Formalizing the Kolmogorov extension Theorem in Lean

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Abstract

We present a formalization in Lean of the Kolmogorov extension theorem, which is a main building block in the construction of the (distribution of) stochastic processes with arbitrary index sets. Our approach is based on mathlib, the mathematical library for Lean. On our way, we provide a formalization of the Caratheodory extension theorem, which allows to extend a set function on a semi-ring to a proper measure on the σ-algebra generated by the semi-ring. A slight generalization of the classical Kolmogorov extension theorem allows us to construct the (distribution of) a stochastic process on a complete and separable pseudo-metric space rather than a metric space.

Keywords: probability, measure theory, Lean, formal mathematics, proof assistant, mathlib

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Mention a precise commit of Mathlib

write about sets or types?

∀ j : J, α j or (j : J) -> α j ?

1 Introduction

One of the main building blocks of modern probability theory are stochastic processes, which are usually defined as any collection of random variables – $(X_t)_{t\in\iota}$ with X_t taking values in some α_t for all $t \in \iota$ – defined on some joint probability space $(\Omega, \mathcal{A}, \mathbf{P})$. (As usual, we will refer to ι as the index set of times.) In order to study such processes, it is fundamental to talk about their joint distribution, i.e. a probability distribution on the product set $\prod_{t\in\iota}\alpha_t$. The usual approach to construct (the distribution of) a stochastic process works as follows: describe properties of the distribution of the stochastic process P_J at some arbitrary but finite number of times $J = \{t_1, ..., t_n\} \subseteq \iota$. The resulting family of probability measures $(P_J)_{J\subseteq\iota \text{ finite}}$ has to be projective in the sense that the projection of P_J to $H \subseteq J$ has to be equal to P_H. In other words, when describing the distribution of the stochastic process at all times in J, and then forgetting all properties for times in $J \setminus H$, results in the description of properties at times in H. One may then ask if this already gives a complete description of the process for all times. For uncountable ι, e.g. $\iota = [0, \infty)$, one is tempted to be pessimistic at first sight since measures – which describe the distribution of the stochastic process – usually only deal well with a countable number of measurable events. However, it is the achievement of Kolmogorov that the finite-dimensional distributions in fact provide a unique description of the distribution of a stochastic process, as long as the underlying family of state spaces $(\alpha_t)_{t\in\iota}$ is nice enough (Polish, i.e. a separable topological space which can be metrized by a complete metric, for example). Assuming a family of topological spaces $(\alpha_t)_{t\in\iota}$, this distribution is a measure on the product-σ-field $\mathcal{F} := \bigotimes_{t\in\iota}\mathcal{B}(\alpha_t)$ (where $\mathcal{B}(\alpha_t)$ is the Borel σ-algebra on α_t). Here, $\mathcal{F}$ is generated by finite projections and hence any element of $\mathcal{F}$ may only depend on at most countably many $t \in \iota$, making this a rather coarse σ-algebra. (In particular, note that this is not the Borel σ-algebra of the product topology for infinite ι.) The result responsible for this insight is usually denoted the Kolmogorov extension theorem, formulated in [11]. We note that a version of the extension result was proved by Daniell in the 1930s, but this paper was not acknowledged by the probabilists of that time [1]. Due to his contribution, the theorem is often called the Daniell-Kolmogorov extension.

The goal of our contribution is a formalization of the proof of that theorem in Lean. Here, we are going to

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use Lean4, since mathlib [12], the mathematical library of Lean has recently been ported to Lean4. The Kolmogorov extension theorem is on the interface between measure theory and probability theory. Here, we rely on a decent amount of formalized mathematics in the measure-theory part of mathlib (outer measures, above all), while not requiring any specific previous formalization of probability theory. (In fact, most of our results are formulated in terms of finite rather than probability measures.)

At first sight, it might be surprising that martingales, a certain class of stochastic process, have already been formalized in Lean [16], although the Kolmogorov extension theorem (on the existence of stochastic processes) is only available by our contribution. Note, however, that a martingale $(X_t)_{t\in\iota}$ is defined as a family of random variables (satisfying some properties) on a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$, while the Kolmogorov extension theorem is on the construction of a probability (or finite) measure $\mathbf{P}$, on which we can define random variables (with certain properties). So, our work complements [16] in the sense that probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$ on which you can define a martingale exist. As an application of our implementation, we construct a probability measure on an infinite product space where all coordinates are independent; see Section 3.

The main steps in our construction, which were previously missing in mathlib, are (more mathematical details are below, and in standard textbooks on Probability Theory, e.g. [9, 10]):

1. a formalization of (the set-system of) semi-rings (see Definition 2.8);
2. a definition of additive contents over semi-rings (see Definition 2.11);
3. a proof that a single probability measure on a Polish space is inner regular with respect to compact sets (see Definition 2.15; in fact, we proved a slight generalization using pseudo-metric spaces, found in Lemma 2.20 below);
4. the classical Carathéodory extension theorem, providing us with a candidate for the measure which we want to construct (see Theorem 3);
5. the Kolmogorov extension theorem, as based on the previous steps (See Theorem 1).

The Kolmogorov extension theorem has been previously formalized in Isabelle/HOL [7]. This formalization only works on Polish spaces (rather than on spaces where every finite measure is inner regular with respect to compact sets, see below), and only in the case where all α_i's are identical. In addition, Isabelle works with simple type theory, while Lean is an implementation of dependent theory.

Possible future work. Let us describe some future projects extending mathlib which become possible by our contribution.

Instances of stochastic processes: An obvious application of Kolmogorov's extension theorem is the construction of basic stochastic processes like Markov chains [9, Section 11], the Poisson process [9, Section 13] and Brownian motion [9, Section 14]. Other Gaussian processes – indexed by $[0, \infty)$ or $\mathbb{R}$ – might as well be constructed the same way. In addition, fields like the Gaussien free field indexed by $\mathbb{R}^d$ [15] can also be given. For these tasks, we would have to define the finite-dimensional distributions (Poisson and normally distributed, respectively), and apply the extension theorem. This task requires the formalization of multi-dimensional Poisson and normal distributions, which – in textbooks – is usually done using characteristic functions. Since these are not yet part of mathlib, we postpone this task to the future.

Sample-path properties: The Kolmogorov extension gives the existence of a distribution of a stochastic process $(X_t)_{t\in\iota}$ with certain properties. Extra work is needed in order to show that – on the same probability space – we can as well define a version (i.e. another process $(Y_t)_{t\in\iota}$ with $\mathbf{P}(X_t = Y_t) = 1$ for all $t \in \iota$) which is right-continuous with left limits (for the Poisson process) or continuous (for Brownian motion). For the former, this follows from some general principles of Markov processes (e.g. [6, Theorem 4.3.6]). For the latter, this requires formalization of the Kolmogorov-Chentsov criterion [9, Theorem 4.23].

Theorem of Ionescu-Tulcea and related results: While Kolmogorov's extension Theorem gives a result for arbitrary ι, but requires some properties of the family $(\alpha_t)_{t\in\iota}$, the Theorem of Ionescu-Tulcea [9, Theorem 8.24] can only deal with countable ι, but has no restrictions on $(\alpha_t)_{t\in\iota}$. The proof of the latter, however, uses transition kernels (and an inductive construction of a content, which can be extended to a measure) rather than projective families. However, the structure of the argument is similar and could use some of our results, like the proof of σ-additivity from continuity at the empty set (see Lemma 2.14). A formalization would complement our work. In addition, using the Theorem of Ionescu-Tulcea one can define infinitely many independent random variables (arbitrary ι) on a joint probability space [9, Theorem 8.24] without requirements on $(\alpha_t)_{t\in\iota}$. All of these results can be used in order to construct countably many independent random variables, which are frequently used in concrete constructions in probability theory. Examples are (one direction of) the Borel-Cantelli-Lemma [9, Theorem 4.28], Kolmogorov's 0-1-law [9, Theorem 4.13], random walks [9, Chapter 12], branching processes [9, Chapter 13], and percolation [10, Chapter 2.4], to name just a few. While the first two are already formalized in mathlib, it is only clear due to the Kolmorov extension theorem (or the Theorem of Ionesu-Tulcea) that infinitely many independent random variables can indeed be defined on a joint probability space. In this sense, our contribution is also important for giving more sense to already formalised parts of mathlib.

2 Formalization of the Daniell-Kolmogorov extension

We start by stating the exact result. We are going to formulate the main result in a modern fashion, as e.g. found in Theorem 2.2 of [14], Theorem 7.7.1 of Volume 2 of [4], Theorem 15.26 of [2], or [5]. Note that these formulations split general assumptions on the underlying space(s) (e.g. a metric property) from the property which is needed in the proof (inner regularity with respect to compact sets). Other – highly readable – references such as [3] state the extension theorem only in special cases such as $\alpha_t = \mathbb{R}$ for all t.

2.1 Formulating the result

Before we can state the result, we begin with some mathematical notions. We start off from metric spaces, but quickly introduce projective families of measures. In all definitions, α will be some set (or type, since Lean is a dependently typed language, `α : Type _`).

