics becomes trivial there: quantum superpositions are still allowed there but become irrelevant.

Black holes then present a major challenge. At first sight they render time reversibility impossible. Objects thrown into a black hole can never be retrieved[3]. Things are often presented as if black holes connect our world to other universes via wormholes[4], or, if one prefers not to refer to these other universes one says that information thrown in can not be retrieved anymore[5]. According to this view information is preserved only if one considers multiply connected universes[6]. This is useless if one wishes to recover quantum mechanical behavior in our universe by itself. However, at closer inspection one finds that any object thrown into a black hole actually does leave some signals behind in our own world[7], and it is conceivable that a unitary theory for our own universe can be built using this as a starting point.[8]

The main difficulty then is to formulate exactly what our degrees of freedom are. Remarkably, it is relatively easy to give a fairly precise estimate of *how many* degrees of freedom we have. This can be deduced in two equivalent ways[9, 10]. One is by considering capture and emission of objects by black holes as scattering experiments. If these are described quantum mechanically one can deduce information about the size of phase space. It must be finite, increasing exponentially with the surface area of the black hole horizon. The other way to deduce the same information is by using thermodynamics. One derives the entropy S of a black hole, finding

$$S = 4\pi M^2 + C, \quad (1)$$

in natural units. The constant C is not known (in fact there could be as yet unknown subdominant terms in this expression, increasing slower than M^2 as the mass increases). In principle C could be infinite (even for small M), which basically corresponds to the remnant theory. I think an infinite C would induce major problems (again implying a deviation from ordinary quantum mechanical behavior) in a consistent theory, so I will henceforth assume C to be bounded.

This entropy is often attributed ‘to the space-time metric itself’, as if there would be yet another separate contribution from the quantized fields in this metric, such as the contribution to the entropy of objects outside the black hole. However, this would not be correct. First of all, the contribution of objects at some distance from the black hole will

always be negligible unless they are really very far away (at $|x| \gg R_{\text{Schwarzschild}}$). But most importantly, the contribution of quantum fields to the entropy *diverges* very near the horizon[10]. The reason for this divergence can be easily understood physically: arbitrary amounts of matter can be thrown in and arbitrary amounts are waiting to radiate out; they contribute infinitely to the total number of physical degrees of freedom.

Yet the black hole entropy had just been argued to be finite, *even with the quantum fields present!* We conclude that one has to attribute the black hole entropy not to the space-time metric itself but to the quantized fields present there (which of course does include the small quantum fluctuations of the metric itself), and then one must choose a cut-off sufficiently close to the horizon such that it exactly matches the known black hole entropy (1). In short, the black hole entropy *includes* the entropy of the quantized fields in its neighborhood[10].

In any quantum theory there is a ‘third law of thermodynamics’ relating the entropy to the total number of degrees of freedom: the dimension of the vector space describing all possible states our system can be in is the exponent of the entropy. For instance in a discrete theory described by n spins that can take only two values (‘Boolean variables’), the dimension \mathcal{N} of Hilbert space is

$$e^S = \mathcal{N} = 2^n , \quad (2)$$

Hence the entropy directly counts the number of Boolean degrees of freedom.

Considering the fact that at the Hawking temperature the contribution to the entropy of fields anywhere in the region $R < |x| \ll R^3$ pales compared to the black hole entropy itself, we can now make an important observation concerning the relevant degrees of freedom of a black hole:

The total number of Boolean degrees of freedom, n , in a region of space-time surrounding a black hole is

$$n = \frac{S}{\ln 2} = \frac{4\pi M^2}{\ln 2} = \frac{A}{4\ln 2} , \quad (3)$$

where A is the horizon area.

We can carry this argument one step further. Consider just any closed spacelike surface, with $S(2)$ topology and total surface area A . Consider all possible field and metric configurations inside this surface. We ask how many mutually orthogonal states there can be. If we want these states to be observable for the outside world we have to insist that the total energy inside the surface be less than 1/4 times its linear dimensions, otherwise our surface would lie within the Schwarzschild radius. Let us first ask how many states would an ordinary quantum field theory allow us to have, given these limits on the volume V and the energy E . Now this is not hard. The most probable state would be a

gas at some temperature $T = 1/\beta$. Its energy would be approximately

$$E = C_1 Z V T^4, \quad (4)$$

where Z is the number of different fundamental particle types with mass less than T and C_1 a numerical constant of order one, all in natural units. The total entropy S is

