DOMINANCE REGIONS FOR AFFINE CLUSTER ALGEBRAS

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Abstract. We study the dominance order for $\boldsymbol{g}$-vectors in affine cluster algebras.

Let λ be an imaginary $\boldsymbol{g}$-vector.

(1) What is the condition that determines whether the separating hyperplane for a mutation map cuts between two imaginary domains of linearity?
(2) The intersection of P_λ with each cone corresponding to an imaginary cluster is convex. Indeed, for any seed t, consider the mutation of that cone along the sequence $\boldsymbol{k}$ corresponding to t. Intersecting this cone with $\eta_{\boldsymbol{k}}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\mathbb{R}^n_{\geqslant 0}$ produces a convex region which mutates back to a convex region under $\left(\eta_{\boldsymbol{k}}^{\widetilde{B}^{t_+}}\right)^{-1}$.
(3) λ does not dominate $c^r\lambda$ for any $r \in \mathbb{Z}$ unless $c^r\lambda = \lambda$. Indeed, if there existed an r so that λ dominates $c^r\lambda$ for all imaginary λ, then by repeatedly applying c^r we would see that $c^r\lambda$ also dominates λ which is impossible unless $\lambda = c^r\lambda$.
(4) For $\lambda \in \mathbb{R}^m$ and sequences $\boldsymbol{k} = (k_N, \ldots, k_1)$ of mutations from t_+ to t, define
$$\mathcal{P}_\lambda^{t_+} = \bigcap_{\boldsymbol{k}} \left(\eta_{\boldsymbol{k}}^{\widetilde{B}^{t_+}}\right)^{-1}\left(\left\{\eta_{\boldsymbol{k}}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\}\right).$$
Then for any sequence $\boldsymbol{\ell}$ joining t_+ to t'_+, we have $\eta_{\boldsymbol{\ell}}^{\widetilde{B}^{t_+}}\mathcal{P}_\lambda^{t_+} = \mathcal{P}_{\lambda'}^{t'_+}$ where $\lambda' = \eta_{\boldsymbol{\ell}}^{\widetilde{B}^{t_+}}(\lambda)$.

For $j \geqslant 1$, we have $(\eta_{c^j}^{\widetilde{B}})^{-1}\left(\{\eta_{c^j}^{\widetilde{B}}(\lambda) + \mu_{c^j}\widetilde{B}\alpha\}\right) = \{\lambda - \widetilde{B}C_{c^{j+1}}\alpha\}$ and $(\eta_{c^{-j}}^{\widetilde{B}})^{-1}\left(\{\eta_{c^{-j}}^{\widetilde{B}}(\lambda) + \mu_{c^{-j}}\widetilde{B}\alpha\}\right) = \{\lambda + \widetilde{B}C_{c^{-j}}\alpha\}$ for λ sufficiently near $\mathfrak{d}_\infty$.

① 1. define imaginary, real, and the imaginary cone $\mathcal{I}$, $\widetilde{B}^+$, δ SS

Theorem 0.1. *Given $\lambda \in \mathbb{R}^m$, let $z \geqslant 0$ be the minimum value such that $\lambda + z\widetilde{B}^+\delta$ is real.* ② *Then the dominance region $\mathcal{P}_\lambda$ is the line segment $\{\lambda + x\widetilde{B}^+\delta : 0 \leqslant x \leqslant z\}$.* 2. We need to say why such an z exists, i.e. $-\widetilde{B}^+\delta$ positively spans the imaginary ray. SS

The uniform formulation of Theorem 0.1 hides in itself two cases: if λ is real then $z = 0$ and $\mathcal{P}_\lambda$ is just the point λ; otherwise $z > 0$ and $\mathcal{P}_\lambda$ is a proper line segment. The former case was already established in [?] in complete generality; we will reprove it here by elementary means as a corollary of a result needed in the proof of the latter case. The proof of Theorem 0.1 is divided into several intermediate claims; we begin by showing that, for imaginary λ, $\mathcal{P}_\lambda \subseteq \mathcal{I}$.

Lemma 0.2. *If λ is imaginary, then $\mathcal{P}_\lambda \subseteq \mathcal{I}$.*

Proof. By definition, $\mathcal{P}_\lambda$ is contained in $\{\lambda + \widetilde{B}^+\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\}$ which is a proper cone since $\widetilde{B}^+$ is full rank. In particular, for sufficiently large z, the dominance region $\mathcal{P}_\lambda$ does not intersect the half space $H = \{\lambda' : \langle -B^+\delta, \lambda'\rangle \geqslant z\}$. ③ Suppose by contradiction that there exists some point $\lambda' \in \mathcal{P}_\lambda \backslash \mathcal{I}$. Since $\mathcal{P}_\lambda$ is stable under mutations $\eta_{\boldsymbol{k}}^{\widetilde{B}^+}(\lambda') \in \mathcal{P}_\lambda$ for any sequence of indices $\boldsymbol{k}$. ④ ⑤ Moreover, since λ' is not in $\mathcal{I}$, for sufficiently large r the sequence $\boldsymbol{k} = (n, \ldots, 1)^r$ satisfies

3. Think about how to write this better. SS

4. WARNING: do this only for sequences fixing λ. SS

5. Define mutation maps. of course we already knew this was to be done! SS

$\eta_{\boldsymbol{k}}^{\widetilde{B}^+}(\lambda') \in H$ contradicting our previous observation. 6 $\square$

For $j \in \mathbb{Z}$, let $\langle j\rangle$ be the element of $[1,n]$ congruent to j mod n. If $j > 0$ let $\boldsymbol{k}_j$ be the sequence $(\langle j\rangle, \langle j-1\rangle, \ldots, \langle 1\rangle)$; it has length j. Let $\boldsymbol{k}_0$ denote the empty sequence. If $j < 0$ let $\boldsymbol{k}_j$ be the sequence $(\langle j+1\rangle, \langle j+2\rangle, \ldots, \langle 0\rangle)$; it has length $-j$.

6. We need to sa more about this: finite/infinite c-orb how limits work. S

Denote by t_j the seed obtained from t_0 mutating along the sequence $\boldsymbol{k}_j$.

Remark 0.3. *If $j \geqslant 0$ the seed t_j corresponds to the c-sorting element whose c-sorting word is the prefix of c^∞ of length j. If $j \leqslant -n$ the seed t_j corresponds to the c^{-1}-sorting element whose c^{-1}-sorting word is the prefix of $c^{-\infty}$ of length $n-j$.*

Set $\mathcal{P}_{\lambda,j} = \left(\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}\right)^{-1}\left(\left\{\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}(\lambda) + \widetilde{B}^{t_j}\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\}\right)$; then $\mathcal{P}_\lambda \subseteq \bigcap_{j\in\mathbb{Z}} \mathcal{P}_{\lambda,j}$.

Lemma 0.4. *For any $j \geqslant 0$, there exists a full-dimensional subset L_j of $\mathbb{R}^m$ such that $\mathcal{I} \subseteq L_j$ and, for any $\lambda \in \mathcal{I}$,*

$$\mathcal{P}_{\lambda,j} \cap L_j = \left\{\lambda + \widetilde{B}^{t_0} C^{t_j}\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap L_j.$$

Proof. First observe that the $\langle i+1\rangle$-st coordinate of $\eta_{\boldsymbol{k}_i}^{\widetilde{B}^{t_0}}(\lambda)$ is non-positive for every $i \in \mathbb{Z}$. Indeed, since $\eta_{\boldsymbol{k}_i}^{\widetilde{B}^{t_0}}(\lambda) \in \mu_{\boldsymbol{k}_i}(\mathcal{I})$, it is of the form Mv for some non-negative vector v and some matrix M whose $\langle i+1\rangle$-st row is a negative elementary unit vector (M is the matrix of the change of coordinates in between positive $\boldsymbol{d}$-vectors and non-initial $\boldsymbol{g}$-vectors). 7 In particular $\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}$ acts on $\mathcal{I}$ as $\widetilde{E}^{t_{j-1}}_{\langle j\rangle,-}\cdots\widetilde{E}^{t_0}_{1,-}$

7. I am trying very hard not to mention the Euler matrix here nor source-sink moves if possible SS

Let L_j be the maximal cone on which $\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}$ and $\widetilde{E}^{t_{j-1}}_{\langle j\rangle,-}\cdots\widetilde{E}^{t_0}_{1,-}$ agree. 8 We compute

8. Should we something to explain why this is not just $\mathcal{I}$? SS

$$\begin{aligned}
\mathcal{P}_{\lambda,j} \cap L_j &= \left(\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}\right)^{-1}\left(\left\{\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}(\lambda) + \widetilde{B}^{t_j}\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\}\right) \cap L_j \\
&= \left(\widetilde{E}^{t_{j-1}}_{\langle j\rangle,-}\cdots\widetilde{E}^{t_0}_{1,-}\right)^{-1}\left(\left\{\widetilde{E}^{t_{j-1}}_{\langle j\rangle,-}\cdots\widetilde{E}^{t_0}_{1,-}(\lambda) + \mu_{\boldsymbol{k}_j}(\widetilde{B}^{t_0})\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\}\right) \cap L_j \\
&= \left\{\lambda + \widetilde{E}^{t_0}_{1,-}\cdots\widetilde{E}^{t_{j-1}}_{\langle j\rangle,-}\mu_{\boldsymbol{k}_j}(\widetilde{B}^{t_0})\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap L_j \\
&= \left\{\lambda + \widetilde{B}^{t_0} F^{t_0}_{1,-}\cdots F^{t_{j-1}}_{\langle j\rangle,-}\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap L_j \\
&= \left\{\lambda + \widetilde{B}^{t_0} C^{t_j}\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap L_j
\end{aligned}$$

where the last identity follows from Lemma 4.1 and the observation that the $\langle i+1\rangle$-st $\boldsymbol{c}$-vector in the seed t_i is positive (it is the leftmost skip for the corresponding c-sorting word). $\square$

Lemma 0.5. *For any $j \leqslant 0$, there exists a full-dimensional subset L_j of $\mathbb{R}^m$ such that $\mathcal{I} \subseteq L_j$ and, for any $\lambda \in \mathcal{I}$,*

$$\mathcal{P}_{\lambda,j} \cap L_j = \left\{\lambda - \widetilde{B}^{t_0} C^{t_{j-n}}\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap L_j.$$

Proof. The proof of this lemma follows closely the proof of the previous one with some important distinctions.

