a space X is Hausdorff if and only if $\Delta(X)$ is closed

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Theorem. A space X is Hausdorff if and only if

$$\{(x,x) \in X \times X \mid x \in X\}$$

is closed in $X \times X$ under the product topology.

Proof. First, some preliminaries: Recall that the diagonal map $\Delta\colon X \to X \times X$ is defined as $x \xmapsto{\Delta} (x,x)$. Also recall that in a topology generated by a basis (like the product topology), a set Y is open if and only if, for every point $y \in Y$, there's a basis element B with $y \in B \subset Y$. Basis elements for $X \times X$ have the form $U \times V$ where U, V are open sets in X.

Now, suppose that X is Hausdorff. We'd like to show its image under Δ is closed. We can do that by showing that its complement $\Delta(X)^c$ is open. $\Delta(X)$ consists of points with equal coordinates, so $\Delta(X)^c$ consists of points (x,y) with x and y distinct.

For any $(x,y) \in \Delta(X)^c$, the Hausdorff condition gives us disjoint open $U, V \subset X$ with $x \in U, y \in V$. Then $U \times V$ is a basis element containing (x,y). U and V have no points in common, so $U \times V$ contains nothing in the image of the diagonal map: $U \times V$ is contained in $\Delta(X)^c$. So $\Delta(X)^c$ is open, making $\Delta(X)$ closed.

Now let's suppose $\Delta(X)$ is closed. Then $\Delta(X)^c$ is open. Given any $(x,y) \in \Delta(X)^c$, there's a basis element $U \times V$ with $(x,y) \in U \times V \subset \Delta(X)^c$. $U \times V$ lying in $\Delta(X)^c$ implies that U and V are disjoint.

If we have $x \neq y$ in X, then (x,y) is in $\Delta(X)^c$. The basis element containing (x,y) gives us open, disjoint U, V with $x \in U, y \in V$. X is Hausdorff, just like we wanted. □