DOMINANCE REGIONS FOR AFFINE CLUSTER ALGEBRAS

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ABSTRACT. NEED THIS

CONTENTS

1. BACKGROUND

We assume the basic definitions of exchange matrices and of matrix mutation. Given a sequence $\mathbf{k} = k_m \cdots k_1$ of indices in $\{1, \ldots, n\}$, we read the sequence from right to left for the purposes of matrix mutation. That is, $\mu_{\mathbf{k}}(B)$ means $\mu_{k_m}(\mu_{k_{m-1}}(\cdots(\mu_{k_1}(B))\cdots))$. We write $\mathbf{k}^{-1}$ for $k_1 \cdots k_m$, the reverse of $\mathbf{k}$. Throughout, we will use without comment the fact that matrix mutation commutes with the maps $B \mapsto -B$ and $B \mapsto B^T$.

Given an exchange matrix B, the ***mutation map*** $\eta_{\mathbf{k}}^B : \mathbb{R}^n \to \mathbb{R}^n$ takes the input vector in $\mathbb{R}^n$, places it as an additional row below B, mutates the resulting matrix according to the sequence $\mathbf{k}$, and outputs the bottom row of the mutated matrix. In this paper, it is convenient to think of vectors in $\mathbb{R}^n$ as column vectors, and also, the mutation maps we need use transposes B^T of exchange matrices. Thus we write maps $\eta_{\mathbf{k}}^{B^T}$. This map takes a vector, places it as an additional *column* to the right of B (not B^T), does mutations according to $\mathbf{k}$, and reads the rightmost column of the mutated matrix.

Given a vector $\lambda \in \mathbb{R}^n$, define $\mathcal{P}_{\lambda,\mathbf{k}}^B = \left(\eta_{\mathbf{k}}^{B^T}\right)^{-1} \left\{\eta_{\mathbf{k}}^{B^T}(\lambda) + B_t\alpha : \alpha \geq 0\right\}$. Define the ***dominance region*** of λ with respect to B to be $\mathcal{P}_\lambda^B = \bigcap_{\mathbf{k}} \mathcal{P}_{\lambda,\mathbf{k}}^B$, where the intersection is over all sequences $\mathbf{k}$.

Lemma 1.1. *If $\lambda' = \eta_{\mathbf{k}}^{B^T}(\lambda)$ and $B' = \mu_{\mathbf{k}}(B)$, then $\eta_{\mathbf{k}}^{B^T}(\mathcal{P}_\lambda^B) = \mathcal{P}_{\lambda'}^{B'}$.*

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Proof. For any $\boldsymbol{\ell}$,

$$\begin{aligned}\eta_{\mathbf{k}}^{B^T}(\mathcal{P}^B_{\lambda,\boldsymbol{\ell}}) &= \eta_{\mathbf{k}}^{B^T}\left(\left(\eta_{\boldsymbol{\ell}}^{B^T}\right)^{-1}\left\{\eta_{\boldsymbol{\ell}}^{B^T}(\lambda)+B_t\alpha:\alpha\geq 0\right\}\right)\\ &= \left(\eta_{\boldsymbol{\ell}}^{B^T}\eta_{\mathbf{k}^{-1}}^{\mu_{\mathbf{k}}(B)^T}\right)^{-1}\left\{\eta_{\boldsymbol{\ell}}^{B^T}(\lambda)+B_t\alpha:\alpha\geq 0\right\}\\ &= \left(\eta_{\boldsymbol{\ell}\mathbf{k}^{-1}}^{\mu_{\mathbf{k}}(B)^T}\right)^{-1}\left\{\eta_{\boldsymbol{\ell}\mathbf{k}^{-1}}^{\mu_{\mathbf{k}}(B)^T}\left(\eta_{\mathbf{k}}^{B^T}(\lambda)\right)+B_t\alpha:\alpha\geq 0\right\}.\end{aligned}$$

Thus $\eta_{\mathbf{k}}^{B^T}(\mathcal{P}^B_\lambda)=\bigcap_{\boldsymbol{\ell}}\mathcal{P}^{B'}_{\lambda',\boldsymbol{\ell}\mathbf{k}^{-1}}=\mathcal{P}^{B'}_{\lambda'}$. □

For seeds t_0 and t and an exchange matrix B, let $C_t^{B;t_0}$ be the matrix whose columns are the C-vectors at t relative to the initial seed t_0 with exchange matrix B. Each column of $C_t^{B;t_0}$ is nonzero and all of its nonzero entries have the same sign. (This is "sign-coherence of C-vectors", which was implicitly conjectured in [**?**] and proved as [**?**, Corollary 5.5].) Thus we will refer to the ***sign*** of a column of $C_t^{B;t_0}$. For $\mathbf{k}=k_m\cdots k_1$, define seeds $t_1,\ldots,t_m$ by $t_0\xrightarrow{k_1}t_1\xrightarrow{k_2}\cdots\xrightarrow{k_m}t_m$. The sequence $\mathbf{k}$ is a ***green sequence*** for an exchange matrix B if column k_ℓ of $C_{t_{\ell-1}}^{B;t_0}$ is *positive* for all ℓ with $1\leq\ell<m$. A ***maximal green sequence*** for B is a green sequence that cannot be extended. That is, the sequence $\mathbf{k}$ is a maximal green sequence if every column of $C_{t_m}^{B;t_0}$ is *negative*. We will call $\mathbf{k}$ a ***red sequence*** for B if it is a green sequence for $-B$. A ***maximal red sequence*** is a red sequence that cannot be extended. (A red sequence relates to antiprincipal coefficients: If we were to define the C-vectors recursively starting with the negative of the identity matrix, the requirement for a red sequence is that the k_ℓ column is negative at every step.)

