

DOMINANCE REGIONS FOR AFFINE CLUSTER ALGEBRAS

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ABSTRACT. NEED THIS

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1. BACKGROUND

Given a sequence $\mathbf{k} = k_m \cdots k_1$ of indices in $\{1, \dots, n\}$, we read the sequence from right to left for the purposes of matrix mutation. That is, $\mu_{\mathbf{k}}(B)$ means $\mu_{k_m}(\mu_{k_{m-1}}(\cdots(\mu_{k_1}(B))\cdots))$. We write \mathbf{k}^{-1} for $k_1 \cdots k_m$, the reverse of \mathbf{k} . Throughout, we will use without comment the fact that matrix mutation commutes with the maps $B \mapsto -B$ and $B \mapsto B^T$.

Given an exchange matrix B , the *mutation map* $\eta_{\mathbf{k}}^B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ takes the input vector in \mathbb{R}^n , places it as an additional row below B , mutates the resulting matrix according to the sequence \mathbf{k} , and outputs the bottom row of the mutated matrix. In this paper, it is convenient to think of vectors in \mathbb{R}^n as column vectors, and also, the mutation maps we need use transposes B^T of exchange matrices. Thus we write maps $\eta_{\mathbf{k}}^{B^T}$. This map takes a vector, places it as an additional *column* to the right of B (not B^T), does mutations according to \mathbf{k} , and reads the rightmost column of the mutated matrix.

Given a vector $\lambda \in \mathbb{R}^n$, define $\mathcal{P}_{\lambda, \mathbf{k}}^B = \left(\eta_{\mathbf{k}}^{B^T}\right)^{-1} \left\{ \eta_{\mathbf{k}}^{B^T}(\lambda) + B_t \alpha : \alpha \geq 0 \right\}$. Define $\mathcal{P}_{\lambda}^B = \bigcap_{\mathbf{k}} \mathcal{P}_{\lambda, \mathbf{k}}^B$, where the intersection is over all sequences \mathbf{k} .

Lemma 1.1. *If $\lambda' = \eta_{\mathbf{k}}^{B^T}$ and $B' = \mu_{\mathbf{k}}(B)$, then $\eta_{\mathbf{k}}^{B^T}(\mathcal{P}_{\lambda}^B) = \mathcal{P}_{\lambda'}^{B'}$.*

Proof. For any ℓ ,

$$\begin{aligned} \eta_{\mathbf{k}}^{B^T}(\mathcal{P}_{\lambda, \ell}^B) &= \eta_{\mathbf{k}}^{B^T} \left(\left(\eta_{\ell}^{B^T} \right)^{-1} \left\{ \eta_{\ell}^{B^T}(\lambda) + B_t \alpha : \alpha \geq 0 \right\} \right) \\ &= \left(\eta_{\ell}^{B^T} \eta_{\mathbf{k}^{-1}}^{\mu_{\mathbf{k}}(B)^T} \right)^{-1} \left\{ \eta_{\ell}^{B^T}(\lambda) + B_t \alpha : \alpha \geq 0 \right\} \\ &= \left(\eta_{\ell \mathbf{k}^{-1}}^{\mu_{\mathbf{k}}(B)^T} \right)^{-1} \left\{ \eta_{\ell \mathbf{k}^{-1}}^{\mu_{\mathbf{k}}(B)^T} \left(\eta_{\mathbf{k}}^{B^T}(\lambda) \right) + B_t \alpha : \alpha \geq 0 \right\}. \end{aligned}$$

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Thus $\eta_{\mathbf{k}}^{B^T}(\mathcal{P}_{\lambda}^B) = \bigcap_{\ell} \mathcal{P}_{\lambda', \ell \mathbf{k}^{-1}}^{B'} = \mathcal{P}_{\lambda'}^{B'}$. \square

For seeds t_0 and t and an exchange matrix B , let $C_t^{B;t_0}$ be the matrix whose columns are the C -vectors at t relative to the initial seed t_0 with exchange matrix B . Each column of $C_t^{B;t_0}$ is nonzero and all of its nonzero entries have the same sign. (This is “sign-coherence of C -vectors”, which was implicitly conjectured in [?] and proved as [?, Corollary 5.5].) Thus we will refer to the *sign* of a column of $C_t^{B;t_0}$. For $\mathbf{k} = k_m \cdots k_1$, define seeds t_1, \dots, t_m by $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m$. The sequence \mathbf{k} is a **green sequence** for an exchange matrix B if column k_{ℓ} of $C_{t_{\ell-1}}^{B;t_0}$ is *positive* for all ℓ with $1 \leq \ell < m$. A **maximal green sequence** for B is a green sequence that cannot be extended. That is, the sequence \mathbf{k} is a maximal green sequence if every column of $C_{t_m}^{B;t_0}$ is *negative*. We will call \mathbf{k} a **red sequence** for B if it is a green sequence for $-B$. A **maximal red sequence** is a red sequence that cannot be extended. (A red sequence relates to antiprincipal coefficients: If we were to define the C -vectors recursively starting with the negative of the identity matrix, the requirement for a red sequence is that the k_{ℓ} column is negative at every step.)

