DOMINANCE REGIONS FOR AFFINE CLUSTER ALGEBRAS

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Abstract. NEED THIS

Contents

1. Background

We assume the basic definitions of exchange matrices and of matrix mutation. Given a sequence $\mathbf{k} = k_m \cdots k_1$ of indices in $\{1, \ldots, n\}$, we read the sequence from right to left for the purposes of matrix mutation. That is, $\mu_{\mathbf{k}}(B)$ means $\mu_{k_m}(\mu_{k_{m-1}}(\cdots(\mu_{k_1}(B))\cdots))$. We write $\mathbf{k}^{-1}$ for $k_1 \cdots k_m$, the reverse of $\mathbf{k}$. Throughout, we will use without comment the fact that matrix mutation commutes with the maps $B \mapsto -B$ and $B \mapsto B^T$.

Given an exchange matrix B, the ***mutation map*** $\eta_{\mathbf{k}}^B : \mathbb{R}^n \to \mathbb{R}^n$ takes the input vector in $\mathbb{R}^n$, places it as an additional row below B, mutates the resulting matrix according to the sequence $\mathbf{k}$, and outputs the bottom row of the mutated matrix. In this paper, it is convenient to think of vectors in $\mathbb{R}^n$ as column vectors, and also, the mutation maps we need use transposes B^T of exchange matrices. Thus we write maps $\eta_{\mathbf{k}}^{B^T}$. This map takes a vector, places it as an additional *column* to the right of B (not B^T), does mutations according to $\mathbf{k}$, and reads the rightmost column of the mutated matrix.

Given a vector $\lambda \in \mathbb{R}^n$, define $\mathcal{P}_{\lambda,\mathbf{k}}^B = \left(\eta_{\mathbf{k}}^{B^T}\right)^{-1} \left\{\eta_{\mathbf{k}}^{B^T}(\lambda) + B_t\alpha : \alpha \in \mathbb{R}^n, \alpha \geq 0\right\}$, where the symbol $\geq$ denotes componentwise comparison. (Throughout the paper, we will define sets indexed by vectors $\alpha \in \mathbb{R}^n$ with $\alpha \geq 0$, or sometimes $\alpha \in \mathbb{R}^m$ with $\alpha \geq 0$. When we can do so without confusion, we will omit the explicit statement that $\alpha \in \mathbb{R}^n$ or $\alpha \in \mathbb{R}^m$.) Define the ***dominance region*** of λ with respect to B to be $\mathcal{P}_\lambda^B = \bigcap_{\mathbf{k}} \mathcal{P}_{\lambda,\mathbf{k}}^B$, where the intersection is over all sequences $\mathbf{k}$.

Lemma 1.1. *If $\lambda' = \eta_{\mathbf{k}}^{B^T}(\lambda)$ and $B' = \mu_{\mathbf{k}}(B)$, then*

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1. $\eta_{\mathbf{k}}^{B^T}(\mathcal{P}_\lambda^B) = \mathcal{P}_{\lambda'}^{B'}$.
2. $\eta_{\mathbf{k}}^{B^T}(\mathcal{P}_{\lambda,\boldsymbol{\ell}}^B) = \mathcal{P}_{\lambda',\boldsymbol{\ell}\mathbf{k}^{-1}}^{B'}$ *for any* $\boldsymbol{\ell}$.

Proof. For any $\boldsymbol{\ell}$,

$$\begin{aligned}
\eta_{\mathbf{k}}^{B^T}(\mathcal{P}_{\lambda,\boldsymbol{\ell}}^B) &= \eta_{\mathbf{k}}^{B^T}\left(\left(\eta_{\boldsymbol{\ell}}^{B^T}\right)^{-1}\left\{\eta_{\boldsymbol{\ell}}^{B^T}(\lambda) + B_t\alpha : \alpha \geq 0\right\}\right)\\
&= \left(\eta_{\boldsymbol{\ell}}^{B^T}\eta_{\mathbf{k}^{-1}}^{\mu_{\mathbf{k}}(B)^T}\right)^{-1}\left\{\eta_{\boldsymbol{\ell}}^{B^T}(\lambda) + B_t\alpha : \alpha \geq 0\right\}\\
&= \left(\eta_{\boldsymbol{\ell}\mathbf{k}^{-1}}^{\mu_{\mathbf{k}}(B)^T}\right)^{-1}\left\{\eta_{\boldsymbol{\ell}\mathbf{k}^{-1}}^{\mu_{\mathbf{k}}(B)^T}\left(\eta_{\mathbf{k}}^{B^T}(\lambda)\right) + B_t\alpha : \alpha \geq 0\right\}\\
&= \mathcal{P}_{\lambda',\boldsymbol{\ell}\mathbf{k}^{-1}}^{B'}.
\end{aligned}$$

Thus $\eta_{\mathbf{k}}^{B^T}(\mathcal{P}_\lambda^B) = \bigcap_{\boldsymbol{\ell}} \mathcal{P}_{\lambda',\boldsymbol{\ell}\mathbf{k}^{-1}}^{B'} = \mathcal{P}_{\lambda'}^{B'}$. □

For seeds t_0 and t and an exchange matrix B, let $C_t^{B;t_0}$ be the matrix whose columns are the C-vectors at t relative to the initial seed t_0 with exchange matrix B. Each column of $C_t^{B;t_0}$ is nonzero and all of its nonzero entries have the same sign. (This is "sign-coherence of C-vectors", which was implicitly conjectured in [1] and proved as [2, Corollary 5.5].) Thus we will refer to the ***sign*** of a column of $C_t^{B;t_0}$. For $\mathbf{k} = k_m \cdots k_1$, define seeds $t_1, \ldots, t_m$ by $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m$. The sequence $\mathbf{k}$ is a ***green sequence*** for an exchange matrix B if column k_ℓ of $C_{t_{\ell-1}}^{B;t_0}$ is *positive* for all ℓ with $1 \leq \ell < m$. A ***maximal green sequence*** for B is a green sequence that cannot be extended. That is, the sequence $\mathbf{k}$ is a maximal green sequence if every column of $C_{t_m}^{B;t_0}$ is *negative*. We will call $\mathbf{k}$ a ***red sequence*** for B if it is a green sequence for $-B$. A ***maximal red sequence*** is a red sequence that cannot be extended. (A red sequence relates to antiprincipal coefficients: If we were to define the C-vectors recursively starting with the negative of the identity matrix, the requirement for a red sequence is that the k_ℓ column is negative at every step.)

Let $G_t^{B;t_0}$ be the matrix whose columns are the $\mathbf{g}$-vectors at t relative to the initial seed t_0 with exchange matrix B. Let $\mathrm{Cone}_t^{B;t_0}$ be the nonnegative linear span of the columns of $G_t^{B;t_0}$. For each $k \in \{1, \ldots, n\}$, the entries in the k^{th} row of $G_t^{B;t_0}$ are not all zero and the nonzero entries have the same sign. (This is "sign-coherence of $\mathbf{g}$-vectors", conjectured as [1, Conjecture 6.13] and proved as [2, Theorem 5.11].) Thus all vectors in $\mathrm{Cone}_t^{B;t_0}$ all have weakly the same sign in the k^{th} position. The inverse of $G_t^{B;t_0}$ is $\left(C_t^{-B^T;t_0}\right)^T$. (This is [3, Theorem 1.2] or [4, Theorem 1.1] and [4, Theorem 3.30].) Thus $\mathrm{Cone}_t^{B;t_0} = \left\{x \in \mathbb{R}^n : x^T C_t^{-B^T;t_0} \geq 0\right\}$, where 0 is a row vector and "$\geq$" means componentwise comparison.

