closed set in a subspace

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In the following, let X be a topological space.

Theorem 1. *Suppose $Y \subseteq X$ is equipped with the subspace topology, and $A \subseteq Y$. Then A is http://planetmath.org/ClosedSetclosed in Y if and only if $A = Y \cap J$ for some closed set $J \subseteq X$.*

Proof. If A is closed in Y, then $Y \setminus A$ is http://planetmath.org/OpenSetopen in Y, and by the definition of the subspace topology, $Y \setminus A = Y \cap U$ for some open $U \subseteq X$. Using http://planetmath.org/SetDifferenceproperties of the set difference, we obtain

$$\begin{aligned} A &= Y \setminus (Y \setminus A) \\ &= Y \setminus (Y \cap U) \\ &= Y \setminus U \\ &= Y \cap U^{\complement}. \end{aligned}$$

On the other hand, if $A = Y \cap J$ for some closed $J \subseteq X$, then $Y \setminus A = Y \setminus (Y \cap J) = Y \cap J^{\complement}$, and so $Y \setminus A$ is open in Y, and therefore A is closed in Y. $\square$

Theorem 2. *Suppose X is a topological space, $C \subseteq X$ is a closed set equipped with the subspace topology, and $A \subseteq C$ is closed in C. Then A is closed in X.*

Proof. This follows from the previous theorem: since A is closed in C, we have $A = C \cap J$ for some closed set $J \subseteq X$, and A is closed in X. $\square$