Let $G_t^{B;t_0}$ be the matrix whose columns are the \mathbf{g} -vectors at t relative to the initial seed t_0 with exchange matrix B . Let $\text{Cone}_t^{B;t_0}$ be the nonnegative linear span of the columns of $G_t^{B;t_0}$. For each $k \in \{1, \dots, n\}$, the entries in the k^{th} row of $G_t^{B;t_0}$ are not all zero and the nonzero entries have the same sign. (This is “sign-coherence of \mathbf{g} -vectors”, conjectured as [?, Conjecture 6.13] and proved as [?, Theorem 5.11].) Thus all vectors in $\text{Cone}_t^{B;t_0}$ all have weakly the same sign in the k^{th} position. The inverse of $G_t^{B;t_0}$ is $(C_t^{-B^T;t_0})^T$. (This is [?, Theorem 1.2] or [?, Theorem 1.1] and [?, Theorem 3.30].) Thus $\text{Cone}_t^{B;t_0} = \{x \in \mathbb{R}^n : x^T C_t^{-B^T;t_0} \geq 0\}$, where 0 is a row vector and “ \geq ” means componentwise comparison.

Given \mathbf{k} with $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m$, let B_i be the exchange matrix at t_i , so that in particular, $B_0 = B$. The map $\eta_{\mathbf{k}}^{B^T}$ is $\eta_{k_m}^{B_m^{T-1}} \circ \cdots \circ \eta_{k_2}^{B_1^T} \circ \eta_{k_1}^{B_0^T}$. The definition of each $\eta_{k_i}^{B_i^{T-1}}$ has two cases, separated by the hyperplane $x_{k_i} = 0$. Two vectors are in the same **domain of definition** of $\eta_{\mathbf{k}}^{B^T}$ if, at every step, the same case applies for the two vectors. (Both cases apply on the hyperplane, so domains of definition are closed.) In particular, $\eta_{\mathbf{k}}^{B^T}$ is linear in each of its domains of definition, but the domains of linearity of $\eta_{\mathbf{k}}^{B^T}$ can be larger than its domains of definition.

There is a fan \mathcal{F}_{B^T} called the **mutation fan** for B^T [?, Definition 5.12]. We will not need the details of the definition, but roughly, the cones of \mathcal{F}_{B^T} are the intersections of domains of definition of all mutation maps $\eta_{\mathbf{k}}^{B^T}$, as \mathbf{k} varies. Thus for each \mathbf{k} , each cone of \mathcal{F}_{B^T} is contained in a domain of definition of $\eta_{\mathbf{k}}^{B^T}$, and the mutation map $\eta_{\mathbf{k}}^{B^T}$ is linear on every cone of \mathcal{F}_{B^T} [?, Proposition 5.3]. Every cone $\text{Cone}_t^{B;t_0}$ is a maximal cone in the mutation fan \mathcal{F}_{B^T} [?, Proposition 8.13]. Thus in particular, the mutation map $\eta_{\mathbf{k}}^{B^T}$ is linear on every cone $\text{Cone}_t^{B;t_0}$. Furthermore, $\text{Cone}_t^{B_m;t_m} = \eta_{\mathbf{k}}^{B^T}(\text{Cone}_t^{B;t_0})$ for every seed t . (This amounts to the initial seed mutation formula for \mathbf{g} -vectors, conjectured as [?, Conjecture 7.12] and shown in [?, Proposition 4.2(v)] to follow from sign-coherence of C -vectors. The restatement in terms of mutation maps is [?, Conjecture 8.11].)

Remark 1.2. As written, [?, Proposition 8.13] is conditional on “sign-coherence of C -vectors”, which was a conjecture but is now a theorem [?, Corollary 5.5].

We will need to relate the cones $\text{Cone}_t^{B;t_0}$ and $\text{Cone}_t^{-B^T;t_0}$. It is immediate from [?, Proposition 7.5] and the skew-symmetry of B that $-B^T$ is a **rescaling** of B , meaning that there is a diagonal matrix Σ with positive entries on the diagonal such that $-B^T = \Sigma^{-1}B\Sigma$. Therefore, [?, Proposition 8.20] says that the i^{th} column of $G_t^{-B^T;t_0}$ is a positive scalar multiple of the i^{th} column of $\Sigma G_t^{B;t_0}$. (In the statement of [?, Proposition 8.20], Σ is multiplied on the right, because there **g**-vectors are row vectors rather than column vectors.) Thus we have the following fact.

