t_1 B_1 initial acyclic

$\lambda \in$ Cambrian fan t s.t. $\lambda \in$ g-cone C_t of t w/respect to t_0.

So exists sortable element (sorting word) $\mathbf{k} = k_1 \cdots k_m$ s.t. $t_1 \xrightarrow{k_1} t_2 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t$

$\lambda_i = \left(\eta^{B_t}_{k_i \cdots k_m}\right)^{-1} \lambda = \eta^{B_i}_{k_m \cdots k_i}(\lambda)$

$\lambda_1 := \eta^{B_1}_{\mathbf{k}^{-1}}(\lambda)$ is in the positive cone C_1. $\mu_{\mathbf{k}^{-1}}(B_1) = B_t$ Note $\left(\eta^{B_1}_{\mathbf{k}^{-1}}\right)^{-1} = \eta^{B_t}_{\mathbf{k}}$.

$\boxed{\mathcal{L}^B_{\mathbf{k},C} = \text{linear map agreeing on } C \text{ with } \eta^B_{\mathbf{k}}. \text{ Must assume } C \text{ is a full-dimensional } B\text{-cone.}}$

Want:

$$\eta^{B_t}_{\mathbf{k}}\{\lambda_1 + B_t\alpha\} \subseteq \mathcal{L}^{B_t}_{\mathbf{k},C_1}\{\lambda_1 + B_t\alpha\} = \{\lambda + B_t C^t \alpha\}$$

↑ ok, sorry, C^t is not a cone. It is the C-matrix at t_0 with respect to B_1 at t_1

Also want?: same thing with "minus" in these two places.?

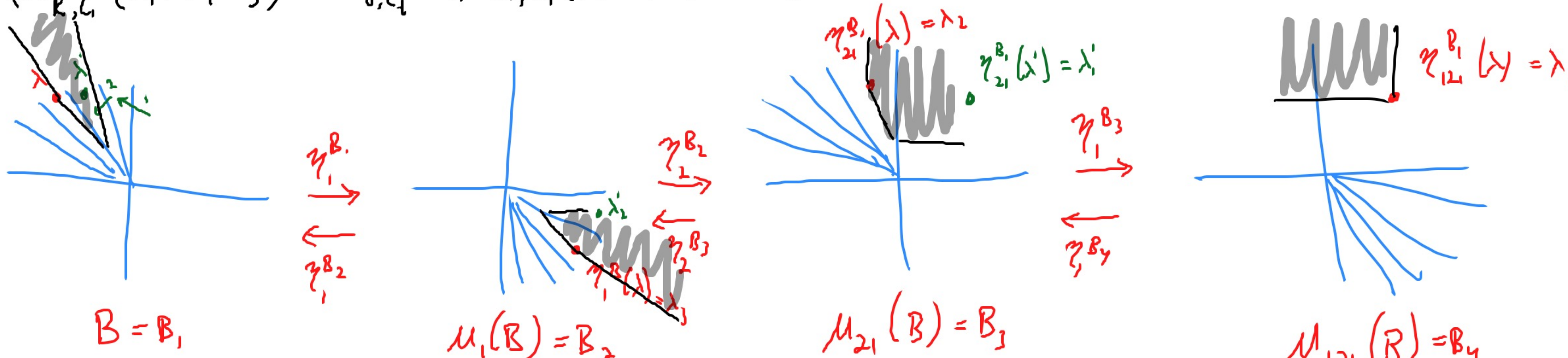
s initial in c:

If $\ell(s\mathbf{k}) < \ell(\mathbf{k})$ (i.e. if s is the first letter of $\mathbf{k}$, i.e. $s = k_1$)

$\eta^{B_1}_s$ is a piecewise-linear automorphism from the mutation fan for B_1 to the mutation fan for B_2.

