DOMINANCE REGIONS FOR AFFINE CLUSTER ALGEBRAS

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ABSTRACT. NEED THIS

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1. Check specific references to scatfan, scatcomb, and affs-cat. N

1. BACKGROUND

Given a sequence $\mathbf{k} = k_m \cdots k_1$ of indices in $\{1, \ldots, n\}$, we read the sequence from right to left for the purposes of matrix mutation. That is, $\mu_{\mathbf{k}}(B)$ means $\mu_{k_m}(\mu_{k_{m-1}}(\cdots(\mu_{k_1}(B))\cdots))$. We write $\mathbf{k}^{-1}$ for $k_1 \cdots k_m$, the reverse of $\mathbf{k}$.

Given an exchange matrix B, the ***mutation map*** $\eta : \mathbb{R}^n \to \mathbb{R}^n$ takes the input vector in $\mathbb{R}^n$, places it as an additional row below B, mutates the resulting matrix according to the sequence $\mathbf{k}$, and outputs the bottom row of the mutated matrix. In this paper, it is convenient to think of vectors in $\mathbb{R}^n$ as column vectors, and also, the mutation maps we need use transposes B^T of exchange matrices. Thus we write maps $\eta_{\mathbf{k}}^{B^T}$. This map takes a vector, places it as an additional *column* to the right of B (not B^T), does mutations according to $\mathbf{k}$, and reads the rightmost column of the mutated matrix.

For seeds t_0 and t and an exchange matrix B, let $C_t^{B;t_0}$ be the matrix whose columns are the C-vectors at t relative to the initial seed t_0 with exchange matrix B. Each column of $C_t^{B;t_0}$ is nonzero and all of its nonzero entries have the same sign. (This is "sign-coherence of C-vectors" which was implicitly conjectured in [**?**] and proved as [**?**, Corollary 5.5].) Thus we will refer to the ***sign*** of a column of $C_t^{B;t_0}$. For $\mathbf{k} = k_m \cdots k_1$, define seeds $t_1, \ldots, t_m$ by $t_0 \overset{k_1}{\longrightarrow} t_1 \overset{k_2}{\longrightarrow} \cdots \overset{k_m}{\longrightarrow} t_m$. The sequence $\mathbf{k}$ is a ***green sequence*** for an exchange matrix B if column k_ℓ of $C_{t_{\ell-1}}^{B;t_0}$ is *positive* for all ℓ with $1 \leq \ell < m$. We will call the sequence $\mathbf{k}$ a ***red sequence*** for B if it is a green sequence for $-B$. (A red sequence relates to antiprincipal coefficients: If we were to define the C-vectors recursively starting with the negative of the identity matrix, the requirement for a red sequence is that the k_ℓ column is negative at every step.)

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Let $G_t^{B;t_0}$ be the matrix whose columns are the **g**-vectors at t relative to the initial seed t_0 with exchange matrix B. Let $\mathrm{Cone}_t^{B;t_0}$ be the nonnegative linear span of the columns of $G_t^{B;t_0}$. For each $k \in \{1,\ldots,n\}$, the entries in the k^{th} row of $G_t^{B;t_0}$ are not all zero and the nonzero entries have the same sign. (This is "sign-coherence of **g**-vectors", conjectured as [**?**, Conjecture 6.13] and proved as [**?**, Theorem 5.11].) Thus all vectors in $\mathrm{Cone}_t^{B;t_0}$ all have weakly the same sign in the k^{th} position. The inverse of $G_t^{B;t_0}$ is $\left(C_t^{-B_0^T;t_0}\right)^T$. (This is [**?**, Theorem 1.2] or [**?**, Theorem 1.1] and [**?**, Theorem 3.30].) Thus $\mathrm{Cone}_t^{B;t_0} = \left\{x \in \mathbb{R}^n : x^T C_t^{-B^T;t_0} \geq 0\right\}$, where 0 is a row vector and "$\geq$" means componentwise comparison.

We will need to relate the cones $\mathrm{Cone}_t^{B;t_0}$ and $\mathrm{Cone}_t^{-B^T;t_0}$. It is immediate from [**?**, Proposition 7.5] and the skew-symmetry of B that $-B^T$ is a ***rescaling*** of B, meaning that there is a diagonal matrix Σ with positive entries on the diagonal such that $-B^T = \Sigma^{-1}B\Sigma$. Therefore, [**?**, Proposition 8.20] says that the i^{th} column of $G_t^{-B^T;t_0}$ is a scalar positive multiple of the i^{th} column of $\Sigma G_t^{B;t_0}$. (In the statement of [**?**, Proposition 8.20], Σ is multiplied on the right, because there **g**-vectors are row vectors rather than column vectors.) Thus we have the following fact.