Lemma 1.3. *The k^{th} entries of vectors in $\text{Cone}_t^{B;t_0}$ have the same sign as the k^{th} entries of vectors in $\text{Cone}_t^{-B^T;t_0}$.*

For $k \in \{1, \dots, n\}$, let J_k be the $n \times n$ matrix that agrees with the identity matrix except that J_k has -1 in position kk . For an $n \times n$ matrix M and $k \in \{1, \dots, n\}$, let $M^{\bullet k}$ be the matrix that agrees with M in column k and has zeros everywhere outside of column k . Let $M^{k\bullet}$ be the matrix that agrees with M in row k and has zeros everywhere outside of row k .

Given a real number a , let $[a]_+$ denote $\max(a, 0)$. Given a matrix $M = [m_{ij}]$, define $[M]_+$ to be the matrix whose ij -entry is $[m_{ij}]_+$. Given an exchange matrix B , an index $k \in \{1, \dots, n\}$ and a sign $\varepsilon \in \{\pm 1\}$, define matrices

$$\begin{aligned} E_{\varepsilon,k}^B &= J_k + [\varepsilon B]_+^{\bullet k} \\ F_{\varepsilon,k}^B &= J_k + [-\varepsilon B]_+^{k\bullet}. \end{aligned}$$

Each matrix $E_{\varepsilon,k}^B$ is its own inverse, and each $F_{\varepsilon,k}^B$ is its own inverse. The following is essentially a result of [?], although it is not stated there in this form. ①

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Lemma 1.4. *For $k \in \{1, \dots, n\}$ and either choice of $\varepsilon \in \{\pm 1\}$, the mutation of B at k is $\mu_k(B) = E_{\varepsilon,k}^B B F_{\varepsilon,k}^B$.*

Proof. We expand the product $(J_k + [\varepsilon B]_+^{\bullet k})B(J_k + [-\varepsilon B]_+^{k\bullet})$ to four terms. The term $[\varepsilon B]_+^{\bullet k}B[-\varepsilon B]_+^{k\bullet}$ is zero because $b_{kk} = 0$. The term $[\varepsilon B]_+^{\bullet k}B J_k$ is $[\varepsilon B]_+^{\bullet k}B^{k\bullet}J_k$, which equals $[\varepsilon B]_+^{\bullet k}B^{k\bullet}$. Similarly, the term $J_k B[-\varepsilon B]_+^{k\bullet}$ equals $B^{\bullet k}[-\varepsilon B]_+^{k\bullet}$. Both Thus the ij -entry of $E_{\varepsilon,k}^B B F_{\varepsilon,k}^B$ is

$$\begin{aligned} &\begin{cases} -b_{ij} & \text{if } k \in \{i, j\} \\ b_{ij} & \text{otherwise} \end{cases} + \begin{cases} |b_{ik}|b_{kj} & \text{if } \text{sgn } b_{ik} = \varepsilon \\ 0 & \text{otherwise} \end{cases} + \begin{cases} b_{ik}|b_{kj}| & \text{if } \text{sgn } b_{kj} = -\varepsilon \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

This coincides with the ij -entry of $\mu_k(B)$. \square

Given a matrix M , write $M_{\text{col}(i)}$ for the i^{th} column of M . We observe that $(MN)_{\text{col } i} = M(N)_{\text{col } i}$.

Lemma 1.5. *Suppose $B = [b_{ij}]$ is an exchange matrix, let $k \in \{1, \dots, n\}$, and choose a sign $\varepsilon \in \{\pm 1\}$.*

1. $(E_{\varepsilon,k}^B B)_{\text{col } i} = J_k(B)_{\text{col } i} + b_{ki}([\varepsilon B]_+)^{\text{col } k}$.
2. $(E_{\varepsilon,k}^B B)_{\text{col } k} = (E_{-\varepsilon,k}^B B)_{\text{col } k} = B_{\text{col } k}$.
3. $(E_{-\varepsilon,k}^B B)_{\text{col } i} = (E_{\varepsilon,k}^B B)_{\text{col } i} - \varepsilon b_{ki}B_{\text{col } k}$.

