```python
from scipy import *
```

Round-Robin Problem

Problem Description

设有n个运动员要进行网球循环赛。设计 一个满足下列条件的比赛日程表：

- 每个选手必须与其他 $n-1$ 个选手各赛一次；
- 每个选手一天只能赛一次；
- 当 n 是偶数时，循环赛进行 $n-1$ 天。
- 当 n 是奇数时，循环赛进行 n 天。

Problem Analysis

我们在课堂上讨论过这个问题的一个特殊的情况：$n=2^k$。

这个特殊的情况保证我们每次都能将选手等分，这样一来我们就可以利用**分治**的策略解决问题：每次把参赛选手平均分成两组，先递归地让他们进行内部的循环赛（并且这两组的情况完全对称）；内部赛结束后，两组选手再与彼方小组的成员比赛。这时候两组人数相同，只需依次进行比赛即可。如下图所示。

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
3	4	1	2	7	8	5	6
4	3	2	1	8	7	6	5
5	6	7	8	1	2	3	4
6	5	8	7	2	1	4	3
7	8	5	6	3	4	1	2
8	7	6	5	4	3	2	1

对于一般的 n, 我们并不能保证每次都能**等分**。事实上，只要 n 不是 2 的幂次，在划分过程中一定会出现奇数的情况。为了模拟上面的情况，在出现奇数时，我们采用**引入一个虚拟选手**的做法，使选手人数重新变为偶数，从而可以等分。

对于一个规模为 n 的问题，安排的日程表中出现虚拟选手意味着与他交手的选手在当轮轮空。而当我们要从规模为 $\lfloor \frac{n}{2} \rfloor$ 的问题得到 n 个选手的循环赛日程表时，由于两组的对称性，**虚拟选手出现的位置将完全对称**，那么我们可以给同时轮空的两个选手安排一场比赛。之后再使两组的选手依次与对组选手交手。这样我们实现了比赛时间的最小化。

Data structure

我们考虑用**一个矩阵**来存储这个日程表。其中每一行代表一个选手，每一列代表一天，矩阵中的元素代表对手。 例如，矩阵第 i 行,第 j 列上的元素，就是选手 i 在第 j 天比赛的对手。

	day 1	**day 2**	**day 3**	**day 4**	**day 5**
1					
2					
3					
4					
5					
6					

Algorithm example

这里我们以6个选手为例，详述我们的算法。我们将仅描述从 3 个选手的子问题合并到 6 个选手的原问题的步骤。首先我们对 3 个选手的情况求解，结果里出现了虚拟选手； $1-3$ 号与 $4-6$ 号情况完全一致。如下图：

$$
\begin{array}{c|cccccc}
 & \text{Day 1} & \text{Day 2} & \text{Day 3} & \text{Day 4} & \text{Day 5} & \text{Day 6} \\
\hline
1 & 2 & 3 & - & & & \\
2 & 1 & - & 3 & & & \\
3 & - & 1 & 2 & & & \\
4 & 5 & 6 & - & & & \\
5 & 4 & - & 6 & & & \\
6 & - & 4 & 5 & & &
\end{array}
$$

由于对称性，我们注意到 i 号和 $i+3$ 号总是同时轮空，因此当他们轮空时，我们可以给他们安排比赛。如下图：

	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6
1	2	3	4			
2	1	5	3			
3	6	1	2			
4	5	6	1			
5	4	2	6			
6	3	4	5			

接下来要进行的是组间的比赛。对于第一个组中的 i，由于 i 和 $i+3$ 可能已经比赛过，因此，第 i 个选手的第 4 天的对手从 $i+4$ 开始，依次增加，如果超过了 6, 我们再填上 $i+3$。每行皆如此。如下图：

	Day 1	Day 2	Day 3	Day 4	Day 5	Day 6
1	2	3	4	5		
2	1	5	3	6		
3	6	1	2	4		
4	5	6	1	3		
5	4	2	6	1		
6	3	4	5	2		