Lemma 1.1. *The k^{th} entries of vectors in $\mathrm{Cone}_t^{B;t_0}$ have the same sign as the k^{th} entries of vectors in $\mathrm{Cone}_t^{-B^T;t_0}$.*

For $k \in \{1,\ldots,n\}$, let J_k be the $n \times n$ matrix that agrees with the identity matrix except that J_k has -1 in position kk. For an $n \times n$ matrix M and $k \in \{1,\ldots,n\}$, let $M^{\bullet k}$ be the matrix that agrees with M in column k and has zeros everywhere outside of column k. Let $M^{k\bullet}$ be the matrix that agrees with M in row k and has zeros everywhere outside of row k.

Given a real number a, let $[a]_+$ denote $\max(a,0)$. Given a matrix $M = [m_{ij}]$, define $[M]_+$ to be the matrix whose ij-entry is $[m_{ij}]_+$. Given an exchange matrix B, an index $k \in \{1,\ldots,n\}$ and a sign $\varepsilon \in \{\pm 1\}$, define matrices

$$E_{\varepsilon,k}^B = J_k + [\varepsilon B]_+^{\bullet k}$$
$$F_{\varepsilon,k}^B = J_k + [-\varepsilon B]_+^{k\bullet}.$$

Each matrix $E_{\varepsilon,k}^B$ is its own inverse, and each $F_{\varepsilon,k}^B$ is its own inverse. The following is essentially a result of [**?**], although it is not stated there in this form. ②

2. Do I have this attribution right? N

Lemma 1.2. *For any $k \in \{1,\ldots,n\}$ and $\varepsilon \in \{\pm 1\}$, the mutation of B at k is $\mu_k(B) = E_{\varepsilon,k}^B B F_{\varepsilon,k}^B$.*

Proof. We expand the product $(J_k + [\varepsilon B]_+^{\bullet k})B(J_k + [-\varepsilon B]_+^{k\bullet})$ to four terms. The term $[\varepsilon B]_+^{\bullet k} B [-\varepsilon B]_+^{k\bullet}$ is zero because $b_{kk} = 0$. The term $[\varepsilon B]_+^{\bullet k} B J_k$ is $[\varepsilon B]_+^{\bullet k} B^{k\bullet} J_k$, which equals $[\varepsilon B]_+^{\bullet k} B^{k\bullet}$. Similarly, the term $J_k B [-\varepsilon B]_+^{k\bullet}$ equals $B^{\bullet k}[-\varepsilon B]_+^{k\bullet}$ Both Thus the ij-entry of $E_{\varepsilon,k}^B B F_{\varepsilon,k}^B$ is

$$\begin{Bmatrix} -b_{ij} & \text{if } k \in \{i,j\} \\ b_{ij} & \text{otherwise} \end{Bmatrix} + \begin{Bmatrix} |b_{ik}|b_{kj} & \text{if } \operatorname{sgn} b_{ik} = \varepsilon \\ 0 & \text{otherwise} \end{Bmatrix} + \begin{Bmatrix} b_{ik}|b_{kj}| & \text{if } \operatorname{sgn} b_{kj} = -\varepsilon \\ 0 & \text{otherwise} \end{Bmatrix}.$$

This coincides with the ij-entry of $\mu_k(B)$. $\square$

Given a matrix M, write $M_{\mathrm{col}(i)}$ for the i^{th} column of M. We observe that $(MN)_{\mathrm{col}\,i} = M(N)_{\mathrm{col}\,i}$.

Lemma 1.3. *Suppose $B = [b_{ij}]$ is an exchange matrix, let $k \in \{1, \ldots, n\}$, and choose a sign $\varepsilon \in \{\pm 1\}$.*

1. $(E^B_{\varepsilon,k} B)_{\mathrm{col}\, i} = J_k(B)_{\mathrm{col}\, i} + b_{ki}([\varepsilon B]_+)_{\mathrm{col}\, k}$.
2. $(E^B_{\varepsilon,k} B)_{\mathrm{col}\, k} = (E^B_{-\varepsilon,k} B)_{\mathrm{col}\, k} = B_{\mathrm{col}\, k}$.
3. $(E^B_{-\varepsilon,k} B)_{\mathrm{col}\, i} = (E^B_{\varepsilon,k} B)_{\mathrm{col}\, i} - \varepsilon b_{ki} B_{\mathrm{col}\, k}$.