$$\eta^{B_1}_s\left(\eta^{B_t}_{\mathbf{k}}\{\lambda_1 + B_t\alpha\}\right) = \eta^{B_1}_s \eta^{B_t}_{k_1 \cdots k_m}\{\lambda_1 + B_t\alpha\} = \eta^{B_t}_{k_2 \cdots k_m}\{\lambda_1 + B_t\alpha\} \text{ so: } \eta^{B_t}_{\mathbf{k}}\{\lambda_1 + B_t\alpha\} = \left(\eta^{B_1}_s\right)^{-1} \eta^{B_t}_{k_2 \cdots k_m}\{\lambda_1 + B_t\alpha\}$$

$$\mathcal{L}^{B_1}_{s,C_t}\left(\mathcal{L}^{B_t}_{\mathbf{k},C_1}\{\lambda_1 + B_t\alpha\}\right) = \mathcal{L}^{B_1}_{s,C_t}\mathcal{L}^{B_t}_{k_1 \cdots k_m, C_1}\{\lambda_1 + B_t\alpha\} = \mathcal{L}^{B_t}_{k_2 \cdots k_m}\{\lambda_1 + B_t\alpha\} \text{ so } \mathcal{L}^{B_t}_{\mathbf{k},C_1}\{\lambda_1 + B_t\alpha\} = \left(\mathcal{L}^{B_1}_{s,C_t}\right)^{-1}\mathcal{L}^{B_t}_{k_2 \cdots k_m, C_1}\{\lambda_1 + B_t\alpha\}$$



By induction on $\ell(k)$ (considering vector $\eta_s^{B_1}(\lambda)$ with initial seed t_2 and B_2):

$$\eta^{B_t}_{k_2\cdots k_m}\{\lambda_1 + B_t\alpha\} \subseteq \mathcal{L}^{B_t}_{k_2\cdots k_m, C_1}\{\lambda_1 + B_t\alpha\} = \{(\eta_s^{B_1})^{-1}\lambda + B_2\, C^t_{\text{w/ res to } B_2, t_2}\,\alpha\}$$

Lemma 1: $\left(\mathcal{L}^{B_1}_{s, C_1}\right)^{-1}\{(\eta_s^{B_1})^{-1}\lambda + B_2\, C^t_{\cdots B_2 t_2}\,\alpha\} = \{\lambda + B_1\, C^t_{\cdots B_1 t_1}\,\alpha\}$

↳ This will just be applying an E, using the EBF trick, and quoting Cambrian framework results.

Lemma 2: $\left(\eta_s^{B_1}\right)^{-1}\mathcal{L}^{B_t}_{k_2\cdots k_m, C_1}\{\lambda + B_t\alpha\} \subseteq \left(\mathcal{L}^{B_1}_{s, C_t}\right)^{-1}\mathcal{L}^{B_t}_{k_2\cdots k_m, C_1}\{\lambda_1 + B_t\alpha\}$

↳ Hopefully this is easy because this is a source-sink move, maybe using the fact that we know the connection to C-vectors.

These two lemmas complete the case $\ell(sk) < \ell(k)$. if we can prove them!

If $\ell(sk) > \ell(k)$ then s does not occur in $k_1 \cdots k_m$

Write $B_{\langle \mathrm{col}\, s\rangle}$ or $B_{\langle \mathrm{row}\, s\rangle}$ or $B_{\langle {\mathrm{row} \atop \mathrm{col}}\, s\rangle}$ for deleting that row and/or column.

$$\eta_k^{B_{t\langle s\rangle}}\left(\left(B_t\right)_s\right) = \left(B_1\right)_s$$

because $\mu_k(B_t) = B_1$

So also

$$\mathcal{L}_{k,C_1'}^{B_{t\langle s\rangle}}\left(\left(B_t\right)_s\right) = \left(B_1\right)_s$$

Remove row s and column s from B_1

Why?

!

in pos root lattice in $\Phi_{\langle s\rangle}$

$$\eta_k^{B_t}\{\lambda_1 + B_t\alpha\} = \eta_k^{B_{t\langle s\rangle}}\left\{\lambda_1 + B_{t\langle s\rangle}\alpha' + (B_t)_s \cdot \begin{pmatrix}\text{nonneg.}\\ \text{integer}\end{pmatrix}\right\}$$

$$\mathcal{L}_{k,C_1}^{B_t}\{\lambda_1 + B_t\alpha\} = \mathcal{L}_{k,C_1'}^{B_{t\langle s\rangle}}\left\{\lambda_1 + B_{t\langle s\rangle}\alpha' + (B_t)_s \cdot \begin{pmatrix}\text{non neg.}\\ \text{integer}\end{pmatrix}\right\}$$

pos. entries in directions besides s

$$\{\lambda + B_1 C^t\alpha\} = \left\{\lambda + B_1\left[{}^{1}C^t_{B_1\langle s\rangle,\, t_1}\right]\alpha\right\} = \left\{\lambda + (B_1)_s\begin{pmatrix}\text{nonneg}\\ \text{intgr}\end{pmatrix} + B_1' C^t_{B_1\langle s\rangle,\, t_1}\alpha'\right\}$$