Let $G_t^{B;t_0}$ be the matrix whose columns are the $\mathbf{g}$-vectors at t relative to the initial seed t_0 with exchange matrix B. Let $\mathrm{Cone}_t^{B;t_0}$ be the nonnegative linear span of the columns of $G_t^{B;t_0}$. For each $k\in\{1,\ldots,n\}$, the entries in the k^{th} row of $G_t^{B;t_0}$ are not all zero and the nonzero entries have the same sign. (This is "sign-coherence of $\mathbf{g}$-vectors", conjectured as [**?**, Conjecture 6.13] and proved as [**?**, Theorem 5.11].) Thus all vectors in $\mathrm{Cone}_t^{B;t_0}$ all have weakly the same sign in the k^{th} position. The inverse of $G_t^{B;t_0}$ is $\left(C_t^{-B^T;t_0}\right)^T$. (This is [**?**, Theorem 1.2] or [**?**, Theorem 1.1] and [**?**, Theorem 3.30].) Thus $\mathrm{Cone}_t^{B;t_0}=\left\{x\in\mathbb{R}^n:x^TC_t^{-B^T;t_0}\geq 0\right\}$, where 0 is a row vector and "$\geq$" means componentwise comparison.

Given $\mathbf{k}$ with $t_0\xrightarrow{k_1}t_1\xrightarrow{k_2}\cdots\xrightarrow{k_m}t_m$, let B_i be the exchange matrix at t_i, so that in particular, $B_0=B$. The map $\eta_{\mathbf{k}}^{B^T}$ is $\eta_{k_m}^{B_{m-1}^T}\circ\cdots\circ\eta_{k_2}^{B_1^T}\circ\eta_{k_1}^{B_0^T}$. The definition of each $\eta_{k_i}^{B_{i-1}^T}$ has two cases, separated by the hyperplane $x_{k_i}=0$. Two vectors are in the same ***domain of definition*** of $\eta_{\mathbf{k}}^{B^T}$ if, at every step, the same case applies for the two vectors. (Both cases apply on the hyperplane, so domains of definition are closed.) In particular, $\eta_{\mathbf{k}}^{B^T}$ is linear in each of its domains of definition, but the domains of linearity of $\eta_{\mathbf{k}}^{B^T}$ can be larger than its domains of definition.

There is a fan $\mathcal{F}_{B^T}$ called the ***mutation fan*** for B^T [**?**, Definition 5.12]. We will not need the details of the definition, but roughly, the cones of $\mathcal{F}_{B^T}$ are the intersections of domains of definition of all mutation maps $\eta_{\mathbf{k}}^{B^T}$, as $\mathbf{k}$ varies. Thus for each $\mathbf{k}$, each cone of $\mathcal{F}_{B^T}$ is contained in a domain of definition of $\eta_{\mathbf{k}}^{B^T}$, and the mutation map $\eta_{\mathbf{k}}^{B^T}$ is linear on every cone of $\mathcal{F}_{B^T}$ [**?**, Proposition 5.3]. Every cone $\mathrm{Cone}_t^{B;t_0}$ is a maximal cone in the mutation fan $\mathcal{F}_{B^T}$ [**?**, Proposition 8.13]. Thus in particular, the mutation map $\eta_{\mathbf{k}}^{B^T}$ is linear on every cone $\mathrm{Cone}_t^{B;t_0}$. Furthermore, $\mathrm{Cone}_t^{B_m;t_m} = \eta_{\mathbf{k}}^{B^T}\left(\mathrm{Cone}_t^{B;t_0}\right)$ for every seed t. (This amounts to the initial seed mutation formula for $\mathbf{g}$-vectors, conjectured as [**?**, Conjecture 7.12] and shown in [**?**, Proposition 4.2(v)] to follow from sign-coherence of C-vectors. The restatement in terms of mutation maps is [**?**, Conjecture 8.11].)

Remark 1.2. As written, [**?**, Proposition 8.13] is conditional on "sign-coherence of C-vectors", which was a conjecture but is now a theorem [**?**, Corollary 5.5].

We will need to relate the cones $\mathrm{Cone}_t^{B;t_0}$ and $\mathrm{Cone}_t^{-B^T;t_0}$. It is immediate from [**?**, Proposition 7.5] and the skew-symmetry of B that $-B^T$ is a ***rescaling*** of B, meaning that there is a diagonal matrix Σ with positive entries on the diagonal such that $-B^T = \Sigma^{-1}B\Sigma$. Therefore, [**?**, Proposition 8.20] says that the i^{th} column of $G_t^{-B^T;t_0}$ is a positive scalar multiple of the i^{th} column of $\Sigma G_t^{B;t_0}$. (In the statement of [**?**, Proposition 8.20], Σ is multiplied on the right, because there $\mathbf{g}$-vectors are row vectors rather than column vectors.) Thus we have the following fact.

Lemma 1.3. *The k^{th} entries of vectors in $\mathrm{Cone}_t^{B;t_0}$ have the same sign as the k^{th} entries of vectors in $\mathrm{Cone}_t^{-B^T;t_0}$.*

For $k \in \{1,\ldots,n\}$, let J_k be the $n\times n$ matrix that agrees with the identity matrix except that J_k has -1 in position kk. For an $n \times n$ matrix M and $k \in \{1,\ldots,n\}$, let $M^{\bullet k}$ be the matrix that agrees with M in column k and has zeros everywhere outside of column k. Let $M^{k\bullet}$ be the matrix that agrees with M in row k and has zeros everywhere outside of row k.