Proof. The first two assertions follow immediately from the fact that $(MN)_{\mathrm{col}\, i} = M(N)_{\mathrm{col}\, i}$ and the fact that $b_{kk} = 0$. The first assertion (for ε and $-\varepsilon$) implies that $(E^B_{-\varepsilon,k} B)_{\mathrm{col}\, i} = (E^B_{\varepsilon,k} B)_{\mathrm{col}\, i} - b_{ki}([\varepsilon B]_+ - [-\varepsilon B]_+)_{\mathrm{col}\, k}$. The third assertion follows. □

We will also need the following simple fact about nonnegative linear spans. Given a set S of vectors, let $\underset{\mathbf{span}}{\mathbf{pos}}(S)$ denote the nonnegative linear span of S. For $k \in \{1, \ldots, n\}$ and $\varepsilon \in \{\pm 1\}$, let $S_{k,\varepsilon}$ be the set of vectors in S whose k^{th} entry has sign strictly agreeing with ε.

Lemma 1.4. *Suppose λ is a vector in $\mathbb{R}^n$ whose k^{th} λ_k has $\varepsilon\lambda_k \leq 0$. Then*

$$\left\{\lambda + \underset{\mathbf{span}}{\mathbf{pos}}(S)\right\} \cap \{x \in \mathbb{R}^n : \varepsilon x_k \geq 0\}$$
$$= \left\{\lambda + \underset{\mathbf{span}}{\mathbf{pos}}(S)\right\} \cap \{x \in \mathbb{R}^n : x_k = 0\} + \underset{\mathbf{span}}{\mathbf{pos}}(S_{k,\varepsilon}).$$

Proof. The set on the right side is certainly contained in the set on the right side. If x is an element of the left side, then x is λ plus a nonzero element y of $\underset{\mathbf{span}}{\mathbf{pos}}(S_{k,\varepsilon})$ plus an element z of $\underset{\mathbf{span}}{\mathbf{pos}}(S \setminus S_{k,\varepsilon})$. Since the sign of $\varepsilon x \geq 0$ and $\varepsilon\lambda \leq 0$, there exists t with $0 \leq t \leq 1$ such that $\lambda + ty + z$ has k^{th} entry 0. We see that $x = (\lambda + ty + z) + (1 - t)y$ is an element of the right side. □

2. First main result

Let B_0 be an exchange matrix. For a sequence $\mathbf{k} = k_m \cdots k_1$ of indices, define seeds $t_1, \ldots, t_m = t$ by $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. Given a vector $\lambda \in \mathbb{R}^n$, we want to understand $\mathcal{P}^{B_0}_{\lambda,\mathbf{k}} = \eta^{B_t^T}_{\mathbf{k}^{-1}}\left\{\eta^{B_0^T}_{\mathbf{k}}(\lambda) + B_t\alpha : \alpha \geq 0\right\}$.
START ALTERNATIVE WORDING

There may be more than one sequence connecting t_0 to t.
NOT TRUE:
The mutation map $\eta^{B_0^T}_{\mathbf{k}}$ depends only on t, not on the choice of $\mathbf{k}$.
HOWEVER, we can rescue the following (by showing that different $\mathbf{k}$s with the same t are related by a global permutation of rows/coumns):
Thus we define $\mathcal{P}^{B_0}_{\lambda,t}$ to be $\mathcal{P}^{B_0}_{\lambda,\mathbf{k}}$ for any sequence $\mathbf{k} = k_m \cdots k_1$ with $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. Our first main result is about $\mathcal{P}^{B_0}_{\lambda,t}$ in the case where λ is in $\mathrm{Cone}^{B_0;t_0}_t$.

Our first main result is about $\mathcal{P}^{B_0}_{\lambda,\mathbf{k}}$ in the case where λ is in $\mathrm{Cone}^{B_0;t_0}_t$ for some sequence $\mathbf{k}$.