By induction on rank:

$$\eta_k^{B_{t\langle s\rangle}}\{\lambda_1 + B_{t\langle s\rangle}\alpha'\} \subseteq \mathcal{L}_{k,C_1'}^{B_{t\langle s\rangle}}\{\lambda_1 + B_{t\langle s\rangle}\alpha'\} = \{\lambda + B_1' C^t_{B_1\langle s\rangle\, t_1}\alpha'\}$$

<u>If $\ell(sk) > \ell(k)$</u> then s does not occur in $k_1 \cdots k_m$ and treat row s

If we ignore column s of B_t (write $B_{t\langle s\rangle}$)

as a coefficient.

in pos root lattice in $\Phi_{\langle s\rangle}$

$$\eta_k^{B_t}\{\lambda_1 + B_t\alpha\} = \eta_k^{B_{t\langle s\rangle}}\{\lambda_1 + B_{t\langle s\rangle}\alpha' + (B_t)_s \cdot \begin{pmatrix}\text{nonneg.}\\ \text{integer}\end{pmatrix}\}$$

$$\mathcal{L}_{k,c_1}^{B_t}\{\lambda_1 + B_t\alpha\} = \mathcal{L}_{k,c_1'}^{B_{t\langle s\rangle}}\{\lambda_1 + B_{t\langle s\rangle}\alpha' + (B_t)_s \cdot \begin{pmatrix}\text{non neg.}\\ \text{integer}\end{pmatrix}\}$$

pos. entries in directions besides s.

$$\eta_k^{B_{t\langle s\rangle}}\left((B_t)_s\right) = (B_1)_s$$

because $\mu_k(B_t) = B_1$

So also

$$\mathcal{L}_{k,c_1'}^{B_{t\langle s\rangle}}\left((B_t)_s\right) = (B_1)_s$$

Remove row s and column s from B_1

$$\{\lambda + B_1C^t\alpha\} = \{\lambda + B_1\begin{bmatrix}1 & \\ & C^t_{B_1\langle s\rangle, t_1}\end{bmatrix}\alpha\} = \{\lambda + (B_1)_s\begin{pmatrix}\text{nonneg}\\ \text{intgr}\end{pmatrix} + B_1' C^t_{B_1\langle s\rangle, t_1}\alpha'\}$$

By induction on rank:

$$\eta_k^{B_{t\langle s\rangle}}\{\lambda_1 + B_{t\langle s\rangle}\alpha'\} \subseteq \mathcal{L}_{k,c_1'}^{B_{t\langle s\rangle}}\{\lambda_1 + B_{t\langle s\rangle}\alpha'\} = \{\lambda + B_1' C^t_{B_1\langle s\rangle t_1}\alpha'\}$$

This is <u>not</u> true, is it?

But we need it if these are to be equal.

OK, what we're doing here:

- Since we're doing the "add a column" mutation maps, we're actually doing the mutation fan for B^T (B_1^T, B_k^T, etc.)
- This mutation fan has the g-vector fan for $\underline{\underline{B}}$ as a subfan, which in turn has the c-Cambrian fan as a subfan (with c such that B is $\begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}$).

(Fan Qin's dominance order is also defined with "add a column" mutation maps.)

$$k \rightarrow \begin{array}{c} \\ \begin{bmatrix} & & \\ & b_{ik} & \\ & & \end{bmatrix} \\ \uparrow \\ k \end{array} \begin{array}{l} g \\ g^k_i \\ g \end{array}$$

$[B^T] \leftrightarrow [B]$ ⋮

.....

Ex: $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $c = s_1 s_2$

c-Camb fan

$$\begin{array}{c} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{array} \begin{array}{l} a \\ b \end{array} \xrightarrow{1} \begin{array}{c} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{array} \begin{array}{l} -a \\ * \end{array}$$

$$* = \begin{cases} b + a & \text{if } a \leq 0 \\ b & \text{if } a \geq 0 \end{cases}$$

This $\underline{\underline{\text{is}}}$ the right map to send the to the c-Cambrian fan to another mutation fan