Given a real number a, let $[a]_+$ denote $\max(a,0)$. Given a matrix $M = [m_{ij}]$, define $[M]_+$ to be the matrix whose ij-entry is $[m_{ij}]_+$. Given an exchange matrix B, an index $k \in \{1,\ldots,n\}$ and a sign $\varepsilon \in \{\pm1\}$, define matrices

$$\begin{aligned} E_{\varepsilon,k}^B &= J_k + [\varepsilon B]_+^{\bullet k} \\ F_{\varepsilon,k}^B &= J_k + [-\varepsilon B]_+^{k\bullet}. \end{aligned}$$

Each matrix $E_{\varepsilon,k}^B$ is its own inverse, and each $F_{\varepsilon,k}^B$ is its own inverse. The following is essentially a result of [**?**], although it is not stated there in this form. 1 1. Do I have this attribution right? N

Lemma 1.4. *For $k \in \{1,\ldots,n\}$ and either choice of $\varepsilon \in \{\pm1\}$, the mutation of B at k is $\mu_k(B) = E_{\varepsilon,k}^B B F_{\varepsilon,k}^B$.*

Proof. We expand the product $(J_k + [\varepsilon B]_+^{\bullet k})B(J_k + [-\varepsilon B]_+^{k\bullet})$ to four terms. The term $[\varepsilon B]_+^{\bullet k}B[-\varepsilon B]_+^{k\bullet}$ is zero because $b_{kk} = 0$. The term $[\varepsilon B]_+^{\bullet k}BJ_k$ is $[\varepsilon B]_+^{\bullet k}B^{k\bullet}J_k$, which equals $[\varepsilon B]_+^{\bullet k}B^{k\bullet}$. Similarly, the term $J_kB[-\varepsilon B]_+^{k\bullet}$ equals $B^{\bullet k}[-\varepsilon B]_+^{k\bullet}$ Both Thus the ij-entry of $E_{\varepsilon,k}^B B F_{\varepsilon,k}^B$ is

$$\begin{Bmatrix} -b_{ij} & \text{if } k \in \{i,j\} \\ b_{ij} & \text{otherwise} \end{Bmatrix} + \begin{Bmatrix} |b_{ik}|b_{kj} & \text{if } \operatorname{sgn} b_{ik} = \varepsilon \\ 0 & \text{otherwise} \end{Bmatrix} + \begin{Bmatrix} b_{ik}|b_{kj}| & \text{if } \operatorname{sgn} b_{kj} = -\varepsilon \\ 0 & \text{otherwise} \end{Bmatrix}.$$

This coincides with the ij-entry of $\mu_k(B)$. □

Given a matrix M, write $M_{\mathrm{col}(i)}$ for the i^{th} column of M. We observe that $(MN)_{\mathrm{col}\,i} = M(N)_{\mathrm{col}\,i}$.

Lemma 1.5. *Suppose $B = [b_{ij}]$ is an exchange matrix, let $k \in \{1, \dots, n\}$, and choose a sign $\varepsilon \in \{\pm 1\}$.*

1. $(E^B_{\varepsilon,k} B)_{\mathrm{col}\, i} = J_k(B)_{\mathrm{col}\, i} + b_{ki}([\varepsilon B]_+)_{\mathrm{col}\, k}$.
2. $(E^B_{\varepsilon,k} B)_{\mathrm{col}\, k} = (E^B_{-\varepsilon,k} B)_{\mathrm{col}\, k} = B_{\mathrm{col}\, k}$.
3. $(E^B_{-\varepsilon,k} B)_{\mathrm{col}\, i} = (E^B_{\varepsilon,k} B)_{\mathrm{col}\, i} - \varepsilon b_{ki} B_{\mathrm{col}\, k}$.

Proof. The first two assertions follow immediately from the fact that $(MN)_{\mathrm{col}\, i} = M(N)_{\mathrm{col}\, i}$ and the fact that $b_{kk} = 0$. The first assertion (for ε and $-\varepsilon$) implies that $(E^B_{-\varepsilon,k} B)_{\mathrm{col}\, i} = (E^B_{\varepsilon,k} B)_{\mathrm{col}\, i} - b_{ki}([\varepsilon B]_+ - [-\varepsilon B]_+)_{\mathrm{col}\, k}$. The third assertion follows. □

We will also need the following simple fact about nonnegative linear spans. Given a set S of vectors, let $\underset{\mathbf{span}}{\mathbf{pos}}(S)$ denote the nonnegative linear span of S. For $k \in \{1, \dots, n\}$ and $\varepsilon \in \{\pm 1\}$, let $S_{k,\varepsilon}$ be the set of vectors in S whose k^{th} entry has sign strictly agreeing with ε.

Lemma 1.6. *Suppose λ is a vector in $\mathbb{R}^n$ whose k^{th} λ_k has $\varepsilon\lambda_k \le 0$. Then*

$$\left\{\lambda + \underset{\mathbf{span}}{\mathbf{pos}}(S)\right\} \cap \{x \in \mathbb{R}^n : \varepsilon x_k \ge 0\}$$
$$= \left\{\lambda + \underset{\mathbf{span}}{\mathbf{pos}}(S)\right\} \cap \{x \in \mathbb{R}^n : x_k = 0\} + \underset{\mathbf{span}}{\mathbf{pos}}(S_{k,\varepsilon}).$$

Proof. The set on the right side is certainly contained in the set on the right side. If x is an element of the left side, then x is λ plus a nonzero element y of $\underset{\mathbf{span}}{\mathbf{pos}}(S_{k,\varepsilon})$ plus an element z of $\underset{\mathbf{span}}{\mathbf{pos}}(S \setminus S_{k,\varepsilon})$. Since the sign of $\varepsilon x \ge 0$ and $\varepsilon\lambda \le 0$, there exists t with $0 \le t \le 1$ such that $\lambda + ty + z$ has k^{th} entry 0. We see that $x = (\lambda + ty + z) + (1-t)y$ is an element of the right side. □

2. First main result

Let B_0 be an exchange matrix. For a sequence $\mathbf{k} = k_m \cdots k_1$ of indices, define seeds $t_1, \dots, t_m = t$ by $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. We will prove the following theorem.

