t_1 B_1 initial acyclic

$\lambda \in$ Cambrian fan t s.t. $\lambda \in$ g-cone C_t of $\mathbf{t}$ w/respect to t_0.

So exists sortable element (sorting word) $\mathbf{k} = k_1 \cdots k_m$ s.t. $t_1 \xrightarrow{k_1} t_2 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t$

$\lambda_i = \left(\eta^{B_t}_{k_i \cdots k_m}\right)^{-1} \lambda = \eta^{B_i}_{k_m \cdots k_i}(\lambda)$

$\lambda_1 := \eta^{B_1}_{\mathbf{k}^{-1}}(\lambda)$ is in the positive cone C_1. $\mu_{\mathbf{k}^{-1}}(B_1) = B_t$ Note $\left(\eta^{B_1}_{\mathbf{k}^{-1}}\right)^{-1} = \eta^{B_t}_{\mathbf{k}}$.

$\mathcal{L}^{B}_{\mathbf{k},C}$ = linear map agreeing on C with $\eta^B_{\mathbf{k}}$. Must assume C is a full-dimensional B-cone.

Want: $\eta^{B_t}_{\mathbf{k}}\{\lambda_1 + B_t \alpha\} \subseteq \mathcal{L}^{B_t}_{\mathbf{k},C_1}\{\lambda_1 + B_t \alpha\} = \{\lambda + B_1 C^t \alpha\}$

ok, sorry, C^t is not a cone. It is the C-matrix at t with respect to B_1 at t_1

Also want?: same thing with "minus" in these two places.?

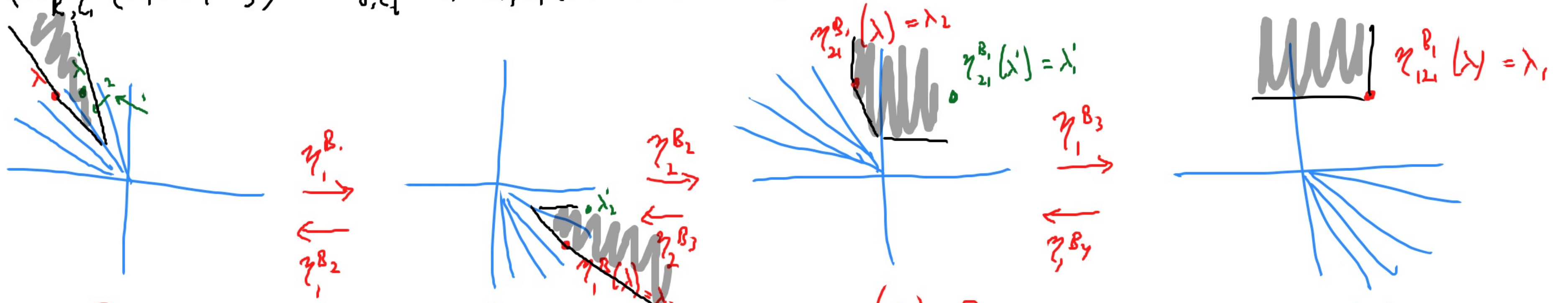
s initial in c:

If $\ell(s\mathbf{k}) < \ell(\mathbf{k})$ (i.e. if s is the first letter of $\mathbf{k}$, i.e. $s = k_1$)

$\eta^{B_1}_s$ is a piecewise-linear automorphism from the mutation fan for B_1 to the mutation fan for B_2.

$\eta^{B_1}_s\left(\eta^{B_t}_{\mathbf{k}}\{\lambda_1 + B_t \alpha\}\right) = \eta^{B_1}_s \eta^{B_t}_{k_1 \cdots k_m}\{\lambda_1 + B_t \alpha\} = \eta^{B_t}_{k_2 \cdots k_m}\{\lambda_1 + B_t \alpha\}$ so: $\eta^{B_t}_{\mathbf{k}}\{\lambda_1 + B_t \alpha\} = \left(\eta^{B_1}_s\right)^{-1} \eta^{B_t}_{k_2 \cdots k_m}\{\lambda_1 + B_t \alpha\}$

$\mathcal{L}^{B_1}_{s,C_t}\left(\mathcal{L}^{B_t}_{\mathbf{k},C_1}\{\lambda_1 + B_t \alpha\}\right) = \mathcal{L}^{B_1}_{s,C_t} \mathcal{L}^{B_t}_{k_1 \cdots k_m, C_1}\{\lambda_1 + B_t \alpha\} = \mathcal{L}^{B_t}_{k_2 \cdots k_m}\{\lambda_1 + B_t \alpha\}$ so $\mathcal{L}^{B_t}_{\mathbf{k},C_1}\{\lambda_1 + B_t \alpha\} = \left(\mathcal{L}^{B_1}_{s,C_t}\right)^{-1} \mathcal{L}^{B_t}_{k_2 \cdots k_m, C_1}\{\lambda_1 + B_t \alpha\}$



$B = B_1$ $\mu_1(B) = B_2$ $\mu_{21}(B) = B_3$ $\mu_{121}(B) = B_4$

By induction on $\ell(k)$ (considering vector $\eta_s^{B_1}(\lambda)$ with initial seed t_2 and B_2):

$$\eta^{B_t}_{k_2 \cdots k_m}\{\lambda_1 + B_t \alpha\} \subseteq \mathcal{L}^{B_t}_{k_2 \cdots k_m, C_1}\{\lambda_1 + B_t \alpha\} = \{(\eta_s^{B_1})^{-1}\lambda + B_2 C^t_{\text{w/ res to } B_2, t_2} \alpha\}$$

<u>Lemma 1</u>: $\left(\mathcal{L}^{B_1}_{s, C_1}\right)^{-1}\{(\eta_s^{B_1})^{-1}\lambda + B_2 C^t_{\cdots B_2 t_2} \alpha\} = \{\lambda + B_1 C^t_{\cdots B_1 t_1} \alpha\}$

↳ This will just be applying an E, using the EBF trick, and quoting Cambrian framework results.

<u>Lemma 2</u>: $(\eta_s^{B_1})^{-1} \mathcal{L}^{B_t}_{k_2 \cdots k_m, C_1}\{\lambda + B_t \alpha\} \subseteq \left(\mathcal{L}^{B_1}_{s, C_t}\right)^{-1} \mathcal{L}^{B_t}_{k_2 \cdots k_m, C_1}\{\lambda_1 + B_t \alpha\}$

↳ Hopefully this is easy because this is a source-sink move, maybe using the fact that we know the connection to C-vectors.

These two lemmas complete the case $\ell(sk) < \ell(k)$. if we can prove them!