$$S = C_2 Z V T^3, \quad (5)$$

where C_2 is another dimensionless constant. Now the Schwarzschild limit requires that

$$2E < (V/(\frac{4}{3}\pi))^{\frac{1}{3}}, \quad (6)$$

hence, with eq. (4),

$$T < C_3 Z^{-\frac{1}{4}} V^{-\frac{1}{6}}, \quad (7)$$

so that

$$S < C_4 Z^{\frac{1}{4}} V^{\frac{1}{2}} = C_5 Z^{\frac{1}{4}} A^{\frac{3}{4}}. \quad (8)$$

The C_i are all constants of order 1 in natural units. Since in quantum field theories, at sufficiently low temperatures, Z is limited by a dimensionless number we find that this entropy is small compared to that of a black hole, if the area A is sufficiently large.

Next consider a set of N black holes, with masses M_i . They contribute to the energy $\sum_i M_i$. So

$$\sum_i M_i < C_6 A^{\frac{1}{2}}, \quad (9)$$

while their total entropy is given by

$$S = C_7 \sum_i M_i^2, \quad (10)$$

where we note that the contribution of their movements to the entropy is negligible. We see that ineq. (10) is saturated when one black hole has the largest possible size that still fits inside our area. Its entropy is

$$S_{max} = \frac{1}{4}A, \quad (11)$$

and this is as large as we can ever make it. This therefore answers our question concerning the total number of possible states. It is given by eqs. (2) and (3). The single black hole is the limit[11].

The importance of this result can hardly be overestimated. At first sight it is counterintuitive. One would have expected that the number of possible states would grow exponentially with the volume, not the area, as in any ordinary field theory with a cut-off.

So if one would take a regularized fermion theory with cut-off at the Planck scale one would get far too many states. But it is clear why our answer came out this way. Most of the states of a regularized quantum field theory would have so much energy that they would collapse into a black hole before they could dictate the further evolution of the system in time. If we want to avoid this a much more rigorous cut-off than a Planckian one must be called for. Note that if there are any fundamental bosons the number of possible states comes out to be strictly infinite.

But then one may come to appreciate this result after all. It means that, given any closed surface, we can represent all that happens inside it by degrees of freedom on this surface itself. This, one may argue, suggests that quantum gravity should be described entirely by a *topological* quantum field theory, in which all physical degrees of freedom can be projected onto the boundary. One Boolean variable per Planckian surface element should suffice. The fact that the total volume inside is irrelevant may be seen as a blessing since it implies that we do not have to worry about the *metric* inside. The inside metric could be so much curved that an entire universe could be squeezed inside our closed surface, regardless how small it is. Now we see that this possibility will not add to the number of allowed states at all. Our result suggests that one should not worry about creating universes inside test tubes.

The same can be said about wormholes. A wormhole with one end sprouting inside our closed volume and its other end somewhere else could connect the inside of our volume to the outside world, thus adding large quantities of possible states. We now believe that this is not allowed in a decent theory of quantum gravity. In a previous publication [8] it was explained why the functional integral describing black holes probably has to be limited to topologically trivial field configurations only. We have detailed ideas concerning the consistency of such a requirement but will not elaborate on this here. So much for wormholes.

We would like to advocate here a somewhat extreme point of view. We suspect that there simply *are* not more degrees of freedom to talk about than the ones one can draw on a surface, as given by eq. (3) The situation can be compared with a hologram of a three dimensional image on a two-dimensional surface. The image is somewhat blurred because of limitations of the hologram technique, but the blurring is small compared to the uncertainties produced by the usual quantum mechanical fluctuations. The details of the hologram on the surface itself are intricate and contain as much information as is allowed by the finiteness of the wavelength of light - read the Planck length.

It is tempting to take the limit where the surface area goes to infinity, and the surface is locally approximately flat. Our variables on the surface then apparently determine all physical events at one side (the black hole side) of the surface. But since the entropy

of a black hole also refers to all physical fields outside the horizon the *same* degrees of freedom determine what happens at this side. Apparently one must conclude that a two-dimensional surface drawn in a three-space can contain all information concerning the entire three-space. In fact, this should hold for *any* two-surface that ranges to infinity. This suggests that physical degrees of freedom in three-space are not independent but, if considered at Planckian scale, they must be infinitely correlated.

