Formalizing the Kolmogorov extension Theorem in mathlib4

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Abstract

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TODO

- Make sure that the line of argument for code and mauscript are in line.

1 Introduction

One of the main building blocks of modern probability theory are stochastic processes, which are usually defined as any collection of random variables – $(X_t)_{t\in\iota}$ with X_t taking values in some α_t, say – defined on some joint probability space $(\Omega, \mathcal{A}, \mathbf{P})$. In order to study such processes, it is fundamental to talk about their distribution, i.e. a probability distribution on the product set $\prod_{t\in\iota}\alpha_t$. The usual approach to construct (the distribution of) a stochastic process works as follows: describe properties of the distribution of the stochastic process P_J at some finite number of times $J = \{t_1, ..., t_n\} \subseteq \iota$. The resulting family of probability measures $(P_J)_{J\subseteq\iota \text{ finite}}$ has to be *projective* in the sense that the projection of P_J to $H \subseteq J$ has to be equal to P_H. In other words, when describing the distribution of the stochastic process at all times in J, and then forgetting (i.e. projecting) all properties for times in $J \setminus H$, results in the description of properties at times in H. One may then ask if this already gives a complete description of the process for all times. For uncountable ι, e.g. $\iota = [0, \infty)$, one is tempted to be pessimistic at first sight since measures – which describe the distribution of the stochastic process – usually only deal well with a countable number of measurable events. However, it is the achievement of Kolmogorov that the finite-dimensional distributions in fact provide a unique description of the distribution of a stochastic process, as long as the underlying family of state spaces $(\alpha_t)_{t\in\iota}$ is nice enough. This means that these spaces are usually required to be complete and separable metric (in particular topological) spaces. This distrubution is a measure on the product-σ-field $\mathcal{F} := \bigotimes_{t\in\iota}\mathcal{B}(\alpha_t)$ (assuming that $\mathcal{B}(\alpha_t)$ is the Borel σ-algebra on α_t). Here, $\mathcal{F}$ is generated by finite projections and hence any element of $\mathcal{F}$ may only depend on at most countably many $t \in \iota$, making this a rather coarse σ-algebra. (In particular, note that this is not the Borel σ-algebra of the product topology.) The result responsible for this insight is usually denoted the *Kolmogorov extension theorem*, formulated in [11]. We note that a version of the extension result was proved by Daniell in the 1930s, but this paper was not acknowledged by the probabilists of that time [2]. Due to his contribution, the theorem is often called the *Daniell-Kolmogorov extension.*

The goal of our contribution is a formalization of the proof of that theorem in `mathlib4`, the mathematical library of `Lean3` [12]. The Kolmogorov extension theorem is on the interface between measure theory and probability theory. Here, we rely on a decent amount of formalized mathematics in the measure-theory part of `mathlib4` (outer measures, above all), while not requiring any specific previous formalization of probability theory. (In fact, most of our results are formulated in terms of finite rather than probability measures.)

At first sight, it might be surprising that martingales, a certain class of stochastic process, have already been formalized in Lean [16], although the Kolmogorov extension theorem (on the existence of stochastic processes) is only available by our contribution. Note, however, that a

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martingale $(X_t)_{t\in\iota}$ is defined as a family of random variables (satisfying some properties) on a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$, while the Kolmogorov extension theorem is on the construction of a probability (or finite) measure $\mathbf{P}$, on which we can define random variables (with certain properties). So, our work complements [16] in the sense that probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$ on which you can define a martingale exist. However, we do not give any specific example for the construction of a specific instance of such a space.

The main steps in our construction, which were previously missing in mathlib4, are (more mathematical details are below):

1. a formalization of (the set-system of) semi-rings (see Definition 2.8 below);
2. adding some main lemmas to measure theory, e.g. continuity of a measure at the emptyset (see e.g. Lemma 2.11, Proposition 2.13, Lemma 2.15 below);
3. a proof that single probability measures on a Polish space is inner regular with respect to compact sets (see Definition 2.14 for a definition; in fact, we proved a slight generalization using pseudo-metric spaces, found in Lemma 2.18 below);
4. the classical Carathéodory extension theorem, providing us a candidate for the measure which we want to construct (see Theorem 3);
5. the Kolmogorov extension theorem, as based on the previous steps.

The Kolmogorov extension theorem has been previously formalized in Isabelle/HOL [1]. However, this formalization only works on Polish spaces (rather than on spaces where every finite measure is inner regular with respect to compact sets, see below), and only in the case where all α_i's are identical. However, type theory as the basis of the formalization, as well as several mathematical concepts (e.g. projective families of measures or the continuity of a measure in the emptyset) are the same in both formalisations in these two interactive theorem provers.

Possible future work. Let us describe some future projects extending mathlib4 which become possible by our contribution.
Instances of stochastic processes: An obvious application of Kolmogorov's extension theorem is the construction of basic stochastic processes like Markov chains [9, Section 11], the Poisson process [9, Section 13] and Brownian motion [9, Section 14]. (Note that other Gaussian processes – indexed by $[0, \infty)$ or $\mathbb{R}$ might as well be constructed the same way. In addition, fields like the Gaussien free field indexed by $\mathbb{R}^d$ [15] can also be given.) For these tasks, we would have to define the finite-dimensional distributions (Poisson and normally distributed, respectively), and apply the extension theorem. This task requires the formalization of multi-dimensional Poisson and normal distributions, which – in textbooks – is usually done using characteristic functions. Since these are not yet part of mathlib4, we postpone this task to the future.
Sample-path properties: The Kolmogorov extension gives the existence of a distribution of a stochastic process $(X_t)_{t\in\iota}$ with certain properties. Extra work is needed in order to show that – on the same probability space – we can as well define a version (i.e. another process $(Y_t)_{t\in\iota}$ with $\mathbf{P}(X_t = Y_t) = 1$ for all $t \in \iota$) which is right-continuous with left limits (for the Poisson process) or continuous (for Brownian motion), respectively. For the former, this follows at least from some general principles of Markov processes (e.g. [6, Theorem 4.3.6]). For the latter, this requires formalization of the Kolmogorov-Chentsov criterion [9, Theorem 4.23].