Theorem 2.1. *Suppose $\mathbf{k} = k_m \cdots k_1$ and $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. If $\mathbf{k}^{-1} = k_1 \cdots k_m$ is a red sequence for B_t, then for any λ in the domain of definition of $\eta^{B_0^T}_{\mathbf{k}}$ that contains $\mathrm{Cone}^{B_0;t_0}_t$,*

$$\mathcal{P}^{B_0}_{\lambda,\mathbf{k}} \subseteq \left\{\lambda + G^{B_0;t_0}_t B_t \alpha : \alpha \ge 0\right\} = \left\{\lambda + B_0 C^{B_0;t_0}_t \alpha : \alpha \ge 0\right\}.$$

Since $\left(\eta^{B_0^T}_{\mathbf{k}}\right)^{-1} = \eta^{B_t^T}_{\mathbf{k}^{-1}}$, we have $\mathcal{P}^{B_0}_{\lambda,\mathbf{k}} = \eta^{B_t^T}_{\mathbf{k}^{-1}}\left\{\eta^{B_0^T}_{\mathbf{k}}(\lambda) + B_t\alpha : \alpha \ge 0\right\}$. Let D be the domain of definition of $\eta^{B_0^T}_{\mathbf{k}}$ that contains $\mathrm{Cone}^{B_0;t_0}_t$. Then $\eta^{B_t^T}_{\mathbf{k}^{-1}}$ is linear on $\eta^{B_0^T}_{\mathbf{k}}(D)$. Let $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}$ be the linear map that agrees with $\eta^{B_t^T}_{\mathbf{k}^{-1}}$ on $\eta^{B_0^T}_{\mathbf{k}}(D)$.

Proposition 2.2. *The matrix for $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}$, acting on column vectors, is $G^{B_0;t_0}_t$.*

Proof. By [**?**, Proposition 8.13], $\mathrm{Cone}^{B_0;t_0}_t = \eta^{B_t^T}_{\mathbf{k}^{-1}}((\mathbb{R}_{\ge 0})^n)$, and therefore also $\eta^{B_0^T}_{\mathbf{k}}\left(\mathrm{Cone}^{B_0;t_0}_t\right) = (\mathbb{R}_{\ge 0})^n$. The proof of [**?**, Proposition 8.13] shows not only an equality of cones, but also that $\eta^{B_t^T}_{\mathbf{k}^{-1}}$ takes the extreme ray of $(\mathbb{R}_{\ge 0})^n$ spanned by e_i

to the extreme ray of $\mathrm{Cone}_t^{B_0;t_0}$ spanned by the i^{th} $\mathbf{g}$-vector at t relative to $B_0;t_0$, where the total order on these $\mathbf{g}$-vectors at t is obtained from the order $e_1,\ldots,e_n$ on $\mathbf{g}$-vectors at t_0 by the sequence $\mathbf{k}$ of mutations. $\square$

We now apply a result of [**?**], namely that $G_t^{B_0;t_0}B_t = B_0C_t^{B_0;t_0}$. This fact follows from the proof of [**?**, Proposition 1.3], or from [**?**, (6.14)], as explained in [**?**, Remark 2.1]. Since $G_t^{B_0;t_0}$ is the matrix for $\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}$ and since $\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}\eta_{\mathbf{k}}^{B_0^T}(\lambda)=\lambda$, we have the following proposition.

Proposition 2.3.

$$\begin{aligned}\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}\Big\{\eta_{\mathbf{k}}^{B_0^T}(\lambda)+B_t\alpha:\alpha\geq 0\Big\} &= \Big\{\lambda+G_t^{B_0;t_0}B_t\alpha:\alpha\geq 0\Big\}\\ &= \Big\{\lambda+B_0C_t^{B_0;t_0}\alpha:\alpha\geq 0\Big\}.\end{aligned}$$

In light of Proposition 2.3, the conclusion of Theorem 2.1 is equivalent to

$$\mathcal{P}_{\lambda,\mathbf{k}}^{B_0}\subseteq\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}\Big\{\eta_{\mathbf{k}}^{B_0^T}(\lambda)+B_t\alpha:\alpha\geq 0\Big\}.$$

Proof of Theorem 2.1. We will prove that $P_{\lambda,\mathbf{k}}^{B_0}\subseteq\Big\{\lambda+B_0C_t^{B_0;t_0}\alpha:\alpha\geq 0\Big\}$, by induction on m (the length of $\mathbf{k}$). The base case, where $\mathbf{k}=\emptyset$, is true because $C_{t_0}^{B_0;t_0}$ is the identity matrix and $\mathcal{P}_{\lambda,\emptyset}=\{\lambda+B_0\alpha:\alpha\geq 0\}$.

[**?**, Proposition 1.4] says that $C_t^{B_0;t_0}=F_{\varepsilon,k_1}^{B_1}C_t^{B_1;t_1}$, where ε is the sign of the k_1-column of $C_{t_1}^{-B_t;t}$. (The hypothesis that $\mathbf{k}^{-1}$ is a red sequence for B_t determines ε, but we leave ε unspecified for now in order to highlight later where this hypothesis is relevant.) By Lemma 1.4 and because $E_{\varepsilon,k_1}^{B_1}$ and $F_{\varepsilon,k_1}^{B_1}$ are their own inverses,

$$\begin{aligned}\Big\{\lambda+B_0C_t^{B_0;t_0}\alpha:\alpha\geq 0\Big\} &= \Big\{\lambda+B_0F_{\varepsilon,k_1}^{B_1}C_t^{B_1;t_1}\alpha:\alpha\geq 0\Big\}\\ &= \Big\{\lambda+E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\alpha:\alpha\geq 0\Big\}\\ &= E_{\varepsilon,k_1}^{B_1}\Big\{E_{\varepsilon,k_1}^{B_1}\lambda+B_1C_t^{B_1;t_1}\alpha:\alpha\geq 0\Big\}.\end{aligned}\tag{2.1}$$