Definition 2.1 (Metric and topological spaces)**.** *1. A pseudo-metric on α is a symmetric map $r : \alpha \times \alpha \to [0, \infty)$ satisfying the triangle inequality, i.e. $r(x, z) \leq r(x, y) + r(y, z)$ for all $x, y, z \in \alpha$.*
If r also satisfies $r(x, y) = 0$ iff $x = y$, we call it a metric. If $r : \alpha \times \alpha \to [0, \infty]$ (i.e. $r(x, y) = \infty$ is allowed), we call r an extended (pseudo-)metric.
2. Let r be an (extended pseudo-)metric. A sequence $x_1, x_2, ... \in \alpha$ is called Cauchy if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $r(x_n, x_m) < \varepsilon$ for all $m, n > N$. The (extended pseudo-) metric is called complete if every Cauchy-sequency has a limit in α.
3. Some[1] $\mathcal{O} \subseteq 2^\alpha$ is called a topology, if it satisfies (i) $\emptyset, \alpha \in \mathcal{O}$, (ii) $\mathcal{O}$ is a π-system, i.e. it is stable under finite intersections, i.e. if $A, B \in \mathcal{O}$, then $A \cap B \in \mathcal{O}$, and (iii) $\mathcal{O}$ is stable under arbitrary unions, i.e. if $A_i \in \mathcal{O}$ for all $i \in \iota$ and ι is arbitrary, then $\bigcup_{i \in \iota} A_i \in \mathcal{O}$.

re-order that next re-mark? I'll make a pro-posal

Remark 2.2 (Metric and topological spaces in mathlib)**.** *1. In Lean, we have the* `class PseudoEMetricSpace (α : Type u) extends EDist`*, where* `EDist` *comes with an extended distance* `edist: α → α → ℝ≥0∞` *(the notation* `ℝ≥0∞` *stands for $[0, \infty]$), and the extension to* `PseudoEMetricSpace` *consists of the components,* `edist_self`*,* `edist_comm`*,* `edist_triangle` *providing the properties of the pseudo-metric, and* `toUniformSpace`*,* `uniformity_edist` *defining a uniform space from the extended pseudo-metric. Such a space does not come with a metric, but with a filter on $\alpha \times \alpha$, which describes which points in α are near. For example, the diagonal of $\alpha \times \alpha$ is a subset of all sets in the uniformity; see [8] for details. We note that a uniform space with a countably generated uniformity filter is pseudometrizable, i.e. there exists a pseudo-metric-space structure that generates the same uniformity; see* `UniformSpace.pseudoMetrizableSpace`*, which formalizes a result stated in [13].*
2. For topological spaces, we have `class TopologicalSpace(α : Type u) : Type u`*, which comes with* `IsOpen : Prop`*,* `isOpen_univ : IsOpen Set.univ`*,* `isOpen_inter` *and* `isOpen_sUnion`*, which are exactly the properties of a topological space described above.*
3. Any (extended pseudo-)metric on α defines a topology, namely the topology generated by $\mathcal{H} := \{\{y : r(x, y) < \varepsilon\} : x \in \alpha, \varepsilon \in (0, \infty)\}$. Actually, in mathlib, as we have seen above, every extended pseudo-metric defines a uniform space, and a uniform space is a `class UniformSpace(α : Type u) extends TopologicalSpace , UniformSpace.Core` *(where the uniformity is defined in* `UniformSpace.Core`*), which connects the* `uniformity` *with the topology using* `∀ (s : Set α), IsOpen s ↔ ∀ (x : α), x ∈ s → {p | p.fst = x → p.snd ∈ s} ∈ uniformity`*. In particular, a uniform space defines a topological space, which can be used by the typeclass system.*

Remark 2.3 (Generated topology)**.** *1. The intersection of any number of topologies is again a topology. For this reason, if $\mathcal{H} \subseteq 2^\alpha$, we define the topology $\mathcal{O} := \bigcap_{\mathcal{F} \supseteq \mathcal{H} \text{ topology}} \mathcal{F}$; see* `TopologicalSpace.generateFrom`*. This is called the topology generated by $\mathcal{H}$. If $\mathcal{H}$ is closed under finite intersections, we call $\mathcal{H}$ a basis for $\mathcal{O}$; see* `TopologicalSpace.IsTopologicalBasis`*.*
2. We call the topology (generated from an extended pseudo-metric) separable (see `TopologicalSpace.SeparableSpace`*) if there is a countable $s \subseteq \iota$ such that $\inf\{r(x, y) : y \in s\} = 0$ for all $x \in \alpha$. (More generally, a* `TopologicalSpace α` *is a* `SeparableSpace` *iff* `∃ s, Set.Countable s ∧ Dense s`*, where the latter means* `∀ (x : α), x ∈ closure s`*, i.e. the space equals its own closure.)*
If there is a countable basis of the topology, it is separable; see `TopologicalSpace.SecondCountableTopology.to_separableSpace`*.*

Finally, we can introduce measures. A measure is defined on a σ-algebra, which we introduce next.

Definition 2.4 (σ-algebras and measures)**.** *1. We call $\mathcal{F} \subseteq 2^\alpha$ a σ-algebra (on α) if (i) $\emptyset \in \mathcal{F}$, (ii) $\mathcal{F}$ is stable under complements, i.e. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,*

[1] We denote by 2^α the power set of α, i.e. the set of all subsets of α.

(iii) $\mathcal{F}$ is stable under countable unions, i.e. $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^\infty A_n \in \mathcal{F}$. We call $(\alpha, \mathcal{F})$ a measurable space.
2. For some σ-algebra $\mathcal{F}$ on α, a function $\mu : \mathcal{F} \to [0, \infty]$ is called a measure, if (i) $\mu(\emptyset) = 0$, (ii) μ is countably additive, i.e. for $A_1, A_2, \ldots \in \mathcal{F}$ pairwise disjoint, we have $\mu\Big(\bigcup_{n=1}^\infty A_n\Big) = \sum_{n=1}^\infty \mu(A_n)$. We call $(\alpha, \mathcal{F}, \mu)$ a measure space.
In addition, μ is called finite if $\mu(\alpha) < \infty$ and a probability measure if $\mu(\alpha) = 1$.

Remark 2.5 (Measur(abl)e spaces in mathlib)**.** *1. The **class** MeasurableSpace α is very similar to a topological space in mathlib, since it comes with MeasurableSet' : Set α → **Prop**, measurableSet_empty : MeasurableSet' ∅, measurableSet_compl : (s : Set α) → MeasurableSet' s → MeasurableSet' sᶜ, and measurableSet_iUnion, i.e. properties (i)-(iii) from Definition 2.4.1.*
*2. We are using outer measures in our construction. In mathlib, this is **structure** MeasureTheory.OuterMeasure α which is a function measureOf : Set α → ℝ≥0∞ such that measureOf ∅ = 0 (i.e. the empty set has measure 0), ∀ {s₁ s₂ : Set α}, s₁ ⊆ s₂ → measureOf s₁ ≤ measureOf s₂ (i.e. monotonicity) and ∀ (s : ℕ → Set α), measureOf (⋃ (i : ℕ), s i) ≤ ∑' (i : ℕ), measureOf (s i) (which we call σ-subadditivity).*
Note that an outer measure is defined on 2^α, whereas a measure is only defined on a subset (the σ-algebra of measurable sets).
3. A measure defined on a type α on which we have a MeasurableSpace α is denoted by the type MeasureTheory.Measure α

Remark 2.6 (Generated σ-algebra, image measure)**.** *We will frequently need two basic results:*

1. The intersection of any number of σ-algebras is again a σ-algebra. For this reason, if $\mathcal{H} \subseteq 2^\alpha$, we define the σ-algebra $\sigma(\mathcal{H}) := \bigcap_{\mathcal{F} \supseteq \mathcal{H}\ \sigma\text{-algebra}} \mathcal{F}$. This is called the σ-algebra generated from $\mathcal{H}$; see MeasurableSpace.generateFrom.
In particular, if $\mathcal{O}$ is a topology on α, we call $\mathcal{B} := \sigma(\mathcal{O})$ the Borel σ-algebra on α. In mathlib, this is

```
def borel (α : Type u) [TopologicalSpace α] :
  MeasurableSpace α :=
  generateFrom { s : Set α | IsOpen s }
```

In addition, mathlib provides a similar notion, which is

```
class OpensMeasurableSpace (α : Type*)
  [TopologicalSpace α] [h : MeasurableSpace α] :
  Prop where borel_le : borel α ≤ h
```

Here, all open sets are measurable, so the σ-algebra defining h might be larger than the Borel σ-algebra. We will need both notions in our proofs.
2. Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{G})$ be measurable spaces. Some $f : \alpha \to \beta$ is called measurable (with respect to $\mathcal{F}$ and $\mathcal{G}$) if $f^{-1}\mathcal{G} \subseteq \mathcal{F}$. In mathlib, see Measurable.
In this case, if μ is a measure on $\mathcal{F}$, the measure $\nu : \mathcal{G} \to [0, \infty], \nu(B) := \mu(f^{-1}(B))$ is called the image (or push-forward) measure of μ under f. We write $\nu := f_*\mu$; see MeasureTheory.Measure.map.