Proof. The first two assertions follow immediately from the fact that $(MN)_{\text{col } i} = M(N)_{\text{col } i}$ and the fact that $b_{kk} = 0$. The first assertion (for ε and $-\varepsilon$) implies that $(E_{-\varepsilon, k}^B B)_{\text{col } i} = (E_{\varepsilon, k}^B B)_{\text{col } i} - b_{ki}([\varepsilon B]_+ - [-\varepsilon B]_+)_{\text{col } k}$. The third assertion follows. \square

We will also need the following simple fact about nonnegative linear spans. Given a set S of vectors, let $\text{span}^{\text{pos}}(S)$ denote the nonnegative linear span of S . For $k \in \{1, \dots, n\}$ and $\varepsilon \in \{\pm 1\}$, let $S_{k, \varepsilon}$ be the set of vectors in S whose k^{th} entry has sign strictly agreeing with ε .

Lemma 1.6. *Suppose λ is a vector in \mathbb{R}^n whose k^{th} λ_k has $\varepsilon \lambda_k \leq 0$. Then*

$$\begin{aligned} \left\{ \lambda + \text{span}^{\text{pos}}(S) \right\} \cap \{x \in \mathbb{R}^n : \varepsilon x_k \geq 0\} \\ = \left\{ \lambda + \text{span}^{\text{pos}}(S) \right\} \cap \{x \in \mathbb{R}^n : x_k = 0\} + \text{span}^{\text{pos}}(S_{k, \varepsilon}). \end{aligned}$$

Proof. The set on the right side is certainly contained in the set on the left side. If x is an element of the left side, then x is λ plus a nonzero element y of $\text{span}^{\text{pos}}(S_{k, \varepsilon})$ plus an element z of $\text{span}^{\text{pos}}(S \setminus S_{k, \varepsilon})$. Since the sign of $\varepsilon x \geq 0$ and $\varepsilon \lambda \leq 0$, there exists t with $0 \leq t \leq 1$ such that $\lambda + ty + z$ has k^{th} entry 0. We see that $x = (\lambda + ty + z) + (1 - t)y$ is an element of the right side. \square

2. FIRST MAIN RESULT

Let B_0 be an exchange matrix. For a sequence $\mathbf{k} = k_m \cdots k_1$ of indices, define seeds $t_1, \dots, t_m = t$ by $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \dots \xrightarrow{k_m} t_m = t$. We will prove the following theorem and corollary.

Theorem 2.1. *Suppose $\mathbf{k} = k_m \cdots k_1$ and $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \dots \xrightarrow{k_m} t_m = t$. If $\mathbf{k}^{-1} = k_1 \cdots k_m$ is a red sequence for B_t , then for any λ in the domain of definition of $\eta_{\mathbf{k}}^{B_0^T}$ that contains $\text{Cone}_t^{B_0; t_0}$,*

$$\mathcal{P}_{\lambda, \mathbf{k}}^{B_0} \subseteq \left\{ \lambda + G_t^{B_0; t_0} B_t \alpha : \alpha \geq 0 \right\} = \left\{ \lambda + B_0 C_t^{B_0; t_0} \alpha : \alpha \geq 0 \right\}.$$

Corollary 2.2. *Suppose t is a seed in the exchange graph for $B_0; t_0$ and take $\lambda \in \text{Cone}_t^{B_0; t_0}$. If there exists a maximal red sequence for B_t , then $\mathcal{P}_{\lambda}^{B_0} = \{\lambda\}$.*

Since $\left(\eta_{\mathbf{k}}^{B_0^T} \right)^{-1} = \eta_{\mathbf{k}^{-1}}^{B_t^T}$, we have $\mathcal{P}_{\lambda, \mathbf{k}}^{B_0} = \eta_{\mathbf{k}^{-1}}^{B_t^T} \left\{ \eta_{\mathbf{k}}^{B_0^T}(\lambda) + B_t \alpha : \alpha \geq 0 \right\}$. Let D be the domain of definition of $\eta_{\mathbf{k}}^{B_0^T}$ that contains $\text{Cone}_t^{B_0; t_0}$. Then $\eta_{\mathbf{k}^{-1}}^{B_t^T}$ is linear on $\eta_{\mathbf{k}}^{B_0^T}(D)$. Let $\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}$ be the linear map that agrees with $\eta_{\mathbf{k}^{-1}}^{B_t^T}$ on $\eta_{\mathbf{k}}^{B_0^T}(D)$.