Given $\mathbf{k}$ with $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m$, let B_i be the exchange matrix at t_i, so that in particular, $B_0 = B$. The map $\eta_{\mathbf{k}}^{B^T}$ is $\eta_{k_m}^{B_{m-1}^T} \circ \cdots \circ \eta_{k_2}^{B_1^T} \circ \eta_{k_1}^{B_0^T}$. The definition of each $\eta_{k_i}^{B_{i-1}^T}$ has two cases, separated by the hyperplane $x_{k_i} = 0$. Two vectors are in the same ***domain of definition*** of $\eta_{\mathbf{k}}^{B^T}$ if, at every step, the same case applies for the two vectors. (Both cases apply on the hyperplane, so domains of definition are closed.) In particular, $\eta_{\mathbf{k}}^{B^T}$ is linear in each of its domains of definition, but the domains of linearity of $\eta_{\mathbf{k}}^{B^T}$ can be larger than its domains of definition.

There is a fan $\mathcal{F}_{B^T}$ called the ***mutation fan*** for B^T [5, Definition 5.12]. We will not need the details of the definition, but roughly, the cones of $\mathcal{F}_{B^T}$ are the intersections of domains of definition of all mutation maps $\eta_{\mathbf{k}}^{B^T}$, as $\mathbf{k}$ varies. Thus for each $\mathbf{k}$, each cone of $\mathcal{F}_{B^T}$ is contained in a domain of definition of $\eta_{\mathbf{k}}^{B^T}$, and the mutation map $\eta_{\mathbf{k}}^{B^T}$ is linear on every cone of $\mathcal{F}_{B^T}$ [5, Proposition 5.3]. Every cone $\mathrm{Cone}_t^{B;t_0}$ is a maximal cone in the mutation fan $\mathcal{F}_{B^T}$ [5, Proposition 8.13]. Thus in particular, the mutation map $\eta_{\mathbf{k}}^{B^T}$ is linear on every cone $\mathrm{Cone}_t^{B;t_0}$. Furthermore, $\mathrm{Cone}_t^{B_m;t_m} = \eta_{\mathbf{k}}^{B^T}\left(\mathrm{Cone}_t^{B;t_0}\right)$ for every seed t. (This amounts to the initial seed mutation formula for $\mathbf{g}$-vectors, conjectured as [1, Conjecture 7.12] and shown in [3, Proposition 4.2(v)] to follow from sign-coherence of C-vectors. The restatement in terms of mutation maps is [5, Conjecture 8.11].)

Remark 1.2. As written, [5, Proposition 8.13] is conditional on "sign-coherence of C-vectors", which was a conjecture but is now a theorem [2, Corollary 5.5].

We will need to relate the cones $\mathrm{Cone}_t^{B;t_0}$ and $\mathrm{Cone}_t^{-B^T;t_0}$. It is immediate from [5, Proposition 7.5] and the skew-symmetry of B that $-B^T$ is a ***rescaling*** of B, meaning that there is a diagonal matrix Σ with positive entries on the diagonal such that $-B^T = \Sigma^{-1}B\Sigma$. Therefore, [5, Proposition 8.20] says that the i^{th} column of $G_t^{-B^T;t_0}$ is a positive scalar multiple of the i^{th} column of $\Sigma G_t^{B;t_0}$. (In the statement of [5, Proposition 8.20], Σ is multiplied on the right, because there $\mathbf{g}$-vectors are row vectors rather than column vectors.) Thus we have the following fact.

Lemma 1.3. *The k^{th} entries of vectors in $\mathrm{Cone}_t^{B;t_0}$ have the same sign as the k^{th} entries of vectors in $\mathrm{Cone}_t^{-B^T;t_0}$.*

For $k \in \{1,\ldots,n\}$, let J_k be the $n\times n$ matrix that agrees with the identity matrix except that J_k has -1 in position kk. For an $n\times n$ matrix M and $k\in\{1,\ldots,n\}$, let $M^{\bullet k}$ be the matrix that agrees with M in column k and has zeros everywhere outside of column k. Let $M^{k\bullet}$ be the matrix that agrees with M in row k and has zeros everywhere outside of row k.

Given a real number a, let $[a]_+$ denote $\max(a,0)$. Given a matrix $M=[m_{ij}]$, define $[M]_+$ to be the matrix whose ij-entry is $[m_{ij}]_+$. Given an exchange matrix B, an index $k\in\{1,\ldots,n\}$ and a sign $\varepsilon\in\{\pm1\}$, define matrices

$$\begin{aligned} E^B_{\varepsilon,k} &= J_k + [\varepsilon B]_+^{\bullet k} \\ F^B_{\varepsilon,k} &= J_k + [-\varepsilon B]_+^{k\bullet}. \end{aligned}$$

Each matrix $E^B_{\varepsilon,k}$ is its own inverse, and each $F^B_{\varepsilon,k}$ is its own inverse. The following is essentially a result of [3], although it is not stated there in this form. 1

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Lemma 1.4. *For $k\in\{1,\ldots,n\}$ and either choice of $\varepsilon\in\{\pm1\}$, the mutation of B at k is $\mu_k(B) = E^B_{\varepsilon,k}BF^B_{\varepsilon,k}$.*

Proof. We expand the product $(J_k + [\varepsilon B]_+^{\bullet k})B(J_k+[-\varepsilon B]_+^{k\bullet})$ to four terms. The term $[\varepsilon B]_+^{\bullet k}B[-\varepsilon B]_+^{k\bullet}$ is zero because $b_{kk}=0$. The term $[\varepsilon B]_+^{\bullet k}BJ_k$ is $[\varepsilon B]_+^{\bullet k}B^{k\bullet}J_k$, which equals $[\varepsilon B]_+^{\bullet k}B^{k\bullet}$. Similarly, the term $J_kB[-\varepsilon B]_+^{k\bullet}$ equals $B^{\bullet k}[-\varepsilon B]_+^{k\bullet}$ Both Thus the ij-entry of $E^B_{\varepsilon,k}BF^B_{\varepsilon,k}$ is

$$\begin{Bmatrix} -b_{ij} & \text{if } k\in\{i,j\} \\ b_{ij} & \text{otherwise} \end{Bmatrix} + \begin{Bmatrix} |b_{ik}|b_{kj} & \text{if } \operatorname{sgn} b_{ik}=\varepsilon \\ 0 & \text{otherwise} \end{Bmatrix} + \begin{Bmatrix} b_{ik}|b_{kj}| & \text{if } \operatorname{sgn} b_{kj}=-\varepsilon \\ 0 & \text{otherwise} \end{Bmatrix}.$$

This coincides with the ij-entry of $\mu_k(B)$. $\square$

Given a matrix M, write $M_{\mathrm{col}(i)}$ for the i^{th} column of M. We observe that $(MN)_{\mathrm{col}\,i} = M(N)_{\mathrm{col}\,i}$.

Lemma 1.5. *Suppose $B = [b_{ij}]$ is an exchange matrix, let $k \in \{1, \ldots, n\}$, and choose a sign $\varepsilon \in \{\pm 1\}$.*

1. $(E^B_{\varepsilon,k}B)_{\mathrm{col}\,i} = J_k(B)_{\mathrm{col}\,i} + b_{ki}([\varepsilon B]_+)_{\mathrm{col}\,k}$.
2. $(E^B_{\varepsilon,k}B)_{\mathrm{col}\,k} = (E^B_{-\varepsilon,k}B)_{\mathrm{col}\,k} = B_{\mathrm{col}\,k}$.
3. $(E^B_{-\varepsilon,k}B)_{\mathrm{col}\,i} = (E^B_{\varepsilon,k}B)_{\mathrm{col}\,i} - \varepsilon b_{ki} B_{\mathrm{col}\,k}$.