We now come to our contribution. The extension theorem is a statement about extending a set-function on a product space to a (finite) measure, where the product space can come with an arbitraty index set. The next definition covers the important concept of a projective family of measures. In short, we define measures on any finite subset of indices in a consistent way. Finite sets (of some type ι) are formalized with Finset ι. defined as a Multiset (a

Definition 2.7 (Projective family and projective limit)**.**
1. For some set ι, we will write $J \subseteq_f \iota$ if $J \subseteq \iota$ and J is finite.
2. Let ι be some (index) set and $(\alpha_i)_{i \in \iota}$ a family of sets. For $J \subseteq \iota$, we denote $\alpha_J := \prod_{j \in J} \alpha_j$ and $\pi_J : \alpha_\iota \to \alpha_J$ the projection. For $H \subseteq J \subseteq \iota$, we write π_H^J for the projection $\alpha_J \to \alpha_H$.
3. Let $\mathcal{F}_i$ be a σ-algebra in α_i, $i \in \iota$. For $J \subseteq_f \iota$, let $\mathcal{F}_J$ be the product-σ-algebra on α_J, and $\mathcal{F}_\iota$ be the σ-algebra generated by cylinder sets $\{\pi_J^{-1} \prod_{j \in J} A_j : J \subseteq_f \iota, A_j \in \sigma(E_j), j \in J\}$.
4. A family $(P_J)_{J \subseteq_f I}$, where P_J is a finite measure on $\mathcal{F}_J$, is called projective if

$$P_H = (\pi_H^J)_* P_J$$

for all $H \subseteq J \subseteq_f I$. (Recall that $A \mapsto (\pi_H^J)_* P_J(A) := P_J((\pi_H^J)^{-1} A)$ is called the image measure of P_J under π_H^J.)
5. If, for some projective family $(P_J)_{J \subseteq_f \iota}$, there is a finite measure P_ι on $\mathcal{F}_\iota$ with $P_J = (\pi_J)_* P_\iota$ for all $J \subseteq_f \iota$, then we call P_ι projective limit of $(P_J)_{J \subseteq_f \iota}$.

In contrast to statements already part of mathlib, in the sequel we highlight definitions and statements which are part of our own code. In the formalization of the above, we use **variable** {ι : **Type** _} {α : ι → **Type** _}, which fixes the index set ι and all spaces $\alpha_t, t \in \iota$ as global variables. In addition, note that the projective property we use here works as long as we have a preorder (which is the subset relation on Finset ι below).

```
def IsProjective
  [Preorder ι]
```

```
(P : ∀ j : ι, α j) (π : ∀ {i j : ι}, j ≤ i → α i → α j) : Prop :=
∀ (i j) (hji : j ≤ i), P j = π hji (P i)
```

With this, we can define the projetive family as follows. The typeclass inference system of Lean automatically uses the subset relation to generate [Preorder (Finset ι)] when isProjective is called.

```
def IsProjectiveMeasureFamily
  [∀ i, MeasurableSpace (α i)]
  (P : ∀ J : Finset ι, Measure (∀ j : J, α j)) :
  Prop :=
  IsProjective P (fun I _ hJI μ => μ.map
  fun x : ∀ i : I, α i => fun j => x ⟨j, hJI j.2⟩ :
  ∀ (I J : Finset ι) (_ : J ⊆ I), Measure (∀ i : I, α i)
  → Measure (∀ j : J, α j))
```

It is worth understanding the precise connection of isProjective and isProjectiveMeasureFamily. In the latter, the first variable of isProjective is the family P of finite measures for all finite subsets of ι. The second variable is the functions which maps two sets I J : Finset ι and a proof hJI of J ⊆ I together with P I (which is μ in the statement) to the image measure on J, which is μ.map (fun x : ∀ i : I, α i => fun j => x ⟨j, hJI j.2⟩). The map defined by the =>-notation maps every $(x_i)_{i\in I}$ to a function of j, whose type is the subtype of I, consisting of a value and a proof of J ⊆ I. In other words, this is $(x_j)_{j\in J}$.
Now we are ready to formulate the Kolmogorov extension theorem:

Theorem 1 (Kolmogorov extension). *For all $t \in \iota$, let α_t be a separable, complete pseudo-extended-metric space and $\mathcal{F}_t$ the Borel σ-algebra generated by its topology. Let $(P_J)_{J\subseteq_f \iota}$ be a projective family of finite measures and P be defined on $\mathcal{A} := \bigcup_{J\subseteq_f \iota} \mathcal{F}_J$ given by $P(A) = P_J(\pi_J A)$ for $A \in \mathcal{F}_J$. Then, there is a unique extension of P to $\sigma(\mathcal{A})$.*

Rather than giving the formalization of this theorem, we give the definition of the resulting measure (which is the projective limit). We give the formalized proof at the end of this section, since we first have to provide a formalization of all tools needed in the proof.

```
def projectiveLimitWithWeakestHypotheses
  [∀ i, PseudoEMetricSpace (α i)]
  [∀ i, BorelSpace (α i)]
  [∀ i, SecondCountableTopology (α i)]
  [∀ i, CompleteSpace (α i)] [∀ i, Nonempty (α i)]
  (P : ∀ J : Finset ι, Measure (∀ j : J, α j))
  [∀ i, IsFiniteMeasure (P i)]
  (hP : IsProjectiveMeasureFamily P) :
  Measure (∀ i, α i)
```

We note that we extend the standard assumption that all α_t are separable, complete metric spaces (or Polish, i.e. separable and metrizable through a complete metric) to cover the case of extended pseudemetric spaces. Such spaces do not satisfy the frequently used Hausdorff (or t2) property, i.e. there can be $x \neq y$ such that all open balls around x and y overlap. We note that this generalization was only possible since underlying results in mathlib were already provided on the same level of generality: More precisely, isCompact_iff_totallyBounded_isComplete, which shows that a set $A \subseteq \alpha$ is compact iff it is complete and totally bounded (see note 4), requires α to be a uniform space (recall that every metric space is uniform); recall such spaces from Remark 2.2, and see note 4 for a definition of total boundedness. Since we also require the underlying space(s) to be second countable (see also in the proof of Lemma 2.20), we have to make a countabiliy assumption, which leads us to (extended) pseudo-metric spaces by UniformSpace.pseudoMetrizableSpace; see also Remark 2.2 for some more details.

2.2 Extending a set function

In the formulation of Theorem 1, we extend P, which is defined on a union of σ-algebras. However, unions of σ-algebras in general are not σ-algebras, but they can be used to define the σ-algebra generated by the union. So, we need to extend P to the σ-algebra generated by $\mathcal{A}$. This is exactly what Carathéodory's extension theorem is made for. In fact, we implemented this result in greater generality than needed for the proof of the extension theorem. Note that $\mathcal{A}$ in Theorem 1 is a ring of sets (see the next definition) containing the whole set. (This is sometimes called a field of sets.) We will work with the weaker semi-ring as introduced next in the formulation of Carathéodory's extension theorem; see e.g. [10, Definition 1.9].

Definition 2.8 (Semi-ring, ring). *Let α be some set. We call $\mathcal{H} \subseteq 2^\alpha$ a <u>semi-ring</u>, if it is (i) a π-system (i.e. closed under $\cap$) and (ii) for all $A, B \in \mathcal{H}$ there is $\mathcal{K} \subseteq_f \mathcal{H}$ with[2] $B \setminus A = \biguplus_{K\in\mathcal{K}} K$.*
We call $\mathcal{H} \subseteq 2^\alpha$ a <u>ring</u>, if it is closed under $\cup$ and under set-differences.

