a space is connected under the ordered topology if and only if it is a linear continuum.

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Let X be totally ordered by the strict total order $<$, and let it have the order topology.

Suppose X is not a linear continuum. Then either X is not bounded-complete, or the order on X is not a dense total order.

Suppose X is not bounded-complete. Let A be a nonempty subset of X that is bounded above by b, but has no least upper bound. Let U be the set of upper bounds of A. If $x \in U$, then x is not a least upper bound of A, so there is a $z \in X$ such that $z < x$ and z is an upper bound of A. Then the set $\{y \in X \mid y > z\}$ is open and contains x. Furthermore, all of its elements exceed z, so it is a subset of U. Thus, U is open. U contains b, so it is not empty. Let $x \in X \setminus U$. Then x is not an upper bound of A, so there is an $a \in A$ such that $x < a$. The set $\{y \in X \mid y < a\}$ is open, and contains no upper bounds of A, so it is a subset of $X \setminus U$. Thus $X \setminus U$ contains a neighborhood of each of its points, and is therefore open. Since U and $X \setminus U$ are open, X is not connected.

Suppose the ordering of X is not dense, so there are a and b in X with $a < b$ so that there is no c in X with $a < c < b$ (there is a gap between a and b). Let $U = \{x \in X \mid x < b\}$ and let $V = \{x \in X \mid x > a\}$. Because there are no elements between a and b, $U \cap V = \emptyset$. By transitivity and trichotomy, $U \cup V = X$. U and V are both open. $a \in U$ and $b \in V$, so neither U nor V is empty. Thus, U and V separate X, so X is not connected.

Therefore, if X is connected, then X is a linear continuum.

Now suppose that X is disconnected and bounded-complete, and that U and V are (nonempty, open and closed) sets separating X. Suppose that $a \in U$, and suppose also that there is an element $b \in V$ such that $a < b$ (if there is none, swap the names of U and V, or reverse the ordering). $Z = \{x \in V \mid x > a\}$ is open (it is the intersection of two open sets), and contains b (so it is not empty). Z is bounded below by a, so it has a greatest lower bound z.

If $z \in U$, then, since U is open, there is an interval in U containing z, which must, to exclude b, be of the form $\{x \mid x < k\}$ or of the form $\{x \mid j < x < k\}$, for some k and perhaps j. But then k would be a lower bound of Z, contradicting the fact that z is the infimum of Z.

If $z \in V$, then, since V is open, there is an interval in V containing z, which must, to exclude a, be of the form $\{x \mid j < x\}$ or of the form $\{x \mid j < x < k\}$ for some j and perhaps k. In either case, $a < j < z$ and the set $W = \{x \mid j < x < z\}$ is a subset of V. If $x \in W$, then $x \in V$ and $x > a$, so $x \in Z$, contradicting the fact that z is the infimum of Z. Thus, there are no elements of X between j and z, so the order on X is not dense. This proves that if X is a linear continuum, then X is connected.