Theorem 2.1. *Fix an exchange pattern with B_0 at t_0. For some vertex t, suppose there exists a red sequence for B_t that ends at t_0. Then for $\lambda \in \mathrm{Cone}^{B_0;t_0}_t$,*

$$\mathcal{P}^{B_0}_{\lambda,t} \subseteq \left\{\lambda + B_0 C^{B_0;t_0}_t \alpha : \alpha \geq 0\right\}.$$

END ALTERNATIVE WORDING

The map $\eta^{B_t^T}_{\mathbf{k}^{-1}}$ is linear on the cone $(\mathbb{R}_{\geq 0})^n$. 3 Let D be the domain of linearity of $\eta^{B_t^T}_{\mathbf{k}^{-1}}$ containing $(\mathbb{R}_{\geq 0})^n$ and let $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}$ be the linear map that agrees with $\eta^{B_t^T}_{\mathbf{k}^{-1}}$ on D.

3. Probably need to explain why. B-cones and such. N

Theorem 2.2. *Suppose* $\mathbf{k} = k_m \cdots k_1$ *and* $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. *Let* $\lambda \in \mathrm{Cone}_t^{B_0;t_0}$. *If* $\mathbf{k}^{-1} = k_1 \cdots k_m$ *is a red sequence for* B_t, *then*

$$\mathcal{P}^{B_0}_{\lambda,\mathbf{k}} \subseteq \Big\{\lambda + B_0 C_t^{B_0;t_0}\alpha : \alpha \geq 0\Big\}.$$

Moving towards the proof of Theorem 2.2, we first determine the matrix for $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}$ acting on column vectors. By [?, Proposition 8.13], $\mathrm{Cone}_t^{B_0;t_0} = \eta^{B_t^T}_{\mathbf{k}^{-1}}((\mathbb{R}_{\geq 0})^n)$. Thus $\eta^{B_0^T}_{\mathbf{k}}\left(\mathrm{Cone}_t^{B_0;t_0}\right) = (\mathbb{R}_{\geq 0})^n$. The proof of [?, Proposition 8.13] shows not only an equality of cones, but also that $\eta^{B_t^T}_{\mathbf{k}^{-1}}$ takes the extreme ray of $(\mathbb{R}_{\geq 0})^n$ spanned by e_i to the extreme ray of $\mathrm{Cone}_t^{B_0;t_0}$ spanned by the i^{th} $\mathbf{g}$-vector at t relative to $B_0;t_0$, where the total order on these $\mathbf{g}$-vectors at t is obtained from the order $e_1,\ldots,e_n$ on $\mathbf{g}$-vectors at t_0 by the sequence $\mathbf{k}$ of mutations. Thus we have the following proposition.

Proposition 2.3. *The matrix for* $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}$, *acting on column vectors, is* $G_t^{B_0;t_0}$.

Remark 2.4. As written, [?, Proposition 8.13] is conditional on "sign-coherence of C-vectors", which was a conjecture but is now a theorem [?, Corollary 5.5].

We now apply a result of [?], namely that $G_t^{B_0;t_0}B_t = B_0 C_t^{B_0;t_0}$. This fact follows from the proof of [?, Proposition 1.3], or from [?, (6.14)], as explained in [?, Remark 2.1]. Since $G_t^{B_0;t_0}$ is the matrix for $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}$ and since $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}\eta^{B_0^T}_{\mathbf{k}}(\lambda) = \lambda$, we rewrite the right side of the containment in Theorem 2.2 as follows.

Proposition 2.5. $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}\Big\{\eta^{B_0^T}_{\mathbf{k}}(\lambda) + B_t\alpha : \alpha \geq 0\Big\} = \Big\{\lambda + B_0 C_t^{B_0;t_0}\alpha : \alpha \geq 0\Big\}.$

In light of Proposition 2.5, Theorem 2.2 is equivalent to

$$\mathcal{P}^{B_0}_{\lambda,\mathbf{k}} \subseteq \mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}\Big\{\eta^{B_0^T}_{\mathbf{k}}(\lambda) + B_t\alpha : \alpha \geq 0\Big\}.$$

Proposition 2.5 also immediately implies the following statement that is weaker than Theorem 2.2.

Proposition 2.6. $\mathcal{P}^{B_0}_{\lambda,\mathbf{k}} \cap D = \Big\{\lambda + B_0 C_t^{B_0;t_0}\alpha : \alpha \geq 0\Big\} \cap D.$

We now prove our first main result.