2 Formalization of the Daniell-Kolmogorov extension

We start by stating the exact result. We are going to formulate the main result in a modern fashion, as e.g. found in Theorem 2.2 of [14], Theorem 7.7.1 of Volume 2 of [4], Theorem 15.26 of [7], or [5]. Note that these formulations split general assumptions on the underlying space(s) (e.g. a metric property) from the property which is needed in the proof (inner regularity with respect to compact sets). Other – highly readable – references such as [3] state the extension theorem only in special cases such as $\alpha_t = \mathbb{R}$ for all t. Whenever we feel necessary, we rely on a standard textbooks on Probability Theory [9, 10] for general purpose.

Before we can state the result, we begin with some mathematical notions. We start off slowly from metric spaces, but quickly introduce projective families of measures. In all definitions, α will be some set (or type, since Lean4 is a dependently typed language, `α : Type _`).

Definition 2.1 (Metric and topological spaces). *1. A pseudo-metric on α is a symmetric map $r : \alpha \times \alpha \to [0, \infty)$ satisfying the triangle inequality, i.e. $r(x, z) \le r(x, y) + r(y, z)$ for all $x, y, z \in \alpha$. If r also satisfies $r(x, y) = 0$ iff $x = y$, we call it a metric. If $r : \alpha \times \alpha \to [0, \infty]$ (i.e. $r(x, y) = \infty$ is allowed), we call r an extended (pseudo-)metric.*
2. Let r be an (extended pseudo-)metric. A sequence $x_1, x_2, \ldots \in \alpha$ is called Cauchy if for all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $r(x_n, x_m) < \varepsilon$ for all $m, n > N$. The (pseudo-extended) metric is called complete if every Cauchy-sequency has a limit in α.
3. Some[1] $\mathcal{O} \subseteq 2^\alpha$ is called a topology, if it satisfies (i) $\emptyset, \alpha \in \mathcal{O}$, (ii) $\mathcal{O}$ is stable under arbitrary unions,

[1] We denote by 2^α the power set of α, i.e. the set of all subsets of α.

i.e. if $A_i \in \mathcal{O}$ for all $i \in \iota$ and ι is arbitrary, then $\bigcup_{i\in\iota} A_i \in \mathcal{O}$, (iii) $\mathcal{O}$ is stable under finite intersections, i.e. if $A_1, ..., A_n \in \mathcal{O}$, then $\bigcap_{i\in\iota} A_i \in \mathcal{O}$.

Remark 2.2 (Metric and topological spaces in mathlib4)**.**

1. In Lean4, we have the **class** PseudoEMetricSpace (α : **Type** u) **extends** EDist*, which comes with the components,* edist_self, edist_comm, edist_triangle *providing the properties of the pseudo-metric, and* toUniformSpace, uniformity_edist *defining a uniform space from the extended pseudo-metric. Such a space does not come with a metric, but with a filter on $\alpha \times \alpha$, which describes which points in α are near. For example, the diagonal of $\alpha \times \alpha$ is a subset of all sets in the uniformity; see [8] for details. We note that uniform spaces with a countably generated uniformity filter is pseudometrizable, i.e. there exists a pseudo-metric-space structure that generates the same uniformity; see* UniformSpace.pseudoMetrizableSpace*, which formalizes a result stated in [13].*
2. As for topological spaces, we have **class** TopologicalSpace(α : **Type** u) : **Type** u*, which comes with* isOpen, isOpen_univ, isOpen_inter *and* isOpen_sUnion*, which are exactly the properties of a topological space described above.*
3. Any (extended pseudo-)metric on α defines a topology, namely the topology generated by $\mathcal{H} := \{\{y : r(x,y) < \varepsilon\} : x \in \alpha, \varepsilon \in (0,\infty)\}$. Actually, in mathlib4*, as we saw above, every extended pseudo-metric defines a uniform space, and a uniform space is a* **class** UniformSpace(α : **Type** u) **extends** TopologicalSpace , UniformSpace.Core *with xxx* to_topological_space *giving a topological space,* to_core *which stores the uniformity filter, and* is_open_uniformity*, which gives all open sets. In particular, a uniform space defines a topological space, which can be used by the typeclass system.*

Remark 2.3 (Generated topology)**.** *1. The intersection of any number of topologies is again a topology. For this reason, if $\mathcal{H} \subseteq 2^\alpha$, we define the topology $\mathcal{O} := \bigcap_{\mathcal{F}\supseteq\mathcal{H}\ topology} \mathcal{F}$; see* TopologicalSpace.generateFrom*. This is called the topology generated by $\mathcal{H}$. If $\mathcal{H}$ is closed under finite intersections, we call $\mathcal{H}$ a basis for $\mathcal{O}$; see* TopologicalSpace.IsTopologicalBasis*.*
2. We call the topology (generated from an extended pseudo-metric) separable (see TopologicalSpace.SeparableSpace*) if there is a countable $s \subseteq \iota$ such that $\inf\{r(x,y) : y \in s\} = 0$ for all $x \in \alpha$.*
If there is a countable basis of the topology, it is separable; see TopologicalSpace.SecondCountableTopology.to_separableSpace*.*

Finally, we can introduce measures. Still, we need even more set-systems, i.e. σ-algebras.

Definition 2.4 (σ-algebras and measures)**.** *1. We call $\mathcal{F} \subseteq 2^\alpha$ a σ-algebra (on α) if (i) $\alpha \in \mathcal{F}$, (ii) $\mathcal{F}$ is stable under complements, i.e. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, (iii) $\mathcal{F}$ is stable under countable unions, i.e. $A_1, A_2, ... \in \mathcal{F} \Rightarrow \bigcup_{n=1}^\infty A_n \in \mathcal{F}$. We call $(\alpha, \mathcal{F})$ a measurable space.*
2. For some σ-algebra $\mathcal{F}$ on α, a function $\mu : \mathcal{F} \to [0,\infty]$ is called a measure, if[2] (i) $\mu(\emptyset) = 0$, (ii) μ is countably additive, i.e. for $A_1, A_2, ... \in \mathcal{F}$ pairwise disjoint, we have $\mu\Big(\bigcup_{n=1}^\infty A_n\Big) = \sum_{n=1}^\infty \mu(A_n)$. We call $(\alpha, \mathcal{F}, \mu)$ a measure space.
In addition, μ is called finite if $\mu(\alpha) < \infty$ and a probability measure if $\mu(\alpha) = 1$.