For any $i \in \mathbb{Z}$ the $\langle i\rangle$-th coordinate of $\eta_{\boldsymbol{k}_i}^{\widetilde{B}^{t_0}}(\lambda)$ is non-negative. Indeed since λ and $-\frac{1}{2}\widetilde{B}^{t_0}\delta$ belong to the same maximal cone, the $\langle i\rangle$-th coordinates of $\eta_{\boldsymbol{k}_i}^{\widetilde{B}^{t_0}}(\lambda)$ and $\eta_{\boldsymbol{k}_i}^{\widetilde{B}^{t_0}}(-\frac{1}{2}\widetilde{B}^{t_0}\delta) = -\frac{1}{2}\widetilde{B}^{t_i}\delta$ weekly agree in sign. But the $\langle i\rangle$-th row of $\widetilde{B}^{t_i}$ contains only non-positive integers while δ is a vector of positive integers and the claim follows. 9 In particular $\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}$ acts on $\mathcal{I}$ as $\widetilde{E}^{t_{j+1}}_{\langle j+1\rangle,+}\cdots\widetilde{E}^{t_0}_{n,+}$. Let

9. Maybe we should use this argument also in the other proposition to avoid talking about the Euler matrix. SS

L_j be the maximal cone on which $\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}$ and $\widetilde{E}^{t_{j+1}}_{\langle j+1\rangle,+}\cdots\widetilde{E}^{t_0}_{n,+}$ agree. As before we compute

$$\begin{aligned}\mathcal{P}_{\lambda,j}\cap L_j &= \left(\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}\right)^{-1}\left(\left\{\eta_{\boldsymbol{k}_j}^{\widetilde{B}^{t_0}}(\lambda)+\widetilde{B}^{t_j}\alpha:\alpha\in\mathbb{R}^n_{\geqslant 0}\right\}\right)\cap L_j\\ &= \left(\widetilde{E}^{t_{j+1}}_{\langle j+1\rangle,+}\cdots\widetilde{E}^{t_0}_{n,+}\right)^{-1}\left(\left\{\widetilde{E}^{t_{j+1}}_{\langle j+1\rangle,+}\cdots\widetilde{E}^{t_0}_{n,+}(\lambda)+\mu_{\boldsymbol{k}_j}(\widetilde{B}^{t_0})\alpha:\alpha\in\mathbb{R}^n_{\geqslant 0}\right\}\right)\cap L_j\\ &= \left\{\lambda+\widetilde{E}^{t_{j+1}}_{\langle j+1\rangle,+}\cdots\widetilde{E}^{t_0}_{n,+}\mu_{\boldsymbol{k}_j}(\widetilde{B}^{t_+})\alpha:\alpha\in\mathbb{R}^n_{\geqslant 0}\right\}\cap L_j\\ &= \left\{\lambda+\widetilde{B}^{t_0}F^{t_{j+1}}_{\langle j+1\rangle,+}\cdots F^{t_0}_{n,+}\alpha:\alpha\in\mathbb{R}^n_{\geqslant 0}\right\}\cap L_j.\end{aligned}$$

Now observe that the first n $\boldsymbol{c}$-vectors encountered while mutating along $\boldsymbol{k}_j$ are positive while the remaining ones are negative. (Provided that $j<-n$, otherwise they are all positive.) Note also that $F^{t_{1-n}}_{1,-}\cdots F^{t_0}_{n,-}=-1$ and that $F^{t_i}_{\langle i\rangle,\varepsilon}=F^{t_{i-n}}_{\langle i-n\rangle,\varepsilon}$ for any i and ε. We get

$$F^{t_{j+1}}_{\langle j+1\rangle,+}\cdots F^{t_0}_{n,+}=F^{t_{j-n+1}}_{\langle j-n+1\rangle,+}\cdots F^{t_{-n}}_{n,+}=-F^{t_{j-n+1}}_{\langle j-n+1\rangle,+}\cdots F^{t_{-n}}_{n,+}F^{t_{1-n}}_{1,-}\cdots F^{t_0}_{n,-}=-C^{t_{j-n}}$$

again by Lemma 4.1 and we are done. □

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Proposition 0.6. *Let λ be imaginary with z minimal such that $\lambda+z\widetilde{B}^{t_0}\delta$ is real. Then $\mathcal{P}_\lambda$ is contained in the half line $\{\lambda+x\widetilde{B}^{t_0}\delta:x\leqslant z\}$.*

10. Note to self: if λ is a cluster monomial in t then exactly the same argument shows that $\mathcal{P}_{\lambda,j}\cap K_t$ has the same expression. This should imply that $\mathcal{P}_\lambda\cap K_t$ is just the point λ but it is not enough to conclude that $\mathcal{P}_\lambda=\lambda$ because we do not know that $\mathcal{P}_\lambda$ is connected. SS

Proof. Since $\mathcal{P}_\lambda\subseteq\bigcap_{j\in\mathbb{Z}}\mathcal{P}_{\lambda,nj}$ it suffices to show that $\bigcap_{j\in\mathbb{Z}}\mathcal{P}_{\lambda,nj}$ is a half line. Moreover, since by Lemma 0.2 $\mathcal{P}_\lambda\subseteq\mathcal{I}$, it suffices to show that $\bigcap_{j\in\mathbb{Z}}(\mathcal{P}_{\lambda,nj}\cap L_j)$ is a half line. In view of Lemmas 0.4 and 0.5, since $\widetilde{B}^{t_0}$ has full rank, it sufficies to show that

$$\bigcap_{j\in\mathbb{Z}}\left\{(-1)^{\mathrm{sgn}(j)}C^{t_{nj}}\alpha:\alpha\in\mathbb{R}^n_{\geqslant 0}\right\}$$

is the line spanned by δ.

By Remark 0.3, if $j\geqslant 0$ the i-th column of $C^{t_{nj}}$ is the root $c^j\alpha_i$. Similarly, if $j\leqslant -1$ the i-th column of $-C^{t_j}$ is the root $c^{j+1}\alpha_i$. AAARGHHH This screws everything up, we cant use it to show that if $c^j\alpha_i$ is in a finite orbit then there is a $j'<0$ such that $-c^j\alpha_i\in -C^{t_{nj'}}$!!!!!! □

From now on the superscipt t will mean that object to which it is applied is expressed in terms of the seed t. Whenever $t=t_+$ and there is no ambiguity we will drop the superscript. We will also drop the superscript when the reference seed is irrelevant.

Let λ be a $\boldsymbol{g}$-vector in the interior of $\mathcal{I}$ and let t be any seed adjacent to the imaginary cluster containing λ i.e. suppose that the cluster expansion of λ is $\sum_{i=1}^{n-2}a_i\xi_i-a_0B_{t_0}\delta$ (with ξ_i in a finite c-orbit, $a_i\geqslant 0$ for all $i\geqslant 1$, and $a_0>0$) and that the $\boldsymbol{g}$-vectors at t are $\{\xi_1,\dots,\xi_n\}$. We now describe the exchange matrix at t.

First observe that, $\{\xi_i\}_{i\in[1,n-2]}$ are $\boldsymbol{g}$-vectors in finite c-orbits and $\{\xi_{n-1},\xi_n\}$ are $\boldsymbol{g}$-vectors in infinite orbits. Since λ is in the interior of $\mathcal{I}$ the set $\{\xi_i\}_{i\in[1,n-2]}$ must contain at least one $\boldsymbol{g}$-vector from the top layer in each tube and if a tube has size ℓ then it must contain precisely $\ell-1$ $\boldsymbol{g}$-vectors from it. Since $\boldsymbol{g}$-vectors from the top layer of a tube are incompatible with one another, the set $\{\xi_i\}_{i\in[1,n-2]}$ must contain exactly one $\boldsymbol{g}$-vector from the top layer in each tube. Let p_i+1 be the size of the i-th tube (so that $p_1+p_2+p_3=n-2$). To fix ideas we assume that $0\leqslant p_1\leqslant p_2\leqslant p_3$. Up to relabelling we can assume that each subset $\{\xi_1,\dots,\xi_{p_1}\}$, $\{\xi_{p_1+1},\dots,\xi_{p_1+p_2}\}$, and $\{\xi_{p_1+p_2+1},\dots,\xi_{p_1+p_2+p_3}\}$ consists of $\boldsymbol{g}$-vectors from the same tube and that ξ_{p_1}, $\xi_{p_1+p_2}$, and

$\xi_{p_1+p_2+p_3}$ are the $\boldsymbol{g}$-vectors in the top layer of each. We will therefore consider the decomposition of the exchange matrix at t in blocks B_{ij} given by the partition $(p_1, p_2, p_3, 2) \vdash n$.

We claim that the block B_{ii} is an exchange matrix of type A_{p_i} for all $i \in [1,2,3]$. Indeed observe first that any tube is of type A (c.f. [**?**][Proposition 4.4 (2)]) hence the corresponding symmetrizer is constant so that B_{ii} is symmetric. Moreover the (coefficient-free) exchange relations for $j < n-2$, $i \neq p_1, p_1 + p_2$ are of the form

$$x_{\xi_j} x_\zeta = x_{\xi_j+\zeta} + x_{\xi_j \uplus \zeta} \tag{1}$$

11. The standard notation for symmetric difference would have been Δ but it is definitely an overloaded symbol. I decided to follow CFZ's notation for the finite type case of which this is an instance I guess SS

for some ζ the $\boldsymbol{g}$-vector of a cluster variable, and $\xi_j \uplus \zeta$ (the symmetric difference of ξ_j and ζ 11) the $\boldsymbol{g}$-vector of a cluster monoimial, both in the same tube where ξ_j lives. Finally, the exchange relations among those listed in (1) that are supported on the i-th tube coincide with those in a cluster algebra of finite type A_{p_i} in which one cluster variable has been frozen (the bijection being given by the identification of the $\boldsymbol{d}$-vectors with respect to the simples at the base of the tube with the $\boldsymbol{d}$-vectors with respect to the fan triangulation of a $(p_i + 3)$-gon). Our claim follows.