Any ring is a semi-ring since $A \cap B = A \setminus (A \setminus B)$, i.e. every ring is a π-system. Using variable {α : Type _} {s t : Set α}, we define a semi-ring as follows:

```
structure SetSemiring (C : Set (Set α)) : Prop where
  empty_mem : ∅ ∈ C
  inter_mem : ∀ (s) (_ : s ∈ C) (t) (_ : t ∈ C), s ∩ t ∈ C
  diff_eq_Union' :
```

[2] We write $A \uplus B$ for $A \cup B$ if $A \cap B = \emptyset$.

```
  ∀ (s) (_ : s ∈ C) (t) (_ : t ∈ C),
  ∃ (I : Finset (Set α)) (_h_ss : ↑I ⊆ C)
  (_h_dis : PairwiseDisjoint (I : Set (Set α)) id),
   t \ s = ⋃₀ I
```

Let us remark that (i) Lean indicates a coercion by ↑, which in this case is from Finset to Set and (ii) we have PairwiseDisjoint (s : Set ι) (f : ι → α) iff the images of any distinct elements of ι under f are different. (Hence, if f = id, the usual definition of pairwise disjoint sets unfolds.) We do not show the formalization of rings here. The import ring of sets in our formalization is the ring of measurable cylinders on a product space, which are defined as follows:

```
def cylinder
  (s : Finset ι) (S : Set (∀ i : s, α i)) :
  Set ((i : ι) → α i) :=
  (fun f : (i : ι) → α i => fun i : s => f i) ⁻¹' S
```

In this definition, s is a finite subset of the index set ι, and for some S in the finite product $\prod_{i\in s}\alpha_i$, consider the projection $\pi_s : \prod_{i\in\iota}\alpha_i \longrightarrow \prod_{i\in s}\alpha_i$, and consider the preimage of S. (This is what the last line in previous definition gives.) Note, however, that s (comprising the potential coordinates where the cylinder deviates from the whole set) need not be unique. However, we have

```
theorem mem_cylinders
  (t : Set ((i : ι) → α i)) :
  t ∈ cylinders α ↔
  ∃ (s S : _) (_ : MeasurableSet S), t = cylinder s S
```

By the exists-statement, from this result, we are able to choose a Finset ι, which can be used. This leads to the definition of cylinders.finset, which will be needed below.

From the definition of a cylinder, we can define the set of all measurable cylinders.

```
def cylinders : Set (Set ((i : ι) → α i)) := ⋃ (s) (S) (_ :
MeasurableSet S), {cylinder s S}
```

The set-system of cylinders is in fact a field, hence a ring and hence a semi-ring. This means we can prove the following:

```
theorem setField_cylinders : SetField (cylinders α)
theorem setRing_cylinders :  SetRing (cylinders α)
theorem setSemiringCylinders : SetSemiring (cylinders α)
```

From the field/ring/semi-ring of cylinders, we have to define the generated σ-algebra. This uses the following:

```
theorem generateFrom_cylinders :
  MeasurableSpace.generateFrom (cylinders α)
  = MeasurableSpace.pi

instance MeasurableSpace.pi
  [m : ∀ a, MeasurableSpace (π a)] :
  MeasurableSpace (∀ a, π a) :=
  iSup a, (m a).comap fun b => b a
```

Here, comap {β : Type} (f : α → β) (m : MeasurableSpace β) is the measurable space consisting of pre-images of measurable subsets of β, and the set of σ-algebras on the product $\prod_{i\in\iota}\alpha_i$ (or on any other space) is a complete lattice, i.e. the subset defined by the comaps has a supremum, which defines the σ-algebra generated by the cylinders.

Let us state an important lemma on semi-rings:

Lemma 2.9. *Let $\mathcal{H}$ be a semi-ring, $\mathcal{I} \subseteq_f \mathcal{H}$, $A \in \mathcal{H}$. Then, there is $\mathcal{K} \subseteq_f \mathcal{H}$ such that $\mathcal{K}$ contains pairwise disjoint sets and $A \setminus \bigcup_{I\in\mathcal{I}} I = \biguplus_{K\in\mathcal{K}} K$.*

Proof. We proceed by induction on $\mathcal{I}$. If $\mathcal{I}$ is a singleton, the assertion is true by the definition of a semi-ring. If it holds for some $\mathcal{I}$ (i.e. there is $\mathcal{K} \subseteq_f \mathcal{H}$ with $A \setminus \bigcup_{I\in\mathcal{I}} I = \biguplus_{K\in\mathcal{K}} K$), let us consider $\mathcal{I}' = \{J\} \cup \mathcal{I}$ for some $J \notin \mathcal{I}$. For each $K \in \mathcal{K}$, Write $K \setminus J = \biguplus_{J_K\in\mathcal{J}_K} J_K$ for some $\mathcal{J}_K \subseteq_f \mathcal{H}$ (which exists by the definition of a semi-ring). Then, write

$$A \setminus \bigcup_{I'\in\mathcal{I}'} I = \Big(A \setminus \bigcup_{I\in\mathcal{I}} I\Big) \setminus J = \biguplus_{K\in\mathcal{K}} K \setminus J = \biguplus_{K\in\mathcal{K}} \biguplus_{J_K\in\mathcal{J}_K} J_K.$$

This concludes the proof, since the latter disjoint union is over a finite set. □

The proof as well as its formalization is somewhat straight-forward, but requires induction over finite sets.

```
theorem exists_disjoint_finset_diff_eq
  (hC : SetSemiring C) (hs : s ∈ C)
  (I : Finset (Set α)) (hI : ↑I ⊆ C) :
  ∃ (J : Finset (Set α)) (_h_ss : ↑J ⊆ C)
  (_h_dis : PairwiseDisjoint (J : Set (Set α)) id),
  s \ ⋃₀ I = ⋃₀ J
```

The existence-statement of the above lemma actually gives rise to a definition. Here, we use Exists.choose in order to extract an element from an ∃-statement. In addition, we do not allow $\emptyset \in \mathcal{K}$ without loss of generality:

```
def diff₀ (hC : SetSemiring C)
  (hs : s ∈ C) (I : Finset (Set α)) (hI : ↑I ⊆ C)
  [DecidableEq (Set α)] : Finset (Set α) :=
  (hC.exists_disjoint_finset_diff_eq hs I hI).choose \ {∅}
```

that exclusion was useful for something. Find out what that is

Extending Lemma 2.9, we would like to write a finite union of members of a semi-ring as a finite union of disjoint sets.

Is this section going into too much detail?

Lemma 2.10. *Let $\mathcal{H}$ be a semi-ring and $A_1, \dots, A_m \in \mathcal{H}$. Then, there are $\mathcal{K}_1, \dots, \mathcal{K}_m \subseteq_f \mathcal{H}$ disjoint such that $\bigcup_{n=1}^m \mathcal{K}_n$ contains disjoint sets and $\bigcup_{m=1}^n A_m = \biguplus_{m=1}^n \biguplus_{K \in \mathcal{K}_n} K$.*

Proof. Indeed, we may write $\bigcup_{m=1}^n A_m = \biguplus_{n=1}^m \Big(A_n \setminus \bigcup_{i=1}^{n-1} A_i \Big)$. Then, the result follows by applying Lemma 2.9 to $A_n \setminus \bigcup_{i=1}^{n-1} A_i$, $n = 1, \dots, m$. □

The formalized version again gives rise to a definition:

```
def indexedDiff₀ (hC : SetSemiring C)
    (J : Finset (Set α)) (hJ : ↑J ⊆ C)
    (n : Fin J.card) : Finset (Set α) :=
  hC.diff₀ (ordered_mem' hJ n) (finsetLT J n)
    (finsetLT_subset' J hJ n)
```

Here, recall that `Fin n` is the subtype of `Nat` consisting of all numbers `<n`. In addition, `ordered_mem' (hJ : ↑J ⊆ C) (n : Fin (card J)) : ↑(ordered J) n ∈ C` gives a proof that the `n`th element of `J` is in `C`, `finsetLT (J : Finset (Set α)) (n : Fin (card J)) : Finset (Set α)` is the set consisting of the `n` sets in `J` numbered $0, \dots, n-1$, and `finsetLT_subset'` gives a proof that `↑(finsetLT J n) ⊆ C`. So, in terms of Lemma 2.10, this gives, for $n = 1, \dots, m$, a construction for $\mathcal{K}_n \subseteq_f \mathcal{H}$.

2.3 Carathéodory's extension theorem

Measures are set functions defined on a σ-algebra (i.e. an algebra stable under countable unions), satisfying some properties which we recall below. Mostly, defining such a measure on the full σ-algebra is not possible, but defining a set function on a semi-ring is possible. As an example, consider Lebesgue-measure on $\mathbb{R}$, with the Borel σ-algebra on $\mathbb{R}$, which is $\sigma(\mathcal{O})$, where $\mathcal{O}$ is the set of open sets, and $\sigma(\mathcal{O})$ is the smallest σ-algebra containing all open sets. Since this set system is defined only abstractly, it is hard to know which volume each of these sets should be assigned to at first sight. However, the volume of an interval is easy, since it may be defined by the length of the interval. So, in order to construct measures from a set function m on a (semi-)ring $\mathcal{H}$ (e.g. the set of all semi-open intervals), it has been a fundamental insight of Carathéodory that one may start by defining a set function (outer measure) μ on 2^α, and then show μ extends m and defines a measure on $\sigma(\mathcal{H})$. (Note that the set of semi-open intervals generates the Borel σ-algebra on $\mathbb{R}$.) We will follow this abstract construction, and start by stating some basic concepts.

