# Problems associated with using weak instruments for estimation of causal effects

COMP3224 - Homework 5

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## 1 Introduction

In this paper we shall discuss the problems described by Bound et al. (1995) that arise when using 'weak instruments' for estimation of causal relationships between an endogenous variable and a response variable of interest [1]. The Bound et al. paper demonstrates their analysis on a paper by Angrist and Krueger (1991) which uses instrumental variables to estimate the effect of education on wages, asserting that the results they found may in fact be biased [2]. We will give some mathematical background behind the issues of weak instruments, demonstrate results with simulations, and address the Angrist and Krueger paper.

## 2 Inconsistency

### 2.1 Analytical results

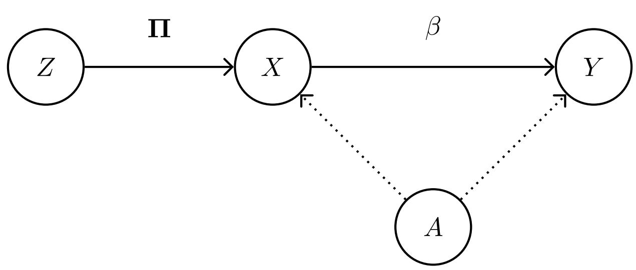


Figure 1: The canonical DAG for instrumental variable problems

Figure 1 exhibits a typical problem in which we may wish to estimate the causal effect of a treatment variable $X$ on a response $Y$, but are confronted with a back-door pathway $X \leftarrow A \rightarrow Y$ via some set of confounding variables $A$. In this case, we call $X$ an endogenous treatment variable. The problem can be expressed by the following set of equations, similarly to Bound et al. (1995):

$$\begin{aligned} \boldsymbol{y} &= \beta \boldsymbol{x} + \boldsymbol{\varepsilon} \\ \boldsymbol{x} &= \boldsymbol{Z\Pi} + \boldsymbol{\epsilon} \end{aligned} \tag{1}$$

Where $\boldsymbol{y}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{x}$ and $\boldsymbol{\epsilon}$ are $N \times 1$ observation vectors of random variables $Y$, $\varepsilon$, $X$ and $\epsilon$; $\beta$ is a constant scalar value; $\boldsymbol{Z}$ is an $N \times K$ matrix where each row is an independent observation of the instruments $z_1, z_2, ..., z_k$ and $\mathbf{\Pi}$ is a $K \times 1$ vector of constants. Note that we have assumed independence between $\boldsymbol{z}$ and $\epsilon$ and some dependence between $X$ and $\varepsilon$ as a result of the endogeneity of $X$. We have not expanded these equations to include the common exogenous variables $A$ as Bound et al. (1995) notes that this will complicate the algebra while having no effect on the results we obtain.

Clearly $\beta$ is our parameter of interest, so we will begin by demonstrating the conditions under which the ordinary least squares (OLS) estimation of this parameter is consistent:

$$
\begin{aligned}
\hat{\beta}_{\text{ols}} &= \frac{Cov(X,Y)}{Var(X)} \\
&= \frac{Cov(X,\beta X+\varepsilon)}{Var(X)} \\
&= \frac{Cov(X,\beta X)}{Var(X)} + \frac{Cov(X,\varepsilon)}{Var(X)} \\
&= \frac{\beta Var(X)}{Var(X)} + \frac{Cov(X,\varepsilon)}{Var(X)} \\
&= \beta + \frac{Cov(X,\varepsilon)}{Var(X)} \\
\implies \text{plim } \hat{\beta}_{\text{ols}} &= \beta + \frac{\sigma_{X,\varepsilon}}{\sigma_X^2}
\end{aligned}
\tag{2}
$$

We then conclude that $\hat{\beta}_{\text{ols}}$ is a consistent estimator of $\beta$ if and only if $\sigma_X^2 \neq 0$ and $\sigma_{X,\varepsilon} = 0$. However, under our assumption that $X$ is endogenous, the second condition can not hold and therefore we conclude an OLS estimation to be inconsistent. As a consequenc of this inconsistency, we shall turn our attention to the 2-stage least square (2SLS) estimator. The 2SLS estimator is obtained, intuitively, by two steps of OLS:

