compactness is preserved under a continuous map

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Theorem [?, ?] Suppose $f\colon X \to Y$ is a continuous map between topological spaces X and Y. If X is compact and f is surjective, then Y is compact.

The inclusion map $[0,1] \hookrightarrow [0,2)$ shows that the requirement for f to be surjective cannot be omitted. If X is compact and f is continuous we can always conclude, however, that $f(X)$ is compact, since http://planetmath.org/IfFcolonXtoYIsCon
$f(X)$ is continuous.

Proof of theorem. (Following [?].) Suppose $\{V_\alpha \mid \alpha \in I\}$ is an arbitrary open cover for $f(X)$. Since f is continuous, it follows that

$$\{f^{-1}(V_\alpha) \mid \alpha \in I\}$$

is a collection of open sets in X. Since $A \subseteq f^{-1}f(A)$ for any $A \subseteq X$, and since the inverse commutes with unions (see http://planetmath.org/InverseImagethis page), we have

$$\begin{aligned} X &\subseteq f^{-1}f(X) \\ &= f^{-1}\big(\bigcup_{\alpha\in I}(V_\alpha)\big) \\ &= \bigcup_{\alpha\in I} f^{-1}(V_\alpha). \end{aligned}$$

Thus $\{f^{-1}(V_\alpha) \mid \alpha \in I\}$ is an open cover for X. Since X is compact, there exists a finite subset $J \subseteq I$ such that $\{f^{-1}(V_\alpha) \mid \alpha \in J\}$ is a finite open cover for X. Since f is a surjection, we have $ff^{-1}(A) = A$ for any $A \subseteq Y$ (see http://planetmath.org/InverseImagethis page). Thus

$$\begin{aligned} f(X) &= f\big(\bigcup_{i\in J} f^{-1}(V_\alpha)\big) \\ &= ff^{-1}\bigcup_{i\in J} f^{-1}(V_\alpha) \\ &= \bigcup_{i\in J} V_\alpha. \end{aligned}$$

Thus $\{V_\alpha \mid \alpha \in J\}$ is an open cover for $f(X)$, and $f(X)$ is compact. □

A shorter proof can be given using the http://planetmath.org/ASpaceIsCompactIfAndOnlyIf
of compactness by the finite intersection property:

Shorter proof. Suppose $\{A_i \mid i \in I\}$ is a collection of closed subsets of Y with the finite intersection property. Then $\{f^{-1}(A_i) \mid i \in I\}$ is a collection of closed subsets of X with the finite intersection property, because if $F \subseteq I$ is finite then

$$\bigcap_{i \in F} f^{-1}(A_i) = f^{-1}\left(\bigcap_{i \in F} A_i\right),$$

which is nonempty as f is a surjection. As X is compact, we have

$$f^{-1}\left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} f^{-1}(A_i) \neq \varnothing$$

and so $\bigcap_{i \in I} A_i \neq \varnothing$. Therefore Y is compact. □

References

[1] I.M. Singer, J.A.Thorpe, *Lecture Notes on Elementary Topology and Geometry*, Springer-Verlag, 1967.

[2] J.L. Kelley, *General Topology*, D. van Nostrand Company, Inc., 1955.

[3] G.J. Jameson, *Topology and Normed Spaces*, Chapman and Hall, 1974.