If one could determine the equation of motion of these variables on the two-plane one would also possess the remedy for the black hole information paradox. In a Rindler space we could put our information surface at the origin of the Rindler coordinate frame. The transformation rules for our variables under Lorentz transformations would correspond to the Rindler equations of motion, and pure states would evolve into pure states while the spectrum density of the black hole would continue to be the one dictated by its entropy.

The infinite correlations in three-space could also present a new starting point for resolving the Einstein-Rosen-Podolsky paradox. In terms of present day quantum field theoretical degrees of freedom it is not possible to interpret quantum mechanics deterministically. But certain cellular automaton models can give a faint resemblance to quantum behavior. The discreteness of our degrees of freedom on the two-surface strongly remind us of a 2+1 dimensional cellular automaton. But this would constitute a speculation on top of the previous one. There are various technical problems one has to face if such considerations were to be persecuted further.

In cellular automaton models for quantum mechanics the problem is not the Copenhagen interpretation[1]. This comes out quite naturally. The more tantalizing problem is how to understand the stability of our vacuum state[13]. Models can be constructed producing Hamiltonians that can be realistic but for which there are nearly always negative energy eigenstates. The absence of negative energy states in the real world is probably related to the presence of the gravitational force, for which the time coordinate is handled very differently from non-gravitational theories. But natural curvature of space and time is equally difficult to realize in cellular automaton models of this sort[12]; we just suspect that these problems are related but exactly how is not understood.

But there are other problems as well. One of these is the presence of the group of Lorentz transformations and the fact that this symmetry group is non-compact. This is of course at the heart of the black hole horizon problem. There, Lorentz transformations are playing the role of time translations. In any theory where the physical degrees of freedom are discrete it is extremely difficult to reproduce anything resembling Lorentz invariance¹. As for other invariances such as rotational invariance, one can usually realize

¹ In principle one could think of realizing one of the non-trivial discrete subgroups of the Lorentz group, but in practice this seems to be impossible to reconcile with locality requirements.

symmetry under one of their finite subgroups and then it is not unnatural to suspect that complete invariance will be recovered in the thermodynamic limit as a consequence of renormalization group effects.

The evolution law of the physical degrees of freedom on a 2-surface is another mystery. Ultimately one wishes to recover full invariance with respect to the Poincaré group (which is a precisely defined invariance group for all states that have asymptotic in- and out-states, as the ones used in an S matrix formalism, but not in quantum cosmology). Now this implies that one not only needs an evolution law for translations in the time direction but also a quite similar looking law for translations in the direction orthogonal to the surface. These two evolution laws should commute with each other in spite of their complete independence. We will show how to reproduce such a feature in cellular automaton models, however, as we will see, requiring these models to possess physically interesting, that is, sufficiently non-trivial interactions, will remain a difficult obstacle.

The question whether models exist in 3+1 dimensions that are such that the data on a two-dimensional surface will determine all observables elsewhere is an extremely intriguing one. For definiteness, let us consider a rectangular lattice in 4-space. Momentarily we will ignore requirements such as Lorentz covariance; it is sufficient to require that signals do not go faster than some limited speed c . The prototype of our models is a cellular automaton. The data f on every site on the lattice can be represented by an integer modulo some number p . This p will often be taken to be a prime number. The time evolution is defined by some local law. Conventionally, one imposes that the value of $f(\mathbf{x}, t)$ be a given function of the values of $f(\mathbf{x}_i, t - 1)$, where $\{\mathbf{x}_i\}$ are a finite set of nearest neighbors of \mathbf{x} .

Now such a model is deterministic in the classical sense. Quantization can be introduced either by allowing $f(\mathbf{x}_i, t)$ to be operators in Hilbert space, or even more simply, by declaring all states of the cellular automaton to be basis elements of Hilbert space. "Quantization" is then trivial. There are problems with this latter proposal in that the Hamiltonian then does not seem to possess a well-defined ground state. Again, let's not dwell on that.

In order to obtain "dimensional reduction" we will have to postulate a further constraint: the values of f on a sheet should fix the values elsewhere. This we can realize in principle by also postulating a law of evolution in the z direction. Since further discussion of this 4 dimensional problem becomes a bit intricate it is illustrative to remove one dimension and treat space as 2 dimensional, space-time as 3 dimensional. Here we will require that the data on a *line* should be sufficient to determine the data elsewhere. Part of the rectangular lattice is depicted in Fig. 1.