The map $\eta_{\mathbf{k}}^{B_0^T}$ is linear on $\mathrm{Cone}_t^{B_0;t_0}$. This map is $\eta_{\mathbf{k}}^{B_0^T}=\eta_{k_m}^{B_{m-1}^T}\circ\cdots\circ\eta_{k_2}^{B_1^T}\circ\eta_{k_1}^{B_0^T}$. The map $\eta_{k_1}^{B_0^T}$ restricts to a linear map from $\mathrm{Cone}_t^{B_0;t_0}$ to $\mathrm{Cone}_t^{B_1;t_1}$. The inverse of $\eta_{k_1}^{B_0^T}$ is $\eta_{k_1}^{B_1^T}$. We claim that $E_{\varepsilon,k_1}^{B_1}$ is the matrix for the linear map on column vectors that agrees with $\eta_{k_1}^{B_1^T}$ on $\mathrm{Cone}_t^{B_1;t_1}$. Since $E_{\varepsilon,k_1}^{B_1}$ is its own inverse, the claim is equivalent to saying that implies that $E_{\varepsilon,k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_0^T}$ on $\mathrm{Cone}_t^{B_0;t_0}$.

By [**?**, (1.13)], ε is the sign of the k_1-column of $\big(G_t^{-B_1^T;t_1}\big)^T$. That is, ε is the sign of the k_1-row of $G_t^{-B_1^T;t_1}$, or in other words, the sign of the k_1-entry of vectors in $\mathrm{Cone}_t^{-B_1^T;t_1}$. By Lemma 1.3, ε is the sign of the k_1-entry of vectors in $\mathrm{Cone}_t^{B_1;t_1}$, which is the sign that determines how $\eta_{k_1}^{B_1^T}$ acts on $\mathrm{Cone}_t^{B_1;t_1}$. One easily checks that the action of $\eta_{k_1}^{B_1^T}$ on vectors whose k_1-entry has sign ε is precisely the action of $E_{\varepsilon,k_1}^{B_1}$.

Let $\lambda' = \eta_{k_1}^{B_0^T}(\lambda)$, so that λ' is in the same domain of definition of $\eta_{k_m\cdots k_2}^{B_1^T}$ as $\mathrm{Cone}_t^{B_1;t_1}$ and so that $\lambda' = E_{\varepsilon,k_1}^{B_1}\lambda$. By induction on m,

$$\eta_{k_2\cdots k_m}^{B_t^T}\Big\{\eta_{k_m\cdots k_2}^{B_1^T}(\lambda') + B_t\alpha : \alpha \ge 0\Big\} \subseteq \Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \ge 0\Big\}.$$

Applying the homeomorphism $\eta_{k_1}^{B_1^T}$ to both sides, we obtain

$$\eta_{\mathbf{k}^{-1}}^{B_t^T}\Big\{\eta_{\mathbf{k}}^{B_0^T}(\lambda') + B_t\alpha : \alpha \ge 0\Big\} \subseteq \eta_{k_1}^{B_1^T}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \ge 0\Big\}.$$

In light of (2.1), we can complete the proof by showing that

$$\eta_{k_1}^{B_1^T}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \ge 0\Big\} \subseteq E_{\varepsilon,k_1}^{B_1}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \ge 0\Big\}.$$

We have seen that $E_{\varepsilon,k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_1^T}$ on the set $\{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\}$. We can similarly check that $E_{-\varepsilon,k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_1^T}$ on $\{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = -\varepsilon\}$. Thus $\eta_{k_1}^{B_1^T}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \ge 0\Big\}$ is

$$(U \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = -\varepsilon\}) \cup (V \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\}),$$

where

$$\begin{aligned} U &= E_{\varepsilon,k_1}^{B_1}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \ge 0\Big\} = E_{\varepsilon,k_1}^{B_1}\lambda' + \underset{\textbf{span}}{\textbf{pos}}\Big\{\Big(E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}\Big\}_{i=1}^n \\ V &= E_{-\varepsilon,k_1}^{B_1}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \ge 0\Big\} = E_{-\varepsilon,k_1}^{B_1}\lambda' + \underset{\textbf{span}}{\textbf{pos}}\Big\{\Big(E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}\Big\}_{i=1}^n, \end{aligned}$$

where $\underset{\textbf{span}}{\textbf{pos}}$ denotes the nonnegative linear span of a set of vectors.

We need to show that $V \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\} \subseteq U$. Since $\eta_{k_1}^{B_1^T}$ is a homeomorphism, $U \cap \{x \in \mathbb{R}^n : x_{k_1} = 0\} = V \cap \{x \in \mathbb{R}^n : x_{k_1} = 0\}$. By Lemma 1.6, any vector in $V \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\}$ equals a vector in $V \cap \{x \in \mathbb{R}^n : x_{k_1} = 0\}$ plus a positive combination of vectors $\Big(E_{-\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$ whose k_1-entry has sign ε. Therefore, it suffices to show that every vector $\Big(E_{-\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$ whose k_1-entry has sign ε is in $\underset{\textbf{span}}{\textbf{pos}}\Big\{\Big(E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}\Big\}_{i=1}^n$.