Equation (1) together with sign-skew-symmetry also imply that the blocks B_{ij} with $i \neq j \in [1,2,3]$ can only have a non-zero entry in the bottom right corner.

Similarly blocks B_{i4} for $i \in [1,2,3]$ can only be non-zero in the last row and blocks B_{4j} for $j \in [1,2,3]$ can only be non-zero in the last column. Since the exchanges of ξ_{p_1}, $\xi_{p_1+p_2}$, and $\xi_{p_1+p_2+p_3}$ are not imaginary each of these matrices has at least one non-zero entry.

Next we claim that B_{44}, up to a possible relabelling, is of the form

$$\begin{bmatrix} 0 & a \\ -b & 0 \end{bmatrix}$$

with $a > 0$, $b > 0$ and $ab = 4$. This follows from the observation that the subalgebra generated by the cluster variables in all seeds containig $\{x_{\xi_i}\}_{i\in[1,n-2]}$ is an affine cluster algebra of rank 2. Indeed let ℓ be the minimum positive power such that c^ℓ fixes pointwise $\mathcal{I}$. Then the collection

$$\left\{x_{c^{j\ell}\xi_i}\right\}_{j\in\mathbb{Z}, i\in[1,n]}$$

is infinite (because both ξ_{n-1} and ξ_n are in infinite c-orbits) and it is contained in the subalgebra which thus contains infinitely many cluster variables. Since it is a subalgebra of an affine type cluster algebra it is also affine and our claim follows.

For each $i \in [p_1, p_1 + p_2, p_1 + p_2 + p_3]$ consider now the submatrix of the exchange matrix at t consisting of the rows and columns $i, n-1, n$. As its 2×2 submatrix on rows and columns $n-1$ and n is affine all of its non-diagonal entries must be non-zero. Otherwise it would be an exchange matrix of wild type. By direct inspection we see that the only possibilities are then those in Table 1. Note that, as the sign patterns are the same regardless of the chosen tube mutating in directions $n-1$ and n do not affect the exchange matrix at t outside of rows and columns $n-1$ and t. Moreover, by direct inspection, it is easy to see that these two mutations only change the signs in both the last two columns and the last two rows.

Some further considerations on the root lengths, together with [**?**][Table 1], show that submatrices of types $A_4^{(2)}$, $G_2^{(1)}$, and $D_4^{(3)}$ can only appear in cases having a signle tube so that the exchange matrix at t is completely determined in these cases. It remains to consider the possibly non-zero entries in the matrices B_{ij} with $i \neq j \in [1,2,3]$. By taking the 4×4 submatrix consisting of the rows and columns corresponding to indices of the $\boldsymbol{g}$-vectors in the top layer of the i-th and j-th tube together with $n-1$ and n we see directly that a non-zero entry in B_{ij} would produce a cluster algebra of wild type.

Table 1. Possible submatrices

Type	matrix	Type	matrix
$A_2^{(1)}$	$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$		
$C_2^{(1)}$	$\begin{bmatrix} 0 & 2 & -2 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$	$D_3^{(2)}$	$\begin{bmatrix} 0 & 1 & -1 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$
$G_2^{(1)}$	$\begin{bmatrix} 0 & 3 & -3 \\ -1 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$	$D_4^{(3)}$	$\begin{bmatrix} 0 & 1 & -1 \\ -3 & 0 & 2 \\ 3 & -2 & 0 \end{bmatrix}$
$A_4^{(2)}$	$\begin{bmatrix} 0 & 1 & -2 \\ -2 & 0 & 4 \\ 1 & -1 & 0 \end{bmatrix}$	$A_4^{(2)}$	$\begin{bmatrix} 0 & 2 & -1 \\ -1 & 0 & 1 \\ 2 & -4 & 0 \end{bmatrix}$

To summarize the exchange matrix at t, after relabelling, is of the form

$$\begin{bmatrix} B_{11} & 0 & 0 & B_{14} \\ 0 & B_{22} & 0 & B_{24} \\ 0 & 0 & B_{33} & B_{34} \\ B_{41} & B_{42} & B_{43} & B_{44} \end{bmatrix}$$

with B_{11}, B_{22}, B_{33} of finite type A, B_{i4} non zero only in the last row, B_{4j} non zero only in the last column, and such that the non-zero rows and columns in

$$\begin{bmatrix} 0 & B_{i4} \\ B_{4i} & B_{44} \end{bmatrix}$$

form one of the matrices from Table 1.

Using Proposition 0.6, the fact that $\mathcal{P}_\lambda \subseteq \{\lambda + \widetilde{B}^+\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\}$, and the fact that $\widetilde{B}^+$ is full rank, we deduce immediately the following upper bound for $\mathcal{P}_\lambda$.

Corollary 0.7. *Let λ be imaginary with z minimal such that $\lambda + z\widetilde{B}^+\delta$ is real. Then $\mathcal{P}_\lambda$ is contained in the line segment $\{\lambda + x\widetilde{B}^+\delta : 0 \leqslant x \leqslant z\}$*

Lemma 0.8. *There exists $r > 0$ so that $c^{-p}s_n \dots s_{\ell+1}\alpha_\ell$ has full support for any $p \geqslant r$ and any ℓ.*

Proof. By [**?**, Theorem 1.2(1)] and [**?**, Section 1], the set $\{c^q s_n \dots s_{\ell+1}\alpha_\ell : q \in \mathbb{Z}\}$ is infinite. On the other hand, there are only finitely many roots without full support. $\square$

Suppose t is a seed corresponding to a c-sortable element v. The construction of Cambrian frameworks in [**?**] 12 provides the following description of the columns of C^t. Let $s_{a_1} \dots s_{a_N}$ be the c-sorting word of v. For an index i consider the longest prefix $s_{a_1} \dots s_{a_p}$ of $s_{a_1} \dots s_{a_N}$ such that any instance of s_i in the corresponding prefix of c^∞ is also in $s_{a_1} \dots s_{a_p}$. Then the i-th column of C^t is the root $s_{a_1} \dots s_{a_p}\alpha_i$. This root is positive if and only if the word $s_{a_1} \dots s_{a_p}s_i$ is reduced. 12. after Prop 5.4 DR

This description of C^t is instrumental in proving the next two results.

Proposition 0.9. *Let t be a seed corresponding to a c-sortable element v 13 whose c-sorting word starts with c^r for r as in Lemma 0.8. Then the columns of C^t are not roots of the form $\pm c^{-p} s_n \dots s_{\ell+1}\alpha_\ell$ with $p \geqslant 0$.* 13. Explain this spondence SS

14. do we need a reference for this? SS

Proof. Let k be the index such that s_k is the leftmost reflection of c^∞ omitted in $s_{a_1} \dots s_{a_N}$. Since 14 c^p is reduced for any p, the k-th column of C^t is a positive root of the form $c^q s_1 \cdots s_{k-1}\alpha_k$ and, following [**?**, Theorem 1.2(1)] and [**?**, Section 1], it is not of the form $c^{-p} s_n \dots s_{\ell+1}\alpha_\ell$ for $p \geqslant 0$.

All the other columns of C^t will be roots of the form $c^q s_{b_1} \cdots s_{b_{i-1}}\alpha_{b_i}$ for $q \geqslant r$ with $k \neq b_j$ for any j. Suppose that one such root were also of the form $\pm c^{-p} s_n \dots s_{\ell+1}\alpha_\ell$ for $p \geqslant 0$ and some index ℓ. Then $\pm c^{-p-q} s_n \dots s_{\ell+1}\alpha_\ell = s_{b_1} \cdots s_{b_{i-1}}\alpha_{b_i}$ would be a root without full support. But $p + q \geqslant r$, contradicting Lemma 0.8. □

Proposition 0.10. *Let t be the seed associated to c^r for r as in Lemma 0.8. Then C^t contains at least one column of the form $c^q s_1 \cdots s_{k-1}\alpha_k$ and at least one column of the form $-c^q s_1 \cdots s_{k-1}\alpha_k$. The remaining columns that are not of the form $\pm c^q s_1 \cdots s_{k-1}\alpha_k$ lie in finite c-orbits.*

Proof. The first column of C^t is $c^r\alpha_1$ and the last column of C^t is $c^r\alpha_n = -c^{r-1} s_1 \cdots s_{n-1}\alpha_n$. By Proposition 0.9, any column of C^t which is not of the form $\pm c^q s_1 \cdots s_{k-1}\alpha_k$ must lie in a finite c-orbit. □

For $t \in \mathbb{T}_n$, define $K^t := \{G_t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\}$. For $\lambda \in \widetilde{K}^t := K^t \times \mathbb{R}^{m-n}$, set

$$L^t_{\lambda,\pm} := \{\lambda \pm \widetilde{B}^{t_+} C^t \alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\}.$$

Lemma 0.11. *For $\lambda \in \widetilde{K}^t$, we have $\mathcal{P}_\lambda \cap \widetilde{K}^t \subseteq L^t_{\lambda,+} \cap L^t_{\lambda,-}$.*

Proof. Suppose t is obtained from t^+ by mutating along the sequence $\boldsymbol{k}^+ = (k_N, \dots, k_1)$. Let $\boldsymbol{\ell}$ denote a red-to-green sequence for B^t, it exists following [**?**, Corollary 3.2.2] since B^{t_+} is acyclic and hence has a maximal green sequence. Write $\boldsymbol{k}^-$ for the concatenation of $\boldsymbol{\ell}$ and $\boldsymbol{k}^+$.