1. Obtain the OLS estimator from regression of the first stage $\hat{\mathbf{\Pi}} = \left(\boldsymbol{Z}^T\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}^T\boldsymbol{x}$ and take the fitted values $\hat{\boldsymbol{x}} = Z\hat{\mathbf{\Pi}} = \boldsymbol{Z}\left(\boldsymbol{Z}^T\boldsymbol{Z}\right)^{-1}\boldsymbol{Z}^T\boldsymbol{x} = \mathbf{P}_{\boldsymbol{Z}}\boldsymbol{x}$, where $\mathbf{P}_{\boldsymbol{Z}}$ is the projection matrix (and hence is symmetric and idempotent).

2. Use the fitted values from the first stage estimation in place of $\boldsymbol{x}$ in the OLS estimator of the second stage $\hat{\beta}_{\text{iv}} = \left(\hat{\boldsymbol{x}}^T\hat{\boldsymbol{x}}\right)^{-1}\hat{\boldsymbol{x}}^T\boldsymbol{y} = \left(\boldsymbol{x}^T\mathbf{P}_{\boldsymbol{Z}}^T\mathbf{P}_{\boldsymbol{Z}}\boldsymbol{x}\right)^{-1}\boldsymbol{x}^T\mathbf{P}_{\boldsymbol{Z}}^T\boldsymbol{y} = \left(\boldsymbol{x}^T\mathbf{P}_{\boldsymbol{Z}}\boldsymbol{x}\right)^{-1}\boldsymbol{x}^T\mathbf{P}_{\boldsymbol{Z}}\boldsymbol{y}$ (simplifications result from the aforementioned properties of the projection matrix).

By the same logic as Equation 2 we can derive the conditions under which $\beta_{iv}$ is consistent:

$$
\begin{aligned}
\hat{\beta}_{\text{iv}} &= \frac{Cov(\hat{X},Y)}{Var(\hat{X})} \\
&= \frac{Cov(\hat{X},\beta\hat{X}+\varepsilon)}{Var(\hat{x})} \\
&= \frac{Cov(\hat{X},\beta\hat{X})}{Var(\hat{X})} + \frac{Cov(\hat{X},\varepsilon)}{Var(\hat{X})} \\
&= \beta + \frac{Cov(\hat{X},\varepsilon)}{Var(\hat{X})} \\
\implies \text{plim } \hat{\beta}_{\text{iv}} &= \beta + \frac{\sigma_{\hat{X},\varepsilon}}{\sigma_{\hat{X}}^2}
\end{aligned}
\tag{3}
$$

We then find that $\hat{\beta}_{\text{iv}}$ is a consistent estimator of $\beta$ if and only if $\sigma_{\hat{X}}^2 \neq 0$ and $\sigma_{\hat{X},\varepsilon} = 0$. To give some intuition behind these statements, firstly note that $\sigma_{\hat{X}}^2 \neq 0$ implies an association between $X$ and $\boldsymbol{z}$. Secondly, $\sigma_{\hat{X},\varepsilon} = 0$ implies that $\hat{\beta}_{\text{iv}}$ is consistent only when there is no association between $\boldsymbol{z}$ and $\varepsilon$ (i.e. $\boldsymbol{z}$ affects $Y$ only through $X$), and we can note that if there is such an association, the inconsistency of $\hat{\beta}_{\text{iv}}$ increases as the association increases.