As a temporary shorthand, write b_{ij} for the entries of B_1 and write k for k_1. Suppose $v_i = \Big(E_{-\varepsilon,k}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$ for some i and suppose the k-entry of v_i has sign ε. Write M for $E_{-\varepsilon,k}^{B_1}B_1$ and write N for $E_{\varepsilon,k}^{B_1}B_1$. Lemma 1.5.1 implies that $M_{kj} = -b_{kj}$ for all j. Lemma 1.5.3 implies that if $\varepsilon M_{kj} \ge 0$, then $M_{\mathrm{col}\,j} = N_{\mathrm{col}\,j} + |b_{kj}|N_{\mathrm{col}\,k}$. Similarly, if $\varepsilon M_{kj} \le 0$, then $M_{\mathrm{col}\,j} = N_{\mathrm{col}\,j} - |b_{kj}|N_{\mathrm{col}\,k}$.

Now $v_i = E_{-\varepsilon,k}^{B_1}B_1\Big(C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$, and $\Big(C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$ has a sign $\delta \in \{\pm 1\}$, meaning that it is not zero and all of its nonzero entries have sign δ. (This is "sign-coherence of C-vectors". See Remark 1.2.) Thus there are nonnegative numbers γ_j such that $v_i = \delta\sum_{j=1}^n \gamma_j M_{\mathrm{col}\,j}$. Write $\{1,\ldots,n\} = S \cup T$ with $S \cup T = \emptyset$ such that $\varepsilon M_{kj} \ge 0$

for all $j \in S$ and $\varepsilon M_{kj} \leq 0$ for all $j \in T$. Then

$$\begin{aligned}
v_i &= \delta \sum_{j \in S} \gamma_j M_{\mathrm{col}\, j} + \delta \sum_{j \in T} \gamma_j M_{\mathrm{col}\, j} \\
&= \delta \sum_{j \in S} \gamma_j (N_{\mathrm{col}\, j} + |b_{kj}| N_{\mathrm{col}\, k}) + \delta \sum_{j \in T} \gamma_j (N_{\mathrm{col}\, j} - |b_{kj}| N_{\mathrm{col}\, k}) \\
&= \delta \sum_{j=1}^{n} \gamma_j N_{\mathrm{col}\, j} - \delta \sum_{j=1}^{n} \varepsilon \gamma_j b_{kj} N_{\mathrm{col}\, k} \\
&= N \left(C_t^{B_1;t_1} \right)_{\mathrm{col}\, i} + \delta \sum_{j=1}^{n} \varepsilon \gamma_j M_{kj} N_{\mathrm{col}\, k} \\
&= N \left(C_t^{B_1;t_1} \right)_{\mathrm{col}\, i} + \sigma N_{\mathrm{col}\, k}.
\end{aligned}$$

where $\sigma = \varepsilon\delta \sum_{j=1}^{n} \gamma_j M_{kj}$ is a positive scalar, because $\delta \sum_{j=1}^{n} \gamma_j M_{kj}$ is the k-entry of v_i, which has sign ε.

As noted above, ε is the sign of the k_1-entry of vectors in $\mathrm{Cone}_t^{-B_1^T;t_1}$. Since $\mathrm{Cone}_t^{-B_1^T;t_1} = \left\{ x \in \mathbb{R}^n : x^T C_t^{B_1;t_1} \geq 0 \right\}$, the rows of $\left(C_t^{B_1;t_1} \right)^{-1}$ span the extreme rays of $\mathrm{Cone}_t^{-B_1^T;t_1}$. In particular $\left(C_t^{B_1;t_1} \right)^{-1} (\varepsilon e_k)$ has nonnegative entries. Thus $C_t^{B_1;t_1} \left(C_t^{B_1;t_1} \right)^{-1} (\varepsilon e_k) = \varepsilon e_k$ is a nonnegative linear combination of columns of $C_t^{B_1;t_1}$.

Now, the hypothesis that $\mathbf{k}^{-1}$ is a red sequence for B_t, or equivalently a green sequence for $-B_t$, says that $\varepsilon = +1$, so that e_k is a nonnegative linear combination of columns of $C_t^{B_1;t_1}$. Thus $N_{\mathrm{col}\, k} = N e_k$ is a nonnegative linear combination of columns of $N C_t^{B_1;t_1}$. We have shown that $v_i = N \left(C_t^{B_1;t_1} \right)_{\mathrm{col}\, i} + \sigma N_{\mathrm{col}\, k}$ is a nonnegative linear combination of columns of $N C_t^{B_1;t_1}$. In other words, v_i is in $\underset{\mathbf{span}}{\mathbf{pos}} \left\{ \left(E_{\varepsilon,k_1}^{B_1} B_1 C_t^{B_1;t_1} \right)_{\mathrm{col}\, i} \right\}_{i=1}^{n}$, as desired. □

3. Extending to extended exchange matrices

We follow [**?**] in considering $m \times n$ extended exchange matrices $\tilde{B}$ that are "tall", in the sense that $m \geq n$. We will also consider $m \times m$ matrices related to $\tilde{B}$: Writing $\tilde{B}$ in block form $\left[\begin{smallmatrix} B \\ E \end{smallmatrix} \right]$, let $\mathbf{B}$ be the matrix with block form $\left[\begin{smallmatrix} B & -E^T \\ E & 0 \end{smallmatrix} \right]$. Most importantly, $\mathbf{B}$ is skew-symmetrizable and agrees with $\tilde{B}$ in columns 1 to n. Throughout, if we have defined an extended exchange matrix $\tilde{B}$, without comment we will take B to be the underlying exchange matrix and $\mathbf{B}$ to be the associated $m \times m$ matrix.

