We want to show that our continuous family of generalized minors from [RSW19, Theorem 4.6] coincide with the family of bases constructed in [Qin19, Theorem 1.2.1].

This boils down to show, for any $n > 1$, that

(1) the set of $\mathbf{g}$-vectors that are dominated by $n\omega^\circ$ with respect to any seed is

$$\{(n-2r)\omega^\circ\}_{r\in\left[1,\left\lfloor\frac{1}{2}\right\rfloor\right]}$$

(2) we can choose a specific basis and rewrite the last formula in [RSW19, Proposition 4.4] as in [Qin19, Theorem 1.2.1] with coefficients being functions of the point $\mathbf{a} \in (\mathbb{k}^\times)^n$.

For a partition $\lambda \vdash n$ let e_λ, s_λ, and m_λ denote the elementary, Shur, and monomial symmetric functions associated to the partition λ evaluated at $\mathbf{a}$. Recall that $m_{1^{(n)}} = s_{1^{(n)}} = e_n$.

Lemma 0.1. *For every $r \in \left[1, \left\lfloor\frac{1}{2}\right\rfloor\right]$ we have $m_{1^{(n)}} S_{\mathbf{a},r} = m_{2^{(r)},1^{(n-2r)}}$.*

Proof. This is immediate from the definition. □

Let $F^{\Delta}_{n\omega^\circ}$, $F^{gr}_{n\omega^\circ}$, $F^{tr}_{n\omega^\circ}$, and $F^{ge}_{n\omega^\circ}$ be the F-polynomials of the generalized minor, greedy basis element, triangular basis element, and generic basis element with $\mathbf{g}$-vector $n\omega^\circ$. We think of F-polynomials as polynomials in the variables u_1 and u_2.

I did not try yet to prove (1) but it should be straightforward. It suffices to establish (2) for the F-polynomials.

Proposition 0.2.

$$m_{1^{(n)}} F^{\Delta}_{n\omega^\circ} = \sum_{r=0}^{\lfloor\frac{n}{2}\rfloor} m_{2^{(k)},1^{(n-2r)}} u_1^r u_2^r F^{ge}_{(n-2r)\omega^\circ}$$

Proof. From [RSW19, Proposition 4.4] we get the following expression for $F^{\Delta}_{n\omega^\circ}$:

$$F^{\Delta}_{n\omega^\circ} = \sum_{0\le k\le \ell\le n} \sum_{r=0}^{\ell} \binom{\ell-r}{k}\binom{n-2r}{\ell-r} S_{\mathbf{a},r} u_1^\ell u_2^{\ell-k}.$$

From the second binomial coefficient we deduce that $n-2r$ is positive so the second sum runs up to $\lfloor\frac{n}{2}\rfloor$. We compute

$$\begin{aligned}
m_{1^{(n)}} F^{\Delta}_{n\omega^\circ} &= \sum_{0\le k\le \ell\le n} \sum_{r=0}^{\lfloor\frac{n}{2}\rfloor} \binom{\ell-r}{k}\binom{n-2r}{\ell-r} m_{2^{(r)},1^{(n-2r)}} u_1^\ell u_2^{\ell-k} \\
&= \sum_{r=0}^{\lfloor\frac{n}{2}\rfloor} m_{2^{(r)},1^{(n-2r)}} u_1^r u_2^r \sum_{0\le k\le \ell\le n} \binom{\ell-r}{k}\binom{n-2r}{\ell-r} u_1^{\ell-r} u_2^{\ell-r-k} \\
&= \sum_{r=0}^{\lfloor\frac{n}{2}\rfloor} m_{2^{(r)},1^{(n-2r)}} u_1^r u_2^r F^{ge}_{(n-2r)\omega^\circ}.
\end{aligned}$$

The last identity follows immediately by comparing with the formula for generic basis elements in [RSW19]. 1 □

1. Thank you Dylan for fixing my stupidity. S

We do not need the following two formulas but I think they are still neat.

Conjecture 0.3.

$$
\begin{aligned}
m_{1^{(n)}} F^{\Delta}_{n\omega^\circ} &= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} s_{2^{(k)},1^{(n-2r)}} u_1^r u_2^r F^{tr}_{(n-2r)\omega^\circ} \\
&= \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} e_{r,n-r} u_1^r u_2^r F^{gr}_{(n-2r)\omega^\circ}
\end{aligned}
$$

References

[Qin19] Fan Qin, *Bases for upper cluster algebras and tropical points*, arXiv e-prints (2019), arXiv:1902.09507.

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