Proof. The first two assertions follow immediately from the fact that $(MN)_{\mathrm{col}\,i} = M(N)_{\mathrm{col}\,i}$ and the fact that $b_{kk} = 0$. The first assertion (for ε and $-\varepsilon$) implies that $(E^B_{-\varepsilon,k}B)_{\mathrm{col}\,i} = (E^B_{\varepsilon,k}B)_{\mathrm{col}\,i} - b_{ki}([\varepsilon B]_+ - [-\varepsilon B]_+)_{\mathrm{col}\,k}$. The third assertion follows. $\square$

We will also need the following simple fact about nonnegative linear spans. Given a set S of vectors, let $\underset{\mathbf{span}}{\mathbf{pos}}(S)$ denote the nonnegative linear span of S. For $k \in \{1, \ldots, n\}$ and $\varepsilon \in \{\pm 1\}$, let $S_{k,\varepsilon}$ be the set of vectors in S whose k^{th} entry has sign strictly agreeing with ε.

Lemma 1.6. *Suppose λ is a vector in $\mathbb{R}^n$ whose k^{th} λ_k has $\varepsilon\lambda_k \le 0$. Then*

$$\left\{\lambda + \underset{\mathbf{span}}{\mathbf{pos}}(S)\right\} \cap \{x \in \mathbb{R}^n : \varepsilon x_k \ge 0\}$$
$$= \left\{\lambda + \underset{\mathbf{span}}{\mathbf{pos}}(S)\right\} \cap \{x \in \mathbb{R}^n : x_k = 0\} + \underset{\mathbf{span}}{\mathbf{pos}}(S_{k,\varepsilon}).$$

Proof. The set on the right side is certainly contained in the set on the right side. If x is an element of the left side, then x is λ plus a nonzero element y of $\underset{\mathbf{span}}{\mathbf{pos}}(S_{k,\varepsilon})$ plus an element z of $\underset{\mathbf{span}}{\mathbf{pos}}(S \setminus S_{k,\varepsilon})$. Since the sign of $\varepsilon x \ge 0$ and $\varepsilon\lambda \le 0$, there exists t with $0 \le t \le 1$ such that $\lambda + ty + z$ has k^{th} entry 0. We see that $x = (\lambda + ty + z) + (1 - t)y$ is an element of the right side. $\square$

2. First main result

Let B_0 be an exchange matrix. For a sequence $\mathbf{k} = k_m \cdots k_1$ of indices, define seeds $t_1, \ldots, t_m = t$ by $t_0 \overset{k_1}{\text{———}} t_1 \overset{k_2}{\text{———}} \cdots \overset{k_m}{\text{———}} t_m = t$. We will prove the following theorem.

Theorem 2.1. *Suppose $\mathbf{k} = k_m \cdots k_1$ and $t_0 \overset{k_1}{\text{———}} t_1 \overset{k_2}{\text{———}} \cdots \overset{k_m}{\text{———}} t_m = t$. If $\mathbf{k}^{-1} = k_1 \cdots k_m$ is a red sequence for B_t, then for any λ in the domain of definition of $\eta^{B_0^T}_{\mathbf{k}}$ that contains $\mathrm{Cone}^{B_0;t_0}_t$,*

$$\mathcal{P}^{B_0}_{\lambda,\mathbf{k}} \subseteq \left\{\lambda + G^{B_0;t_0}_t B_t \alpha : \alpha \in \mathbb{R}^n, \alpha \ge 0\right\} = \left\{\lambda + B_0 C^{B_0;t_0}_t \alpha : \alpha \in \mathbb{R}^n, \alpha \ge 0\right\}.$$

Since $\left(\eta^{B_0^T}_{\mathbf{k}}\right)^{-1} = \eta^{B_t^T}_{\mathbf{k}^{-1}}$, we have $\mathcal{P}^{B_0}_{\lambda,\mathbf{k}} = \eta^{B_t^T}_{\mathbf{k}^{-1}}\left\{\eta^{B_0^T}_{\mathbf{k}}(\lambda) + B_t\alpha : \alpha \ge 0\right\}$. Let D be the domain of definition of $\eta^{B_0^T}_{\mathbf{k}}$ that contains $\mathrm{Cone}^{B_0;t_0}_t$. Then $\eta^{B_t^T}_{\mathbf{k}^{-1}}$ is linear on $\eta^{B_0^T}_{\mathbf{k}}(D)$. Let $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}$ be the linear map that agrees with $\eta^{B_t^T}_{\mathbf{k}^{-1}}$ on $\eta^{B_0^T}_{\mathbf{k}}(D)$.

Proposition 2.2. *The matrix for $\mathcal{L}^{B_t^T}_{\mathbf{k}^{-1}}$, acting on column vectors, is $G^{B_0;t_0}_t$.*

Proof. By [5, Proposition 8.13], $\mathrm{Cone}_t^{B_0;t_0} = \eta_{\mathbf{k}^{-1}}^{B_t^T}((\mathbb{R}_{\geq 0})^n)$, and therefore also $\eta_{\mathbf{k}}^{B_0^T}\left(\mathrm{Cone}_t^{B_0;t_0}\right) = (\mathbb{R}_{\geq 0})^n$. The proof of [5, Proposition 8.13] shows not only an equality of cones, but also that $\eta_{\mathbf{k}^{-1}}^{B_t^T}$ takes the extreme ray of $(\mathbb{R}_{\geq 0})^n$ spanned by e_i to the extreme ray of $\mathrm{Cone}_t^{B_0;t_0}$ spanned by the i^{th} $\mathbf{g}$-vector at t relative to $B_0;t_0$, where the total order on these $\mathbf{g}$-vectors at t is obtained from the order $e_1,\ldots,e_n$ on $\mathbf{g}$-vectors at t_0 by the sequence $\mathbf{k}$ of mutations. $\square$

We now apply a result of [3], namely that $G_t^{B_0;t_0}B_t = B_0C_t^{B_0;t_0}$. This fact follows from the proof of [3, Proposition 1.3], or from [1, (6.14)], as explained in [3, Remark 2.1]. Since $G_t^{B_0;t_0}$ is the matrix for $\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}$ and since $\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}\eta_{\mathbf{k}}^{B_0^T}(\lambda) = \lambda$, we have the following proposition.

Proposition 2.3.

$$\begin{aligned}\mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}\left\{\eta_{\mathbf{k}}^{B_0^T}(\lambda) + B_t\alpha : \alpha\in\mathbb{R}^n, \alpha\geq 0\right\} &= \left\{\lambda + G_t^{B_0;t_0}B_t\alpha : \alpha\in\mathbb{R}^n, \alpha\geq 0\right\}\\ &= \left\{\lambda + B_0C_t^{B_0;t_0}\alpha : \alpha\in\mathbb{R}^n, \alpha\geq 0\right\}.\end{aligned}$$

In light of Proposition 2.3, the conclusion of Theorem 2.1 is equivalent to

$$\mathcal{P}_{\lambda,\mathbf{k}}^{B_0} \subseteq \mathcal{L}_{\mathbf{k}^{-1}}^{B_t^T}\left\{\eta_{\mathbf{k}}^{B_0^T}(\lambda) + B_t\alpha : \alpha\geq 0\right\}.$$

Proof of Theorem 2.1. We will prove that $P_{\lambda,\mathbf{k}}^{B_0} \subseteq \left\{\lambda + B_0C_t^{B_0;t_0}\alpha : \alpha\geq 0\right\}$, by induction on m (the length of $\mathbf{k}$). The base case, where $\mathbf{k}=\emptyset$, is true because $C_{t_0}^{B_0;t_0}$ is the identity matrix and $\mathcal{P}_{\lambda,\emptyset} = \{\lambda + B_0\alpha : \alpha\geq 0\}$.