Remark 2.5 (Measur(abl)e spaces in mathlib4)**.** *1. The* **class** MeasurableSpace(α : **Type** u_7) : **Type** u_7 *is very similar to a topological space in* mathlib4*, since it comes with* MeasurableSet'*,* measurableSet_empty*,* measurableSet_compl*, and* measurableSet_iUnion*.*
2. For a measure space, we require a measurable space as a variable, and add the set function: **structure** MeasureTheory.Measure(α : **Type** u_6) [inst : MeasurableSpace α] **extends** MeasureTheory.OuterMeasure*. Here σ-additivity of a* Measure α *reads xxx note on outer measures*

```
m_iUnion : ∀ ⦃f : ℕ → Set α⦄,
  (∀ (i : ℕ), MeasurableSet (f i)) →
  Pairwise (Disjoint on f) →
  ↑toOuterMeasure (⋃ (i : ℕ), f i) =
  ∑' (i : ℕ), ↑toOuterMeasure (f i)
```

Remark 2.6 (Generated σ-algebra, image measure)**.** *We will frequently need two basic results:*
1. The intersection of any number of σ-algebras is again a σ-algebra. For this reason, if $\mathcal{H} \subseteq 2^\alpha$, we define the σ-algebra $\sigma(\mathcal{H}) := \bigcap_{\mathcal{F}\supseteq\mathcal{H}\ \sigma\text{-algebra}} \mathcal{F}$. This is called the σ-algebra generated from $\mathcal{H}$; see MeasurableSpace.generateFrom*.*
In particular, if $\mathcal{O}$ is a topology on α, we call $\mathcal{B} := \sigma(\mathcal{O})$ the Borel σ-algebra on α. In mathlib4*, this is*

```
def borel (α : Type u) [TopologicalSpace α] :
  MeasurableSpace α :=
  generateFrom { s : Set α | IsOpen s }
```

In addition, mathlib4 *provides a similar notion, which is*

```
class OpensMeasurableSpace (α : Type*)
  [TopologicalSpace α] [h : MeasurableSpace α] :
  Prop where borel_le : borel α ≤ h
```

[2] Since $\alpha \in \mathcal{F}$ and $\mathcal{F}$ is stable under complements, we also have that $\emptyset \in \mathcal{F}$

Here, all open sets are measurable, so the σ-algebra defining h *might be larger than the Borel σ-algebra. We will need both notions in our proofs.*
2. Let $(\alpha, \mathcal{F})$ and $(\beta, \mathcal{G})$ be measurable spaces. Some $f : \alpha \to \beta$ is called measurable (with respect to $\mathcal{F}$ and $\mathcal{G}$) if $f^{-1}\mathcal{G} \subseteq \mathcal{F}$.
In this case, if μ is a measure on $\mathcal{F}$, the measure $\nu : \mathcal{G} \to [0, \infty], \nu(B) := \mu(f^{-1}(B))$ is called the image (or push-forward) measure of μ under f. We write $\nu := f_*\mu$; see MeasureTheory.Measure.map.

The extension theorem is on finite measures on product spaces. In particular, the product is taken over a set of arbitrary cardinality. The next definition covers the important concept of a projective family of measures. In short term, we define measures on any finite subset of indices in a consistent way. Recall that finite sets (of some type ι are formalized with `Finset ι`. Such a `Finset` is defined as a `Multiset` (a quotient type of the `List`-type with respect to permutations of the list) together with a proof that it contains no duplicates.

Definition 2.7 (Projective family and projective limit)**.**
1. For some set ι, we will write $J \subseteq_f \iota$ if $J \subseteq \iota$ and J is finite.
2. Let ι be some (index) set and $(\alpha_i)_{i\in\iota}$ a family of sets. For $J \subseteq \iota$, we denote $\alpha_J := \prod_{j\in J} \alpha_j$ and $\pi_J : \alpha_\iota \to \alpha_J$ the projection. For $H \subseteq J \subseteq \iota$, we write π_H^J for the projection $\alpha_J \to \alpha_H$.
3. Let $\mathcal{F}_i$ be a σ-algebra in α_i, $i \in \iota$. For $J \subseteq_f \iota$, let $\mathcal{F}_J$ be the product-σ-algebra on α_J, and $\mathcal{F}_\iota$ be the σ-algebra generated by cylinder sets $\{\pi_J^{-1} \prod_{j\in J} A_j : J \subseteq_f \iota, A_j \in \sigma(E_j), j \in J\}$.
4. A family $(P_J)_{J\subseteq_f I}$, where P_J is a finite measure on $\mathcal{F}^J$, is called projective if

$$P_H = (\pi_H^J)_* P_J$$

for all $H \subseteq J \subseteq_f I$. (Recall that $A \mapsto (\pi_H^J)_* P_J(A) := P_J((\pi_H^J)^{-1}A)$ is called the image measure of P_J under π_H^J.)
5. If, for some projective family $(P_J)_{J\subseteq_f \iota}$, there is a finite measure P_ι on $\mathcal{F}_\iota$ with $P_J = (\pi_J)_* P_\iota$ for all $J \subseteq_f \iota$, then we call P_ι projective limit of $(P_J)_{J\subseteq_f \iota}$.