Definition 2.11. *For some set α, let $\mathcal{H} \subseteq 2^\alpha$ and call any $m : \mathcal{H} \to [0, \infty]$ a content.*
1. m is called additive if for $\mathcal{K} \subseteq_f \mathcal{H}$ pariwise disjoint and $\bigcup_{K \in \mathcal{K}} K \in \mathcal{H}$, we have $m\Big(\bigcup_{K \in \mathcal{K}} K\Big) = \sum_{K \in \mathcal{K}} m(K)$. If the same holds for[3] $\mathcal{K} \subseteq_c \mathcal{H}$ pariwise disjoint, we say that m is σ-additive.
2. The set-function m is called sub-additive if for $\mathcal{K} \subseteq_f \mathcal{H}$ and $\bigcup_{K \in \mathcal{K}} K \in \mathcal{H}$, we have $m\Big(\bigcup_{K \in \mathcal{K}} K\Big) \le \sum_{K \in \mathcal{K}} m(K)$. (Note that elements of $\mathcal{K}$ need not be disjoint.) Here, σ-sub-additivity is defined in the obvious way using $\mathcal{K} \subseteq_c \mathcal{H}$.
3. If $m(A) \le m(B)$ for $A \subseteq B$ and $A, B \in \mathcal{H}$, we say that m is monotone.
4. If $\mathcal{H}$ is a σ-algebra and m is σ-additive with $m(\emptyset) = 0$, we call m a measure.
5. If $\mathcal{H} = 2^\alpha$ and m is monotone and σ-sub-additive with $m(\emptyset) = 0$, we call m an outer measure.

For additive contents, we need two definitions. `AddContent` defines a set-function `toFun` on `Set α`, whereas `extendContent` comes with `m : ∀ s : Set α, s ∈ C → ℝ≥0∞`.

```
structure AddContent (C : Set (Set α)) where
  toFun : Set α → ℝ≥0∞
  empty' : toFun ∅ = 0
  add' : ∀ (I : Finset (Set α))
    (_h_ss : ↑I ⊆ C) (_h_dis : PairwiseDisjoint
      (I : Set (Set α)) id)
    (_h_mem : ⋃₀ ↑I ∈ C),
    toFun (⋃₀ I) = ∑ u in I, toFun u

def extendContent
    (hC : SetSemiring C)
    (m : ∀ s : Set α, s ∈ C → ℝ≥0∞)
    (m_empty : m ∅ hC.empty_mem = 0)
    (m_add :
      ∀ (I : Finset (Set α)) (h_ss : ↑I ⊆ C)
      (_h_dis : PairwiseDisjoint (I : Set (Set α)) id)
      (h_mem : ⋃₀ ↑I ∈ C),
      m (⋃₀ I) h_mem = ∑ u : I, m u (h_ss u.prop)) :
    AddContent C
```

~~In these formalizations, we introduced $m(\emptyset) = 0$ as a separate hypothesis with empty'. Note that this also follows formally from $m(\emptyset) = m(\emptyset \cup \emptyset) = 2m(\emptyset)$. However, since Finsets are assumed to have distinct elements, we cannot have $\emptyset$ twice in I.~~

For the concrete application we have in mind, we introduce a definition which uses a specific semi-ring, the measurable cylinders, based on a projective family

[3] We write $A \subseteq_c B$ if A is a countable subset of B.

of measures. That is the goal of the two definitions `kolmogorovFun` and `kolContent` below:

```
def kolmogorovFun
  (P : ∀ J : Finset ι, Measure (∀ j : J, α j))
  (s : Set (∀ i, α i))
  (hs : s ∈ cylinders α) : ℝ≥0∞ :=
 P (cylinders.finset hs) (cylinders.set hs)

def kolContent
  (hP : IsProjectiveMeasureFamily P) :
  AddContent (cylinders α) :=
 extendContent setSemiringCylinders
  (kolmogorovFun P) (kolmogorovFun_empty hP)
  (kolmogorovFun_additive hP)
```

Here, `kolmogorovFun_empty` and `kolmogorovFun_additive` give proofs for `m_empty` and `m_add` in `extendContent` as applied to a `kolmogorovFun`.
From the next lemma, we will need monotonicity and sub-additivity, as well as σ-additive $\Rightarrow$ σ-sub-additive in the proof of the Carathéodory extension, Theorem 3. We do not show any details about its formalization here.

Lemma 2.12 (Set-functions on semi-rings)**.** *Let $\mathcal{H}$ be a semi-ring and $m : \mathcal{H} \to [0, \infty]$ additive. Then, m is monotone and sub-additive. In addition, m is σ-additive iff it is σ-sub-additive.*

Proof. We start by monotonicity. Let $A, B \in \mathcal{H}$ with $A \subseteq B$ and $\mathcal{K} \subseteq_f \mathcal{H}$ with $B \setminus A = \biguplus_{K \in \mathcal{K}} K$. Therefore, we can write $m(A) \leq m(A) + \sum_{K \in \mathcal{K}} m(K) = m(B)$.

Next, we claim that for $\biguplus_{I \in \mathcal{I}} I \subseteq A$ with all sets belonging to $\mathcal{H}$, we have $\sum_{I \in \mathcal{I}} m(I) \leq m(A)$. For this, write $A \setminus \biguplus_{I \in \mathcal{I}} I = \biguplus_{K \in \mathcal{K}} K$ as in Lemma 2.9. Then,

$$m(A) = m\Big(\biguplus_{I \in \mathcal{I}} I \uplus \biguplus_{K \in \mathcal{K}} K\Big) = \sum_{I \in \mathcal{I}} m(I) + \sum_{K \in \mathcal{K}} m(K) \geq \sum_{I \in \mathcal{I}} m(I).$$

For sub-additivity, let $\mathcal{I} \subseteq_f \mathcal{H}$ with $\bigcup_{I \in \mathcal{I}} I \in \mathcal{H}$. Without loss of generality, we write $\mathcal{I} = \{I_1, ..., I_n\}$ for some n. We need to show $m\Big(\bigcup_{k=1}^n I_i\Big) \leq \sum_{k=1}^n m(I_k)$. For $k = 2, ..., n$, we write

$$\bigcup_{k=1}^n I_k = \biguplus_{k=1}^n \Big(I_k \setminus \bigcup_{j=1}^{k-1} I_j\Big) = \biguplus_{k=1}^n \biguplus_{K_k \in \mathcal{K}_k} K_k$$

with $\mathcal{K}_k$ as in Lemma 2.9. So, since $\biguplus_{K_k \in \mathcal{K}_k} K_k \subseteq I_k \in \mathcal{H}$,

$$m\Big(\bigcup_{k=1}^n I_i\Big) = \sum_{k=1}^n \sum_{K_k \in \mathcal{K}_k} m(K_k) \leq \sum_{k=1}^n m(I_k).$$

Now, we show that m is σ-additive $\iff$ it is σ-sub-additive.
'$\Rightarrow$': Here, just copy the proof of sub-additivity, but using countable $\mathcal{I}$, i.e. $n = \infty$. For '$\Leftarrow$', let $\mathcal{I} \subseteq_c \mathcal{H}$ and consist of disjoint sets with $A = \biguplus_{I \in \mathcal{I}} I \in \mathcal{H}$. Since m is monotone and for any $\mathcal{I}' \subseteq_f \mathcal{I}$, we have $\biguplus_{I \in \mathcal{I}'} I \subseteq A$ (hence $\sum_{I \in \mathcal{I}'} m(I') \leq m(A)$),

$$\sum_{I \in \mathcal{I}} m(I) = \sup_{\mathcal{I}' \subseteq_f \mathcal{I}} \sum_{I \in \mathcal{I}'} m(I) \leq m(A) \leq \sum_{I \in \mathcal{I}} m(I)$$

by σ-sub-additivity. So, σ-additivity follows. □

Although some material on outer measures was covered in mathlib already, the classical extension theorem (extending a set function m on a semi-ring $\mathcal{H}$)) was not provided yet. In particular, this result states that the outer measure coincides with m on $\mathcal{H}$. All statements on equality of μ and m present in mathlib at the time of writing have too many requirements: they all require m to be a σ-additive function defined on a σ-algebra.

The next result extends a set function on a semi-ring to an outer measure.

Proposition 2.13 (Outer measure induced by a set function on a semi-ring)**.** *Let $\mathcal{H}$ be a semi-ring and $m : \mathcal{H} \to \mathbb{R}_+$ additive. For $A \subseteq E$ let*

$$\mu(A) := \inf_{\mathcal{G} \in \mathcal{U}(A)} \sum_{G \in \mathcal{G}} m(G)$$

where

$$\mathcal{U}(A) := \{\mathcal{G} \subseteq_c \mathcal{H}, A \subseteq \bigcup_{G \in \mathcal{G}} G\}$$

is the set of countable coverings of A. Then, μ is an outer measure.

Let us now formulate the classical results by Carathéodory. The first leads to the definition of the measurable space `OuterMeasure.caratheodory`, covered in mathlib. See e.g. [9, Theorem 2.1]

Theorem 2 (μ-measurable sets are a σ-algebra)**.** *Let μ be an outer measure on E and $\mathcal{F}$ the set of μ-measurable sets, i.e. the set of sets A satisfying*

$$\mu(B) = \mu(B \cap A) + \mu(B \cap A^c), \qquad B \subseteq E. \qquad (1)$$

Then, $\mathcal{F}$ is a σ-Algebra and $\mu|_{\mathcal{F}}$ is a measure. In addition, $\{N \subseteq \Omega : \mu(N) = 0\} \subseteq \mathcal{F}$, i.e. $\mathcal{F}$ is complete.