这样，我们通过这个例子完成了对算法的描述。

Implementation

```
def roundRobin(n):
    '''
    Scheduling for the round robin tournament.

    Parameters:
    n -- the number of players. Could be any positive integers.

    Returns:

    schedule -- the schedule for the round robin tournament.
    Each row represent the opponents of a player, listed in time order.
    When n is even, schedule should be an n*(n-1) matrix;
    When it is odd, the size should be (n+1)*n with a virtual player.
    The virtual player's number exceeds n.
    '''
    if n & 1:
        n += 1
```

```
    # base case
    if n == 2:
        return array([[2],[1]]) # 2>n, so it represent a virtual player




    # n is now an even number, the tournament need n-1 days to complete
    schedule = zeros((n, n-1))

    # divide
    half = n // 2
    half_case = roundRobin(half)

    # conquer
    # print('half_case shape', half_case.shape)
    # first list the inner schedule into the total
    for i in range(half):
        schedule[i, 0: half_case.shape[1]] = half_case[i,:]
        #print(half_case.shape)
        assert(i + half < n)
        schedule[i + half, 0:half_case.shape[1]] = half_case[i, :] + half

    # eliminate the needless virtual players
    for i in range(half):
        bye = schedule[i,:] > half  #originally draw a bye

        # due to the symmetry of i and i + half
        schedule[i, bye] = i + half + 1
        schedule[i + half, bye] = i + 1


    # games between groups
    for i in range(half):
        x = i + half + 2
        if x > n:
            x -= half

        for j in range(half_case.shape[1], n - 1):
            schedule[i, j] = x
            schedule[x - 1, j] = i + 1
            x += 1
            if x > n:
                x -= half

    return schedule
```

如果 n 为偶数，上述函数返回的就是最终的结果矩阵；如果 n 为奇数，这个函数返回的矩阵的前 n 行（去掉虚拟选手的赛程）是我们结果矩阵，且如果矩阵中出现了大于 n 的数字，表明这一轮该选手轮空。为了方便观察，我们写如下函数：

```
def show_mat(mat, n):
```

```
    '''
    show the schedule matrix.

    Parameters:
    mat -- the result matrix from the function roundRobin.
    n -- the number of players.

    returns:
    None.
    We simply print the result in a beautified way.
    '''


    for i in range(n):
        x = list(array(mat[i,:], dtype = int))
        virtual = [i for i in range(len(x)) if x[i] > n]

        if len(virtual) > 0:
            x[squeeze(virtual)] = None # None implies no match that day

        print('The schedule for player', i + 1, 'is:', x)
```