[3, Proposition 1.4] says that $C_t^{B_0;t_0} = F_{\varepsilon,k_1}^{B_1}C_t^{B_1;t_1}$, where ε is the sign of the k_1-column of $C_{t_1}^{-B_t;t}$. (The hypothesis that $\mathbf{k}^{-1}$ is a red sequence for B_t determines ε, but we leave ε unspecified for now in order to highlight later where this hypothesis is relevant.) By Lemma 1.4 and because $E_{\varepsilon,k_1}^{B_1}$ and $F_{\varepsilon,k_1}^{B_1}$ are their own inverses,

$$\begin{aligned}\left\{\lambda + B_0C_t^{B_0;t_0}\alpha : \alpha\geq 0\right\} &= \left\{\lambda + B_0F_{\varepsilon,k_1}^{B_1}C_t^{B_1;t_1}\alpha : \alpha\geq 0\right\}\\ &= \left\{\lambda + E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\alpha : \alpha\geq 0\right\}\\ &= E_{\varepsilon,k_1}^{B_1}\left\{E_{\varepsilon,k_1}^{B_1}\lambda + B_1C_t^{B_1;t_1}\alpha : \alpha\geq 0\right\}.\end{aligned} \tag{2.1}$$

The map $\eta_{\mathbf{k}}^{B_0^T}$ is linear on $\mathrm{Cone}_t^{B_0;t_0}$. This map is $\eta_{\mathbf{k}}^{B_0^T} = \eta_{k_m}^{B_{m-1}^T}\circ\cdots\circ\eta_{k_2}^{B_1^T}\circ\eta_{k_1}^{B_0^T}$. The map $\eta_{k_1}^{B_0^T}$ restricts to a linear map from $\mathrm{Cone}_t^{B_0;t_0}$ to $\mathrm{Cone}_t^{B_1;t_1}$. The inverse of $\eta_{k_1}^{B_0^T}$ is $\eta_{k_1}^{B_1^T}$. We claim that $E_{\varepsilon,k_1}^{B_1}$ is the matrix for the linear map on column vectors that agrees with $\eta_{k_1}^{B_1^T}$ on $\mathrm{Cone}_t^{B_1;t_1}$. Since $E_{\varepsilon,k_1}^{B_1}$ is its own inverse, the claim is equivalent to saying that implies that $E_{\varepsilon,k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_0^T}$ on $\mathrm{Cone}_t^{B_0;t_0}$.

By [3, (1.13)], ε is the sign of the k_1-column of $\left(G_t^{-B_1^T;t_1}\right)^T$. That is, ε is the sign of the k_1-row of $G_t^{-B_1^T;t_1}$, or in other words, the sign of the k_1-entry of vectors in $\mathrm{Cone}_t^{-B_1^T;t_1}$. By Lemma 1.3, ε is the sign of the k_1-entry of vectors in $\mathrm{Cone}_t^{B_1;t_1}$, which is the sign that determines how $\eta_{k_1}^{B_1^T}$ acts on $\mathrm{Cone}_t^{B_1;t_1}$. One easily checks that the action of $\eta_{k_1}^{B_1^T}$ on vectors whose k_1-entry has sign ε is precisely the action of $E_{\varepsilon,k_1}^{B_1}$.

Let $\lambda' = \eta_{k_1}^{B_0^T}(\lambda)$, so that λ' is in the same domain of definition of $\eta_{k_m\cdots k_2}^{B_1^T}$ as $\mathrm{Cone}_t^{B_1;t_1}$ and so that $\lambda' = E_{\varepsilon,k_1}^{B_1}\lambda$. By induction on m,

$$\eta_{k_2\cdots k_m}^{B_t^T}\Big\{\eta_{k_m\cdots k_2}^{B_1^T}(\lambda') + B_t\alpha : \alpha \geq 0\Big\} \subseteq \Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\}.$$

Applying the homeomorphism $\eta_{k_1}^{B_1^T}$ to both sides, we obtain

$$\eta_{\mathbf{k}^{-1}}^{B_t^T}\Big\{\eta_{\mathbf{k}}^{B_0^T}(\lambda') + B_t\alpha : \alpha \geq 0\Big\} \subseteq \eta_{k_1}^{B_1^T}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\}.$$

In light of (2.1), we can complete the proof by showing that

$$\eta_{k_1}^{B_1^T}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\} \subseteq E_{\varepsilon,k_1}^{B_1}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\}.$$

We have seen that $E_{\varepsilon,k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_1^T}$ on the set $\{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\}$. We can similarly check that $E_{-\varepsilon,k_1}^{B_1}$ is the linear map that agrees with $\eta_{k_1}^{B_1^T}$ on $\{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = -\varepsilon\}$. Thus $\eta_{k_1}^{B_1^T}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\}$ is

$$(U \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = -\varepsilon\}) \cup (V \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\}),$$

where

$$U = E_{\varepsilon,k_1}^{B_1}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\} = E_{\varepsilon,k_1}^{B_1}\lambda' + \underset{\mathbf{span}}{\mathbf{pos}}\Big\{\Big(E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}\Big\}_{i=1}^n$$
$$V = E_{-\varepsilon,k_1}^{B_1}\Big\{\lambda' + B_1C_t^{B_1;t_1}\alpha : \alpha \geq 0\Big\} = E_{-\varepsilon,k_1}^{B_1}\lambda' + \underset{\mathbf{span}}{\mathbf{pos}}\Big\{\Big(E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}\Big\}_{i=1}^n,$$

where $\underset{\mathbf{span}}{\mathbf{pos}}$ denotes the nonnegative linear span of a set of vectors.

We need to show that $V \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\} \subseteq U$. Since $\eta_{k_1}^{B_1^T}$ is a homeomorphism, $U \cap \{x \in \mathbb{R}^n : x_{k_1} = 0\} = V \cap \{x \in \mathbb{R}^n : x_{k_1} = 0\}$. By Lemma 1.6, any vector in $V \cap \{x \in \mathbb{R}^n : \operatorname{sgn} x_{k_1} = \varepsilon\}$ equals a vector in $V \cap \{x \in \mathbb{R}^n : x_{k_1} = 0\}$ plus a positive combination of vectors $\Big(E_{-\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$ whose k_1-entry has sign ε. Therefore, it suffices to show that every vector $\Big(E_{-\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$ whose k_1-entry has sign ε is in $\underset{\mathbf{span}}{\mathbf{pos}}\Big\{\Big(E_{\varepsilon,k_1}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}\Big\}_{i=1}^n$.

As a temporary shorthand, write b_{ij} for the entries of B_1 and write k for k_1. Suppose $v_i = \Big(E_{-\varepsilon,k}^{B_1}B_1C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$ for some i and suppose the k-entry of v_i has sign ε. Write M for $E_{-\varepsilon,k}^{B_1}B_1$ and write N for $E_{\varepsilon,k}^{B_1}B_1$. Lemma 1.5.1 implies that $M_{kj} = -b_{kj}$ for all j. Lemma 1.5.3 implies that if $\varepsilon M_{kj} \geq 0$, then $M_{\mathrm{col}\,j} = N_{\mathrm{col}\,j} + |b_{kj}|N_{\mathrm{col}\,k}$. Similarly, if $\varepsilon M_{kj} \leq 0$, then $M_{\mathrm{col}\,j} = N_{\mathrm{col}\,j} - |b_{kj}|N_{\mathrm{col}\,k}$.