The matrix $\mathbf{B}$ defines mutation maps $\eta_{\mathbf{k}}^{\mathbf{B}^T}$ that act on $\mathbb{R}^m$ rather than $\mathbb{R}^n$, but without exception we will only consider mutations in positions $1, \ldots, n$. Also, given $\mathbf{B}$, a sequence $\mathbf{k} = k_m \cdots k_1$ of indices in $\{1, \ldots, n\}$, and seeds $t_1, \ldots, t_m$ by $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$, there are associated matrices of $\mathbf{g}$-vectors and C-vectors, which we write as $\mathbf{G}_t^{\mathbf{B};t_0}$ and $\mathbf{C}_t^{\mathbf{B};t_0}$. Since $\mathbf{k}$ only contains indices in

$\{1,\ldots,n\}$, these matrices have block forms

$$\mathbf{G}_t^{\mathbf{B};t_0} = \begin{bmatrix} G_t^{B;t_0} & 0 \\ * & I_{m-n} \end{bmatrix} \quad\text{and}\quad \mathbf{C}_t^{\mathbf{B};t_0} = \begin{bmatrix} C_t^{B;t_0} & * \\ 0 & I_{m-n} \end{bmatrix},$$

where I_{m-n} is the identity matrix.

Given a vector $\lambda \in \mathbb{R}^m$, define $\mathcal{P}^{\tilde{B}}_{\lambda,\mathbf{k}} = \left(\eta_{\mathbf{k}}^{\mathbf{B}^T}\right)^{-1}\left\{\eta_{\mathbf{k}}^{\mathbf{B}^T}(\lambda) + \tilde{B}_t\alpha : \alpha \in \mathbb{R}^m, \alpha \geq 0\right\}$. Define the ***dominance region*** $\mathcal{P}^{\tilde{B}}_\lambda$ of λ with respect to $\tilde{B}$ to be the intersection $\bigcap_{\mathbf{k}} \mathcal{P}^B_{\lambda,\mathbf{k}}$ all sequences $\mathbf{k}$ of indices in $\{1,\ldots,n\}$.

Since $\mathbf{k}$ consists only of indices in $\{1,\ldots,n\}$, the domains of definition of $\eta_{\mathbf{k}}^{\mathbf{B}^T}$ are determined by the domains of definition of $\eta_{\mathbf{k}}^{B^T}$. Specifically, each domain of definition of $\eta_{\mathbf{k}}^{\mathbf{B}^T}$ is the set of vectors whose projection to $\mathbb{R}^n$ (ignoring the last $m-n$ entries) is a domain of definition of $\eta_{\mathbf{k}}^{B^T}$. Accordingly, we define $\mathrm{Cone}_t^{\tilde{B};t_0}$ to be the set of vectors in $\mathbb{R}^m$ whose projection to $\mathbb{R}^n$ is in $\mathrm{Cone}_t^{B;t_0}$.

To understand dominance regions $\mathcal{P}^{\tilde{B}}_\lambda$, it is enough to consider the case where λ has nonzero entries only in positions $1,\ldots,n$. Other dominance regions are obtained by translation, as explained in the following lemma. The lemma is an immediate consequence of the fact that domains of definition of $\eta_{\mathbf{k}}^{\mathbf{B}^T}$ depend only on the first n coordinates.

Lemma 3.1. *If λ and λ' are vectors in $\mathbb{R}^m$ that agree in the first n coordinates, then $\mathcal{P}^{\tilde{B}}_{\lambda'} = \mathcal{P}^{\tilde{B}}_\lambda - \lambda + \lambda'$.*

Lemma 1.1 immediately implies the following lemma.

Lemma 3.2. *If $\lambda' = \eta_{\mathbf{k}}^{\mathbf{B}^T}$ and $\tilde{B}' = \mu_{\mathbf{k}}(\tilde{B})$, then $\eta_{\mathbf{k}}^{\mathbf{B}^T}(\mathcal{P}^{\tilde{B}}_\lambda) = \mathcal{P}^{\tilde{B}'}_{\lambda'}$.*

We will prove the following extension of Theorem 2.1 and an important corollary.

Theorem 3.3. *Suppose $\mathbf{k} = k_m \cdots k_1$ is a sequence of indices in $\{1,\ldots,n\}$ and $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. If $\mathbf{k}^{-1} = k_1 \cdots k_m$ is a red sequence for B_t, then for any λ in the domain of definition of $\eta_{\mathbf{k}}^{\mathbf{B}_0^T}$ that contains $\mathrm{Cone}_t^{B_0;t_0}$,*

$$\mathcal{P}^{\tilde{B}_0}_{\lambda,\mathbf{k}} \subseteq \left\{\lambda + \mathbf{G}_t^{\mathbf{B}_0;t_0}\tilde{B}_t\alpha : \alpha \geq 0\right\} = \left\{\lambda + \tilde{B}_0 C_t^{B_0;t_0}\alpha : \alpha \geq 0\right\}.$$

Proof. First, we notice that $\mathbf{k}^{-1} = k_1 \cdots k_m$ is a red sequence for $\mathbf{B}_t$, or in other words, $\mathbf{k}$ is a green sequence for $-\mathbf{B}_t$. Indeed, since $\mathbf{C}_{t_{\ell-1}}^{-\mathbf{B};t_0} = \begin{bmatrix} C_{t_{\ell-1}}^{-\mathbf{B};t_0} & * \\ 0 & I_{m-n} \end{bmatrix}$, the sign of column k_ℓ of $\mathbf{C}_{t_{\ell-1}}^{-\mathbf{B};t_0}$ equals the sign of column k_ℓ of $C_{t_{\ell-1}}^{-\mathbf{B};t_0}$ whenever $1 \leq \ell < k$. Thus Theorem 2.1 says that

$$\mathcal{P}^{\mathbf{B}_0}_{\lambda,\mathbf{k}} \subseteq \left\{\lambda + \mathbf{G}_t^{\mathbf{B}_0;t_0}\mathbf{B}_t\alpha : \alpha \in \mathbb{R}^m, \alpha \geq 0\right\} = \left\{\lambda + \mathbf{B}_0\mathbf{C}_t^{\mathbf{B}_0;t_0}\alpha : \alpha \in \mathbb{R}^m, \alpha \geq 0\right\}.$$

The assertion of Theorem 3.3 is that the same holds even when, in each term, the conditions $\alpha \in \mathbb{R}^m, \alpha \geq 0$ are strengthened by requiring that α is zero in coordinates $n+1,\ldots,m$.