First observe that $\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1}$ acts linearly on $\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}$. Then, since $\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1} = \eta^{\widetilde{B}^t}_{(\boldsymbol{k}^+)^{op}}$, we see from [**?**, Equation (1.13)] that the action on $\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}$ is given by the matrix $\widetilde{G}^t$, i.e.

$$\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1} (\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}) = \widetilde{G}^t (\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}) = \widetilde{K}^t.$$

By Lemma 2.2, we have $\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^-}\right)^{-1} = \left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1} \left(\eta^{\widetilde{B}^t}_{\boldsymbol{\ell}}\right)^{-1}$. But $\eta^{\widetilde{B}^t}_{\boldsymbol{\ell}} (\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}) = (\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n})$ and so $\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^-}\right)^{-1} (\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n}) = -\widetilde{G}^t (\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n}) = \widetilde{K}^t$.

It follows that

$$\begin{aligned}
S_{\boldsymbol{k}^+,\lambda} \cap \widetilde{K}^t &= \left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1} \left\{\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&= \left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1} \left(\left\{\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \left(\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}\right)\right) \\
&= \widetilde{G}^t \left(\left\{\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \left(\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}\right)\right) \\
&= \left\{\lambda + \widetilde{G}^t \widetilde{B}^t \alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&= \left\{\lambda + \widetilde{B}^{t_+} C^t \alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&\subseteq L^t_{\lambda,+},
\end{aligned}$$

where the last equality uses Corollary 4.4. Similarly, we have

$$\begin{aligned}
S_{\boldsymbol{k}^-,\lambda} \cap \widetilde{K}^t &= \left(\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}\right)^{-1} \left\{\left(\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha\right) : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&= \left(\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}\right)^{-1} \left(\left\{\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \left(\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n}\right)\right) \\
&= -\widetilde{G}^t \left(\left\{\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \left(\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n}\right)\right) \\
&= \left\{\lambda - \widetilde{G}^t\widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&= \left\{\lambda - \widetilde{B}^{t_+}C^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&\subseteq L^t_{\lambda,-}.
\end{aligned}$$

Since $\mathcal{P}_\lambda \subseteq S_{\boldsymbol{k}^\pm,\lambda}$, the claim follows. □

Lemma 0.12. *For any root β the product $(\beta^\vee)^T B^+\delta$ is*

- *positive if $\beta = c^p s_1 \dots s_{k-1}\alpha_k$ for some $p \in \mathbb{Z}$ and some k,*
- *negative if $\beta = c^{-p} s_n \dots s_{k+1}\alpha_k$ for some $p \in \mathbb{Z}$ and some k,*
- *zero otherwise.*

Proof. Begin by observing that the product $(\beta^\vee)^T B^+\gamma$ is invariant under source-sink moves. Indeed, $(s_1\beta^\vee)^T B^+(s_1\gamma) = (\beta^\vee)^T(E^{t_+}_{1,-}B^+F^{t_+}_{1,-})\gamma = (\beta^\vee)^T\mu_1(B^+)\gamma$ and similarly for s_n.

Using this invariance, it suffices to establish the first claim for $p = 0$ and $k = 1$. Since δ is in the kernel of A, we have that $(\alpha_1^\vee)^T B^+\delta = (\alpha_1^\vee)^T(A + B^+)\delta$. The result then follows immediately from the observation that $A + B^+$ is a lower-triangular matrix with positive entries on the diagonal and that δ has full support.

Similarly it suffices to establish the second claim only for $p = 1$ and $k = 1$. By the same computation we just did we have $(-\alpha_1^\vee)^T B^+\delta = -(\alpha_1^\vee)^T(A + B^+)\delta < 0$.

For the remaining case, since B^+ is skew-symmetrizable, $(\delta^\vee)^T B^+\delta = 0$. By assumption, the root β is in a finite c-orbit (c.f. [**?**, Proposition 1.9 and Section 1, final Remark]) so that there is a positive ℓ such that $\sum_{i=0}^{\ell} c^i\beta = q\delta$ for some $q \neq 0$ (cf. [**?**]). By the invariance under source-sink moves we have that $(c\beta^\vee)^T B^+\delta = (\beta^\vee)^T B^+\delta$ and we can compute

$$0 = q\,(\delta^\vee)^T B^+\delta = \left(\sum_{i=0}^{\ell} c^i\beta^\vee\right)^T B^+\delta = \ell\,(\beta^\vee)^T B^+\delta$$

to conclude that $(\beta^\vee)^T B^+\delta = 0$. □

Lemma 0.13. *Suppose t is obtained from t_+ by mutating along the sequence $\boldsymbol{k} = (k_N, \dots, k_1)$ passing through the seeds $t_+ = t_1, \dots, t_{N+1} = t$. Consider seeds $t'_1, \dots, t'_{N+1}$ with t'_{i+1} obtained from t'_i by mutation in direction k_i and t'_1 obtained from t_+ by mutating along the sequence $(n, \dots, 1)$ sufficiently many times that $\left(G^{\vee,t'_i}\right)^T\delta$ is a nonnegative vector for every i.* 15 16

15. i.e. on the positive side of $\delta^\perp$. This sentence needs to be justified; somewhere we need to discuss the plane $\delta^\perp$ and how it cuts the g-ector fan. SS

16. Think about $\vee$ or no $\vee$ in the G matrix here. DR

For $1 \leqslant i \leqslant N$, let ε_i be the opposite of the sign of the k_i-th column of $C^{\vee,t'_i}$. Set $\delta_1 = \delta$ and define $\delta_{i+1} = F^{t_i}_{k_i,\varepsilon_i} \cdots F^{t_1}_{k_1,\varepsilon_1}\delta$. Then the following hold.

1. *δ_i is a nonnegative vector for $1 \leqslant i \leqslant N + 1$;*
2. *For $1 \leqslant i \leqslant N$, if the k_i-th entry of $-\widetilde{B}^{t_i}\delta_i$ is nonzero then its sign is ε_i.*
3. *$\eta_{\boldsymbol{k}}^{\widetilde{B}^+}\left(-\widetilde{B}^+\delta\right) = -\widetilde{B}^t\delta_{N+1}$.*

Remark 0.14. *Something about $\left(G^{\vee,t'_i}\right)^T\delta$ being the absolute value of the kernel of the quasi-Cartan companion A^{t_N}. Maybe mention something about companion bases not existing in affine types.*

Proof. First observe that $F^{t_i}_{k_i,\varepsilon_i} = F^{t'_i}_{k_i,\varepsilon_i}$ for each i. By Lemma 4.7, $F^{t'_i}_{k_i,\varepsilon_i}\cdots F^{t'_1}_{k_1,\varepsilon_1} = (G^{\vee,t'_{i+1}})^T M$ for some matrix M such that $M\delta = \delta$ since t'_1 is obtained from t_+ by repeatedly mutating along the sequence $(n,\dots,1)$. Therefore $\delta_i = (G^{\vee,t'_i})^T\delta$ is a nonnegative vector by assumption.

The first n entries of $\widetilde{B}^{t_i}\delta_i$ coincide with the entries of $B^{t_i}\delta_i$. We apply Corollary 4.8 to get

$$B^{t_i}\delta_i = B^{t'_i}\delta_i = B^{t'_i}(G^{\vee,t'_i})^T\delta = \left(C^{\vee,t'_i}\right)^T B^+\delta.$$

Redefining the seeds t'_i, if needed, to ensure that the hypotheses of Proposition 0.9 are satisfied and combining Lemma 0.12 with Lemma 0.9 we see that the k_i-th sign of $\widetilde{B}^{t_i}\delta_i$ weakly agrees with the sign of the k_i-th column of $C^{\vee,t'_i}$.

To conclude, we are now able to compute

$$\begin{aligned}\eta^{\widetilde{B}^+}_{\boldsymbol{k}}\left(-\widetilde{B}^+\delta\right) &= -\widetilde{E}^{t_N}_{k_N,\varepsilon_N}\cdots\widetilde{E}^{t_1}_{k_1,\varepsilon_1}\widetilde{B}^+\delta\\ &= -\widetilde{B}^t F^{t_N}_{k_N,\varepsilon_N}\cdots F^{t_1}_{k_1,\varepsilon_1}\delta\\ &= -\widetilde{B}^t\delta_{N+1}.\end{aligned}$$

□

Corollary 0.15. *Suppose t is obtained from t_+ by mutating along the sequence $\boldsymbol{k} = (k_N,\dots,k_1)$ passing through the seeds $t_+ = t_1,\dots,t_{N+1} = t$. Then the imaginary cone at t_{N+1} is contained in the hyperplane orthogonal to $\widetilde{\delta}_{N+1}$.*

17. This should be defined SS

18. do we want to change notation: $\mathfrak{d}_\infty$ instead of $\mathcal{I}$? SS

Proof. We work by induction on N, the claim being known for $N = 0$. Recall that $\mathcal{I}^{t_N}$ (17) (18) is cut into (finitely many) domains of linearity for the mutation maps and that $-B^{t_N}\delta_N$ is contained in each of them. Therefore the k_N-th coordinate of any $\lambda\in\mathcal{I}^{t_N}$ weekly agrees in sign with the k_N-th coordinate of $-B^{t_N}\delta_N$.