In the formalization, we use `variable {ι : Type _} {α : ι → Type _}`, which fixes the index set ι and all spaces $\alpha_t, t \in \iota$. In addition, note that the projective property we use here works as long as we have a preorder (which is the subset relation on `Finset ι` below).

```
def IsProjective [Preorder ι]
  (P : ∀ j : ι, α j) (π : ∀ {i j : ι}, j ≤ i → α i → α j) : Prop :=
  ∀ (i j) (hji : j ≤ i), P j = π hji (P i)
```

With this, we can define the projetive family as follows. Note that `Lean4` automatically uses the subset relation as `[Preorder (Finset ι)]` when `isProjective` is called.

```
def IsProjectiveMeasureFamily
  [∀ i, MeasurableSpace (α i)]
  (P : ∀ J : Finset ι, Measure (∀ j : J, α j)) :
  Prop :=
  IsProjective P (fun I _ hJI μ => μ.map
  fun x : ∀ i : I, α i => fun j => x ⟨j, hJI j.2⟩ :
  ∀ (I J : Finset ι) (_ : J ⊆ I), Measure (∀ i : I, α i)
  → Measure (∀ j : J, α j))
```

It is worth understanding the precise connection ot `isProjective` and `isProjectiveMeasureFamily`. In the latter, the first variable of `isProjective` is the family `P` of finite measures for all finite subsets of ι. The second variable is the functions which maps two sets `I J : Finset ι` and a proof `hJI` of `J ⊆ I` together with `P I` to the image measure on `J`, which is `μ.map (fun x : ∀ i : I, α i => fun j => x ⟨j, hJI j.2⟩)`. The map defined by the `=>`-notation maps every $(x_i)_{i\in I}$ to a function of `j`, whose type is the subtype of `I`, consisting of a value and a proof of `J ⊆ I`. In other words, this is $(x_j)_{j\in J}$.
Now we are ready to formulate the Kolmogorov extension theorem:

Theorem 1 (Kolmogorov extension)**.** *For all $t \in \iota$, let α_t be a separable, complete pseudo-extended-metric space and $\mathcal{F}_t$ the Borel σ-algebra generated by its topology. Let $(P_J)_{J\subseteq_f \iota}$ be a projective family of finite measures and P be defined on $\mathcal{A} := \bigcup_{J\subseteq_f \iota} \mathcal{F}_J$ given by $P(A) = P_J(A)$ for $A \in \mathcal{F}_J$. Then, there is a unique extension of P to $\sigma(\mathcal{A})$.*

Rather than giving the formalization of this theorem, we give the definition of the resulting measure (which is the projective limit). We give the formalized proof at the end of this section, since we first have to provide a formalization of all tools needed in the proof.

```
noncomputable def projectiveLimitWithWeakestHypotheses
  [∀ i, PseudoEMetricSpace (α i)]
  [∀ i, BorelSpace (α i)]
  [∀ i, TopologicalSpace.SecondCountableTopology (α i)]
  [∀ i, CompleteSpace (α i)] [Nonempty (∀ i, α i)]
  (P : ∀ J : Finset ι, Measure (∀ j : J, α j))
  [∀ i, IsFiniteMeasure (P i)]
  (hP : IsProjectiveMeasureFamily P) :
  Measure (∀ i, α i)
```

We note that we extend the standard assumption that all α_t are separable, complete metric spaces (or Polish, i.e. separable and metrizable through a complete metric) to cover the case of extended pseude-metric spaces. Such spaces do not satisfy the frequently used Hausdorff (or t2) property, i.e. there can be $x \neq y$ such that all open balls around x and y overlap.

2.1 Extending a set function

In the formulation of Theorem 1, we extend P, which is defined on a union of σ-algebras. However, unions of σ-algebras in general are not σ-algebras, but they can be used to define the σ-algebra generated by the union. So, we need to extend P to the σ-algebra generated by $\mathcal{A}$. This is exactly what Carathéodory's extension theorem is made for. In fact, we implemented this result in greater generality than needed for the proof of the extension theorem. Note that $\mathcal{A}$ in Theorem 1 is a ring of sets (see the next definition) containing the whole set. We will work with the weaker semi-ring as introduced next in the formulation of Carathéodory's extension theorem.

Definition 2.8 (Semi-ring, ring)**.** *Let α be some set. We call $\mathcal{H} \subseteq 2^\alpha$ a* semi-ring*, if it is (i) a π-system (i.e. closed under $\cap$) and (ii) for all $A, B \in \mathcal{H}$ there is $\mathcal{K} \subseteq_f \mathcal{H}$ with*[3] *$B \setminus A = \biguplus_{K \in \mathcal{K}} K$.*
We call $\mathcal{H} \subseteq 2^\alpha$ a ring*, if it is closed under $\cup$ and under set-differences.*

Using `variable {α : Type _} {C : Set (Set α)} {s t : Set α}`, we define a semi-ring as follows:

```
structure SetSemiring (C : Set (Set α)) : Prop where
  empty_mem : ∅ ∈ C
  inter_mem : ∀ (s) (_ : s ∈ C) (t) (_ : t ∈ C), s ∩ t ∈ C
  diff_eq_Union' :
    ∀ (s) (_ : s ∈ C) (t) (_ : t ∈ C),
    ∃ (I : Finset (Set α)) (_h_ss : ↑I ⊆ C)
    (_h_dis : PairwiseDisjoint (I : Set (Set α)) id),
     t \ s = ⋃₀ I
```

Recall that `Lean4` indicates a coercion by `↑`, which in this case is from `Finset` to `Set`.
We do not show the formalization of rings here. Note, however, that any ring is a semi-ring since $A \cap B = A \setminus (A \setminus B)$, i.e. every ring is a π-system. In our formlaization, we also use the notion of a field $\mathcal{H}$, which is a ring with $\alpha \in \mathcal{H}$. Since we a re dealing with product spaces, we will use cylinders, which are defined as follows:

```
def cylinder (s : Finset ι) (S : Set (∀ i : s, α i)) :
  Set ((i : ι) → α i) :=
  (fun f : (i : ι) → α i => fun i : s => f i) ⁻¹' S
```

In this definition, `s` is a finite subset of the index set ι, and for some S in the finite product $\prod_{i \in s} \alpha_i$, consider the projection $\pi_s : \prod_{i \in \iota} \alpha_i \to \prod_{i \in s} \alpha_i$, and consider the preimage of S. (This is what the last line in previous definition gives.) From this, we can defined the set of all measurable cylinders.