Proposition 2.3. *The matrix for $\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}$, acting on column vectors, is $G_t^{B_0; t_0}$.*

Proof. By [?, Proposition 8.13], $\text{Cone}_t^{B_0; t_0} = \eta_{\mathbf{k}^{-1}}^{B_t^T}((\mathbb{R}_{\geq 0})^n)$, and therefore also $\eta_{\mathbf{k}}^{B_0^T}(\text{Cone}_t^{B_0; t_0}) = (\mathbb{R}_{\geq 0})^n$. The proof of [?, Proposition 8.13] shows not only an equality of cones, but also that $\eta_{\mathbf{k}^{-1}}^{B_t^T}$ takes the extreme ray of $(\mathbb{R}_{\geq 0})^n$ spanned by e_i to the extreme ray of $\text{Cone}_t^{B_0; t_0}$ spanned by the i^{th} \mathbf{g} -vector at t relative to $B_0; t_0$, where the total order on these \mathbf{g} -vectors at t is obtained from the order e_1, \dots, e_n on \mathbf{g} -vectors at t_0 by the sequence \mathbf{k} of mutations. \square

We now apply a result of [?], namely that $G_t^{B_0;t_0} B_t = B_0 C_t^{B_0;t_0}$. This fact follows from the proof of [?, Proposition 1.3], or from [?, (6.14)], as explained in [?, Remark 2.1]. Since $G_t^{B_0;t_0}$ is the matrix for $\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}$ and since $\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T} \eta_{\mathbf{k}}^{B_0^T}(\lambda) = \lambda$, we have the following proposition.

Proposition 2.4.

$$\begin{aligned} \mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T} \left\{ \eta_{\mathbf{k}}^{B_0^T}(\lambda) + B_t \alpha : \alpha \geq 0 \right\} &= \left\{ \lambda + G_t^{B_0;t_0} B_t \alpha : \alpha \geq 0 \right\} \\ &= \left\{ \lambda + B_0 C_t^{B_0;t_0} \alpha : \alpha \geq 0 \right\}. \end{aligned}$$

In light of Proposition 2.4, the conclusion of Theorem 2.1 is equivalent to

$$\mathcal{P}_{\lambda, \mathbf{k}}^{B_0} \subseteq \mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T} \left\{ \eta_{\mathbf{k}}^{B_0^T}(\lambda) + B_t \alpha : \alpha \geq 0 \right\}.$$

Proof of Theorem 2.1. We will prove that $\mathcal{P}_{\lambda, \mathbf{k}}^{B_0} \subseteq \left\{ \lambda + B_0 C_t^{B_0;t_0} \alpha : \alpha \geq 0 \right\}$, by induction on m (the length of \mathbf{k}). The base case, where $\mathbf{k} = \emptyset$, is true because $C_{t_0}^{B_0;t_0}$ is the identity matrix and $\mathcal{P}_{\lambda, \emptyset} = \left\{ \lambda + B_0 \alpha : \alpha \geq 0 \right\}$.

[?, Proposition 1.4] says that $C_t^{B_0;t_0} = F_{\varepsilon, k_1}^{B_1} C_t^{B_1;t_1}$, where ε is the sign of the k_1 -column of $C_{t_1}^{-B_t;t}$. (The hypothesis that \mathbf{k}^{-1} is a red sequence for B_t determines ε , but we leave ε unspecified for now in order to highlight later where this hypothesis is relevant.) By Lemma 1.4 and because $E_{\varepsilon, k_1}^{B_1}$ and $F_{\varepsilon, k_1}^{B_1}$ are their own inverses,

$$\begin{aligned} \left\{ \lambda + B_0 C_t^{B_0;t_0} \alpha : \alpha \geq 0 \right\} &= \left\{ \lambda + B_0 F_{\varepsilon, k_1}^{B_1} C_t^{B_1;t_1} \alpha : \alpha \geq 0 \right\} \\ (2.1) \quad &= \left\{ \lambda + E_{\varepsilon, k_1}^{B_1} B_1 C_t^{B_1;t_1} \alpha : \alpha \geq 0 \right\} \\ &= E_{\varepsilon, k_1}^{B_1} \left\{ E_{\varepsilon, k_1}^{B_1} \lambda + B_1 C_t^{B_1;t_1} \alpha : \alpha \geq 0 \right\}. \end{aligned}$$

The map $\eta_{\mathbf{k}}^{B_0^T}$ is linear on $\text{Cone}_t^{B_0;t_0}$. This map is $\eta_{\mathbf{k}}^{B_0^T} = \eta_{k_m}^{B_m-1} \circ \dots \circ \eta_{k_2}^{B_1^T} \circ \eta_{k_1}^{B_0^T}$. The map $\eta_{k_1}^{B_0^T}$ restricts to a linear map from $\text{Cone}_t^{B_0;t_0}$ to $\text{Cone}_t^{B_1;t_1}$. The inverse of $\eta_{k_1}^{B_0^T}$ is $\eta_{k_1}^{B_1^T}$. We claim that $E_{\varepsilon, k_1}^{B_1}$ is the matrix for the linear map on column vectors that agrees with $\eta_{k_1}^{B_1^T}$ on $\text{Cone}_t^{B_1;t_1}$. Since $E_{\varepsilon, k_1}^{B_1}$ is its own inverse, the claim is equivalent to saying that implies that $E_{\varepsilon, k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_0^T}$ on $\text{Cone}_t^{B_0;t_0}$.