If the k_N-th entry of $-B^{t_N}\delta_N$ is nonzero it must have sign ε_N, then for any $\lambda\in\mathcal{I}^{t_N}$ we have $\eta^{\widetilde{B}^{t_N}}_{k_N}(\lambda) = \widetilde{E}^{t_N}_{k_N,\varepsilon_N}\lambda$ and we compute

$$\begin{aligned}\langle\widetilde{\delta}_{N+1},\eta^{\widetilde{B}^{t_N}}_{k_N}(\lambda)\rangle &= \langle\widetilde{F}^{t_N}_{k_N,\varepsilon_N}\widetilde{\delta}_N,\widetilde{E}^{t_N}_{k_N,\varepsilon_N}\lambda\rangle\\ &= \left(\widetilde{F}^{t_N}_{k_N,\varepsilon_N}\widetilde{\delta}_N\right)^T\widetilde{D}\widetilde{E}^{t_N}_{k_N,\varepsilon_N}\lambda\\ &= \widetilde{\delta}_N^T\left(\widetilde{F}^{t_N}_{k_N,\varepsilon_N}\right)^T\widetilde{D}\widetilde{E}^{t_N}_{k_N,\varepsilon_N}\lambda\\ &= \widetilde{\delta}_N^T\widetilde{D}\widetilde{E}^{t_N}_{k_N,\varepsilon_N}\widetilde{E}^{t_N}_{k_N,\varepsilon_N}\lambda\\ &= \widetilde{\delta}_N^T\widetilde{D}\lambda\\ &= \langle\widetilde{\delta}_N,\lambda\rangle\\ &= 0\end{aligned}$$

where we used the definition of pairing and the fact that, for any t, k, and ε, $\left(\widetilde{F}^t_{k,\varepsilon}\right)^T\widetilde{D} = \widetilde{D}\widetilde{E}^t_{k,\varepsilon}$.
(19)

19. Note to self: The symmetrized D is there because the dual basis to weights is coroots SS

If instead the k_N-th entry of $-B^{t_N}\delta_N$ is zero then $\widetilde{F}^{t_N}_{k_N,+}\widetilde{\delta}_N = \widetilde{\delta}_{N+1} = \widetilde{F}^{t_N}_{k_N,-}\widetilde{\delta}_N$ since the k_N-th row of $-B^{t_N}$ is equal to the k_N-th row of $F^{t_N}_{k_N,+} - F^{t_N}_{k_N,-}$. For $\lambda \in \mathcal{I}^{t_N}$ let ε the sign such that $\eta^{\widetilde{B}^{t_N}}_{k_N}(\lambda) = \widetilde{E}^{t_N}_{k_N,\varepsilon}\lambda$ and compute as before

$$\langle \widetilde{\delta}_{N+1}, \eta^{\widetilde{B}^{t_N}}_{k_N}(\lambda)\rangle = \langle \widetilde{F}^{t_N}_{k_N,\varepsilon}\widetilde{\delta}_N, \widetilde{E}^{t_N}_{k_N,\varepsilon}\lambda\rangle = \langle \widetilde{\delta}_N, \widetilde{E}^{t_N}_{k_N,\varepsilon}\widetilde{E}^{t_N}_{k_N,\varepsilon}\lambda\rangle = \langle \widetilde{\delta}_N, \lambda\rangle = 0.$$

□

Proposition 0.16. *Let λ be imaginary with z minimal such that $\lambda + z\widetilde{B}^+\delta$ is real. Then $\mathcal{P}_\lambda$ contains the line segment $\{\lambda + x\widetilde{B}^+\delta : 0 \leqslant x \leqslant z\}$.*

Proof. Suppose t is obtained from t_+ by mutating along the sequence $\boldsymbol{k}$. To establish the claim we need to show that, for $0 \leqslant x \leqslant z$,

$$\eta^{\widetilde{B}^+}_{\boldsymbol{k}}(\lambda + x\widetilde{B}^+\delta) \in \left\{\eta^{\widetilde{B}^+}_{\boldsymbol{k}}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\}.$$

Since the vectors λ, $\lambda + x\widetilde{B}^+\delta$, and $-\widetilde{B}^+\delta$ all live in the same cone of the mutation fan, 20 we have that 20. we need to decide which presentation we will use and quote afftheta SS

$$\eta^{\widetilde{B}^+}_{\boldsymbol{k}}(\lambda + x\widetilde{B}^+\delta) = \eta^{\widetilde{B}^+}_{\boldsymbol{k}}(\lambda) - x\eta^{\widetilde{B}^+}_{\boldsymbol{k}}(-\widetilde{B}^+\delta)$$

and our task reduces to showing that

$$\eta^{\widetilde{B}^+}_{\boldsymbol{k}}(-\widetilde{B}^+\delta) = -\widetilde{B}^t\alpha$$

for some positive vector $\alpha \in \mathbb{R}^n_{\geqslant 0}$. Lemma 0.13 gives an explicit formula for such a vector α and completes the proof. □

This concludes the proof of Theorem 0.1.

Some old stuff we no longer need

For $t \in \mathbb{T}_n$, define $K^t := \{G_t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\}$. For $\lambda \in \widetilde{K}^t := K^t \times \mathbb{R}^{m-n}$, set

$$L^t_{\lambda,\pm} := \{\lambda \pm \widetilde{B}^{t_+}C^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\}.$$

Lemma 0.17. *For $\lambda \in \widetilde{K}^t$, we have $\mathcal{P}_\lambda \cap \widetilde{K}^t \subseteq L^t_{\lambda,+} \cap L^t_{\lambda,-}$.*

Proof. Suppose t is obtained from t^+ by mutating along the sequence $\boldsymbol{k}^+ = (k_N, \ldots, k_1)$. Set $\boldsymbol{k}^- = (n, \ldots, 1, k_N, \ldots, k_1)$.

First observe that $\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1}$ acts linearly on $\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}$. Then, since $\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1} = \eta^{\widetilde{B}^t}_{(\boldsymbol{k}^+)^{op}}$, we see from [?, Equation (1.13)] that the action on $\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}$ is given by the matrix $\widetilde{G}^t$, i.e.

$$\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1}(\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}) = \widetilde{G}^t(\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}) = \widetilde{K}^t.$$

By Lemma 2.2, we have $\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^-}\right)^{-1} = \left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^+}\right)^{-1}\left(\eta^{\widetilde{B}^t}_{(n,\cdots,1)}\right)^{-1}$. But $\eta^{\widetilde{B}^t}_{(n,\cdots,1)}(\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}) \overset{?}{=} (\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n})$ and so $\left(\eta^{\widetilde{B}^{t_+}}_{\boldsymbol{k}^-}\right)^{-1}(\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n}) = -\widetilde{G}^t(\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n}) = \widetilde{K}^t$.

It follows that

$$\begin{aligned}
S_{\boldsymbol{k}^+,\lambda} \cap \widetilde{K}^t &= \left(\eta_{\boldsymbol{k}^+}^{\widetilde{B}^{t_+}}\right)^{-1} \left\{\eta_{\boldsymbol{k}^+}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&= \left(\eta_{\boldsymbol{k}^+}^{\widetilde{B}^{t_+}}\right)^{-1} \left(\left\{\eta_{\boldsymbol{k}^+}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \left(\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}\right)\right) \\
&= \widetilde{G}^t \left(\left\{\eta_{\boldsymbol{k}^+}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \left(\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{m-n}\right)\right) \\
&= \left\{\lambda + \widetilde{G}^t\widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&= \left\{\lambda + \widetilde{B}^{t_+}C^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&\subseteq L^t_{\lambda,+},
\end{aligned}$$

where the last equality uses Corollary 4.4. Similarly, we have

$$\begin{aligned}
S_{\boldsymbol{k}^-,\lambda} \cap \widetilde{K}^t &= \left(\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}\right)^{-1} \left\{\left(\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha\right) : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&= \left(\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}\right)^{-1} \left(\left\{\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \left(\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n}\right)\right) \\
&= -\widetilde{G}^t \left(\left\{\eta_{\boldsymbol{k}^-}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \left(\mathbb{R}^n_{\leqslant 0} \times \mathbb{R}^{m-n}\right)\right) \\
&= \left\{\lambda - \widetilde{G}^t\widetilde{B}^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&= \left\{\lambda - \widetilde{B}^{t_+}C^t\alpha : \alpha \in \mathbb{R}^n_{\geqslant 0}\right\} \cap \widetilde{K}^t \\
&\subseteq L^t_{\lambda,-}.
\end{aligned}$$

Since $\mathcal{P}_\lambda \subseteq S_{\boldsymbol{k}^\pm,\lambda}$, the claim follows. □

Below here is the stuff we wrote so far.

1. Introduction

Cluster algebras are recursively defined commutative rings. Since their discovery by Fomin and Zelevinsky through an intensive study of dual canonical bases [?], cluster algebras have found application throughout mathematics, including Lie theory [?], representation theory [?], Teichmüller theory [?], and mathematical physics [?]. See [?] for a more exhaustive description of the deep connections found to cluster algebras.

A guiding question in the theory has always been to understand possible bases of a cluster algebra. Qin put bounds on how the pointed bases can be related.

2. Mutation Maps and Dominance

Fix $m \geqslant n$. Let $\widetilde{B} = (b_{ij})$ be an $m \times n$ exchange matrix with principal $n \times n$ submatrix B. Note that our exchange matrices are tall, which matches the convention of [?]. Then B is skew-symmetrizable with DB skew-symmetric for some diagonal integer matrix $D = \mathrm{diag}(d_1, \dots, d_n)$.

For $b \in \mathbb{R}$, write $[b]_+ = \max(b, 0)$. Given a sign $\varepsilon \in \{\pm\}$ and $1 \leqslant k \leqslant n$, define an $m \times m$ matrix $\widetilde{E}_{k,\varepsilon} = (e_{ij})$ with

$$e_{ij} = \begin{cases} 1 & \text{if } i = j \neq k; \\ -1 & \text{if } i = j = k; \\ [\varepsilon b_{ik}]_+ & \text{if } i \neq j = k; \\ 0 & \text{otherwise;} \end{cases} \tag{2}$$

and an $n \times n$ matrix $F_{k,\varepsilon} = (f_{ij})$ with

$$f_{ij} = \begin{cases} 1 & \text{if } k \neq i = j; \\ -1 & \text{if } k = i = j; \\ [-\varepsilon b_{kj}]_+ & \text{if } k = i \neq j; \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Observe that $\widetilde{E}_{k,\varepsilon}^2 = \mathbb{1}_m$ and $F_{k,\varepsilon}^2 = \mathbb{1}_n$ for any choice of ε. Then define $\mu_k \widetilde{B} = \widetilde{E}_{k,\varepsilon} \widetilde{B} F_{k,\varepsilon}$. Using the identity $b_{ij} = [b_{ij}]_+ - [-b_{ij}]_+$ it is easy to see that $\mu_k \widetilde{B}$ doesn't depend on the choice of sign ε. Moreover, the principal part $\mu_k B$ of $\mu_k \widetilde{B}$ is again skew-symmetrizable using the same matrix D.