[3] We write $A \uplus B$ for $A \cup B$ if $A \cap B = \emptyset$.

```
def cylinders : Set (Set ((i : ι) → α i)) :=
 ⋃ (s) (S) (_ : MeasurableSet S), {cylinder s S}
```

The set-system of cylinders is in fact a field, hence a ring and hence a semi-ring:

```
theorem setField_cylinders : SetField (cylinders α) :=
 { empty_mem := empty_mem_cylinders α
  univ_mem := univ_mem_cylinders α
  union_mem := union_mem_cylinders
  diff_mem := diff_mem_cylinders }
```

```
theorem setRing_cylinders :
  MeasureTheory.SetRing (cylinders α) :=
  setField_cylinders.toSetRing
```

```
theorem setSemiringCylinders :
  MeasureTheory.SetSemiring (cylinders α) :=
  setField_cylinders.setSemiring
```

```
theorem generateFrom_cylinders :
  MeasurableSpace.generateFrom (cylinders α)
  = MeasurableSpace.pi
```

measurable product space

```
instance MeasurableSpace.pi
  [m : ∀ a, MeasurableSpace (π a)] :
  MeasurableSpace (∀ a, π a) :=
  ▯ a, (m a).comap fun b => b a
```

Let us state an important lemma on semi-rings:

Lemma 2.9. *Let $\mathcal{H}$ be a semiring, $\mathcal{I} \subseteq_f \mathcal{H}$, $A \in \mathcal{H}$. Then, there is $\mathcal{K} \subseteq_f \mathcal{H}$ such that all sets in $\mathcal{K}$ are pairwise disjoint and $A \setminus \bigcup_{I \in \mathcal{I}} I = \biguplus_{K \in \mathcal{K}} K$.*

The formalization of the proof is somewhat straightforward, but requires induction over finite sets.

```
theorem exists_disjoint_finset_diff_eq
  (hC : SetSemiring C) (hs : s ∈ C)
  (I : Finset (Set α)) (hI : ↑I ⊆ C) :
  ∃ (J : Finset (Set α)) (_h_ss : ↑J ⊆ C)
  (_h_dis : PairwiseDisjoint (J : Set (Set α)) id),
  s \ ⋃₀ I = ⋃₀ J
```

Thes existance-statement of the above lemma actually gives rise to a definition:

```
noncomputable def diff₀ (hC : SetSemiring C)
  (hs : s ∈ C) (I : Finset (Set α)) (hI : ↑I ⊆ C)
  [DecidableEq (Set α)] : Finset (Set α) :=
  (hC.exists_disjoint_finset_diff_eq hs I hI).choose \ {∅}
```

Complementing Lemma 2.9, we actually need another result for some constructions needed in the proofs: For some $A_1, ..., A_m$ and $n \neq m$, we need finite sets $\mathcal{K}_1, ..., \mathcal{K}_m$ such that $A_n \setminus \bigcup_{i=1}^{n-1} A_i = \biguplus_{K \in \mathcal{K}_n} K$ for $n = 1, ..., m$. Using this, we can write

$$\bigcup_{n=1}^{m} A_n = A_1 \uplus (A_2 \setminus A_1) \uplus \cdots \uplus \Big(A_m \setminus \bigcup_{i=1}^{m-1} A_i\Big) = \bigcup_{n=1}^{m} \biguplus_{K \in \mathcal{K}_n} K,$$

i.e. we can write a union of sets in the semi-ring as a disjoint union of sets in the semi-ring. The formalized version is as follows:

```
noncomputable def indexedDiff₀ (hC : SetSemiring C)
  (J : Finset (Set α)) (hJ : ↑J ⊆ C)
  (n : Fin J.card) : Finset (Set α) :=
  hC.diff₀ (ordered_mem' hJ n) (finsetLT J n)
  (finsetLT_subset' J hJ n)
```

2.2 Carathéodory's extension theorem

Measures are set functions defined on a σ-algebra (i.e. an algebra stable under countable unions), satisfying some properties which we recall below. Mostly, defining such a measure on the full σ-algebra is not possible, but defining a set function on a semi-ring is possible. As an example, consider Lebesgue-measure on $\mathbb{R}$, with the Borel σ-algebra on $\mathbb{R}$, which is $\sigma(\mathcal{O})$, where $\mathcal{O}$ is the set of open sets, and $\sigma(\mathcal{O})$ is the smallest σ-algebra containing all open sets. Since this set system is defined only abstractly, it is hard to know which volume each of these sets should be assigned to at first sight. However, the volume of an interval is easy, since it may be defined by the length of the interval. So, in order to construct measures from a set function m on a (semi-)ring $\mathcal{H}$ (e.g. the set of all semi-open intervals), it has been a fundamental insight of Carathéodory that one may start by defining a set function (outer measure) μ on 2^α, and then show μ extends m and defines a measure on $\sigma(\mathcal{H})$. (Note that the set of semi-open intervals generates the Borel σ-algebra on $\mathbb{R}$.) We will follow this construction, and start by stating some basic concepts.

Definition 2.10. *For some set α, let $\mathcal{H} \subseteq 2^\alpha$ and $m : \mathcal{H} \to [0, \infty]$.*

1. m is called additive if for $\mathcal{K} \subseteq_f \mathcal{H}$ pariwise disjoint and $\bigcup_{K \in \mathcal{K}} K \in \mathcal{H}$, we have $m\Big(\bigcup_{K \in \mathcal{K}} K\Big) = \sum_{K \in \mathcal{K}} m(K)$. If the same holds for[4] $\mathcal{K} \subseteq_c \mathcal{H}$ pariwise disjoint, we say that m is σ-additive.

2. The set-function m is called sub-additive if for $\mathcal{K} \subseteq_f \mathcal{H}$ and $\bigcup_{K \in \mathcal{K}} K \in \mathcal{H}$, we have $m\Big(\bigcup_{K \in \mathcal{K}} K\Big) \leq \sum_{K \in \mathcal{K}} m(K)$. (Note that elements of $\mathcal{K}$ need not be disjoint.) Here, σ-sub-additivity is defined in the obvious way using $\mathcal{K} \subseteq_c \mathcal{H}$.