Now $v_i = E_{-\varepsilon,k}^{B_1}B_1\Big(C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$, and $\Big(C_t^{B_1;t_1}\Big)_{\mathrm{col}\,i}$ has a sign $\delta \in \{\pm 1\}$, meaning that it is not zero and all of its nonzero entries have sign δ. (This is "sign-coherence of C-vectors". See Remark 1.2.) Thus there are nonnegative numbers γ_j such that $v_i = \delta\sum_{j=1}^n \gamma_j M_{\mathrm{col}\,j}$. Write $\{1,\ldots,n\} = S \cup T$ with $S \cup T = \emptyset$ such that $\varepsilon M_{kj} \geq 0$

for all $j \in S$ and $\varepsilon M_{kj} \leq 0$ for all $j \in T$. Then

$$\begin{aligned}
v_i &= \delta \sum_{j\in S} \gamma_j M_{\mathrm{col}\,j} + \delta \sum_{j\in T} \gamma_j M_{\mathrm{col}\,j} \\
&= \delta \sum_{j\in S} \gamma_j (N_{\mathrm{col}\,j} + |b_{kj}| N_{\mathrm{col}\,k}) + \delta \sum_{j\in T} \gamma_j (N_{\mathrm{col}\,j} - |b_{kj}| N_{\mathrm{col}\,k}) \\
&= \delta \sum_{j=1}^{n} \gamma_j N_{\mathrm{col}\,j} - \delta \sum_{j=1}^{n} \varepsilon \gamma_j b_{kj} N_{\mathrm{col}\,k} \\
&= N \left(C_t^{B_1;t_1} \right)_{\mathrm{col}\,i} + \delta \sum_{j=1}^{n} \varepsilon \gamma_j M_{kj} N_{\mathrm{col}\,k} \\
&= N \left(C_t^{B_1;t_1} \right)_{\mathrm{col}\,i} + \sigma N_{\mathrm{col}\,k}.
\end{aligned}$$

where $\sigma = \varepsilon\delta \sum_{j=1}^{n} \gamma_j M_{kj}$ is a positive scalar, because $\delta \sum_{j=1}^{n} \gamma_j M_{kj}$ is the k-entry of v_i, which has sign ε.

As noted above, ε is the sign of the k_1-entry of vectors in $\mathrm{Cone}_t^{-B_1^T;t_1}$. Since $\mathrm{Cone}_t^{-B_1^T;t_1} = \left\{ x \in \mathbb{R}^n : x^T C_t^{B_1;t_1} \geq 0 \right\}$, the rows of $\left(C_t^{B_1;t_1} \right)^{-1}$ span the extreme rays of $\mathrm{Cone}_t^{-B_1^T;t_1}$. In particular $\left(C_t^{B_1;t_1} \right)^{-1} (\varepsilon e_k)$ has nonnegative entries. Thus $C_t^{B_1;t_1} \left(C_t^{B_1;t_1} \right)^{-1} (\varepsilon e_k) = \varepsilon e_k$ is a nonnegative linear combination of columns of $C_t^{B_1;t_1}$.

Now, the hypothesis that $\mathbf{k}^{-1}$ is a red sequence for B_t, or equivalently a green sequence for $-B_t$, says that $\varepsilon = +1$, so that e_k is a nonnegative linear combination of columns of $C_t^{B_1;t_1}$. Thus $N_{\mathrm{col}\,k} = N e_k$ is a nonnegative linear combination of columns of $N C_t^{B_1;t_1}$. We have shown that $v_i = N \left(C_t^{B_1;t_1} \right)_{\mathrm{col}\,i} + \sigma N_{\mathrm{col}\,k}$ is a nonnegative linear combination of columns of $N C_t^{B_1;t_1}$. In other words, v_i is in $\underset{\mathbf{span}}{\mathbf{pos}} \left\{ \left(E_{\varepsilon,k_1}^{B_1} B_1 C_t^{B_1;t_1} \right)_{\mathrm{col}\,i} \right\}_{i=1}^{n}$, as desired. □

3. Extending to extended exchange matrices

We follow [1] in considering $m \times n$ extended exchange matrices $\tilde{B}$ that are "tall", in the sense that $m \geq n$. We will also consider $m \times m$ matrices related to $\tilde{B}$: Writing $\tilde{B}$ in block form $\left[\begin{smallmatrix} B \\ E \end{smallmatrix}\right]$, let $\mathbf{B}$ be the matrix with block form $\left[\begin{smallmatrix} B & -E^T \\ E & 0 \end{smallmatrix}\right]$. Most importantly, $\mathbf{B}$ is skew-symmetrizable and agrees with $\tilde{B}$ in columns 1 to n. Throughout, if we have defined an extended exchange matrix $\tilde{B}$, without comment we will take B to be the underlying exchange matrix and $\mathbf{B}$ to be the associated $m \times m$ matrix.

The matrix $\mathbf{B}$ defines mutation maps $\eta_{\mathbf{k}}^{\mathbf{B}^T}$ that act on $\mathbb{R}^m$ rather than $\mathbb{R}^n$, but without exception we will only consider mutations in positions $1, \ldots, n$. Also, given $\mathbf{B}$, a sequence $\mathbf{k} = k_m \cdots k_1$ of indices in $\{1, \ldots, n\}$, and seeds $t_1, \ldots, t_m$ by $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$, there are associated matrices of $\mathbf{g}$-vectors and C-vectors, which we write as $\mathbf{G}_t^{\mathbf{B};t_0}$ and $\mathbf{C}_t^{\mathbf{B};t_0}$. Since $\mathbf{k}$ only contains indices in

$\{1,\ldots,n\}$, these matrices have block forms

$$\mathbf{G}_t^{\mathbf{B};t_0} = \begin{bmatrix} G_t^{B;t_0} & 0 \\ H_t^{\tilde{B};t_0} & I_{m-n} \end{bmatrix} \quad \text{and} \quad \mathbf{C}_t^{\mathbf{B};t_0} = \begin{bmatrix} C_t^{B;t_0} & D_t^{\tilde{B};t_0} \\ 0 & I_{m-n} \end{bmatrix},$$

where $H_t^{\tilde{B};t_0}$ is an $(m-n)\times n$ matrix, $D_t^{\tilde{B};t_0}$ is an $n\times(m-n)$ matrix, and I_{m-n} is the identity matrix.

Given a vector $\lambda\in\mathbb{R}^m$, define $\mathcal{P}^{\tilde{B}}_{\lambda,\mathbf{k}} = \left(\eta_{\mathbf{k}}^{\mathbf{B}^T}\right)^{-1}\left\{\eta_{\mathbf{k}}^{\mathbf{B}^T}(\lambda)+\tilde{B}_t\alpha : \alpha\in\mathbb{R}^n, \alpha\geq 0\right\}$. Define the ***dominance region*** $\mathcal{P}^{\tilde{B}}_{\lambda}$ of λ with respect to $\tilde{B}$ to be the intersection $\bigcap_{\mathbf{k}}\mathcal{P}^{B}_{\lambda,\mathbf{k}}$ all sequences $\mathbf{k}$ of indices in $\{1,\ldots,n\}$.