Bound et al. (1995) takes these inferences a step further, and presents the relative inconsistency of the 2SLS estimator to the OLS estimator by subtracting $\beta$ from both sides of the probability limits in Equations 2 and 3 and dividing as follows:

$$
\begin{aligned}
\frac{\text{plim } \hat{\beta}_{\text{iv}} - \beta}{\text{plim } \hat{\beta}_{\text{ols}} - \beta} &= \frac{\sigma_{\hat{X},\varepsilon}/\sigma_{\hat{X}}^2}{\sigma_{X,\varepsilon}/\sigma_X^2} \\
&= \frac{\sigma_{\hat{X},\varepsilon}/\sigma_{X,\varepsilon}}{\sigma_{\hat{X}}^2/\sigma_X^2} \\
&= \frac{\sigma_{\hat{X},\varepsilon}/\sigma_{X,\varepsilon}}{R^2_{X,\boldsymbol{z}}}
\end{aligned}
\tag{4}
$$

We notice that the denominator is the population $R^2$ value for a regression of $X$ on $\boldsymbol{z}$. Here we will informally define weak instruments as instrumental variables with low values of $R^2_{X,\boldsymbol{z}}$. Recalling that $\sigma_{X,\varepsilon}$ is necessarily non-zero as $X$ is endogenous, we can infer that when there is an association between $\boldsymbol{z}$ and $\varepsilon$ such that $\sigma_{\hat{X},\varepsilon} \neq 0$, the effect of this term on the inconsistency of the IV estimator is increased when the instrument is weak. Of particular concern is the notion that we may find large inconsistencies from only very small correlations between the instrument and error if the instrument is weak enough, severely restricting the scope of problems in which weak instruments can yield consistent estimates.

## 2.2 Empirical comparison of OLS and IV estimation on large samples

The results we have obtained so far are demonstrated to be correct by simulation. 20 million data points were simulated from two systems in order to approximate the asymptotic results under large data sets. The first system is described by the following structural equations:

$$\begin{aligned} Z &= \varepsilon_Z \\ A &= 0.1Z + \varepsilon_A \\ X &= 2A + 5Z + \varepsilon_X \\ Y &= 2X + 2A + \varepsilon_Y \end{aligned} \tag{5}$$

Where each error term is given by a standard Gaussian. This system uses a single instrumental variable $Z$ which has a weak association with the common exogenous variable $A$ but a strong association with the endogenous variable $X$ ($R^2 = 0.844$). Our parameter of interest is $\beta = 2$. Using the methods of OLS and 2SLS we obtained results of $\hat{\beta}_{\text{ols}} = 2.157$ and $\hat{\beta}_{\text{iv}} = 2.038$. Clearly we see that the OLS estimation is inconsistent, while the 2SLS estimation appears consistent.

The second system is given by similar set of structural equations to the first system, but with a much weaker association between the instrumental variable and the endogenous variable:

$$\begin{aligned} Z &= \varepsilon_Z \\ A &= 0.1Z + \varepsilon_A \\ X &= 2A + 0.1Z + \varepsilon_X \\ Y &= 2X + 2A + \varepsilon_Y \end{aligned} \tag{6}$$

In this system the association between $Z$ and $X$ is much lower ($R^2 = 0.018$), and hence we consider $Z$ a weak instrument. Again we find an inconsistent estimate with OLS of $\hat{\beta}_{\text{ols}} = 2.798$, but this time we see that the 2LS estimate is also largely inconsistent, with $\hat{\beta}_{\text{iv}} = 2.667$. This fits our analytical conclusion that when there is an association between the instrumental variables and the common exogenous variables, even if this association is very small, the effect on the inconsistency of the IV estimation is amplified by weaker instruments.

The results on both systems have been displayed with partial regression plots in Figure 2.