Given a matrix M, denote by $M_{\bullet k}$ (resp. $M_{k\bullet}$) the k-th column (resp. k-th row) of M and write $[M]_{\bullet k}$ (resp $[M]_{k\bullet}$) the square matrix whose k-th column (resp. k-th row) matches that of M with all other entries being zero.

Lemma 2.1. *For $\varepsilon \in \{\pm 1\}$ and $1 \leqslant k \leqslant n$, we have*

1. $\widetilde{E}_{k,-\varepsilon} \widetilde{E}_{k,\varepsilon} = \mathbb{1}_m + \varepsilon [\widetilde{B}]_{\bullet k}$, $\widetilde{E}_{k,\varepsilon} - \widetilde{E}_{k,-\varepsilon} = \varepsilon [\widetilde{B}]_{\bullet k}$, *and* $\widetilde{E}_{k,\varepsilon} [\widetilde{B}]_{\bullet k} = [\widetilde{B}]_{\bullet k}$*;*
2. $F_{k,-\varepsilon} F_{k,\varepsilon} = \mathbb{1}_n + \varepsilon [B]_{k\bullet}$, $F_{k,\varepsilon} - F_{k,-\varepsilon} = -\varepsilon [B]_{k\bullet}$, *and* $F_{k,\varepsilon} [B]_{k\bullet} = -[B]_{k\bullet}$.

Proof. This is immediate from the equality $\varepsilon b_{ij} = [\varepsilon b_{ij}]_+ - [-\varepsilon b_{ij}]_+$. □

To record sequences of these matrix mutations, we introduce the n-regular rooted tree $\mathbb{T}_n$ with root vertex t_0 and edges labeled by $\{1, \ldots, n\}$. Associate $m \times n$ matrices $\widetilde{B}^t$ with principal part B^t to the vertices $t \in \mathbb{T}_n$ so that:

- $\widetilde{B}^{t_0} = \widetilde{B}$;
- if $t, t' \in \mathbb{T}_n$ are joined by an edge labeled k, then $\widetilde{B}^{t'} = \mu_k \widetilde{B}^t$.

Given a sequence $\boldsymbol{k} = (k_N, \ldots, k_1)$ with $k_i \in \{1, \ldots, n\}$ write $\mu_{\boldsymbol{k}}$ for the iterated matrix mutation $\mu_{k_N} \circ \cdots \circ \mu_{k_1}$. Then more directly, when t' is obtained from t by following edges labeled by $\boldsymbol{k} = (k_N, \ldots, k_1)$, we have $\widetilde{B}^{t'} = \mu_{\boldsymbol{k}} \widetilde{B}$.

A skew-symmetric matrix $B = (b_{ij})$ is <u>acyclic</u> if there is no sequence $i_1, \ldots, i_r, i_{r+1} = i_1$ so that $b_{i_\ell i_{\ell+1}} > 0$ for $1 \leqslant \ell \leqslant r$. In the case when B is acyclic, there exists a permutation σ of $\{1, \ldots, n\}$ so that $r < r'$ implies $b_{\sigma_r \sigma_{r'}} \geqslant 0$. We also associate to B a Cartan matrix $A = (a_{ij})$ with $a_{ii} = 2$ and $a_{ij} = -|b_{ij}|$i for $i \neq j$. We say that the mutation pattern is of <u>affine type</u> if there exists acyclic B^t whose associated Cartan matrix gives rise to an affine Dynkin diagram.

Assume there exists $t_+ \in \mathbb{T}_n$ so that $B^{t_+} = (b_{ij}^{t_+})$ is acyclic with $b_{ij}^{t_+} \geqslant 0$ for $i < j$. This provides a Coxeter element $c = s_1 \cdots s_n$ in the Weyl group associated to A.

Given $\boldsymbol{k} = (k_N, \ldots, k_1)$ and any $\widetilde{B}$, define the piecewise-linear <u>mutation map</u> $\eta_{\boldsymbol{k}}^{\widetilde{B}} : \mathbb{R}^m \to \mathbb{R}^m$ where $\eta_{\boldsymbol{k}}^{\widetilde{B}}(\nu)$ is the last column of $\mu_{\boldsymbol{k}}([\widetilde{B}\ \nu])$.

Lemma 2.2.

(1) $\left(\eta_{\boldsymbol{k}}^{\widetilde{B}}\right)^{-1} = \eta_{\boldsymbol{k}^{op}}^{\mu_{\boldsymbol{k}}\widetilde{B}}$, *where* $\boldsymbol{k}^{op} := (k_1, \ldots, k_N)$.

(2) *Consider a sequence of mutations* $\boldsymbol{k} = \boldsymbol{k}''\boldsymbol{k}'$ *where* $\mu_{\boldsymbol{k}'}$ *goes from* t_+ *to* t' *and* $\boldsymbol{k}''$ *goes from* t' *to* t''. *Then* $\eta_{\boldsymbol{k}}^{\widetilde{B}^{t_+}} = \eta_{\boldsymbol{k}''}^{\widetilde{B}^{t'}}\eta_{\boldsymbol{k}'}^{\widetilde{B}^{t_+}}$.

For $\lambda \in \mathbb{R}^m$ and a sequence $\boldsymbol{k} = (k_N, \ldots, k_1)$ of mutations from t_+ to t, define

$$S_{\boldsymbol{k},\lambda} := \left\{ \left(\eta_{\boldsymbol{k}}^{\widetilde{B}^{t_+}}\right)^{-1} \left(\eta_{\boldsymbol{k}}^{\widetilde{B}^{t_+}}(\lambda) + \widetilde{B}^t\alpha\right) : \alpha \in \mathbb{R}^n_{\geqslant 0} \right\}.$$

Definition 2.3. *For* $\lambda \in \mathbb{R}^m$, *define the* <u>*dominance region*</u>

$$\mathcal{P}_\lambda = \bigcap_{\boldsymbol{k}} S_{\boldsymbol{k},\lambda}.$$

When $\mu \in \mathcal{P}_\lambda$, *we say* <u>$\lambda$ *dominates* μ</u>. *Write* $\mathcal{P}_\lambda^{\mathbb{Z}} := \mathcal{P}_\lambda \cap \widetilde{B}^+ \cdot \mathbb{Z}^n_{\geqslant 0}$.

Theorem 2.4. [?] $\mathcal{P}_{\mathbb{Z}}(\lambda)$ *controls the deformations of a basis element pointed at* $\widetilde{\lambda}$.

3. Weyl group combinatorics of mutations

Let $A^+ = (a_{ij})$ denote the Cartan companion of B^{t_+}. Fix an n-dimensional vector space V with basis $\alpha_1^\vee, \ldots, \alpha_n^\vee$, called the <u>simple coroots</u>. Define <u>simple roots</u> $\alpha_i := d_i\alpha_i^\vee$ which provide another basis of V. We will use the bilinear pairing K on V defined by $K(\alpha_i^\vee, \alpha_j) = a_{ij}$. This defines the <u>simple reflections</u> $s_i(\beta) = \beta - K(\alpha_i^\vee, \beta)\alpha_i$ and the corresponding Weyl group $W = \langle s_1, \ldots, s_n\rangle$ acting linearly on V. Write $S = \{s_1, \ldots, s_n\}$ for the collection of simple reflections. Given $I \subseteq S$, let $W_I = \langle s : s \in I\rangle$ denote the <u>parabolic subgroup</u> generated by I. For $s \in S$, write $[s] = S\backslash\{s\}$.

Let $c = s_1 \cdots s_n$ be the <u>Coxeter element</u> of W associated to B^{t_+}. More generally, any element of W that can be obtained as the product of all elements of S in some order is a <u>Coxeter element</u>. An element $s \in S$ is <u>initial</u> in c if $\ell(sc) < \ell(c)$, in this case sc is a Coxeter element of $W_{[s]}$. When s_k is initial in c, s_kcs_k is the Coxeter element associated to $\mu_kB^{t_+}$. An element $w \in W$ is <u>c-sortable</u> if the following recursive definition holds

- the identity element of W is c-sortable;
- if s is initial in c and $\ell(sw) < \ell(w)$, then w is c-sortable if and only if sw is scs-sortable;
- if s is initial in c and $\ell(sw) > \ell(w)$, then w is c-sortable if and only if $w \in W_{[s]}$ is sc-sortable.

Statement about constructing $C(v)$ from skips.

Definition 3.1. *We say* $\nu, \lambda \in \mathbb{R}^m$ *are in the same* <u>$\widetilde{B}^+$*-class*</u> *if they lie in the same domain of linearity for* $\eta_{\boldsymbol{k}}^{\widetilde{B}^+}$ *for all sequences* $\boldsymbol{k}$. *Define the* <u>*mutation fan*</u> $\mathcal{F}_{\widetilde{B}^+}$ *whose maximal cones are the closures of the* $\widetilde{B}^+$*-classes.*

Remark 3.2. $\mathcal{F}_{\widetilde{B}^+} = \mathcal{F}_{B^+} \times \mathbb{R}^{m-n}$ *since the linearity domain in which* ν *lies depends only on its first* n *coordinates.*

Cite these results for affine types:

- $\mathcal{F}_{B^+}$ is simplicial and complete
- $\mathcal{F}_{B^+}$ is equivalent to the (transposed) scattering diagram fan and the associahedron(?) fan
- A maximal (by inclusion) cone K in $\mathcal{F}_{B^+}$ is <u>real</u> if there exists $t \in \mathbb{T}_n$ such that $K = K_t := G^t \cdot \mathbb{R}^n_{\geqslant 0}$, otherwise it is <u>imaginary</u>.
- There is a finite of imaginary cones.