Since $\mathbf{k}$ consists only of indices in $\{1,\ldots,n\}$, the domains of definition of $\eta_{\mathbf{k}}^{\mathbf{B}^T}$ are determined by the domains of definition of $\eta_{\mathbf{k}}^{B^T}$. Specifically, each domain of definition of $\eta_{\mathbf{k}}^{\mathbf{B}^T}$ is the set of vectors whose projection to $\mathbb{R}^n$ (ignoring the last $m-n$ entries) is a domain of definition of $\eta_{\mathbf{k}}^{B^T}$. Accordingly, we define $\mathrm{Cone}_t^{\tilde{B};t_0}$ to be the set of vectors in $\mathbb{R}^m$ whose projection to $\mathbb{R}^n$ is in $\mathrm{Cone}_t^{B;t_0}$. Since $\mathrm{Cone}_t^{B_m;t_m} = \eta_{\mathbf{k}}^{B^T}\left(\mathrm{Cone}_t^{B;t_0}\right)$ for every seed t, also $\mathrm{Cone}_t^{\tilde{B}_m;t_m} = \eta_{\mathbf{k}}^{\mathbf{B}^T}\left(\mathrm{Cone}_t^{\tilde{B};t_0}\right)$ for every seed t

To understand dominance regions $\mathcal{P}^{\tilde{B}}_{\lambda}$, it is enough to consider the case where λ has nonzero entries only in positions $1,\ldots,n$. Other dominance regions are obtained by translation, as explained in the following lemma. The lemma is an immediate consequence of the fact that domains of definition of $\eta_{\mathbf{k}}^{\mathbf{B}^T}$ depend only on the first n coordinates.

Lemma 3.1. *If λ and λ' are vectors in $\mathbb{R}^m$ that agree in the first n coordinates, then $\mathcal{P}^{\tilde{B}}_{\lambda'} = \mathcal{P}^{\tilde{B}}_{\lambda} - \lambda + \lambda'$.*

Lemma 1.1 immediately implies the following lemma.

Lemma 3.2. *If $\lambda' = \eta_{\mathbf{k}}^{\mathbf{B}^T}$ and $\tilde{B}' = \mu_{\mathbf{k}}(\tilde{B})$, then*

1. $\eta_{\mathbf{k}}^{\mathbf{B}^T}(\mathcal{P}^{\tilde{B}}_{\lambda}) = \mathcal{P}^{\tilde{B}'}_{\lambda'}$.
2. $\eta_{\mathbf{k}}^{\mathbf{B}^T}(\mathcal{P}^{\tilde{B}}_{\lambda,\boldsymbol{\ell}}) = \mathcal{P}^{\tilde{B}'}_{\lambda',\boldsymbol{\ell}\mathbf{k}^{-1}}$ *for any $\boldsymbol{\ell}$.*

We will prove the following extension of Theorem 2.1 and an important corollary.

Theorem 3.3. *Suppose $\mathbf{k} = k_m\cdots k_1$ is a sequence of indices in $\{1,\ldots,n\}$ and $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. If $\mathbf{k}^{-1} = k_1\cdots k_m$ is a red sequence for B_t, then for any λ in the domain of definition of $\eta_{\mathbf{k}}^{\mathbf{B}_0^T}$ that contains $\mathrm{Cone}_t^{B_0;t_0}$,*

$$\mathcal{P}^{\tilde{B}_0}_{\lambda,\mathbf{k}} \subseteq \left\{\lambda + \mathbf{G}_t^{\mathbf{B}_0;t_0}\tilde{B}_t\alpha : \alpha\in\mathbb{R}^n, \alpha\geq 0\right\} = \left\{\lambda + \tilde{B}_0 C_t^{B_0;t_0}\alpha : \alpha\in\mathbb{R}^n, \alpha\geq 0\right\}.$$

Proof. First, we notice that $\mathbf{k}^{-1} = k_1\cdots k_m$ is a red sequence for $\mathbf{B}_t$, or in other words, $\mathbf{k}$ is a green sequence for $-\mathbf{B}_t$. Indeed, since $\mathbf{C}_{t_{\ell-1}}^{-\mathbf{B};t_0} = \begin{bmatrix} C_{t_{\ell-1}}^{-\mathbf{B};t_0} & * \\ 0 & I_{m-n} \end{bmatrix}$, the sign of column k_ℓ of $\mathbf{C}_{t_{\ell-1}}^{-\mathbf{B};t_0}$ equals the sign of column k_ℓ of $C_{t_{\ell-1}}^{-\mathbf{B};t_0}$ whenever $1\leq \ell < k$. Thus Theorem 2.1 says that

$$\mathcal{P}^{\mathbf{B}_0}_{\lambda,\mathbf{k}} \subseteq \left\{\lambda + \mathbf{G}_t^{\mathbf{B}_0;t_0}\mathbf{B}_t\alpha : \alpha\in\mathbb{R}^m, \alpha\geq 0\right\} = \left\{\lambda + \mathbf{B}_0\mathbf{C}_t^{\mathbf{B}_0;t_0}\alpha : \alpha\in\mathbb{R}^m, \alpha\geq 0\right\}.$$

The assertion of Theorem 3.3 is that the same holds even when, in each term, the conditions $\alpha\in\mathbb{R}^m, \alpha\geq 0$ are strengthened by requiring that α is zero in coordinates $n+1,\ldots,m$.

Thus we run through the proof of Theorem 2.1 with $\mathbf{B}$ replacing B and m replacing n throughout and these additional conditions on α in all relevant expressions. There is no effect on the argument until the point of showing that $V \cap \{x \in \mathbb{R}^m : \operatorname{sgn} x_{k_1} = \varepsilon\} \subseteq U$. Here, we need to show that every vector $v_i = \left(E^{\mathbf{B}_1}_{-\varepsilon,k_1}\mathbf{B}_1\mathbf{C}_t^{\mathbf{B}_1;t_1}\right)_{\mathrm{col}\,i}$ with $i \in \{1,\ldots,n\}$ whose k_1-entry has sign ε is contained in $\underset{\mathbf{span}}{\mathbf{pos}}\left\{\left(E^{\mathbf{B}_1}_{\varepsilon,k_1}\mathbf{B}_1\mathbf{C}_t^{\mathbf{B}_1;t_1}\right)_{\mathrm{col}\,i}\right\}_{i=1}^n$. We argue as in the proof of Theorem 2.1 that $v_i = N\left(\mathbf{C}_t^{\mathbf{B}_1;t_1}\right)_{\mathrm{col}\,i} + \sigma N_{\mathrm{col}\,k}$ and that εe_k is a nonnegative linear combination of columns of $\mathbf{C}_t^{\mathbf{B}_1;t_1}$. Since $\mathbf{C}_t^{\mathbf{B};t_0} = \begin{bmatrix} C_t^{B;t_0} & * \\ 0 & I_{m-n}\end{bmatrix}$, we conclude that εe_k is a nonnegative linear combination of columns 1 through n of $\mathbf{C}_t^{\mathbf{B}_1;t_1}$. Thus v_i is a nonnegative linear combination of columns 1 through n of $N\mathbf{C}_t^{\mathbf{B}_1;t_1}$ as desired. □

Corollary 3.4. *Suppose $\tilde{B}_0$ is an extended exchange matrix with linearly independent columns. Suppose t is a seed in the exchange graph for $\tilde{B}_0; t_0$ and take $\lambda \in \mathrm{Cone}_t^{\tilde{B}_0;t_0}$. If there exists a maximal red sequence for B_t, then $\mathcal{P}_\lambda^{\tilde{B}_0} = \{\lambda\}$.*

Proof. Let t' be the seed at the end of the maximal red sequence for B_t. There exists $\boldsymbol{\ell} = \ell_q\ell_{q-1}\cdots\ell_1$ with $t_0 = t'_0 \xrightarrow{\ell_1} t'_1 \xrightarrow{\ell_2} \cdots \xrightarrow{\ell_q} t'_q = t'$. Let $\lambda' = \eta_{\boldsymbol{\ell}}^{\mathbf{B}_0^T}(\lambda)$. Lemma 3.2 says $\eta_{\boldsymbol{\ell}}^{\mathbf{B}_0^T}(\mathcal{P}_\lambda^{\tilde{B}_0}) = \mathcal{P}_{\lambda'}^{\tilde{B}_{t'}}$. Thus it is enough to prove that $\mathcal{P}_{\lambda'}^{\tilde{B}_{t'}} = \{\lambda'\}$. Since $\eta_{\boldsymbol{\ell}}^{\mathbf{B}_0^T}\left(\mathrm{Cone}_t^{\tilde{B}_0;t_0}\right) = \mathrm{Cone}_t^{\tilde{B}_{t'};t'}$, we have reduced the proof to the case where there is a maximal red sequence for B_t starting from t and ending at t_0.