Suppose now that on every plaquette of the lattice a relation among the data f is

imposed. Thus, in the figure there are constraints of the form

$$g_a(f(A), f(B), f(C), f(D)) = 0, \quad (12a)$$

$$g_b(f(E), f(F), f(G), f(H)) = 0, \quad (12b)$$

$$g_c(f(A), f(B), f(F), f(E)) = 0, \quad (12c)$$

$$g_d(f(D), f(C), f(G), f(H)) = 0, \quad (12d)$$

$$g_e(f(A), f(D), f(H), f(E)) = 0, \quad (12e)$$

$$g_f(f(B), f(C), f(G), f(F)) = 0. \quad (12f)$$

We require all these constraints to be such that if in any of these equations three of the four entries are given the fourth will be uniquely determined.

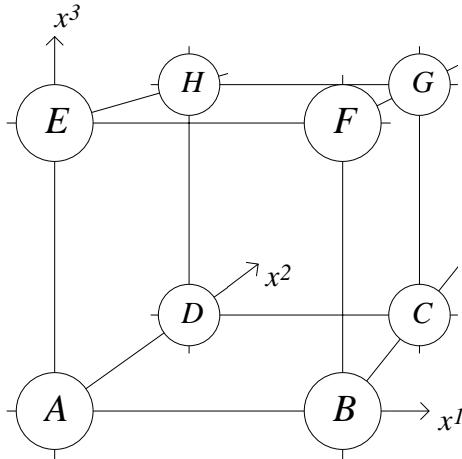


Fig. 1

The lattice sites in x^1, x^2, x^3 coordinates

If we choose the x, y, t coordinates in one of the principle directions of the lattice the situation becomes a bit singular. It is preferable to define

$$\begin{aligned} t &= x^1 + x^2 + x^3, \\ x &= x^1 - x^2, \\ y &= x^3 - \frac{1}{2}(x^1 + x^2). \end{aligned} \quad (13)$$

if on two nonsecutive t layers the data are known successive application of eqs (12a-f) will produce the data in all of space-time. But now consider the series of points $\mathbf{x}(n)$, $n \in \mathbb{Z}$, defined by

$$\begin{aligned} \mathbf{x}(3n+1) &= \mathbf{x}(3n) + \mathbf{e}^1, \\ \mathbf{x}(3n+2) &= \mathbf{x}(3n+1) - \mathbf{e}^2, \\ \mathbf{x}(3n+3) &= \mathbf{x}(3n+2) + \mathbf{e}^3, \end{aligned} \quad (14)$$

where $\mathbf{e}^{1,2,3}$ are the three unit vectors of the lattice (the signs in front of the \mathbf{e}^i can actually be chosen at will). Suppose $f(\mathbf{x}(n))$ are given for all n . Successive application of the six identities (12 a–f) then also fixes all data elsewhere.

However, one cannot choose the constraints (12a–f) any way one likes. This is because the data elsewhere will be *over-determined*. Suppose that in the Figure the data are given at D , H , E and F . They form part of a series (14). By applying four of the six equations (12) the other data on the cube are determined. The remaining two must then be satisfied automatically. If these would not be satisfied our system would be constrained further, in such a way that practically no solutions survive at all. The point is that eqs (12a, b) generate translations along the x^1x^2 plane, eqs. (12c, d) translations along the x^1x^3 plane and (12e, f) translations along the x^2x^3 plane. These three translation operators, viewed as operators in Hilbert space, should commute with each other. Only when they are chosen very meticulously these commutation requirements can be met. One can also say that we have translation operators defined on the one dimensional series of data on the points (14). They define translations in the directions $\mathbf{e}^i \pm \mathbf{e}^j$, which should all commute with each other. Two linear combinations of these, $U_1(\delta t)$ and $U_2(\delta y)$, should generate *independent* translations in directions orthogonal to the line $\sigma(\mathbf{e}^1 - \mathbf{e}^2 + \mathbf{e}^3)$ and a third, U_3 , corresponds to a translation in the direction of the line. The three independent translation operators one then has should commute with each other. Commutation with U_3 is usually easy to implement by choosing the evolution not to depend on the coordinate n , but commutation of U_1 with U_2 is as hard as finding two different yet commuting local Hamiltonians for a quantum system.