- Any maximal imaginary cone K is $n-1$ dimensional and is contained in $\delta^\perp$ (need to define δ and $\perp$)
- There exists $N > 0$ so that c^N (as it acts on weights) fixes K pointwise
- $-\frac{1}{2}B^+ \cdot \delta$ is one of the primitive vectors spanning a ray of K
- Let $\eta_1, \dots, \eta_{n-2}$ be the other $n-2$ primitive vectors spanning rays of K.
- Then there exists a real cone H in $\mathcal{F}_{B^+}$ such that
  - $\eta_1, \dots, \eta^{n-2}$ span rays of H
  - $c^{\ell N} H$ is a real cone for all $\ell \geqslant 0$
  - $\lim_{\ell \to \infty} c^{\ell N} H = K$

  The second condition above defines a sequence $t_\ell \in \mathbb{T}_n$ that converges to K.
- Let η_{n-1}, η_n be the remaining primitive vectors spanning rays of H. Then, in the appropriate scaling limit, both of these vectors limit to $-\frac{1}{2}B^+ \cdot \delta$ under the action of $c^{\ell N}$ as $\ell \to \infty$.

Part of $\mathcal{F}_{B^+}$ is the Cambrian fan
after Prop. 5.4 in [**?**] gives C-matrices explicitly

Theorem 3.3. *Let $\widetilde{\lambda}$ be imaginary with s minimal such that $\widetilde{\lambda} + s\widetilde{B}^+ \cdot \delta$ is real.* 21 *Then the dominance region $\mathcal{P}(\widetilde{\lambda})$ is the segment $\{\widetilde{\lambda} + r\widetilde{B}^+ \cdot \delta : 0 \leqslant r \leqslant s\}$.* 21. We need to say why such an s exists. SS

Remark 3.4. *When $\widetilde{B}^+$ is full rank, $\mathcal{P}_{\mathbb{Z}}(\widetilde{\lambda})$ contains points of the form $\widetilde{\lambda} + r\widetilde{B}^+ \cdot \delta$ with $r \in \mathbb{Z}_{\geqslant 0}$. Without the full rank assumption, this no longer has to hold.*

Proof.

✓ $\mathcal{P}(\widetilde{\lambda})$ is contained in the imaginary cone. $\mathcal{P}(\widetilde{\lambda})$ is contained in $\widetilde{\lambda} + \widetilde{B}^+ \cdot \mathbb{R}^n_{\geqslant 0}$ which is a proper cone since $\widetilde{B}^+$ is full rank. In particular, for sufficiently large r, $\mathcal{P}(\widetilde{\lambda})$ doesn't contain $-r\widetilde{B}^+ \cdot \delta$. Then for any point $\widetilde{\mu}$ outside the closure of the imaginary cone, for sufficiently large ℓ the vector $c^\ell \widetilde{\mu}$ has large magnitude and is arbitrarily close the imaginary ray.

- $\mathcal{P}(\widetilde{\lambda})$ is contained in the ray $\{\widetilde{\lambda} + r\widetilde{B}^+ \cdot \delta : r \geqslant 0\}$.
  - Define green and red regions for real $\widetilde{\lambda}$.
  - Compute the green and red regions as $\widetilde{\lambda} \pm \widetilde{B}^+ \cdot C^t \cdot \mathbb{R}^n_{\geqslant 0}$. Use $\widetilde{G}^t \widetilde{B}^t = \widetilde{B}^+ C^t$.
  - Limits of regions make sense because of continuity after intersecting with domains of linearity.
  - Use limits of C^t to draw the conclusion.

✓ $\mathcal{P}(\widetilde{\lambda})$ contains $\{\widetilde{\lambda} + r\widetilde{B}^+ \cdot \delta : 0 \leqslant r \leqslant s\}$.
  - δ spans the kernel of the Cartan companion A of B^+ as well as any number of Coxeter mutations away
  - after sufficiently many Coxeter mutations, the seed t lies entirely on one side of the imaginary hyperplane and so the kernel κ_t of the quasi-Cartan companion is sign coherent

    proof: Since $A^t := (C^{\vee,t})^T A_0 C^t$ then $G^t A^t (G^{\vee,t})^T := A_0$. Since G^t is invertible $A_0 \cdot \delta = 0$ implies $A^t (G^{\vee,t})^T \cdot \delta = 0$ i.e. $(G^{\vee,t})^T \cdot \delta$ spans the kernel of A^t. After sufficiently many coxeter all the g-vectors at t are on the positive side of $\delta^\perp$ and the result follows.
  - for t' connected to t by mutation in direction k, we want $\eta_k^{\widetilde{B}^t}(\widetilde{B}^t \cdot \kappa_t) = \widetilde{B}^{t'} \cdot \kappa_{t'}$
  - $\eta_k^{\widetilde{B}^t}(\widetilde{B}^t \cdot \kappa_t) = \widetilde{E}^t_{k,\varepsilon} \widetilde{B}^t \cdot \kappa_t = \widetilde{E}^t_{k,\varepsilon} \widetilde{B}^t F^t_{k,\varepsilon} F^t_{k,\varepsilon} \cdot \kappa_t = \widetilde{B}^{t'} \cdot F^t_{k,\varepsilon} \kappa_t$ but $\kappa_{t'} = F^t_{k,\varepsilon_{trop}} \kappa_t$

  - For Cambrian mutations, ε_{trop} is always positive. This follows from the explicit formula for $\boldsymbol{c}$-vectors in [?] after Prop. 5.4.
  - The k-th row of $F^t_{k,+} + F^t_{k,-}$ is the negative of the k-th row of the associated Cartan companion (not quasi)
  - The k-th row of $F^t_{k,-} - F^t_{k,+}$ is the k-th row of $\widetilde{B}^t$
  - The k-th entries of $\widetilde{B}^{t'} \cdot F^t_{k,\varepsilon}\kappa_t$ and $\widetilde{B}^{t'} \cdot \kappa_{t'}$ are the same
  - $A^t := (C^{\vee,t})^T A_0 C^t$
  - $A^{t'} = (F^{t,\vee}_{k,\varepsilon_{trop}})^T A_t F^t_{k,\varepsilon_{trop}}$
- For (sufficiently) Cambrian seeds t, the sign of the k-th entry of $\widetilde{B}^t(G^{\vee,t})^T\delta$ is the same as the sign of the k-th $\boldsymbol{c}$-vector since $\widetilde{B}^t(G^{\vee,t})^T\delta = (C^{\vee,t})^T B^+\delta = -2(C^{\vee,t})^T\nu_c(\delta)$ and ν_c is given by the negative of the Euler matrix.

□

4. $\boldsymbol{c}$-vectors and $\boldsymbol{g}$-vectors

Definition $\boldsymbol{c}$-matrices C^t, for $t \in \mathbb{T}_n$, recursively as follows:

- $C^{t_+} = \mathbb{1}_n$ is the $n \times n$ identity matrix;
- when t and t' are joined by an edge labeled k, $C^{t'} = (c^{t'}_{ij})$ is related to $C^t = (c^t_{ij})$ by

$$c^{t'}_{ij} = \begin{cases} -c^t_{ij} & \text{if } i = k \text{ or } j = k; \\ c^t_{ij} + [-\varepsilon c^t_{ik}]_+ b^t_{kj} + c^t_{ik}[\varepsilon b^t_{kj}]_+ & \text{otherwise;} \end{cases} \tag{4}$$

  for any choice of sign $\varepsilon \in \{\pm 1\}$.

Following [?, ?] it is known that the column of C^t are always sign-coherent, i.e. all entries of each column $C^t_{\bullet k}$ are either all nonnegative or all nonpositive. Write ε^t_k for the sign of the nonzero entries of the k-th column of C^t. Using this choice of sign, the expression in (4) simplifies to

$$c^{t'}_{ij} = \begin{cases} -c^t_{ij} & \text{if } i = k \text{ or } j = k; \\ c^t_{ij} + c^t_{ik}[\varepsilon^t_k b^t_{kj}]_+ & \text{otherwise.} \end{cases} \tag{5}$$

Lemma 4.1. *Suppose t is obtained from t_+ by mutating along the sequence $\boldsymbol{k} = (k_N, \ldots, k_1)$ passing through $t_+ = t_1, \ldots, t_N, t_{N+1} = t$. Then C^t can be factored as $F^{t_1}_{k_1,-\varepsilon^{t_1}_{k_1}} \cdots F^{t_N}_{k_N,-\varepsilon^{t_N}_{k_N}}$.*

Proof. The recursion (5) can be rewritten as $C^{t'} = C^t F^t_{k,-\varepsilon^t_k}$ and the claim follows by induction. □

Define $\boldsymbol{g}$-matrices $\widetilde{G}^t$, for $t \in \mathbb{T}_n$, recursively as follows:

- $\widetilde{G}^{t_+} = \mathbb{1}_m$ is the $m \times m$ identity matrix;
- when t and t' are joined by an edge labeled k, $\widetilde{G}^{t'} = (g^{t'}_{ij})$ is related to $\widetilde{G}^t = (g^t_{ij})$ by

$$g^{t'}_{ij} = \begin{cases} -g^t_{ik} + \sum_{\ell=1}^{m} g^t_{i\ell}[-b^t_{\ell k}\varepsilon^t_k]_+ & \text{if } j = k; \\ g^t_{ij} & \text{otherwise.} \end{cases} \tag{6}$$

Note that the $\boldsymbol{g}$-matrices can also be defined via an arbitrary sign $\varepsilon \in \{\pm 1\}$ as in (4), however such a general expression is unnecessary for our purposes.

Remark 4.2. *Since we only mutate in directions $k \in [1,n]$, $\widetilde{G}^t$ has the following block form:*

$$\begin{bmatrix} G^t & 0 \\ * & \mathbb{1}_{m-n} \end{bmatrix}$$

where G^t is the $n \times n$ $\boldsymbol{g}$-matrix for the coefficient-free case.