By [?, (1.13)], ε is the sign of the k_1 -column of $(G_t^{-B_1^T; t_1})^T$. That is, ε is the sign of the k_1 -row of $G_t^{-B_1^T; t_1}$, or in other words, the sign of the k_1 -entry of vectors in $\text{Cone}_t^{-B_1^T; t_1}$. By Lemma 1.3, ε is the sign of the k_1 -entry of vectors in $\text{Cone}_t^{B_1; t_1}$, which is the sign that determines how $\eta_{k_1}^{B_1^T}$ acts on $\text{Cone}_t^{B_1; t_1}$. One easily checks that the action of $\eta_{k_1}^{B_1^T}$ on vectors whose k_1 -entry has sign ε is precisely the action of $E_{\varepsilon, k_1}^{B_1}$.

Let $\lambda' = \eta_{k_1}^{B_0^T}(\lambda)$, so that λ' is in the same domain of definition of $\eta_{k_m \dots k_2}^{B_1^T}$ as $\text{Cone}_t^{B_1; t_1}$ and so that $\lambda' = E_{\varepsilon, k_1}^{B_1} \lambda$. By induction on m ,

$$\eta_{k_2 \dots k_m}^{B_t^T} \left\{ \eta_{k_m \dots k_2}^{B_1^T}(\lambda') + B_t \alpha : \alpha \geq 0 \right\} \subseteq \left\{ \lambda' + B_1 C_t^{B_1; t_1} \alpha : \alpha \geq 0 \right\}.$$

Applying the homeomorphism $\eta_{k_1}^{B_1^T}$ to both sides, we obtain

$$\eta_{k_1}^{B_1^T} \left\{ \eta_{\mathbf{k}}^{B_0^T}(\lambda') + B_t \alpha : \alpha \geq 0 \right\} \subseteq \eta_{k_1}^{B_1^T} \left\{ \lambda' + B_1 C_t^{B_1; t_1} \alpha : \alpha \geq 0 \right\}.$$

In light of (2.1), we can complete the proof by showing that

$$\eta_{k_1}^{B_1^T} \left\{ \lambda' + B_1 C_t^{B_1; t_1} \alpha : \alpha \geq 0 \right\} \subseteq E_{\varepsilon, k_1}^{B_1} \left\{ \lambda' + B_1 C_t^{B_1; t_1} \alpha : \alpha \geq 0 \right\}.$$

We have seen that $E_{\varepsilon, k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_1^T}$ on the set $\{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\}$. We can similarly check that $E_{-\varepsilon, k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_1^T}$ on $\{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = -\varepsilon\}$. Thus $\eta_{k_1}^{B_1^T} \left\{ \lambda' + B_1 C_t^{B_1; t_1} \alpha : \alpha \geq 0 \right\}$ is

$$(U \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = -\varepsilon\}) \cup (V \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\}),$$

where

$$\begin{aligned} U &= E_{\varepsilon, k_1}^{B_1} \left\{ \lambda' + B_1 C_t^{B_1; t_1} \alpha : \alpha \geq 0 \right\} = E_{\varepsilon, k_1}^{B_1} \lambda' + \operatorname{pos}_{\operatorname{span}} \left\{ \left(E_{\varepsilon, k_1}^{B_1} B_1 C_t^{B_1; t_1} \right)_{\operatorname{col} i} \right\}_{i=1}^n \\ V &= E_{-\varepsilon, k_1}^{B_1} \left\{ \lambda' + B_1 C_t^{B_1; t_1} \alpha : \alpha \geq 0 \right\} = E_{-\varepsilon, k_1}^{B_1} \lambda' + \operatorname{pos}_{\operatorname{span}} \left\{ \left(E_{-\varepsilon, k_1}^{B_1} B_1 C_t^{B_1; t_1} \right)_{\operatorname{col} i} \right\}_{i=1}^n, \end{aligned}$$

where $\operatorname{pos}_{\operatorname{span}}$ denotes the nonnegative linear span of a set of vectors.