Let us phrase the constraints the following way. Consider Fig. 1. One is free to choose $f(A)$, $f(B)$, $f(D)$ and $f(E)$. Then the values of $f(C)$, $f(F)$ and $f(H)$ are uniquely determined by eqs (12a, c) and (e). But the value of $f(G)$ is overdetermined. The three equations (12b, d) and (f) should all yield the same value. Writing the solutions to eqs (12a–f) as

$$f(C) = h_a(f(A), f(B), f(D)), \quad (15a)$$

$$f(G) = h_b(f(E), f(F), f(H)), \quad (15b)$$

$$f(F) = h_c(f(A), f(B), f(E)), \quad (15c)$$

$$f(G) = h_d(f(D), f(C), f(H)), \quad (15d)$$

$$f(H) = h_e(f(A), f(D), f(E)), \quad (15e)$$

$$f(G) = h_f(f(B), f(C), f(F)), \quad (15f)$$

our requirement corresponds to

$$\begin{aligned}
h_b(f(E), h_c(f(A), f(B), f(E)), h_e(f(A), f(D), f(E))) &= \\
h_d(f(D), h_c(f(A), f(B), f(D)), h_e(f(A), f(D), f(E))) &= \\
h_f(f(B), h_c(f(A), f(B), f(D)), h_b(f(A), f(B), f(E))) .
\end{aligned} \tag{16}$$

An easy way to implement these commutation constraints is by choosing the functions g_i to be *linear in the functions f_i modulo p* :

$$g_i(\{f_j\}) = \sum_j A_{ij} f_j + B_i \pmod{p}, \tag{17}$$

where the coefficients A_{ij} must all have an inverse modulo p . From this one gets three equations for $f(G)$:

$$f(G) = K_{1,\alpha} f(A) + K_{2,\alpha} f(B) + K_{3,\alpha} f(D) + K_{4,\alpha} f(E) + K_{5,\alpha} \pmod{p}, \quad \alpha = 1, 2, 3. \tag{18}$$

It is not hard to find sets of coefficients A_{ij} , B_i such that the 10 equations

$$K_{i,1} = K_{i,2} = K_{i,3} \tag{19}$$

are obeyed.

A next step is to attempt to find other realizations of our commutation requirement. Using a computer search program the author generated solutions which at first sight seemed to be quite different, but careful analysis revealed that all solutions found were actually equivalent to the linear ones, eq. (17) after applying permutation operations on the numbers f . This could be seen as a disappointment because one might argue that a linear relation such as eq. (17) is too trivial to be of much physical interest. It implies that solutions can be superimposed onto each other, as if we were describing only “non-interacting” particles. Our challenge at present is to find any set of plaquette relations that does not allow a superposition procedure to obtain new solutions from old ones (note that superposition here means addition modulo a number, and is quite distinct from quantum mechanical superposition which is always allowed).

Our problem could be seen to simplify a bit by using different lattices. Basically what we want to achieve is a cellular automaton with two commuting but essentially different evolution laws. Suppose we have a triangular lattice in the $x - y$ direction as well as the $x - t$ direction, see Fig. 2. We may demand that the data on the x axis alone should determine all others. This implies that we have a relation U_1 determining how the data

look in the y direction, and an evolution operator U_2 in the time direction. We have

$$f(F) = U_1(f(A), f(B)), \quad (20a)$$

$$f(G) = U_1(f(B), f(C)), \quad (20b)$$

$$f(P) = U_2(f(A), f(B)), \quad (20c)$$

$$f(Q) = U_2(f(B), f(C)), \quad (20d)$$

$$f(X) = U_2(f(F), f(G)), \quad (20e)$$

$$f(X) = U_1(f(P), f(Q)), \quad (20f)$$

and for all choices of $f(A)$, $f(B)$ and $f(C)$ the equations (12e) and (12f) should agree. These equations are easier to study than the equations (16). Again there are solutions where U_1 and U_2 are different, providing no direct relations between $f(F)$ and $f(P)$ or $f(G)$ and $f(Q)$, but all solutions we found are equivalent to linear ones modulo a prime number p .