Proof of Theorem 2.2. [?, Proposition 1.4] says that $C_t^{B_0;t_0} = F^{B_1}_{\varepsilon,k_1} C_t^{B_1;t_1}$, where ε is the sign of the k_1-column of $C_{t_1}^{-B_t;t}$. (The hypothesis that $\mathbf{k}^{-1}$ is a red sequence for B_t determines ε, but we leave ε unspecified for now in order to highlight later where this hypothesis is relevant.) By Lemma 1.2 and because $E^{B_1}_{\varepsilon,k_1}$ and $F^{B_1}_{\varepsilon,k_1}$ are their own inverses,

$$\begin{aligned}
\Big\{\lambda + B_0 C_t^{B_0;t_0}\alpha : \alpha \geq 0\Big\} &= \Big\{\lambda + B_0 F^{B_1}_{\varepsilon,k_1} C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\} \\
&= \Big\{\lambda + E^{B_1}_{\varepsilon,k_1} B_1 C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\} \qquad (2.1)\\
&= E^{B_1}_{\varepsilon,k_1}\Big\{E^{B_1}_{\varepsilon,k_1}\lambda + B_1 C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\}.
\end{aligned}$$

The map $\eta_{\mathbf{k}}^{B_0^T}$ is linear on $\mathrm{Cone}_t^{B_0;t_0}$. 4 This map is $\eta_{\mathbf{k}}^{B_0^T}=\eta_{k_m}^{B_{m-1}^T}\circ\cdots\circ\eta_{k_2}^{B_1^T}\circ\eta_{k_1}^{B_0^T}$. 4. Probably need to explain this too. N Since $\eta_{k_2\cdots k_m}^{B_t^T}((\mathbb{R}_{\geq 0})^n)=\mathrm{Cone}_t^{B_1;t_1}$ (again by [?, Proposition 8.13]), we see that $\eta_{k_1}^{B_0^T}$ restricts to a linear map from $\mathrm{Cone}_t^{B_0;t_0}$ to $\mathrm{Cone}_t^{B_1;t_1}$. The inverse of $\eta_{k_1}^{B_0^T}$ is $\eta_{k_1}^{B_1^T}$.

We claim that $E_{\varepsilon,k_1}^{B_1}$ is the matrix for the linear map on column vectors that agrees with $\eta_{k_1}^{B_1^T}$ on $\mathrm{Cone}_t^{B_1;t_1}$. Since $E_{\varepsilon,k_1}^{B_1}$ is its own inverse, the claim is equivalent to saying that implies that $E_{\varepsilon,k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_0^T}$ on $\mathrm{Cone}_t^{B_0;t_0}$.

By [?, (1.13)], ε is the sign of the k_1-column of $\left(G_t^{-B_1^T;t_1}\right)^T$. That is, ε is the sign of the k_1-row of $G_t^{-B_1^T;t_1}$, or in other words, the sign of the k_1-entry of vectors in $\mathrm{Cone}_t^{-B_1^T;t_1}$. By Lemma 1.1, ε is the sign of the k_1-entry of vectors in $\mathrm{Cone}_t^{B_1;t_1}$, which is the sign that determines how $\eta_{k_1}^{B_1^T}$ acts on $\mathrm{Cone}_t^{B_1;t_1}$. We now easily check that the action of $\eta_{k_1}^{B_1^T}$ on vectors whose k_1-entry has sign ε is precisely the action of $E_{\varepsilon,k_1}^{B_1}$.

Let $\lambda'=\eta_{k_1}^{B_0^T}(\lambda)$, so that $\lambda'\in\mathrm{Cone}_t^{B_1;t_1}$ and $\lambda'=E_{\varepsilon,k_1}^{B_1}\lambda$. By induction on m,

$$\eta_{k_2\cdots k_m}^{B_t^T}\left\{\eta_{k_m\cdots k_2}^{B_1^T}(\lambda')+B_t\alpha:\alpha\geq 0\right\}\subseteq\left\{\lambda'+B_1C_t^{B_1;t_1}\alpha:\alpha\geq 0\right\}.$$

Applying the homeomorphism $\eta_{k_1}^{B_1^T}$ to both sides, we obtain

$$\eta_{\mathbf{k}^{-1}}^{B_t^T}\left\{\eta_{\mathbf{k}}^{B_0^T}(\lambda')+B_t\alpha:\alpha\geq 0\right\}\subseteq\eta_{k_1}^{B_1^T}\left\{\lambda'+B_1C_t^{B_1;t_1}\alpha:\alpha\geq 0\right\}.$$