Lemma 4.3. *Suppose t is obtained from t_+ by mutating along the sequence $\boldsymbol{k} = (k_N, \dots, k_1)$ passing through $t_+ = t_1, \dots, t_N, t_{N+1} = t$. Then $\widetilde{G}^t$ can be factored as $\widetilde{E}^{t_1}_{k_1,-\varepsilon^{t_1}_{k_1}} \cdots \widetilde{E}^{t_N}_{k_N,-\varepsilon^{t_N}_{k_N}}$.*

Proof. The recursion (6) can be rewritten as $\widetilde{G}^{t'} = \widetilde{G}^t \widetilde{E}^t_{k,-\varepsilon^t_k}$ and the claim follows by induction. □

Proof. This is immediate from the definition (6). □

Corollary 4.4. *For any $t \in \mathbb{T}_n$, we have*

$$\widetilde{G}^t \widetilde{B}^t = \widetilde{B}^+ C^t.$$

Proof. Suppose t is obtained from t_+ by mutating along the sequence $\boldsymbol{k} = (k_N, \dots, k_1)$ passing through $t_+ = t_1, \dots, t_N, t_{N+1} = t$. Then, by definition, we have

$$\widetilde{B}^t = \mu_{\boldsymbol{k}} \widetilde{B}^+ = \widetilde{E}^{t_N}_{k_N,-\varepsilon^{t_N}_{k_N}} \cdots \widetilde{E}^{t_1}_{k_1,-\varepsilon^{t_1}_{k_1}} \widetilde{B}^+ F^{t_1}_{k_1,-\varepsilon^{t_1}_{k_1}} \cdots F^{t_N}_{k_N,-\varepsilon^{t_N}_{k_N}},$$

and the result follows from Lemma 4.1 and Lemma 4.3 using the identity $(E^t_{k,\varepsilon})^2 = \mathbb{1}_m$. □

It will be convenient to introduce $m \times m$ matrices $\widetilde{C}^{\vee,t}$ and $n \times n$ matrices $G^{\vee,t}$ for $t \in \mathbb{T}_n$ defined recursively by $\widetilde{C}^{\vee,t_+} = \mathbb{1}_m$, $G^{\vee,t_+} = \mathbb{1}_n$, and

$$c^{\vee,t'}_{ij} = \begin{cases} -c^{\vee,t}_{ij} & \text{if } i = k \text{ or } j = k; \\ c^{\vee,t}_{ij} + c^{\vee,t}_{ik}[-b^t_{jk}\varepsilon^t_k]_+ & \text{otherwise;} \end{cases} \tag{7}$$

$$g^{\vee,t'}_{ij} = \begin{cases} -g^{\vee,t}_{ik} + \sum_{\ell=1}^{n} g^{\vee,t}_{i\ell}[\varepsilon^t_k b^t_{k\ell}]_+ & \text{if } j = k; \\ g^{\vee,t}_{ij} & \text{otherwise;} \end{cases} \tag{8}$$

whenever $\widetilde{C}^{\vee,t} = (c^{\vee,t}_{ij})$ (resp. $G^{\vee,t} = (g^{\vee,t}_{ij})$) is related to $\widetilde{C}^{\vee,t'} = (c^{\vee,t'}_{ij})$ (resp. $G^{\vee,t'} = (g^{\vee,t'}_{ij})$) by mutation in direction k.

Remark 4.5. *Since we only mutate in directions $k \in [1,n]$, $\widetilde{C}^{\vee,t}$ has the following block form:*

$$\begin{bmatrix} C^{\vee,t} & * \\ 0 & \mathbb{1}_{m-n} \end{bmatrix}$$

where $C^{\vee,t}$ is the $n \times n$ $\boldsymbol{c}$-matrix for $-B^T$.

Lemma 4.6. *For $1 \leqslant k, \ell \leqslant n$, we have $d_k c^t_{k\ell} = c^{\vee,t}_{k\ell} d_\ell$ and $d_k g^t_{k\ell} = g^{\vee,t}_{k\ell} d_\ell$. In particular, $DC^t = C^{\vee,t} D$ and so the first n columns of $\widetilde{C}^{\vee,t}$ share the same tropical signs with the n columns of C^t.*

Proof. The first claim is an easy induction using (5) and (7) or (6) and (8) together with the identity $d_k b^t_{k\ell} = -d_\ell b^t_{\ell k}$. The second claim is an immediate consequence. □

Lemma 4.7. *Suppose t is obtained from t_+ by mutating along the sequence $\boldsymbol{k} = (k_N, \dots, k_1)$ passing through $t_+ = t_1, \dots, t_N, t_{N+1} = t$. Then the following hold.*

(1) $\widetilde{C}^{\vee,t}$ *can be factored as* $\big(\widetilde{E}^{t_1}_{k_1,-\varepsilon^{t_1}_{k_1}}\big)^T \cdots \big(\widetilde{E}^{t_N}_{k_N,-\varepsilon^{t_N}_{k_N}}\big)^T$. *In particular,* $\big(\widetilde{C}^{\vee,t}\big)^T = \big(\widetilde{G}^t\big)^{-1}$.

(2) $G^{\vee,t}$ *can be factored as* $\big(F^{t_1}_{k_1,-\varepsilon^{t_1}_{k_1}}\big)^T \cdots \big(F^{t_N}_{k_N,-\varepsilon^{t_N}_{k_N}}\big)^T$. *In particular,* $\big(G^{\vee,t}\big)^T = \big(C^t\big)^{-1}$.

Proof. The recursions (7) and (8) can be written as $\widetilde{C}^{\vee,t'} = \widetilde{C}^{\vee,t}\big(\widetilde{E}^t_{k,-\varepsilon^t_k}\big)^T$ and $G^{\vee,t'} = G^{\vee,t}\big(F^t_{k,-\varepsilon^t_k}\big)^T$ respectively. The first claims then follow by induction. The second claims follow from the identities $\big(\widetilde{E}^t_{k,-\varepsilon^t_k}\big)^2 = \mathbb{1}_m$ and $\big(F^t_{k,-\varepsilon^t_k}\big)^2 = \mathbb{1}_n$. □

Corollary 4.8. *For any* $t \in \mathbb{T}_n$, *we have*

$$\widetilde{B}^t\big(G^{\vee,t}\big)^T = \big(\widetilde{C}^{\vee,t}\big)^T\widetilde{B}^+.$$

The following is a well-known result from representation theory of associative algebras concerning the Euler paring that we recast here to save on notation; see [?] for the details.

Remark 4.9. *Say something about quasi-Cartan companions.*

There exists a rank 2 cluster algebra with frozen variables η_i, with $\boldsymbol{g}$-vectors $\boldsymbol{g}_k$ for $k \in \mathbb{Z}$. This gives real clusters $X_k := \{\eta_1, \dots, \eta_{n-2}, \boldsymbol{g}_k, \boldsymbol{g}_{k+1}\}$ such that $\lim_{j\to\infty} c^{j\ell}\boldsymbol{g}_k = \nu_c(\delta)$.

Lemma 4.10. *There exist* $\lambda_k \in \mathrm{Span}_{\geqslant 0}X_k$ *so that* $\lim_{k\to\infty}\lambda_k = \lambda$.

5. Green and Red Regions

Definition 5.1. *Consider real* $\widetilde{\lambda}$, *say* $\widetilde{\lambda} \in \widetilde{K}_t := K_t \times \mathbb{R}^{m-n}$. *Let* t *be connected to* t_+ *by a sequence of edges labeled by* $\boldsymbol{k}^+ = (k_N, \dots k_1)$. *Define the green cone* $S^+_\lambda := S_{\boldsymbol{k}^+,\widetilde{\lambda}} \cap \widetilde{K}_t$. *Similarly, let* $\boldsymbol{k}^- = (n, \dots, 1, k_N, \dots, k_1)$ *and define the red cone* $S^-_\lambda := S_{\boldsymbol{k}^-,\widetilde{\lambda}} \cap \widetilde{K}_t$.

Define $S^\pm_\lambda := \lim_{k\to\infty} S^\pm_{\lambda_k}$.

Lemma 5.2. $\mathcal{P}(\lambda) \subseteq \Lambda^\pm_\lambda$

Lemma 5.3. *Consider* λ *inside the* $\boldsymbol{g}$*-vector fan, say* λ *lies inside the cone spanned by* G_t. *Then the intersection of* $\phi_t^{-1}\mathcal{C}_t(\phi_t\lambda)$ *with this* $\boldsymbol{g}$*-cone is spanned by the columns of* $\widetilde{B}^+C^t$.

Lemma 5.4. *Similar statement for the red region*

22. Possibly multiplied by a Laurent monomial in coefficients DR

Corollary 5.5. *Let* $\lambda \in \mathbb{Z}^m$ *correspond to a cluster monomial* 22. *Then* $\mathcal{P}(\lambda) = \{\lambda\}$.

Remark 5.6. *This uses full rank assumption.*

Lemma 5.7. *For imaginary* λ, $\Lambda^+_\lambda \cap \Lambda^-_\lambda$ *is the line through* λ *in direction* $\nu_c(\delta)$.

Corollary 5.8. *Assume* $\widetilde{B}$ *is affine. For imaginary* λ, $\mathcal{P}(\lambda)$ *is contained in the line through* λ *in direction* $\nu_c(\delta)$.

6. Affine Type

Let B be an acyclic exchange matrix of affine type with DB skew-symmetric. Write A for the Cartan companion of B and note that A has corank 1. Consider B^t mutation equivalent to B and C^t the associated $\boldsymbol{c}$-matrix.

Let A_0 denote the Cartan companion of B and write δ_0 for the positive vector spanning the kernel of A_0. For $t \in \mathbb{T}_n$, let $A^t := (C^{\vee,t})^T A_0 C^t$ denote the Reading-Speyer quasi-Cartan companion of B^t [?, Cor. 3.29].

Theorem 6.1. *Let δ^t be the absolute value of the kernel of A^t. Then the primitive purely imaginary $\boldsymbol{g}$-vector direction at the seed t is $-B^t\delta^t/2$.*

Lemma 6.2. *The matrix $\mathcal{E}^t A^t \mathcal{E}^t$ is an admissible quasi-Cartan companion of B^t of corank 1. Moreover, the kernel of A^t is spanned by a non-negative vector δ^t.*

Define $\zeta^t := -B^t\delta^t$.

Lemma 6.3. *The vector ζ^t is an imaginary $\boldsymbol{g}$-vector.*

Lemma 6.4. *With respect to the seed t, the $\boldsymbol{d}$-vector of the imaginary theta basis element ϑ_{ζ^t} is δ^t.*

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23. Is this worth keeping DR