The second result states that for an outer measure induced by an additive content on a semi-ring, we have $\sigma(\mathcal{H}) \subseteq \mathcal{F}$. In particular, we then have defined a measure on $\sigma(\mathcal{H})$. This result is not yet covered in mathlib. However, for product spaces, in `MeasureTheory.Constructions.Pi`, there is `pi_caratheodory`, which shows that $\sigma(\mathcal{H}) \subseteq \mathcal{F}$ in the construction of a product measure for a finite index set. In addition, there is `pi_pi_aux`, which shows that μ extends m on $\mathcal{H}$ in the same special case. See also [9, Theorem 2.5].

Theorem 3 (Carathéodory extension)**.** *Let $\mathcal{H}$ be a semi-ring and $m : \mathcal{H} \to \mathbb{R}_+$ σ-finite and σ-additive. Furthermore, let μ be the induced outer measure from Proposition 2.13 and $\mathcal{F}$ the σ-algebra from Theorem 2. Then, $\sigma(\mathcal{H}) \subseteq \mathcal{F}$ and μ coincides with m on $\mathcal{H}$.*

Proof. First, m is σ-sub-additive by Lemma 2.12.

Step 1: $\mu|_{\mathcal{H}} = m$: Let $A \in \mathcal{H}$. Choose $\mathcal{K} \subseteq_c \mathcal{H}$ with $A \subseteq \bigcup_{K\in\mathcal{K}} K$ and

$$\mu(A) \geq \sum_{K\in\mathcal{K}} m(K) - \varepsilon.$$

By $A = \bigcup_{K\in\mathcal{K}} K \cap A$,

$$\mu(A) \leq m(A) \leq \sum_{K\in\mathcal{K}} m(K\cap A) \leq \sum_{K\in\mathcal{K}} m(K) \leq \mu(A) + \varepsilon,$$

where we have used σ-sub-additivity of m in the second and monotonicity of m in the third $\leq$ (see Lemma 2.12). With $\varepsilon \to 0$ we find that $\mu(A) = m(A)$.

Step 2: $\sigma(\mathcal{H}) \subseteq \mathcal{F}$: Let $B \subseteq E$, $A \in \mathcal{H}$ and $\varepsilon > 0$. Choose $\mathcal{K} \subseteq_c \mathcal{H}$ such that $B \subseteq \bigcup_{K\in\mathcal{K}} K$ and $\mu(B) \geq \sum_{K\in\mathcal{K}} m(K) - \varepsilon$. Then, by additivity of m,

$$\begin{aligned}\mu(B) + \varepsilon &\geq \sum_{K\in\mathcal{K}} \mu(K) = \sum_{K\in\mathcal{K}} \mu(K\cap A) + \sum_{K\in\mathcal{K}} \mu(K\cap A^c)\\ &\geq \mu(B\cap A) + \mu(B \cap A^c).\end{aligned}$$

By sub-additivity of μ, we find that $\mu(B) \leq \mu(B\cap A) + \mu(B\cap A)$, so letting $\varepsilon \to 0$ leads to $\mu(A) = \mu(E\cap A) + \mu(E\cap A^c)$. This implies that A is $\mathcal{F}$-measurable, and we have shown $\sigma(\mathcal{H}) \subseteq \mathcal{F}$, since $\mathcal{F}$ is a σ-algebra. ◻

Here is a formalization of the measure resulting from Theorem 3:

```
def Measure.ofAddContent
  [mα : MeasurableSpace α] (hC : SetSemiring C)
  (hC_gen : MeasurableSpace.generateFrom C = mα)
  (m : AddContent C)
  (m_sigma_subadd : ∀ ⦃f : ℕ → Set α⦄
    (hf : ∀ i, f i ∈ C) (hf_Union : (⋃ i, f i) ∈ C),
    m (⋃ i, f i) hf_Union ≤ ∑' i, m (f i) (hf i)) :
  Measure α
```

2.4 σ-additivity of set functions

For the proof of Kolmogorov's extension Theorem, note that P as given in Theorem 1 is a finite additive set function on the ring $\mathcal{A}$. In order to use Theorem 3, we therefore have to show σ-additivity. For this, we will use Lemma 2.17 below, i.e. inner regularity of P with respect to compact sets. Before, we will show useful alternative conditions for σ-additivity, which do not make use of any topological structure of the underlying space.

Lemma 2.14 (σ-additivity and continuity at $\emptyset$)**.** *Let $\mathcal{R}$ be a ring and $m : \mathcal{R} \to \mathbb{R}_+$ additive. Then, the following are equivalent:*

1. m is σ-additive;
2. m is σ-sub-additive;
3. m is continuous from below, i.e. for $A, A_1, A_2, \ldots \in \mathcal{R}$ with $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \bigcup_{n=1}^\infty A_n$, we have $m(A) = \lim_{n\to\infty} m(A_n)$.
4. m is continuous from above in $\emptyset$, i.e. for $A_1, A_2, \ldots \in \mathcal{R}$ with $A_1 \supseteq A_2 \supseteq \ldots$ and $\bigcap_{n=1}^\infty A_n = \emptyset$, we have $\lim_{n\to\infty} m(A_n) = 0$.

Proof. 1.⇔2. was already shown in Lemma 2.12, since $\mathcal{R}$ is a semi-ring.

1.⇒3.: For $A_1, A_2, \ldots$ as in 3., we write (with $A_0 := \emptyset$)

$$m(A) = \lim_{n\to\infty} m\Big(\biguplus_{k=1}^n A_k \setminus \Big(\Big(\bigcup_{l=1}^{k-1} A_l\Big)\Big)\Big) = \lim_{n\to\infty} m(A_n).$$

3.⇒1.: Let $A_1, A_2, \ldots \in \mathcal{R}$ be disjoint and $A = \biguplus_{n=1}^\infty A_n$. Set $B_n := \bigcup_{k=1}^n A_k$, such that $B_1, B_2, \ldots$ satisfy 3. Hence,

$$\begin{aligned} m\Big(\biguplus_{n=1}^\infty A_n\Big) &= m\Big(\bigcup_{n=1}^\infty B_n\Big) = \lim_{n\to\infty} m\Big(\bigcup_{k=1}^n B_n\Big)\\ &= \lim_{n\to\infty} m\Big(\biguplus_{k=1}^n A_n\Big) = \sum_{n=1}^\infty m(A_n).\end{aligned}$$

3.⇒4.: Let $A_1, A_2, \cdots \in \mathcal{R}$ as in 4. Set $B_n := A_1 \setminus A_n$. Then, $B = A_1, B_1, B_2, \cdots \in \mathcal{R}$ satisfy 3., and therefore

$$m(A_1) = \lim_{n\to\infty} m(B_n) = m(A_1) - \lim_{n\to\infty} m(A_n),$$

and 4. follows.

4.⇒3.: Let $A, A_1, A_2, \cdots \in \mathcal{R}$ as in 3. Let $B_n := A \setminus A_n \in \mathcal{R}, n \in \mathbb{N}$. Then, $\bigcap_{n=1}^\infty B_n = \emptyset$, so

$$0 = \lim_{n\to\infty} \mu(B_n) = \mu(A) - \lim_{n\to\infty} \mu(A_n),$$

and 3. follows. ◻

As an example, we give the formalization of 4. ⇒ 1.:

Rename the todo! And replace by the version about contents?

```
theorem countably_additive_of_todo
  (hC : SetRing C)
  (m : ∀ s : Set α, s ∈ C → ℝ≥0∞)
  (hm_ne_top : ∀ {s} (hs : s ∈ C), m s hs ≠ ∞)
  (hm_add : ∀ {s t : Set α} (hs : s ∈ C) (ht : t ∈ C),
    Disjoint s t →
    m (s ∪ t) (hC.union_mem hs ht) = m s hs + m t ht)
  (hm : ∀ ⦃s : ℕ → Set α⦄ (hs : ∀ n, s n ∈ C),
    Antitone s → (⋂ n, s n) = ∅ →
    Tendsto (fun n => m (s n) (hs n)) atTop (nhds 0))
    ⦃f : ℕ → Set α⦄ (h : ∀ i, f i ∈ C) (hUf : (⋃ i, f i) ∈ C)
```

```
    (h_disj : Pairwise (Disjoint on f)) :
  m (⋃ i, f i) hUf = ∑' i, m (f i) (h i)
```

Next, we will be extending our analysis to the case of a topological space. We define inner (and outer) regularity of set functions.

Definition 2.15 (Inner regularity)**.** *Let α be some set, equipped with a topology, and m be a set-function on some $\mathcal{H} \subseteq 2^\alpha$.*

1. Let $p, q : 2^\alpha \to \{true, false\}$. Then, m is called inner regular with respect to p and q, if

$$m(A) = \sup\{m(F) : p(F) = true, F \subseteq A\}$$

for all $A \in \mathcal{H}$ with $q(A) = true$.