In light of (2.1), we can complete the proof by showing that

$$\eta_{k_1}^{B_1^T}\left\{\lambda'+B_1C_t^{B_1;t_1}\alpha:\alpha\geq 0\right\}\subseteq E_{\varepsilon,k_1}^{B_1}\left\{\lambda'+B_1C_t^{B_1;t_1}\alpha:\alpha\geq 0\right\}.$$

We have seen that $E_{\varepsilon,k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_1^T}$ on the set $\{x\in\mathbb{R}^n:\operatorname{sgn}x_{k_1}=\varepsilon\}$. We can similarly check that $E_{-\varepsilon,k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_1^T}$ on $\{x\in\mathbb{R}^n:\operatorname{sgn}x_{k_1}=-\varepsilon\}$. Thus $\eta_{k_1}^{B_1^T}\left\{\lambda'+B_1C_t^{B_1;t_1}\alpha:\alpha\geq 0\right\}$ is

$$(U\cap\{x\in\mathbb{R}^n:\operatorname{sgn}x_{k_1}=-\varepsilon\})\cup(V\cap\{x\in\mathbb{R}^n:\operatorname{sgn}x_{k_1}=\varepsilon\}),$$

where

$$U=E_{\varepsilon,k_1}^{B_1}\left\{\lambda'+B_1C_t^{B_1;t_1}\alpha:\alpha\geq 0\right\}=E_{\varepsilon,k_1}^{B_1}\lambda'+\underset{\mathbf{span}}{\mathbf{pos}}\left\{\left(E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\right)_{\mathrm{col}\,i}\right\}_{i=1}^n$$
$$V=E_{-\varepsilon,k_1}^{B_1}\left\{\lambda'+B_1C_t^{B_1;t_1}\alpha:\alpha\geq 0\right\}=E_{-\varepsilon,k_1}^{B_1}\lambda'+\underset{\mathbf{span}}{\mathbf{pos}}\left\{\left(E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\right)_{\mathrm{col}\,i}\right\}_{i=1}^n,$$

where $\underset{\mathbf{span}}{\mathbf{pos}}$ denotes the nonnegative linear span of a set of vectors.

We need to show that $V\cap\{x\in\mathbb{R}^n:\operatorname{sgn}x_{k_1}=\varepsilon\}\subseteq U$. Since $\eta_{k_1}^{B_1^T}$ is a homeomorphism, $U\cap\{x\in\mathbb{R}^n:x_{k_1}=0\}=V\cap\{x\in\mathbb{R}^n:x_{k_1}=0\}$. By Lemma 1.4, any vector in $V\cap\{x\in\mathbb{R}^n:\operatorname{sgn}x_{k_1}=\varepsilon\}$ equals a vector in $V\cap\{x\in\mathbb{R}^n:x_{k_1}=0\}$ plus a positive combination of vectors $\left(E_{-\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\right)_{\mathrm{col}\,i}$ whose k_1-entry has sign ε. Therefore, it suffices to show that every vector $\left(E_{-\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\right)_{\mathrm{col}\,i}$ whose k_1-entry has sign ε is in $\underset{\mathbf{span}}{\mathbf{pos}}\left\{\left(E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\right)_{\mathrm{col}\,i}\right\}_{i=1}^n$.

As a temporary shorthand, write b_{ij} for the entries of B_1 and write k for k_1. Suppose $v_i = \left(E^{B_1}_{-\varepsilon,k} B_1 C^{B_1;t_1}_t\right)_{\mathrm{col}\, i}$ for some i and suppose the k-entry of v_i has sign ε. Write M for $E^{B_1}_{-\varepsilon,k} B_1$ and write N for $E^{B_1}_{\varepsilon,k} B_1$. Lemma 1.3.1 implies that $M_{kj} = -b_{kj}$ for all j. Lemma 1.3.3 implies that if $\varepsilon M_{kj} \geq 0$, then $M_{\mathrm{col}\, j} = N_{\mathrm{col}\, j} + |b_{kj}| N_{\mathrm{col}\, k}$. Similarly, if $\varepsilon M_{kj} \leq 0$, then $M_{\mathrm{col}\, j} = N_{\mathrm{col}\, j} - |b_{kj}| N_{\mathrm{col}\, k}$.