3. If $m(A) \leq m(B)$ for $A \subseteq B$ and $A, B \in \mathcal{H}$, we say that m is monotone.

4. If $\mathcal{H}$ is a σ-algebra and m is σ-additive, we call m a measure.

5. If $\mathcal{H} = 2^\alpha$ and m is monotone and σ-sub-additive with $m(\emptyset) = 0$, we call m an outer measure.

[4] We write $A \subseteq_c B$ if A is a countable subset of B.

An additive $m : \mathcal{H} \to [0, \infty]$ is formalized as follows:

```
structure AddContent (C : Set (Set α)) where
  toFun : Set α → ℝ≥0∞
  empty' : toFun ∅ = 0
  add' : ∀ (I : Finset (Set α))
    (_h_ss : ↑I ⊆ C) (_h_dis : PairwiseDisjoint
      (I : Set (Set α)) id)
    (_h_mem : ⋃₀ ↑I ∈ C),
    toFun (⋃₀ I) = ∑ u in I, toFun u
```

In this formalization, we introduced $m(\emptyset) = 0$ as a separate hypothesis with empty'. Note that this also follows formally from $m(\emptyset) = m(\emptyset \cup \emptyset) = 2m(\emptyset)$. However, sicne Finsets are assumed to have distinct elements, we cannot have $\emptyset$ twice in I.

For the concrete application we have in mind, we need to be more specific which semiring to use, based on a projective family of probability measures. We do this using kolContent, which is based on the following function, we takes a family of measures and a cylinder set as input:

```
noncomputable def kolmogorovFun
  (P : ∀ J : Finset ι, Measure (∀ j : J, α j))
  (s : Set (∀ i, α i))
  (hs : s ∈ cylinders α) : ℝ≥0∞ :=
  P (cylinders.finset hs) (cylinders.set hs)
```

With this, we define an add_content on cylinders. (Note that extend_content in fact defines an additive content on a semiring.)

```
noncomputable def kolContent
  (hP : IsProjectiveMeasureFamily P) :
  AddContent (cylinders α) :=
  extendContent setSemiringCylinders
  (kolmogorovFun P) (kolmogorovFun_empty hP)
  (kolmogorovFun_additive hP)
```

From the next lemma, we will need monotonicity and sub-additivity, as well as σ-additive $\Rightarrow$ σ-sub-additive in the proof of the Carathéodory extension, Theorem 3.

Lemma 2.11 (Set-functions on semi-rings)**.** *Let $\mathcal{H}$ be a semi-ring and $m : \mathcal{H} \to [0, \infty]$ additive. Then, m is monotone and sub-additive. In addition, m is σ-additive iff it is σ-sub-additive.*

Although some material on outer measures was covered in mathlib4 already, the classical extension theorem (extending a set function m on a semiring $\mathcal{H}$)) was not provided yet. In particular, this result states that the outer measure coincides with m on $\mathcal{H}$. All statements on equality of μ and m present in mathlib4 at the time of writing have too many requirements.

The next result talks about an induced_outer_measure in terms of mathlib4.

Theorem 2.5

Proposition 2.12 (Outer measure induced by a set function on a semi-ring)**.** *Let $\mathcal{H}$ be a semi-ring and $m : \mathcal{H} \to \mathbb{R}_+$ additiv. For $A \subseteq E$ let*

$$\mu(A) := \inf_{\mathcal{G} \in \mathcal{U}(A)} \sum_{G \in \mathcal{G}} m(G)$$

where

$$\mathcal{U}(A) := \{\mathcal{G} \subseteq_c \mathcal{H}, A \subseteq \bigcup_{G \in \mathcal{G}} G\}$$

is the set of countable coverings of A. Then, μ is an outer measure.

Let us now formulate the classical results by Carathéodory. The first leads to the definition of the measurable space OuterMeasure.caratheodory, covered in mathlib4. See e.g. [9, Theorem 2.1]

Theorem 2 (**μ**-measurable sets are a **σ**-algebra)**.** *Let μ be an outer measure on E and $\mathcal{F}$ the set of μ-measurable sets, i.e. the set of sets A satisfying*

$$\mu(B) = \mu(B \cap A) + \mu(B \cap A^c), \qquad B \subseteq E. \tag{1}$$

Then, $\mathcal{F}$ is a σ-Algebra and $\mu|_{\mathcal{F}}$ is a measure. In addition, $\mathcal{N} := \{N \subseteq \Omega : \mu(N) = 0\} \subseteq \mathcal{F}$, i.e. $\mathcal{F}$ is complete.

The second result, is not yet covered in mathlib4. However, for product spaces, in measureTheory.Constructions.Pi, there is pi_caratheodory, which shows that $\sigma(\mathcal{H}) \subseteq \mathcal{F}$ in a special case. In addition, there is pi_pi_aux, which shows that μ extends m on $\mathcal{H}$ in the same special case. See also [9, Theorem 2.5].

Theorem 3 (Carathéodory extension)**.** *Let $\mathcal{H}$ be a semiring and $m : \mathcal{H} \to \mathbb{R}_+$ σ-finite and σ-additive. Furthermore, let μ be the induced outer measure from Proposition 2.12 and $\mathcal{F}$ the σ-algebra from Theorem 2. Then, $\sigma(\mathcal{H}) \subseteq \mathcal{F}$ and μ coincides with m on $\mathcal{H}$.*

```
noncomputable def Measure.ofAddSubaddCaratheodory
  (hC : SetSemiring C)
  (m : ∀ s : Set α, s ∈ C → ℝ≥0∞)
  (m_empty : m ∅ hC.empty_mem = 0)
  (m_add : ∀ (I : Finset (Set α)) (h_ss : ↑I ⊆ C)
    (_h_dis : PairwiseDisjoint (I : Set (Set α)) id)
    (h_mem : ⋃₀ ↑I ∈ C), m (⋃₀ I) h_mem = ∑ u : I, m u (h_ss u
  (m_sigma_subadd : ∀ ⦃f : ℕ → Set α⦄
    (hf : ∀ i, f i ∈ C) (hf_Union : (⋃ i, f i) ∈ C),
    m (⋃ i, f i) hf_Union ≤ ∑' i, m (f i) (hf i)) :
    @Measure α (inducedOuterMeasure m
     hC.empty_mem m_empty).caratheodory
```

Here, MeasureTheory.OuterMeasure.caratheodory is already implemented and gives the assertion that $\sigma(\mathcal{H}) \subseteq \mathcal{F}$.