Thus we run through the proof of Theorem 2.1 with $\mathbf{B}$ replacing B and m replacing n throughout and these additional conditions on α in all relevant expressions. There is no effect on the argument until the point of showing that $V \cap \{x \in \mathbb{R}^m : \operatorname{sgn} x_{k_1} = \varepsilon\} \subseteq U$. Here, we need to show that every vector $v_i = \left(E^{\mathbf{B}_1}_{-\varepsilon,k_1}\mathbf{B}_1\mathbf{C}_t^{\mathbf{B}_1;t_1}\right)_{\mathrm{col}\,i}$ with $i \in \{1,\ldots,n\}$ whose k_1-entry has sign ε is contained in $\underset{\mathbf{span}}{\mathbf{pos}}\left\{\left(E^{\mathbf{B}_1}_{\varepsilon,k_1}\mathbf{B}_1\mathbf{C}_t^{\mathbf{B}_1;t_1}\right)_{\mathrm{col}\,i}\right\}_{i=1}^n$. We argue as in the proof of Theorem 2.1 that $v_i = N\left(\mathbf{C}_t^{\mathbf{B}_1;t_1}\right)_{\mathrm{col}\,i} + \sigma N_{\mathrm{col}\,k}$ and that εe_k is a nonnegative linear combination of columns of $\mathbf{C}_t^{\mathbf{B}_1;t_1}$. Since $\mathbf{C}_t^{\mathbf{B};t_0} = \begin{bmatrix} C_t^{B;t_0} & * \\ 0 & I_{m-n}\end{bmatrix}$, we conclude that εe_k is a nonnegative linear combination of columns 1 through n of $\mathbf{C}_t^{\mathbf{B}_1;t_1}$. Thus v_i is a nonnegative linear combination of columns 1 through n of $N\mathbf{C}_t^{\mathbf{B}_1;t_1}$ as desired. □

Corollary 3.4. *Suppose $\tilde{B}_0$ is an extended exchange matrix with linearly independent columns. Suppose t is a seed in the exchange graph for $\tilde{B}_0; t_0$ and take $\lambda \in \mathrm{Cone}_t^{\tilde{B}_0;t_0}$. If there exists a maximal red sequence for B_t, then $\mathcal{P}_\lambda^{\tilde{B}_0} = \{\lambda\}$.*

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Proof. Let t' be the seed at the end of the maximal red sequence for B_t. There exists $\boldsymbol{\ell} = \ell_q\ell_{q-1}\cdots\ell_1$ with $t_0 = t'_0 \xrightarrow{\ell_1} t'_1 \xrightarrow{\ell_2} \cdots \xrightarrow{\ell_q} t'_q = t'$. Let $\lambda' = \eta_{\boldsymbol{\ell}}^{B_0^T}(\lambda)$. Lemma 1.1 says $\eta_{\boldsymbol{\ell}}^{B_0^T}(\mathcal{P}_\lambda^{B_0}) = \mathcal{P}_{\lambda'}^{B_{t'}}$. Thus it is enough to prove that $\mathcal{P}_{\lambda'}^{B_{t'}} = \{\lambda'\}$. Since $\eta_{\boldsymbol{\ell}}^{B_0^T}\left(\mathrm{Cone}_t^{B_0;t_0}\right) = \mathrm{Cone}_t^{B_{t'};t'}$, we have reduced the proof to the case where there is a maximal red sequence for B_t starting from t and ending at t_0.

Working in that reduction, let $\mathbf{k} = k_m\cdots k_1$ be the reverse of the maximal red sequence and define seeds $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. Then Theorem 2.1 says that $\mathcal{P}_{\lambda,\mathbf{k}}^{B_0} \subseteq \left\{\lambda + B_0C_t^{B_0;t_0}\alpha : \alpha \geq 0\right\}$.

Since $\mathbf{k}^{-1}$ is a maximal red sequence for B_t, or in other words a maximal green sequence for $-B_t$, every column of $C_{t_0}^{-B_t;t}$ has negative sign, so $\mathrm{Cone}_{t_0}^{B_t^T;t} = \left\{x \in \mathbb{R}^n : x^TC_{t_0}^{-B_t;t} \geq 0\right\}$ consists of vectors with nonpositive entries. Since $(\mathbb{R}_{\leq 0})^n$ is a cone in the mutation fan $\mathcal{F}_{-B_t}$ (for example, combining [**?**, Proposition 7.1], [**?**, Proposition 8.9], and sign-coherence of C-vectors) and also $\mathrm{Cone}_{t_0}^{B_t^T;t}$ is a cone in $\mathcal{F}_{-B_t}$, we see that $\mathrm{Cone}_{t_0}^{B_t^T;t} = (\mathbb{R}_{\leq 0})^n$. Thus, up to permuting columns, $C_{t_0}^{-B_t;t}$ is the negative of the identity matrix. We see that $\mathcal{P}_{\lambda,\mathbf{k}}^{B_0} \subseteq \{\lambda - B_0\alpha : \alpha \geq 0\}$.

Since also $\mathcal{P}_{\lambda,\emptyset}^{B_0}\{\lambda + B_0\alpha : \alpha \geq 0\}$, and since the columns of B_0 are linearly independent, 2 we conclude that $\mathcal{P}_\lambda^{B_0} = \{\lambda\}$. □

2. Need at least the hypothesis that span of B_t does not contain a line! Here I have just taken "linearly independent columns". We will need to do extended matrices in some form. :(N

References

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