Working in that reduction, let $\mathbf{k} = k_m\cdots k_1$ be the reverse of the maximal red sequence and define seeds $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$. Then Theorem 3.3 says that $\mathcal{P}_{\lambda,\mathbf{k}}^{\tilde{B}_0} \subseteq \left\{\lambda + \tilde{B}_0 C_t^{B_0;t_0}\alpha : \alpha \in \mathbb{R}^n, \alpha \geq 0\right\}$.

Since $\mathbf{k}^{-1}$ is a maximal red sequence for B_t, or in other words a maximal green sequence for $-B_t$, every column of $C_{t_0}^{-B_t;t}$ has negative sign, so $\mathrm{Cone}_{t_0}^{B_t^T;t} = \left\{x \in \mathbb{R}^n : x^T C_{t_0}^{-B_t;t} \geq 0\right\}$ consists of vectors with nonpositive entries. Since $(\mathbb{R}_{\leq 0})^n$ is a cone in the mutation fan $\mathcal{F}_{-B_t}$ (for example, combining [5, Proposition 7.1], [5, Proposition 8.9], and sign-coherence of C-vectors) and also $\mathrm{Cone}_{t_0}^{B_t^T;t}$ is a cone in $\mathcal{F}_{-B_t}$, we see that $\mathrm{Cone}_{t_0}^{B_t^T;t} = (\mathbb{R}_{\leq 0})^n$. Thus, up to permuting columns, $C_{t_0}^{-B_t;t}$ is the negative of the identity matrix. We see that $\mathcal{P}_{\lambda,\mathbf{k}}^{\tilde{B}_0} \subseteq \left\{\lambda - \tilde{B}_0\alpha : \alpha \in \mathbb{R}^n, \alpha \geq 0\right\}$.

Since also $\mathcal{P}_{\lambda,\emptyset}^{\tilde{B}_0}\left\{\lambda + \tilde{B}_0\alpha : \alpha \geq 0\right\}$, and since the columns of $\tilde{B}_0$ are linearly independent, we conclude that $\mathcal{P}_\lambda^{\tilde{B}_0} = \{\lambda\}$. □

4. Affine type

Let B_0 be acyclic of affine type, indexed so that entries above the diagonal are nonnegative. Take λ in the imaginary cone. Let t be any seed such that $\mathrm{Cone}_t^{B_0;t_0}$ has $n-2$ rays on the boundary of the imaginary wall $\mathfrak{d}_\infty$ such that λ is in the imaginary cone spanned by those $n-2$ rays and the imaginary ray. Let $\mathbf{k} = k_m\cdots k_1$ be a sequence such that $t_0 \xrightarrow{k_1} t_1 \xrightarrow{k_2} \cdots \xrightarrow{k_m} t_m = t$.

Let $\tilde{B}_0$ be an extension of B_0 that has linearly independent columns. Computations show that 2

2. Probably need more specific notation about $\mathfrak{d}_\infty$. N

$$\mathcal{P}^{\tilde{B}_0}_{\lambda,\mathbf{k}} \cap \mathcal{P}^{\tilde{B}_0}_{\lambda,\mathbf{k}n(n-1)\cdots 1} \cap \mathfrak{d}_\infty = \Big\{\lambda + x\tilde{B}_0\delta : x \in \mathbb{R}\Big\} \cap \mathfrak{d}_\infty.$$

Let u be the seed reached from t_0 by the sequence $n(n-1)\cdots 1$ and let λ_u be $\eta^{\mathbf{B}_0^T}_{n(n-1)\cdots 1}(\lambda)$. Lemma 3.2 says that $\eta^{\mathbf{B}_0^T}_{n(n-1)\cdots 1}(\mathcal{P}^{\tilde{B}_0}_{\lambda,\mathbf{k}}) = \mathcal{P}^{\tilde{B}_u}_{\lambda_u,\mathbf{k}12\cdots n}$ and

$$\eta^{\mathbf{B}_0^T}_{n(n-1)\cdots 1}(\mathcal{P}^{\tilde{B}_0}_{\lambda,\mathbf{k}n(n-1)\cdots 1}) = \mathcal{P}^{\tilde{B}_u}_{\lambda_u,\mathbf{k}n\cdots 11\cdots n} = \mathcal{P}^{\tilde{B}_u}_{\lambda_u,\mathbf{k}}.$$

The map $\eta^{\mathbf{B}_0^T}_{n(n-1)\cdots 1}$ is linear on $\mathfrak{d}_\infty$ and maps $-\tilde{B}_0\delta$ to $-\tilde{B}_u\delta$, so it maps the set $\Big\{\lambda + x\tilde{B}_0\delta : x \in \mathbb{R}\Big\} \cap \mathfrak{d}_\infty$ to $\Big\{\lambda_u + x\tilde{B}_u\delta : x \in \mathbb{R}\Big\} \cap \mathfrak{d}_\infty$. Thus we will show that

$$\mathcal{P}^{\tilde{B}_u}_{\lambda_u,\mathbf{k}} \cap \mathcal{P}^{\tilde{B}_u}_{\lambda_u,\mathbf{k}12\cdots n} \cap \mathfrak{d}_\infty = \Big\{\lambda_u + x\tilde{B}_u\delta : x \in \mathbb{R}\Big\} \cap \mathfrak{d}_\infty.$$

Let t' be the seed reached from t_0 by the sequence $\mathbf{k}n(n-1)\cdots 1$ or in other words, the seed reached from u by the sequence $\mathbf{k}$. Leaving out repetitions of "$\alpha \geq 0$" for reasons of space,

$$\begin{aligned}
&\mathcal{P}^{\tilde{B}_u}_{\lambda_u,\mathbf{k}} \cap \mathcal{P}^{\tilde{B}_u}_{\lambda_u,\mathbf{k}12\cdots n} \cap \mathfrak{d}_\infty \\
&\qquad = \Big(\eta^{\mathbf{B}_u^T}_{\mathbf{k}}\Big)^{-1}\Big\{\eta^{\mathbf{B}_u^T}_{\mathbf{k}}(\lambda_u) + \tilde{B}_{t'}\alpha\Big\} \cap \Big(\eta^{\mathbf{B}_u^T}_{\mathbf{k}1\cdots n}\Big)^{-1}\Big\{\eta^{\mathbf{B}_u^T}_{\mathbf{k}1\cdots n}(\lambda) + \tilde{B}_t\alpha\Big\} \cap \mathfrak{d}_\infty \\
&\qquad = \eta^{\mathbf{B}_{t'}^T}_{\mathbf{k}^{-1}}\Big\{\eta^{\mathbf{B}_u^T}_{\mathbf{k}}(\lambda_u) + \tilde{B}_{t'}\alpha\Big\} \cap \eta^{\mathbf{B}_t^T}_{n\cdots 1\mathbf{k}^{-1}}\Big\{\eta^{\mathbf{B}_u^T}_{\mathbf{k}1\cdots n}(\lambda_u) + \tilde{B}_t\alpha\Big\} \cap \mathfrak{d}_\infty \\
&\qquad = \eta^{\mathbf{B}_{t'}^T}_{\mathbf{k}^{-1}}\Big\{\eta^{\mathbf{B}_u^T}_{\mathbf{k}}(\lambda_u) + \tilde{B}_{t'}\alpha\Big\} \cap \eta^{\mathbf{B}_0^T}_{n\cdots 1}\Big(\eta^{\mathbf{B}_t^T}_{\mathbf{k}^{-1}}\Big\{\eta^{\mathbf{B}_u^T}_{\mathbf{k}1\cdots n}(\lambda_u) + \tilde{B}_t\alpha\Big\}\Big) \cap \mathfrak{d}_\infty
\end{aligned}$$