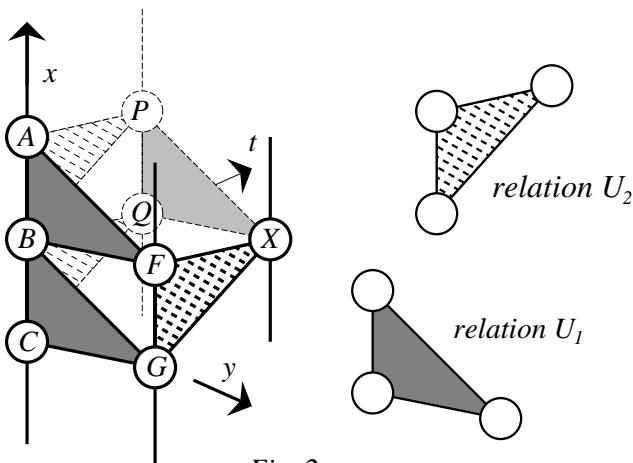


Fig. 2

Cellular automaton with two evolution laws on triangular lattices.

In the real world there is one more dimension. By choosing the initial data to be periodic in that one extra dimension the problem reduces to the three dimensional one. Ultimately one would also like to reobtain local Lorentz invariance and coordinate reparametrization invariance. These imply large symmetry groups that are as yet impossible to implement.

The reason why it is interesting to attempt to find an accidental non-trivial solution of the equations is that many cellular automata, defined on different lattice types, can be transformed into each other, as long as the evolution equations act upon nearest neighbors only. So we actually cover a large class of models. If the real world obeys comparable rules it could be equivalent to such an automaton also, though, admittedly, we consider

such a simple structure unlikely. Indeed, one can also devise interaction schemes among discrete variables that are essentially local but *not* equivalent to cellular automata of the kind considered here.

Even though the problems just mentioned are grave and conceptually difficult to disentangle, they do not seem to be insurmountable. Our basic problem is that there seem to be *too many* symmetry requirements and the Hilbert space of physically realizable states seems to be *too small*. The picture we sketched of the 2+1 (or perhaps it is better to say 2+2) dimensional nature of our micro-universe seems to follow from quite general arguments. Rejecting any of these arguments leads to quite different and perhaps equally if not more difficult problems, and one cannot help observing that one's preferences seem to be related to nearly religious prejudices. Besides the author not many other physicists have tried this particular avenue. We advocate its further pursuit.

References

1. G. 't Hooft, *Nucl. Phys.* **B 342** (1990) 471; *J. Stat. Phys.* **53** (1988) 323.
2. S.W. Hawking, *Phys. Rev.* **D 14** (1976) 2460.
3. S.W. Hawking and G.F.R. Ellis, "The Large Scale Structure of Space-time", Cambridge: Cambridge Univ. Press, 1973.
4. S.W. Hawking, *Phys. Rev.* **D 37** (1988) 904; S. Coleman, *Nucl. Phys.* **B 307** (1988) 864; *ibid.* **B 310** (1988) 643; S.B. Giddings and A. Strominger, *Nucl. Phys.* **B 307** (1988) 854.
5. C.G. Callen, S.B. Giddings, J.A. Harvey and A. Strominger, *Phys. Rev.* **D 45** (1992) R1005.
6. V.P. Frolov and I.G. Novikov, *Phys. Rev.* **D 42** (1990) 1057.
7. T. Dray and G. 't Hooft, *Nucl. Phys.* **B 253** (1985) 173; T. Dray and G. 't Hooft, *Commun. Math. Phys.* **99** (1985) 613; G.'t Hooft, *Nucl. Phys.* **B 335** (1990) 138.
8. C.R. Stephens, G. 't Hooft and B.F. Whiting, "Black hole evaporation without information loss", Utrecht/Gainesville prepr. THU-93/20; UF-RAP-93-11; gr-qc/9310006.
9. G. 't Hooft, "On the Quantization of Space and Time", Proc. of the 4th Seminar on Quantum Gravity, May 25-29, 1987, Moscow, USSR, ed. M.A. Markov et al (World Scientific 1988).
10. G. 't Hooft, *Nucl.Phys.* **B 256** (1985) 727.
11. G. 't Hooft, *Physica Scripta* **T 36** (1991) 247.
12. G. 't Hooft, "A Two-dimensional Model with Discrete General Coordinate-Invariance", in "The Gardener of Eden", Physicalia Magazine, vol **12**, in honour of R. Brout, eds. P. Nicoletopoulos and J. Orloff, Brussels, 1990.
13. G. 't Hooft, K. Isler and S. Kalitzin, *Nucl. Phys.* **B 386** (1992) 495