简单看一下结果，我们挑选了 $n = 4k, 4k+1, 4k+2, 4k+3$ 的多种样例：

```
#case 1
show_mat(roundRobin(1),1)
print('--------------------------------------------------')
#case 2
show_mat(roundRobin(3),3)
print('--------------------------------------------------')
#case 3
show_mat(roundRobin(8), 8)
print('--------------------------------------------------')
#case 4
show_mat(roundRobin(15),15)
print('--------------------------------------------------')
#case 5
show_mat(roundRobin(14),14)
```

```
The schedule for player 1 is: [None]
--------------------------------------------------
The schedule for player 1 is: [2, None, 3]
The schedule for player 2 is: [1, 3, None]
The schedule for player 3 is: [None, 2, 1]
--------------------------------------------------
The schedule for player 1 is: [2, 4, 3, 6, 7, 8, 5]
The schedule for player 2 is: [1, 3, 4, 7, 8, 5, 6]
The schedule for player 3 is: [4, 2, 1, 8, 5, 6, 7]
The schedule for player 4 is: [3, 1, 2, 5, 6, 7, 8]
The schedule for player 5 is: [6, 8, 7, 4, 3, 2, 1]
The schedule for player 6 is: [5, 7, 8, 1, 4, 3, 2]

The schedule for player 7 is: [8, 6, 5, 2, 1, 4, 3]
```

```
The schedule for player 8 is: [7, 5, 6, 3, 2, 1, 4]
-------------------------------------------------
The schedule for player 1 is: [2, 4, 3, 6, 7, 8, 5, 10, 11, 12, 13, 14, 15, None, 9]
The schedule for player 2 is: [1, 3, 4, 7, 8, 5, 6, 11, 12, 13, 14, 15, None, 9, 10]
The schedule for player 3 is: [4, 2, 1, 8, 5, 6, 7, 12, 13, 14, 15, None, 9, 10, 11]
The schedule for player 4 is: [3, 1, 2, 5, 6, 7, 8, 13, 14, 15, None, 9, 10, 11, 12]
The schedule for player 5 is: [6, 8, 7, 4, 3, 2, 1, 14, 15, None, 9, 10, 11, 12, 13]
The schedule for player 6 is: [5, 7, 8, 1, 4, 3, 2, 15, None, 9, 10, 11, 12, 13, 14]
The schedule for player 7 is: [8, 6, 5, 2, 1, 4, 3, None, 9, 10, 11, 12, 13, 14, 15]
The schedule for player 8 is: [7, 5, 6, 3, 2, 1, 4, 9, 10, 11, 12, 13, 14, 15, None]
The schedule for player 9 is: [10, 12, 11, 14, 15, None, 13, 8, 7, 6, 5, 4, 3, 2, 1]
The schedule for player 10 is: [9, 11, 12, 15, None, 13, 14, 1, 8, 7, 6, 5, 4, 3, 2]
The schedule for player 11 is: [12, 10, 9, None, 13, 14, 15, 2, 1, 8, 7, 6, 5, 4, 3]
The schedule for player 12 is: [11, 9, 10, 13, 14, 15, None, 3, 2, 1, 8, 7, 6, 5, 4]
The schedule for player 13 is: [14, None, 15, 12, 11, 10, 9, 4, 3, 2, 1, 8, 7, 6, 5]
The schedule for player 14 is: [13, 15, None, 9, 12, 11, 10, 5, 4, 3, 2, 1, 8, 7, 6]
The schedule for player 15 is: [None, 14, 13, 10, 9, 12, 11, 6, 5, 4, 3, 2, 1, 8, 7]
-------------------------------------------------
The schedule for player 1 is: [2, 4, 3, 6, 7, 8, 5, 9, 10, 11, 12, 13, 14]
The schedule for player 2 is: [1, 3, 4, 7, 9, 5, 6, 10, 11, 12, 13, 14, 8]
The schedule for player 3 is: [4, 2, 1, 10, 5, 6, 7, 11, 12, 13, 14, 8, 9]
The schedule for player 4 is: [3, 1, 2, 5, 6, 7, 11, 12, 13, 14, 8, 9, 10]
The schedule for player 5 is: [6, 12, 7, 4, 3, 2, 1, 13, 14, 8, 9, 10, 11]
The schedule for player 6 is: [5, 7, 13, 1, 4, 3, 2, 14, 8, 9, 10, 11, 12]
The schedule for player 7 is: [14, 6, 5, 2, 1, 4, 3, 8, 9, 10, 11, 12, 13]
The schedule for player 8 is: [9, 11, 10, 13, 14, 1, 12, 7, 6, 5, 4, 3, 2]
The schedule for player 9 is: [8, 10, 11, 14, 2, 12, 13, 1, 7, 6, 5, 4, 3]
The schedule for player 10 is: [11, 9, 8, 3, 12, 13, 14, 2, 1, 7, 6, 5, 4]
The schedule for player 11 is: [10, 8, 9, 12, 13, 14, 4, 3, 2, 1, 7, 6, 5]
The schedule for player 12 is: [13, 5, 14, 11, 10, 9, 8, 4, 3, 2, 1, 7, 6]
The schedule for player 13 is: [12, 14, 6, 8, 11, 10, 9, 5, 4, 3, 2, 1, 7]
The schedule for player 14 is: [7, 13, 12, 9, 8, 11, 10, 6, 5, 4, 3, 2, 1]
```

Correctness checking

我们需要验证算法的正确性。 怎样的结果是正确的：

- 每一行是 $[1, n]$ 的不重复数字，且应包含除行数以外的所有数字。
- 每一列除了轮空时有一个数字大于 n, 其他应是 $[1, n]$ 上的不重复数字，并且应该全部出现。
- 对所有的 i, 如果

```
schedule[i,j] = k
```

就有

```
schedule[k-1,j] = i+1
```

```
def check(mat, n):
    '''
    To check the correctness of our algorithm