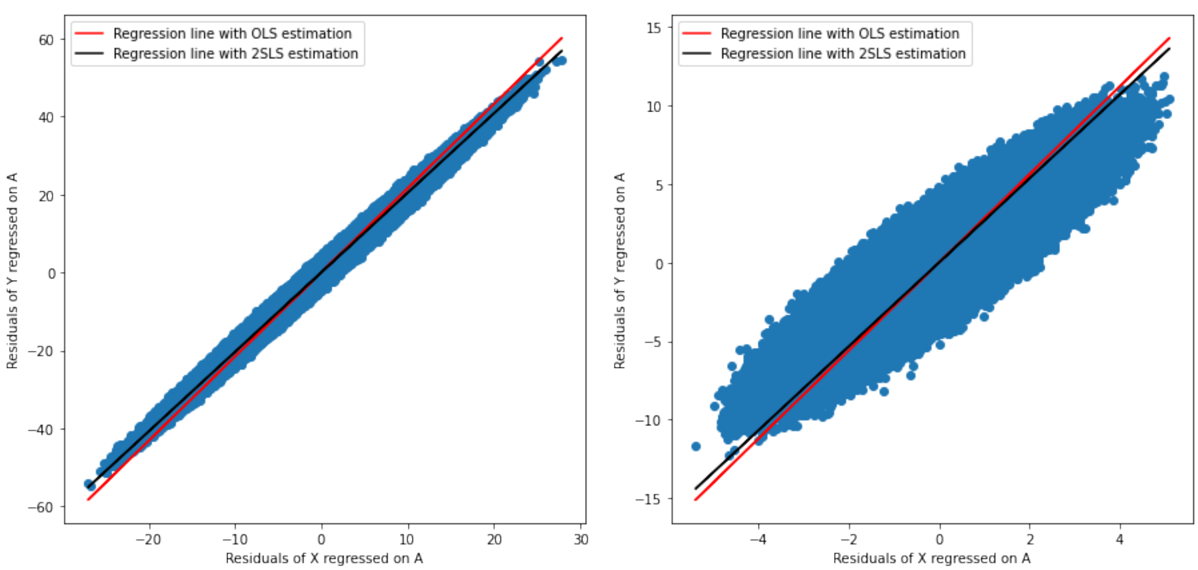


Figure 2: The results of the simulations for system 1 (left) and 2 (right)

# 3 Finite-sample bias

## 3.1 Analytical results

In this section we inherit the general system from Equation 1 and work under the assumption of $E[\varepsilon, \boldsymbol{z}] = 0$, from which it follows that $\hat{\beta}_{\text{iv}}$ is consistent. Bound et al. assert that under the condition of consistency $\hat{\beta}_{\text{iv}}$ is nonetheless biased in the direction of $E[\hat{\beta}_{\text{ols}}]$, and that this bias decreases as sample size and $R^2_{X,\boldsymbol{z}}$ increase.

The paper proposes several methods of deriving approximations or exact values (under certain assumptions) of the bias of the IV estimator relative to the OLS estimator. The mathematical derivations of most of these results is beyond the scope of the paper, but some important results are noteworthy. The notion of a 'concentration parameter' is introduced, defined as $\tau^2 \equiv \boldsymbol{\Pi}^T \boldsymbol{Z}^T \boldsymbol{Z}\boldsymbol{\Pi}/\sigma^2_\epsilon$ [3]. We find that the population analogy of the F statistic on the instruments in the first-stage OLS estimation is $\tau^2/K$, and that the bias of the IV estimator relative to the OLS estimator is approximately $1/(1+\tau^2/K)$ [4]. We can find an approximate formula for this relative bias by first finding the bias of the OLS estimator by subtracting $\beta$ from both sides of the penultimate line of Equation 2 and taking expectations:

$$E[\hat{\beta}_{\text{ols}} - \beta] = \frac{\sigma_{X,\varepsilon}}{\sigma^2_X} \tag{7}$$

Then our approximation for the bias of the IV estimator follows clearly [5]:

$$E[\hat{\beta}_{\text{iv}} - \beta] \approx \frac{\sigma_{X,\varepsilon}}{\sigma^2_X} \frac{1}{1+\tau^2/K} \tag{8}$$

From this equation we see that $E[\hat{\beta}_{\text{iv}} - \beta] \to 0$ as $\tau^2/K \to \infty$ and $E[\hat{\beta}_{\text{iv}} - \beta] \to \sigma_{X,\varepsilon}/\sigma^2_X$ as $\tau^2/K \to 0$. Intuitively this result confirms that the bias of the IV estimator shrinks as the first stage becomes stronger, and implies higher bias for weaker instruments.

# References


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