lattice of topologies

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Author	CWoo (3771)
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Let X be a set. Let L be the set of all topologies on X. We may order L by inclusion. When $\mathcal{T}_1 \subseteq \mathcal{T}_2$, we say that $\mathcal{T}_2$ is `http://planetmath.org/Finer`finer than $\mathcal{T}_1$, or that $\mathcal{T}_2$ refines $\mathcal{T}_1$.

Theorem 1. *L, ordered by inclusion, is a complete lattice.*

Proof. Clearly L is a partially ordered set when ordered by $\subseteq$. Furthermore, given any family of topologies $\mathcal{T}_i$ on X, their intersection $\bigcap \mathcal{T}_i$ also defines a topology on X. Finally, let $\mathcal{B}_i$'s be the corresponding subbases for the $\mathcal{T}_i$'s and let $\mathcal{B} = \bigcup \mathcal{B}_i$. Then $\mathcal{T}$ generated by $\mathcal{B}$ is easily seen to be the supremum of the $\mathcal{T}_i$'s. □

Let L be the lattice of topologies on X. Given $\mathcal{T}_i \in L$, $\mathcal{T} := \bigvee \mathcal{T}_i$ is called the *common refinement* of $\mathcal{T}_i$. By the proof above, this is the coarsest topology that is than each $\mathcal{T}_i$.

If X is non-empty with more than one element, L is also an atomic lattice. Each atom is a topology generated by one non-trivial subset of X (non-trivial being non-empty and not X). The atom has the form $\{\varnothing, A, X\}$, where $\varnothing \subset A \subset X$.

Remark. In general, a *lattice of topologies* on a set X is a sublattice of *the* lattice of topologies L (mentioned above) on X.