induced Alexandroff topology on a poset

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Let $(X, \leq)$ be a poset. For any $x \in X$ define following subset:

$$(-\infty, x] = \{y \in X \mid y \leq x\}.$$

The induced Alexandroff topology τ on X is defined as a topology generated by $\{(-\infty, x]\}_{x\in X}$.

Proposition 1. (X, τ) is a T_0, Alexandroff space.

Proof. Let $x, y \in X$ be such that $x \neq y$. Note that this implies that $x \not\leq y$ or $y \not\leq x$ (because $\leq$ is antisymmetric). Therefore $x \notin (-\infty, y]$ or $y \notin (-\infty, x]$. Thus (X, τ) is T_0.

Now in order to show that (X, τ) is Alexandroff it is enough to show that an arbitrary intersection of base sets is open. So assume that $\{x_i\}_{i\in I}$ is a subset of X such that

$$A = \bigcap_{i\in I} (-\infty, x_i] \neq \emptyset$$

and let $y \in A$. Then (since $\leq$ is transitive) it is clear that

$$(-\infty, y] \subseteq A$$

and thus

$$\bigcap_{i\in I} (-\infty, x_i] = A = \bigcup_{y\in A} (-\infty, y]$$

and therefore the intersection is open, which completes the proof. $\square$

Proposition 2. Let X, Y be posets and $f : X \to Y$ a function. Then f preserves order if and only if f is continuous in induced Alexandroff topologies.

Proof. ,,$\Rightarrow$" Assume that f preserves order and let $A = (-\infty, y] \subseteq Y$ be an open base set. We wish to show that $f^{-1}(A)$ is open in X. So take any $x \in f^{-1}(A)$. Now if $y \leq x$, then $f(y) \leq f(x)$ (since f preserves order) and thus $f(y) \in A$. Therefore $y \in f^{-1}(A)$. Since y was arbitrary we obtain that for any $x \in f^{-1}(A)$ we have $(-\infty, x] \subseteq f^{-1}(A)$ and thus

$$f^{-1}(A) = \bigcup_{x\in f^{-1}(A)} (-\infty, x],$$

which implies that $f^{-1}(A)$ is open.

,,$\Leftarrow$" Assume that f is continuous and let $y \leq x$ for some $x, y \in X$. Assume that $f(y) \not\leq f(x)$. Let $A = (-\infty, f(x)]$. Therefore $f(y) \notin A$, but A is open, so $f^{-1}(A)$ is open (because f is continuous). Thus $(-\infty, x] \subseteq f^{-1}(A)$. But $y \leq x$, so $y \in f^{-1}(A)$. But this implies that $f(y) \in A$. Contradiction. $\square$