In the proof of Kolmogorov's extension theorem, we use the following definition, which is based on Measure.ofAddSubaddCaratheodory.

```
noncomputable def Measure.ofAddContent
  [mα : MeasurableSpace α] (hC : SetSemiring C)
  (hC_gen : MeasurableSpace.generateFrom C = mα)
  (m : AddContent C)
  (m_sigma_subadd :
    ∀ ⦃f : ℕ → Set α⦄ (_hf : ∀ i, f i ∈ C) (_hf_Union : (⋃ i, f i) ∈ C),
     m (⋃ i, f i) ≤ ∑' i, m (f i)) :
     Measure α
```

2.3 σ-additivity of set functions

For the proof of Theorem 1, note that P as given in the result is a finite additive set function on the ring $\mathcal{A}$. In order to use Theorem 3, we therefore have to show σ-additivity. For this, we will use Lemma 2.15 below, i.e. inner regularity of P with respect to compact sets. Before, we will show useful alternative conditions for σ-additivity, which do not make use of any topological structure of the underlying space.

Lemma 2.13 (σ-additivity and continuity at $\emptyset$)**.** *Let $\mathcal{R}$ be a ring and $m : \mathcal{R} \to \mathbb{R}_+$ additive. Then, the following are equivalent:*

1. m is σ-additive;
2. m is σ-sub-additive;
3. m is continuous from below, i.e. for $A, A_1, A_2, \ldots \in \mathcal{R}$ with $A_1 \subseteq A_2 \subseteq \cdots$, we have $m(A) = \lim_{n\to\infty} m(A_n)$.
4. m is continuous from above in $\emptyset$, i.e. for $A_1, A_2, \ldots \in \mathcal{R}$ with $A_1 \supseteq A_2 \supseteq \ldots$ and $\bigcap_{n=1}^\infty A_n = \emptyset$, we have $\lim_{n\to\infty} m(A_n) = 0$.

As an example, we give the formalization of 4. ⇒ 1.:

```
theorem countably_additive_of_todo (hC : SetRing C)
  (m : ∀ s : Set α, s ∈ C → ℝ≥0∞)
  (hm_ne_top : ∀ {s} (hs : s ∈ C), m s hs ≠ ∞)
  (hm_add : ∀ {s t : Set α} (hs : s ∈ C) (ht : t ∈ C),
    Disjoint s t →
    m (s ∪ t) (hC.union_mem hs ht) = m s hs + m t ht)
  (hm : ∀ ⦃s : ℕ → Set α⦄ (hs : ∀ n, s n ∈ C),
    Antitone s → (⋂ n, s n) = ∅ →
    Tendsto (fun n => m (s n) (hs n)) atTop (𝓝 0))
    ⦃f : ℕ → Set α⦄ (h : ∀ i, f i ∈ C) (hUf : (⋃ i, f i) ∈ C)
```

```
  (h_disj : Pairwise (Disjoint on f)) :
  m (⋃ i, f i) hUf = ∑' i, m (f i) (h i)
```

Next, we will be extending our analysis to the case of a topological space. We define inner (and outer) regularity of set functions.

Definition 2.14 (Inner regularity)**.** *Let α be some set, equipped with a topology, and m be a set-function on some $\mathcal{H} \subseteq 2^\alpha$.*

1. Let $p, q : 2^\alpha \to \{true, false\}$. Then, m is called inner regular with respect to p and q, if

$$m(A) = \sup\{m(F) : p(F) = true, F \subseteq A\}$$

for all $A \in \mathcal{H}$ with $q(A) = true$.
2. If $q(A) = true$ iff A is measurable, we neglect the and *q. If $p(A) = true$ iff A is closed (ompact, closed and compact), we say that m is inner regular with respect to closed (compact, compact and closed) sets.*

The above definition closely resembles its formalization in mathlib4:

```
def MeasureTheory.Measure.InnerRegular
  {α : Type u_1} {_ : MeasurableSpace α}
  (μ : Measure α) (p q : Set α → Prop) :=
  ∀ ⦃U⦄, q U → ∀ r < μ U, ∃ K, K ⊆ U ∧ p K ∧ r < μ K
```

For the next result, recall that for compact sets $C_1, C_2, ...$ with $\bigcap_{n=1}^\infty C_n = \emptyset$, there is some N with $\bigcap_{n=1}^N C_n = \emptyset$.

Lemma 2.15. *Let α be a topological space and μ be an additive set function on a ring $\mathcal{R}$ contained by the Borel σ-algebra, and which is inner regular with respect to compact sets. Then, μ is σ-additive.*

Here is the formalization using a kol_content:

```
theorem kolContent_sigma_additive_of_innerRegular
  (hP : IsProjectiveMeasureFamily P)
  (hP_inner : ∀ J,
    (P J).InnerRegular (fun s => IsCompact s ∧ IsClosed s)
    MeasurableSet)
  ⦃f : ℕ → Set (∀ i, α i)⦄ (hf : ∀ i, f i ∈ cylinders α)
  (hf_Union : (⋃ i, f i) ∈ cylinders α)
  (h_disj : Pairwise (Disjoint on f)) :
  kolContent hP (⋃ i, f i) = ∑' i, kolContent hP (f i)
```

Using this, we also have kolContent_countably_subadditive_of_innerRegular by using countably_subadditive_of_countably_additive.