We need to show that $V \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\} \subseteq U$. Since $\eta_{k_1}^{B_1^T}$ is a homeomorphism, $U \cap \{x \in \mathbb{R}^n : x_{k_1} = 0\} = V \cap \{x \in \mathbb{R}^n : x_{k_1} = 0\}$. By Lemma 1.6, any vector in $V \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\}$ equals a vector in $V \cap \{x \in \mathbb{R}^n : x_{k_1} = 0\}$ plus a positive combination of vectors $\left(E_{-\varepsilon, k_1}^{B_1} B_1 C_t^{B_1; t_1} \right)_{\operatorname{col} i}$ whose k_1 -entry has sign ε . Therefore, it suffices to show that every vector $\left(E_{-\varepsilon, k_1}^{B_1} B_1 C_t^{B_1; t_1} \right)_{\operatorname{col} i}$ whose k_1 -entry has sign ε is in $\operatorname{pos}_{\operatorname{span}} \left\{ \left(E_{\varepsilon, k_1}^{B_1} B_1 C_t^{B_1; t_1} \right)_{\operatorname{col} i} \right\}_{i=1}^n$.

As a temporary shorthand, write b_{ij} for the entries of B_1 and write k for k_1 . Suppose $v_i = \left(E_{-\varepsilon, k}^{B_1} B_1 C_t^{B_1; t_1} \right)_{\operatorname{col} i}$ for some i and suppose the k -entry of v_i has sign ε . Write M for $E_{-\varepsilon, k}^{B_1} B_1$ and write N for $E_{\varepsilon, k}^{B_1} B_1$. Lemma 1.5.1 implies that $M_{kj} = -b_{kj}$ for all j . Lemma 1.5.3 implies that if $\varepsilon M_{kj} \geq 0$, then $M_{\operatorname{col} j} = N_{\operatorname{col} j} + |b_{kj}| N_{\operatorname{col} k}$. Similarly, if $\varepsilon M_{kj} \leq 0$, then $M_{\operatorname{col} j} = N_{\operatorname{col} j} - |b_{kj}| N_{\operatorname{col} k}$.

Now $v_i = E_{-\varepsilon, k}^{B_1} B_1 \left(C_t^{B_1; t_1} \right)_{\operatorname{col} i}$, and $\left(C_t^{B_1; t_1} \right)_{\operatorname{col} i}$ has a sign $\delta \in \{\pm 1\}$, meaning that it is not zero and all of its nonzero entries have sign δ . (This is “sign-coherence of C -vectors”. See Remark 1.2.) Thus there are nonnegative numbers γ_j such that $v_i = \delta \sum_{j=1}^n \gamma_j M_{\operatorname{col} j}$. Write $\{1, \dots, n\} = S \cup T$ with $S \cup T = \emptyset$ such that $\varepsilon M_{kj} \geq 0$

for all $j \in S$ and $\varepsilon M_{kj} \leq 0$ for all $j \in T$. Then

$$\begin{aligned}
v_i &= \delta \sum_{j \in S} \gamma_j M_{\text{col } j} + \delta \sum_{j \in T} \gamma_j M_{\text{col } j} \\
&= \delta \sum_{j \in S} \gamma_j (N_{\text{col } j} + |b_{kj}| N_{\text{col } k}) + \delta \sum_{j \in T} \gamma_j (N_{\text{col } j} - |b_{kj}| N_{\text{col } k}) \\
&= \delta \sum_{j=1}^n \gamma_j N_{\text{col } j} - \delta \sum_{j=1}^n \varepsilon \gamma_j b_{kj} N_{\text{col } k} \\
&= N \left(C_t^{B_1; t_1} \right)_{\text{col } j} + \delta \sum_{j=1}^n \varepsilon \gamma_j M_{kj} N_{\text{col } k} \\
&= N \left(C_t^{B_1; t_1} \right)_{\text{col } j} + \sigma N_{\text{col } k}.
\end{aligned}$$

where $\sigma = \varepsilon \delta \sum_{j=1}^n \gamma_j M_{kj}$ is a positive scalar, because $\delta \sum_{j=1}^n \gamma_j M_{kj}$ is the k -entry of v_i , which has sign ε .

As noted above, ε is the sign of the k_1 -entry of vectors in $\text{Cone}_t^{-B_1^T; t_1}$. Since $\text{Cone}_t^{-B_1^T; t_0} = \left\{ x \in \mathbb{R}^n : x^T C_t^{B_1; t_0} \geq 0 \right\}$, the rows of $\left(C_t^{B_1; t_0} \right)^{-1}$ span the extreme rays of $\text{Cone}_t^{-B_1^T; t_1}$. In particular $\left(C_t^{B_1; t_0} \right)^{-1} (\varepsilon e_k)$ has nonnegative entries. Thus $C_t^{B_1; t_0} \left(C_t^{B_1; t_0} \right)^{-1} (\varepsilon e_k) = \varepsilon e_k$ is a nonnegative linear combination of columns of $C_t^{B_1; t_0}$.

