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stalk

Canonical name	Stalk
Date of creation	2013-03-22 12:37:15
Last modified on	2013-03-22 12:37:15
Owner	djao (24)
Last modified by	djao (24)
Numerical id	9
Author	djao (24)
Entry type	Definition
Classification	msc 54B40
Classification	msc 14F05
Classification	msc 18F20
Related topic	Sheaf
Related topic	LocalRing

Let F be a presheaf over a topological space X with values in an abelian category $\mathcal{A}$, and suppose direct limits exist in $\mathcal{A}$. For any point $p \in X$, the *stalk F_p* of F at p is defined to be the object in $\mathcal{A}$ which is the direct limit of the objects $F(U)$ over the directed set of all open sets $U \subset X$ containing p, with respect to the restriction morphisms of F. In other words,

$$F_p := \varinjlim_{U \ni p} F(U)$$

If $\mathcal{A}$ is a category consisting of sets, the stalk F_p can be viewed as the set of all germs of sections of F at the point p. That is, the set F_p consists of all the equivalence classes of ordered pairs (U, s) where $p \in U$ and $s \in F(U)$, under the equivalence relation $(U, s) \sim (V, t)$ if there exists a neighborhood $W \subset U \cap V$ of p such that $\operatorname{res}_{U,W} s = \operatorname{res}_{V,W} t$.

By universal properties of direct limit, a morphism $\phi : F \longrightarrow G$ of presheaves over X induces a morphism $\phi_p : F_p \longrightarrow G_p$ on each stalk F_p of F. Stalks are most useful in the context of sheaves, since they encapsulate all of the local data of the sheaf at the point p (recall that sheaves are basically defined as presheaves which have the property of being completely characterized by their local behavior). Indeed, in many of the standard examples of sheaves that take values in rings (such as the sheaf $\mathcal{D}_X$ of smooth functions, or the sheaf $\mathcal{O}_X$ of regular functions), the ring F_p is a local ring, and much of geometry is devoted to the study of sheaves whose stalks are local rings (so-called "locally ringed spaces").

We mention here a few illustrations of how stalks accurately reflect the local behavior of a sheaf; all of these are drawn from [?].

- A morphism of sheaves $\phi : F \longrightarrow G$ over X is an isomorphism if and only if the induced morphism ϕ_p is an isomorphism on each stalk.

- A sequence $F \longrightarrow G \longrightarrow H$ of morphisms of sheaves over X is an exact sequence at G if and only if the induced morphism $F_p \longrightarrow G_p \longrightarrow H_p$ is exact at each stalk G_p.

- The sheafification F' of a presheaf F has stalk equal to F_p at every point p.

References

[1] Robin Hartshorne, *Algebraic Geometry*, Springer–Verlag New York Inc., 1977 (GTM **52**).