Recall that a finite measure on the Borel σ-algebra is inner regular with respect to closed sets, but also outer regular with respect to open sets.

Lemma 2.16. *Let r be an extended pseudo-metric on α, and the topology on α be given by r. If μ is a finite measure on the Borel σ-algebra $\mathcal{B}$ on α, it is inner regular with respect to closed sets. In fact, we have also outer regularity with respect to open sets, i.e.*

$$\mu(A) = \inf\{\mu(O) : A \subseteq O \text{ open}\}.$$

For the formalization, we use a new result: P.InnerRegular (fun s => IsCompact s ∧ IsClosed s) IsClosed, which holds for every second countable extended pseudo metric space, which is OpensMeasurable. Then, from mathlib4, we use Measure.InnerRegular.of_pseudoEMetricSpace in order to show that every open set is inner regular with respect to closed sets, and Measure.InnerRegular.measurableSet_of_open to show the above lemma.

In the next lemma, we will need that a closed subset of a compact set is compact. (See isCompact_of_isClosed_subset.)

Lemma 2.17. *Let r be an extended pseudo-metric on α, and the topology on α be given by r. If μ is a finite measure on the Borel σ-algebra $\mathcal{B}$ on α, the following are equivalent:*

1. For all $\varepsilon > 0$, there is some closed and compact K with $\mu(K^c) < \varepsilon$.
2. For all $\varepsilon > 0$ and $A \in \mathcal{B}$, there is some closed and compact $K \subseteq A$ with $\mu(A \setminus K) < \varepsilon$.

We apply this result by using that 1. is satisfied for complete and separable extended pseudo-metric spaces:

Lemma 2.18. *Let α be a complete and separable, extended pseudo-metric space, and μ a finite measure on its Borel σ-algebra $\mathcal{B}$. Then, for any $\varepsilon > 0$, there is some $K \subseteq \alpha$ with compact closure and*[5] *$\mu((\bar{K})^c) < \varepsilon$.*

Let us combine the last two results in one combined formalized result:

```
theorem innerRegular_isCompact_isClosed_
  measurableSet_of_complete_countable
  [PseudoEMetricSpace α] [CompleteSpace α]
  [TopologicalSpace.SecondCountableTopology α]
  [BorelSpace α] (P : Measure α) [IsFiniteMeasure P] :
  P.InnerRegular (fun s => IsCompact s ∧ IsClosed s)
    MeasurableSet
```

With the above results, the strategy of the proof of Theorem 1 is as follows: We have to show that under the given conditions, condition 1 of Lemma 2.17 holds. By the equivalent 2., we find that P is inner regular with respect to closed compact sets. In particular, the conditions of Lemma 2.15 are satisfied, showing that P is σ-additive. Then, we proceed as described right before

[5]We write $\bar{K}$ for the closure of K.

Proposition 2.13. So, it remains to show condition 1 of Lemma 2.17 for complete and separable extended pseudo-metric spaces. Note that – as a special case of isCompact_iff_totallyBounded_isComplete from mathlib4, any subset of such a space is compact iff it is complete and totally bounded. Since closed subsets of a complete space are complete, the closure of a totally bounded set is (still totally bounded, hence) compact.

2.4 Proof of Kolmogorov's extension theorem

Now we can put everything together: According to Lemma 2.18, condition 1 of Lemma 2.17 is satisfied for the set function P on the ring $\mathcal{A}$ from Theorem 1. The same lemma shows that the conditions for Lemma 2.15 are satisfied, so P is σ-additive on $\mathcal{A}$. As a consequence, we can use the Carathéodory extension from Theorem 3. This shows all other assertions. The formalization of this proof resembles these arguments. We leave out all instances in the reformulation of the result and its proof (see below Theorem 1 for a full formulation):

```
noncomputable def projectiveLimitWithWeakestHypotheses
  [∀ i, PseudoEMetricSpace (α i)]
  [∀ i, BorelSpace (α i)]
  [∀ i, TopologicalSpace.SecondCountableTopology (α i)]
  [∀ i, CompleteSpace (α i)] [Nonempty (∀ i, α i)]
  (P : ∀ J : Finset ι, Measure (∀ j : J, α j))
  [∀ i, IsFiniteMeasure (P i)]
  (hP : IsProjectiveMeasureFamily P) :
  Measure (∀ i, α i) :=
    Measure.ofAddContent setSemiringCylinders
    generateFrom_cylinders (kolContent hP)
    (kolContent_countably_subadditive_of_innerRegular hP
      fun J => innerRegular_isCompact_isClosed_
      measurableSet_of_complete_countable (P J))
```

3 Concluding thoughts

Using semi-rings in measure theory: Frequently, the construction of measures on some σ-algebra $\mathcal{F}$ (on some space α) uses outer measures, which can be defined on 2^α (the set of all subsets of α). We provide a general framework using Carathéodory's extension theorem, which states an extension of a set-function on a semi-ring by an outer measure, not previously implemented in mathlib4. This could also be used to redefine Stieltjes measures (in particular Lebesgue measure) on $\mathbb{R}$, as well as product measures (on finite products). More precisely, e.g. StieltjesFunction.outer_Ioc and MeasureTheory.constructions.pi.pi_pi_aux are specific examples of the more general statement, that the outer measure extends the set function a the semi-ring. (In the first case, it would be the semi-ring of half-open intervals, in the second case, it would be the semi-ring of cylinder sets on a product space.)

Prohorovs theorem: The proof that single finite measures on a Polish space is tight (i.e. inner regular with respect to compact sets) is a special case of Prohorov's theorem, which states that a set $\mathcal{M}$ of finite measures is relatively compact (in the weak topology on the set of finite measures, i.e. every sequence of measures in the set has a weakly convergent sub-sequence) if and only if $\mathcal{M}$ is tight (i.e. for all $\varepsilon > 0$ there is some compact K such that $\sup_{m\in\mathcal{M}} m(K^c) < \varepsilon$). We treat here the special case of $\mathcal{M}$ being a singleton (hence is a compact set).

xxx design decisions?

Acknowledgements

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