Now, the hypothesis that \mathbf{k}^{-1} is a red sequence for B_t , or equivalently a green sequence for $-B_t$, says that $\varepsilon = +1$, so that e_k is a nonnegative linear combination of columns of $C_t^{B_1; t_0}$. Thus $N_{\text{col } k} = N e_k$ is a nonnegative linear combination of columns of $N C_t^{B_1; t_0}$. We have shown that $v_i = N \left(C_t^{B_1; t_1} \right)_{\text{col } j} + \sigma N_{\text{col } k}$ is a nonnegative linear combination of columns of $N C_t^{B_1; t_0}$. In other words, v_i is in $\text{pos span} \left\{ \left(E_{\varepsilon, k_1}^{B_1} B_1 C_t^{B_1; t_1} \right)_{\text{col } i} \right\}_{i=1}^n$, as desired. \square

Proof of Corollary 2.2. Let t' be the seed at the end of the maximal red sequence for B_t . There exists $\ell = \ell_q \ell_{q-1} \cdots \ell_1$ with $t_0 = t'_0 \xrightarrow{\ell_1} t'_1 \xrightarrow{\ell_2} \cdots \xrightarrow{\ell_q} t'_q = t'$. Let $\lambda' = \eta_\ell^{B_0^T}(\lambda)$. Lemma 1.1 says $\eta_\ell^{B_0^T}(\mathcal{P}_\lambda^{B_0}) = \mathcal{P}_{\lambda'}^{B_{t'}}$. Thus it is enough to prove that $\mathcal{P}_{\lambda'}^{B_{t'}} = \{\lambda'\}$. Since $\eta_\ell^{B_0^T}(\text{Cone}_t^{B_0; t_0}) = \text{Cone}_{t'}^{B_{t'}; t'}$, we have reduced the proof to the case where there is a maximal red sequence for B_t starting from t and ending at t_0 .

Working in that reduction, let $\mathbf{k} = k_m \cdots k_1$ be the reverse of the maximal red sequence and define seeds $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. Then Theorem 2.1 says that $\mathcal{P}_{\lambda, \mathbf{k}}^{B_0} \subseteq \left\{ \lambda + B_0 C_t^{B_0; t_0} \alpha : \alpha \geq 0 \right\}$.

Since \mathbf{k}^{-1} is a maximal red sequence for B_t , or in other words a maximal green sequence for $-B_t$, every column of $C_{t_0}^{-B_t; t}$ has negative sign, so $\text{Cone}_{t_0}^{B_t^T; t} = \left\{ x \in \mathbb{R}^n : x^T C_{t_0}^{-B_t; t} \geq 0 \right\}$ consists of vectors with nonpositive entries. Since $(\mathbb{R}_{\leq 0})^n$ is a cone in the mutation fan \mathcal{F}_{-B_t} (for example, combining [?, Proposition 7.1],

[?, Proposition 8.9], and sign-coherence of C -vectors) and also $\text{Cone}_{t_0}^{B_t^T; t}$ is a cone in \mathcal{F}_{-B_t} , we see that $\text{Cone}_{t_0}^{B_t^T; t} = (\mathbb{R}_{\leq 0})^n$. Thus, up to permuting columns, $C_{t_0}^{-B_t; t}$ is the negative of the identity matrix. We see that $\mathcal{P}_{\lambda, \mathbf{k}}^{B_0} \subseteq \{\lambda - B_0 \alpha : \alpha \geq 0\}$.

Since also $\mathcal{P}_{\lambda, \emptyset}^{B_0} \{\lambda + B_0 \alpha : \alpha \geq 0\}$, and since the columns of B_0 are linearly independent, ② we conclude that $\mathcal{P}_{\lambda}^{B_0} = \{\lambda\}$. \square

BIG QUESTION:

Does this work whenever λ is in the domain of linearity containing $\text{Cone}_t^{B_0; t^0}$, or do you really need $\lambda \in \text{Cone}_t^{B_0; t^0}$?

NR HOMEWORK: Prove that this works.

SS HOMEWORK: Prove that the imaginary cone is in the domain of linearity when t is a “neighboring” seed.

THEN: We use the EBF trick again to rewrite $B_0 C$ as GB_t .

Paper this spring.

REFERENCES

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(D. Rupel) NEED THIS

(S. Stella) NEED THIS

2. Need at least the hypothesis that span of B_t does not contain a line! Here I have just taken “linearly independent columns”. We will need to do extended matrices in some form. :(N