2. If $q(A) = true$ iff A is measurable, we neglect the and q. If $p(A) = true$ iff A is closed (compact, closed and compact), we say that m is inner regular with respect to closed (compact, compact and closed) sets.

The above definition closely resembles its formalization in mathlib:

```
def MeasureTheory.Measure.InnerRegular
  {α : Type u_1}  {_ : MeasurableSpace α}
  (μ : Measure α) (p q : Set α → Prop) :=
  ∀ ⦃U⦄, q U → ∀ r < μ U, ∃ K, K ⊆ U ∧ p K ∧ r < μ K
```

For the next result, recall that for compact sets $C_1, C_2, \dots$ with $\bigcap_{n=1}^\infty C_n = \emptyset$, there is some N with $\bigcap_{n=1}^N C_n = \emptyset$. More generally, compact sets form a compact system, which is defined as follows:

Definition 2.16. *Let $\mathcal{C} \subseteq 2^\alpha$. If, for all $C_1, C_2, \dots$ with $\bigcap_{n=1}^\infty C_n = \emptyset$, there is some N with $\bigcap_{n=1}^N C_n = \emptyset$, we call $\mathcal{C}$ a compact system.*

Here is the formalization:

```
def IsCompactFamily (p : Set α → Prop) : Prop :=
  ∀ C : ℕ → Set α, (∀ i, p (C i)) → ⋂ i, C i = ∅ →
  ∃ (s : Finset ℕ), ⋂ i ∈ s, C i = ∅
```

Lemma 2.17. *Let α be a topological space and μ be an additive set function on a ring $\mathcal{R}$ contained by the Borel σ-algebra, and which is inner regular with respect to a compact system. Then, μ is σ-additive.*

Proof. According to Lemma 2.14, we need to show continuity of μ in $\emptyset$, so let $A_1 \supseteq A_2 \supseteq \dots \in \mathcal{R}$ satisfy $\bigcap_{n=1}^\infty A_n = \emptyset$ and $\varepsilon > 0$. Let $\delta_1, \delta_2, \dots > 0$ with $\sum_{n=1}^\infty \delta_n < \varepsilon$. For each n, let $\mathcal{C} \ni C_n \subseteq A_n \in \mathcal{R}$ with $\mu(A_n) \leq \mu(C_n) + \delta_n$. We have that $\bigcap_{n=1}^\infty C_n \subseteq \bigcap_{n=1}^\infty A_n = \emptyset$, so there is $N \in \mathbb{N}$ with $\bigcap_{n=1}^N C_n = \emptyset$ since $\mathcal{C}$ is a compact system. So, for any $m > N$, we have that

$$\mu(A_m) = \mu\Big(\Big(\bigcap_{n=1}^m A_n\Big) \setminus \Big(\bigcap_{n=1}^m C_n\Big)\Big) \leq \sum_{n=1}^m \mu(A_n \setminus C_n)$$
$$\leq \sum_{n=1}^m \delta_n < \varepsilon.$$

This concludes the proof. ◻

In order to apply this result and show the extension theorem, we need to show that $\{\pi_J^{-1}C : C \in \prod_{j\in J}\alpha_j \text{ compact and closed}\}$ is a compact systems. Note that compact sets are closed in Hausdorff spaces, but we do not have this property since we are working with pseudo-metric spaces. Since Lemma 2.17 gives the σ-additivity of a `kolContent`, which is defined through the projective family `P`, we have:

```
theorem kolContent_sigma_additive_of_innerRegular
  (hP : IsProjectiveMeasureFamily P)
  (hP_inner : ∀ J,
    (P J).InnerRegular (fun s => IsCompact s ∧ IsClosed s)
    MeasurableSet)
  ⦃f : ℕ → Set (∀ i, α i)⦄ (hf : ∀ i, f i ∈ cylinders α)
  (hf_Union : (⋃ i, f i) ∈ cylinders α)
  (h_disj : Pairwise (Disjoint on f)) :
  kolContent hP (⋃ i, f i) = ∑' i, kolContent hP (f i)
```

Note that the assumption `hP_inner` from above translates directly to inner regularity of the `kolContent`, which is defined through the projective family `P`. Moreover, since the Carathéodory extension theorem requires σ-subadditivity (rather than σ-additivity), we can use `countably_subadditive_of_countably_additive` in order to show `kolContent_countably_subadditive_of_innerRegular`, and the assumptions of the last result also imply σ-subadditivity.

What remains to be done is to show conditions under which the projective family is inner regular with respect to compact (and closed) sets. For this, we need some assumption on the underlying spaces. The following lemma provides the generalizations to extended pseudo-metric spaces:

The next result is already implemented in mathlib: Use `Measure.InnerRegular.of_pseudoEMetricSpace` in order to show that every open set is inner regular with respect to closed sets, and `Measure.InnerRegular.measurableSet_of_open` to show that every measurable set is inner regular with respect to closed sets. The claimed outer regularity is part of `MeasureTheory.Measure.InnerRegular.weaklyRegular_of_finite`.

Lemma 2.18. *Let r be an extended pseudo-metric on α, and the topology on α be given by r. If μ is a finite measure on the Borel σ-algebra $\mathcal{B}$ on α, it is inner regular* too much detail? omit the proof?

with respect to closed sets. In fact, we have also outer regularity with respect to open sets, i.e. for all $A \in \mathcal{B}$,

$$\mu(A) = \inf\{\mu(O) : A \subseteq O \text{ open}\}.$$

Proof. It suffices to prove that

$$\mathcal{A} := \Big\{A \in \mathcal{B} : \mu(B) = \sup_{F \subseteq B \text{ closed}} \mu(F) = \sup_{B \subseteq O \text{ open}} \mu(O)\Big\}$$

is a σ-algebra containing all closed sets. So, we proceed in two steps.

Step 1: $\mathcal{A}$ is a σ-algebra: Since μ is finite, we have that $\mu(A^c) = \mu(\alpha) - \mu(A)$ for all A. Using this, we already have that $\mathcal{A}$ is closed under complements. So, we are left with showing that $\mathcal{A}$ is closed under countable unions. For this, let $A_1, A_2, ... \in \mathcal{A}$ and $A := \bigcup_{n=1}^\infty A_n$. Fix $\varepsilon > 0$ and a sequence $\delta_1, \delta_2, ... > 0$ with $\sum_{n=1}^\infty \delta_n < \varepsilon$. For each n, let $F_n \subseteq A_n \subseteq O_n$ with F_n closed, O_n open and $\mu(O_n \setminus F_n) \le \delta_n$. Then, $O := \bigcup_{n=1}^\infty O_n$ is open and

$$0 \le \mu(O) - \mu(A) \le \sum_{n=1}^\infty \mu(O_n \setminus A) \le \sum_{n=1}^\infty \mu(O_n \setminus A_n) < \varepsilon.$$

This shows outer regularity of μ with respect to open sets at A. It remains to show inner regularity with respect to closed sets. For this, note that $\bigcup_{n=1}^N F_n$ is closed for all N and recall from Proposition 2.14 that μ is continuous from below. Therefore, setting $F := \bigcup_{n=1}^\infty F_n$,

$$0 \le \mu(A) - \lim_{N\to\infty} \mu\Big(\bigcup_{n=1}^N F_n\Big) = \mu(A \setminus F) \le \sum_{n=1}^\infty \mu(A_n \setminus F)$$
$$\le \sum_{n=1}^\infty \mu(A_n \setminus F_n) < \varepsilon.$$

This shows inner regularity of μ with respect to closed sets at A.

Step 2: $\mathcal{A}$ contains all closed sets: Let $\varepsilon_n \downarrow 0$ and, for $\varepsilon > 0$ and some $A \subseteq \alpha$, $A^\varepsilon := \{x : \exists y \in A, r(x,y) < \varepsilon\}$. Then, for A closed, we have that $A = \bigcap_{n=1}^\infty A^{\varepsilon_n}$ and A^{ε_n} is open for all n. Clearly, $\mu(A) = \sup\{\mu(F) : F \text{ closed}, F \subseteq A\}$, since A is closed. By continuity of μ, we find that $\mu(A^{\varepsilon_n}) \xrightarrow{n\to\infty} \mu(A)$, therefore $\mu(A) = \inf\{\mu(O) : F \text{ open}, A \subseteq O\}$, and we have shown that $A \in \mathcal{A}$. □

In the next lemma, we will need that a closed subset of a compact set is compact. (See isCompact_of_isClosed_subset.) We describe the formalization of the next lemma only in conjunction with Lemma 2.20.

Lemma 2.19. *Let r be an extended pseudo-metric on α, and the topology on α be given by r. If μ is a finite measure on the Borel σ-algebra $\mathcal{B}$ on α, the following are equivalent:*

1. For all $\varepsilon > 0$, there is some closed and compact K with $\mu(K^c) < \varepsilon$.

2. For all $\varepsilon > 0$ and $A \in \mathcal{B}$, there is some closed and compact $K \subseteq A$ with $\mu(A \setminus K) < \varepsilon$.