```

```
    Parameters:

    mat -- the matrix gained from the function roundRobin.

    n -- the parameter for roundRobin. Actually it is needless, since we can
    determine its value through mat's shape, but with it it is more convenient.

    Returns:

    if return True, it means our algorithm works correctly. Else it is incorrect.
    '''
    # check the properties about rows
    for i in range(n):
        b = in1d(list(range(1, n+1)), mat[i,:]) # check whether the numbers from 1 to n is in
the i-th row of mat


        assert(squeeze(where(b == False)).shape == ()) # to assure there is only 1 number
missing i.e. the row number
        if squeeze(where(b == False)) != i:  # to check if the missing number is the row number
            print(squeeze(where(b == False)))
            return False

        if i == n-1 and n&1 == 0:
            pass # if n is even there is only n-1 columns
        else:
            c = in1d(list(range(1, n+1)), mat[:,i]) # check whether the numbers from 1 to n is
in the i-th column of mat
            if squeeze(where(c == False)).shape[0] != 0:
                return False

            # check whether there at most 1 larger than n
            d = in1d(mat[:,i], list(range(1,n+1)))
            skip = squeeze(where(d == False))
            assert(skip.shape == () or skip.shape == (0,))
            if skip.shape == (0,) and mat[skip, i] <= n:
                return False

        # check whether schedule[i,j] = k implies schedule[k-1,h] = i+1
        for j in range(mat.shape[1]):
            k = int(mat[i,j])
            if mat[k-1,j] != i+1:
                print(i)
                print(mat[k,j])
                return False



    # check the properties about columns


    return True
```

由于循环赛参与者不太可能太多，我们仅检验 n 在 $1-200$ 范围内算法的正确性：

```
result = True
for i in range(1,201):
    result = result and check(roundRobin(i),i)

print(result)
```

```
True
```

这样，我们认为我们的算法是正确的。

Complexity analysis

根据上面的讨论，我们可以得出 $T(n) = T(\frac{n}{2}) + O(n^2)$

根据主定理，我们知道 $T(n) = \Theta(n^2)$ 下面我们通过测对不同大小的情况程序的运行时间来进行测试。

```
import time

def comp_test():
    runtime = []
    for i in range(1,1001):
        tic = time.time()
        roundRobin(i)
        toc = time.time()
        runtime.append(toc - tic)
        if i % 50 == 0:
            print('current:', i)
    return runtime
```

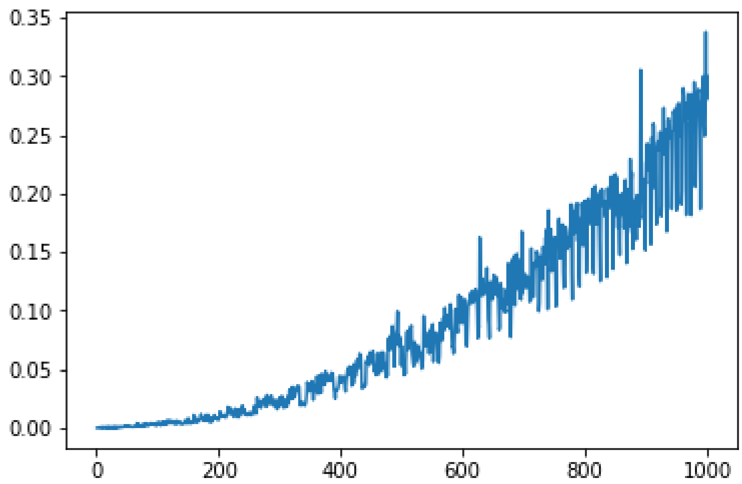
```
runtime = comp_test()
```

```
current: 50
current: 100
current: 150
current: 200
current: 250
current: 300
current: 350
current: 400
current: 450
current: 500
current: 550
current: 600
current: 650
current: 700
current: 750
current: 800

current: 850
```

```
current: 900
current: 950
current: 1000
```

```
from matplotlib.pyplot import *
% matplotlib inline

plot(list(range(1,1001)), runtime)
```

```
[<matplotlib.lines.Line2D at 0x2ab08a85c50>]
```



从上面的图像我们发现代码的运行时间大致与 n 的大小成二次方关系，这验证了我们刚才的分析。