Now $v_i = E^{B_1}_{-\varepsilon,k} B_1 \left(C^{B_1;t_1}_t\right)_{\mathrm{col}\, i}$, and $\left(C^{B_1;t_1}_t\right)_{\mathrm{col}\, i}$ has a sign $\delta \in \{\pm 1\}$, meaning that it is not zero and all of its nonzero entries have sign δ. (This is "sign-coherence of C-vectors". See Remark 2.4.) Thus there are nonnegative numbers γ_j such that $v_i = \delta \sum_{j=1}^n \gamma_j M_{\mathrm{col}\, j}$. Write $\{1, \ldots, n\} = S \cup T$ with $S \cup T = \emptyset$ such that $\varepsilon M_{kj} \geq 0$ for all $j \in S$ and $\varepsilon M_{kj} \leq 0$ for all $j \in T$. Then

$$
\begin{aligned}
v_i &= \delta \sum_{j \in S} \gamma_j M_{\mathrm{col}\, j} + \delta \sum_{j \in T} \gamma_j M_{\mathrm{col}\, j} \\
&= \delta \sum_{j \in S} \gamma_j (N_{\mathrm{col}\, j} + |b_{kj}| N_{\mathrm{col}\, k}) + \delta \sum_{j \in T} \gamma_j (N_{\mathrm{col}\, j} - |b_{kj}| N_{\mathrm{col}\, k}) \\
&= \delta \sum_{j=1}^n \gamma_j N_{\mathrm{col}\, j} - \delta \sum_{j=1}^n \varepsilon \gamma_j b_{kj} N_{\mathrm{col}\, k} \\
&= N \left(C^{B_1;t_1}_t\right)_{\mathrm{col}\, j} + \delta \sum_{j=1}^n \varepsilon \gamma_j M_{kj} N_{\mathrm{col}\, k} \\
&= N \left(C^{B_1;t_1}_t\right)_{\mathrm{col}\, j} + \sigma N_{\mathrm{col}\, k}.
\end{aligned}
$$

where $\sigma = \varepsilon\delta \sum_{j=1}^n \gamma_j M_{kj}$ is a positive scalar, because $\delta \sum_{j=1}^n \gamma_j M_{kj}$ is the k-entry of v_i, which has sign ε.

As noted above, ε is the sign of the k_1-entry of vectors in $\mathrm{Cone}_t^{-B_1^T;t_1}$. Since $\mathrm{Cone}_t^{-B_1^T;t_0} = \left\{x \in \mathbb{R}^n : x^T C^{B_1;t_0}_t \geq 0\right\}$, the rows of $\left(C^{B_1;t_0}_t\right)^{-1}$ span the extreme rays of $\mathrm{Cone}_t^{-B_1^T;t_1}$. In particular $\left(C^{B_1;t_0}_t\right)^{-1} (\varepsilon e_k)$ has nonnegative entries. Thus $C^{B_1;t_0}_t \left(C^{B_1;t_0}_t\right)^{-1} (\varepsilon e_k) = \varepsilon e_k$ is a nonnegative linear combination of columns of $C^{B_1;t_0}_t$.

Now, the hypothesis that $\mathbf{k}^{-1}$ is a red sequence for B_t, or equivalently a green sequence for $-B_t$, says that $\varepsilon = +1$, so that e_k is a nonnegative linear combination of columns of $C^{B_1;t_0}_t$. Thus $N_{\mathrm{col}\, k} = N e_k$ is a nonnegative linear combination of columns of $N C^{B_1;t_0}_t$. We have shown that $v_i = N \left(C^{B_1;t_1}_t\right)_{\mathrm{col}\, j} + \sigma N_{\mathrm{col}\, k}$ is a nonnegative linear combination of columns of $N C^{B_1;t_0}_t$. In other words, v_i is in $\underset{\textbf{span}}{\textbf{pos}} \left\{\left(E^{B_1}_{\varepsilon,k_1} B_1 C^{B_1;t_1}_t\right)_{\mathrm{col}\, i}\right\}_{i=1}^n$, as desired. □

WHERE TO PUT THIS? (define maximal green sequence) (define $\mathcal{P}_\lambda$)

Corollary 2.7. *Suppose t' is a seed in the exchange graph for $B;t$ and take $\lambda \in \mathrm{Cone}^{B;t}_{t'}$. If there exists a maximal green sequence for B, then $\mathcal{P}^B_\lambda = \{\lambda\}$.*

References

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