Proof. 2. ⇒ 1.: This is clear since $\alpha \in \mathcal{B}$. 1. ⇒ 2.: Fix $\varepsilon > 0$ and $A \in \mathcal{B}$. Let K be closed and compact with $\mu(K^c) < \varepsilon/2$. By Lemma 2.18, there is some closed $F \subseteq A$ with $\mu(A) < \mu(F) + \varepsilon/2$. Now, we have that $F \cap K \subseteq A$ is closed and compact, and with $A \setminus (F \cap K) = (A \setminus F) \cup (A \setminus K)$,

$$\mu(A \setminus (F \cap K)) \le \mu(A \setminus F) + \mu(A \setminus K) < \varepsilon/2 + \mu(K^c) < \varepsilon.$$

□

We apply this result by using that 1. is satisfied for complete and separable extended pseudo-metric spaces. Here, for the generalization to extended pseudo-metric spaces, we use that any subset of such a space is compact iff it is complete and totally bounded[4] (see isCompact_iff_totallyBounded_isComplete in mathlib). Since closed subsets of a complete space are complete, the closure of a totally bounded set is (still totally bounded, hence) compact. (This fact is used in the proof below.)

Lemma 2.20. *Let α be a complete and separable, extended pseudo-metric space, and μ a finite measure on its Borel σ-algebra $\mathcal{B}$. Then, for any $\varepsilon > 0$, there is some $K \subseteq \alpha$ with compact closure and*[5] *$\mu((\bar{K})^c) < \varepsilon$.*

Proof. Since α is separable, there is a subset $\{x_1, x_2, ...\} \subseteq_c \alpha$ with closure α. By separability, for each $n \in \mathbb{N}$, $\alpha = \bigcup_{k=1}^\infty B_{\varepsilon_n}(x_k)$. Since μ is continuous from above in $\emptyset$ (see Lemma 2.14)

$$0 = \mu\Big(\Big(\bigcup_{k=1}^\infty B_{\varepsilon_n}(x_k)\Big)^c\Big) = \lim_{N\to\infty} \mu\Big(\Big(\bigcup_{k=1}^N B_{\varepsilon_n}(x_k)\Big)^c\Big).$$

Let $\delta_n \downarrow 0$ be summable with $\sum_{n=1}^\infty \delta_n = 1$ (e.g. $\delta_n = 2^{-n}$). Then, there is some $N_n \in \mathbb{N}$ with $\mu\Big(E \setminus \bigcup_{k=1}^{N_n} B_{\varepsilon_n}(x_k)\Big) < \varepsilon\delta_n$. Now take

$$A := \bigcap_{n=1}^\infty \bigcup_{k=1}^{N_n} B_{\varepsilon_n}(x_k),$$

which by construction is totally bounded (for $r > 0$ take n such that $\varepsilon_n < r$, such that $B_r(x_1), ..., B_r(x_{N_n})$ cover A), and therefore has a compact closure as noted in the lines directly preceding the lemma. In addition, by σ-sub-additivity of μ,

$$\mu((\overline{A})^c) \le \mu(A^c) = \mu\Big(\bigcup_{n=1}^\infty \Big(\Big(\bigcup_{k=1}^{N_n} B_{\varepsilon_n}(x_k)\Big)^c\Big)\Big)$$
$$\le \sum_{n=1}^\infty \mu\Big(\Big(\bigcup_{k=1}^{N_n} B_{\varepsilon_n}(x_k)\Big)^c\Big) < \varepsilon,$$

and we are done. □

[4] A subset of a pseudo-metric space is totally bounded iff, for all $\varepsilon > 0$, it can be covered by finitely many balls of radius ϵ.

[5] We write $\bar{K}$ for the closure of K.

Let us combine the last two results in one combined formalized result. This the gives the desired result that a measure on a second countable, complete extended pseudo-metric space is inner regular with respect to closed compact sets.

```
theorem innerRegular_isCompact_isClosed_
  measurableSet_of_complete_countable
  [PseudoEMetricSpace α] [CompleteSpace α]
  [SecondCountableTopology α]
  [BorelSpace α] (P : Measure α) [IsFiniteMeasure P] :
  P.InnerRegular (fun s => IsCompact s ∧ IsClosed s)
    MeasurableSet
```

2.5 Proof of Kolmogorov's extension theorem

We now describe the proof of Kolmogorov's extension theorem as well as its formalization: For the proof of Theorem 1, we

1. apply Theorem 3 for the ring (hence semi-ring) $\mathcal{A}$ and the set-function P as given in Theorem 1;
2. show σ-additivity (as assumed in Theorem 3) of P by using Lemma 2.17. For the latter, we need to show that P is inner regular with respect to a compact system. Here, note that $\{\pi_J^{-1}C : C \in \prod_{j\in J}\alpha_j \text{ compact and closed}\}$ is a compact system and $P(\pi_J^{-1}C) = P_J(C)$, so we need to show that P_J is inner regular with respect to compact and closed sets, $J \subseteq_f \iota$;
3. use Lemma 2.20 in combination with Lemma 2.19 (1.⇒2.) and the properties of completeness, separability of the underlying extended pseudo-metric spaces in order to see that every P_J has the desired property of being inner regular wrt compact and closed sets. This concludes the proof of Theorem 1.

The formalization of this proof resembles these arguments. We leave out all instances in the reformulation of the result and its proof (see below Theorem 1 for a full formulation):

```
def projectiveLimitWithWeakestHypotheses
  (P : ∀ J : Finset ι, Measure (∀ j : J, α j))
  (hP : IsProjectiveMeasureFamily P) :
  Measure (∀ i, α i) :=
 Measure.ofAddContent setSemiringCylinders
  generateFrom_cylinders (kolContent hP)
  (kolContent_countably_subadditive_of_innerRegular hP
    fun J => innerRegular_isCompact_isClosed_
    measurableSet_of_complete_countable (P J))
```

Let us look closer at the formalized proof: `kolContent_countably_subadditive_of_innerRegular` has the same hypotheses as `kolContent_sigma_additive_of_innerRegular`, except `h_disj`. So, the lemma in the last brackets shows σ-sub-additivity of the content P from Theorem 1. Then, `generateFrom_cylinders` gives the `MeasurableSpace` on which we define the `addContent`.

3 Concluding thoughts

The usability of our contribution must be proved by an example: Adding to our general theory, we have implemented the example of an infinite product measure. (As an application of `projectiveLimitWithWeakestHypotheses`, we have to use complete and separable extended pseudo-metric spaces, but note that an implementation of the Theorem by Ionescu and Tulcea would give the same result without these prerequisites.) Here, we can build on `Measure.pi`, which defines a finite product measure. Using the same automatically inferred instances, we obtain:

```
def independentFamily
  (P : ∀ i, Measure (α i))
  [∀ i, IsProbabilityMeasure (P i)] : Measure (∀ i, α i)
  := projectiveLimitWithWeakestHypotheses
  (fun J : Finset ι => Measure.subset_pi P J)
  (product_isProjective P)

def Measure.subset_pi
  (P : ∀ i, Measure (α i)) (I : Finset ι) :
  Measure ((i : I) → (α i)) :=
  Measure.pi (fun (i : I) => (P (i : ι)))
```

Moreover, we will highlight some paths we imagine can be taken, leading to further extensions of `mathlib`, which build on our contribution.

Using semi-rings in measure theory: Frequently, the construction of measures on some σ-algebra $\mathcal{F}$ (on some space α) uses outer measures, which can be defined on 2^α (the set of all subsets of α). We provide a general framework using Carathéodory's extension theorem, which states an extension of a set-function on a semi-ring by an outer measure, not previously implemented in mathlib. This could also be used to redefine Stieltjes measures (in particular Lebesgue measure) on $\mathbb{R}$, as well as product measures (on finite products). More precisely, e.g. `StieltjesFunction.outer_Ioc` and `MeasureTheory.constructions.pi.pi_pi_aux` are specific examples of the more general statement, that the outer measure extends the set function a the semi-ring. (In the first case, it would be the semi-ring of half-open intervals, in the second case, it would be the semi-ring of cylinder sets on a product space.)

Prohorovs theorem: The proof that single finite measures on a Polish space is tight (i.e. inner regular with respect to compact sets) is a special case of Prohorov's theorem, which states that a set $\mathcal{M}$ of finite measures is relatively compact (in the weak topology on the set of finite measures, i.e. every sequence of measures in the set has a weakly convergent sub-sequence) if and only if $\mathcal{M}$ is tight (i.e. for all $\varepsilon > 0$ there is some compact K such that $\sup_{m \in \mathcal{M}} m(K^c) < \varepsilon$). We treat here the special case of $\mathcal{M}$ being a singleton (hence is a compact set).

Acknowledgements

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