3. If we use this, we need to put the $\mathbf{G}BC$ thing in the background. N

Now, writing $\tilde{B}_u = \left[\begin{smallmatrix} B_u \\ E_u \end{smallmatrix}\right]$, we have 3

$$\begin{aligned}
\tilde{B}_t = \mu_{\mathbf{k}}(\mu_{1\cdots n}(\tilde{B}_u)) = \mu_{\mathbf{k}}(\tilde{B}_0) &= \mu_{\mathbf{k}}\left((\mathbf{G}^{B_u;u}_{t_0})^{-1}\tilde{B}_u C^{B_u;u}_{t_0}\right) \\
&= \mu_{\mathbf{k}}\left(\begin{bmatrix} G^{B_u;i}_{t_0} & 0 \\ H^{\tilde{B}_u;u}_{t_0} & I_{m-n} \end{bmatrix}^{-1}\left[\begin{smallmatrix} B_u \\ E_u \end{smallmatrix}\right] C^{B_u;u}_{t_0}\right) \\
&= \mu_{\mathbf{k}}\left(\begin{bmatrix} (G^{B_u;u}_{t_0})^{-1} & 0 \\ -H^{\tilde{B}_u;u}_{t_0}(G^{B_u;u}_{t_0})^{-1} & I_{m-n} \end{bmatrix}\left[\begin{smallmatrix} B_u \\ E_u \end{smallmatrix}\right] C^{B_u;u}_{t_0}\right) \\
&= \mu_{\mathbf{k}}\left(\begin{bmatrix} B_0 \\ -H^{\tilde{B}_u;u}_{t_0}B_0 + E_u C^{B_u;u}_{t_0} \end{bmatrix}\right)
\end{aligned}$$

We compute that $B_0 = B_u$ and that $C^{B_u;u}_{t_0} = -I_n$. So $\tilde{B}_t = \mu_{\mathbf{k}}\left(\begin{bmatrix} B_u \\ -H^{\tilde{B}_u;u}_{t_0}B_u - E_u \end{bmatrix}\right)$. On the other hand, $\tilde{B}_{t'} = \mu_{\mathbf{k}}(\tilde{B}_u) = \mu_{\mathbf{k}}\left(\left[\begin{smallmatrix} B_u \\ E_u \end{smallmatrix}\right]\right)$.

KEY POINT: On $\mathfrak{d}_\infty$, the map $\eta^{\mathbf{B}_0^T}_{n\cdots 1}$ is linear, and agrees with c or c^{-1} or something. So there is just a chance that we know something.

4.1. **Other ideas.** Let $\lambda_0 = \eta^{\mathbf{B}_0^T}_{\mathbf{k}}$. Lemma 3.2 says that

$$\eta^{\mathbf{B}_0^T}_{\mathbf{k}}(\mathcal{P}^{\tilde{B}_0}_{\lambda,\mathbf{k}n(n-1)\cdots 1}) = \mathcal{P}^{\tilde{B}_t}_{\lambda_0,\mathbf{k}n(n-1)\cdots 1\mathbf{k}^{-1}} = \Big(\eta^{\mathbf{B}_t}_{\mathbf{k}n\cdots 1\mathbf{k}^{-1}}\Big)^{-1}\Big\{\eta^{\mathbf{B}_t}_{\mathbf{k}n\cdots 1\mathbf{k}^{-1}}(\lambda_0) + \tilde{B}_{t'}\alpha : \alpha \geq 0\Big\}.$$

Also,

$$\eta^{\mathbf{B}_0^T}_{\mathbf{k}}(\mathcal{P}^{\tilde{B}_0}_{\lambda,\mathbf{k}}) = \mathcal{P}^{\tilde{B}_t}_{\lambda_0,\mathbf{k}\mathbf{k}^{-1}} = \mathcal{P}^{\tilde{B}_t}_{\lambda_0,\emptyset} = \Big\{\lambda_0 + \tilde{B}_t\alpha : \alpha \geq 0\Big\}.$$

We believe we could prove that $\mathcal{P}_{\lambda,\mathbf{k}}$ only depends on the seed that $\mathbf{k}$ leads to, not the specific $\mathbf{k}$. So does it make sense to consider the sequence (maximal red or maximal green) that connects t and t'.

5. An outline of a plan

We have a characterization of the exchange matrices of ***neighboring seeds*** (seeds that have $n-2$ **g**-vectors in the imaginary wall. I *think* we know that this characterization is "if and only if", i.e. an exchange matrix in an affine exchange pattern has that form if and only if it belongs to a neighboring seed. So we'll call these exchange matrices ***neighboring exchange matrices***.

1. To each neighboring exchange matrix, we need to associate an exchange matrix of rank $n-2$ and show that it is a product of finite type-B exchange matrices of the same sizes as the type-A blocks in the neighboring exchange matrix. So we'll call it the ***type-B companion***.

2. We need to show that the intersection of the nonnegative span of B intersected with the appropriate $\delta^\perp$ is the nonnegative span of $-\delta$ and the columns of the type-B companion. (OK, not really δ, but I mean the hyperplane containing the imaginary wall and the direction of the imaginary ray. This is terrible notation but you know what I mean.)

3. Call the indices of the last column/row of each B_{ii} for $i = 1, 2, 3$ (if they exist) the ***special indices*** and call the two indices of B_{44} the ***affine indices***. (But really, we won't assume that we re-indexed everything to separate the blocks and put the affine indices last. Even so, we can identify special and affine indices.) We need to show that mutating B in non-special indices commutes with mutating the type-B companion in the corresponding indices. (We think of the type-B companion as being indexed by the non-affine indices of B.) I think this should just be straightforward.

4. We then need to show that for each special index i, there is a sequence of mutations of B in i and the affine indices that yields a neighboring seed, and that the type-B companions of these two neighboring seeds are related by mutation at i. We probably need to show this case-by-case in the sense of the table in affine_dominance, but this seems actually doable and possibly not very long. (Exhibiting one sequence and checking will do.) I think that this will change which of the three is special and which are affine.

5. Finally, we need to check that the mutation maps for all of these mutations (in non-affine indices), as they act on the imaginary wall, are the same as the mutation maps for the type-B companion. That is, the mutation maps for B send the imaginary ray to the new imaginary ray and act on the boundary of the imaginary wall just as the mutation maps for the type-B companion act on an $(n-2)$-dimensional plane. Again, we should be able to do this by checking case-by-case in the table.

If we have all that, then I think we're done. Because we would know that the type-B companion has mutation sequences that reduce its dominance regions to points, and the corresponding mutation sequences for B would reduce dominance regions in the imaginary wall to the appropriate line segment.

Idea what the type-B companion might be: Simply scale the special columns by 2 and delete the affine columns and rows. (Or maybe it should be the rows? Or maybe either would work? Or maybe it depends on the type in the table?) Maybe it can't be that simple, but I kinda think it has to be.

Somehow Item 5 is really just the same as Item 4. In either case, the point is to see what happens to a column that is not i and not affine.

References

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