injective map between real numbers is a homeomorphism

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Lemma. Assume that I is an open interval and $f : I \to \mathbb{R}$ is an injective, continuous map. Then $f(I) \subseteq \mathbb{R}$ is an open subset.

Proof. Since f is injective, then of course f is monotonic. Without loss of generality, we may assume that f is increasing. Let $y = f(x) \in f(I)$. Since I is open, then there are $\alpha, \beta \in I$ such that $\alpha < x < \beta$. Therefore $f(\alpha) < y < f(\beta)$ and (because continuous functions are Darboux functions) for any $y' \in \big(f(\alpha), f(\beta)\big)$ there exists $x' \in I$ such that $f(x') = y'$. This shows that $\big(f(\alpha), f(\beta)\big)$ is an open neighbourhood of y contained in $f(I)$ and therefore (since y was arbitrary) $f(I)$ is open. $\square$

Proposition. Assume that I is an open interval and $f : I \to \mathbb{R}$ is an injective, continuous map. Then f is a homeomorphism onto image.

Proof. Of course, it is enough to show that f is an open map. But if $U \subseteq I$ is open, then there are disjoint, open intervals I_α such that

$$U = \bigcup_\alpha I_\alpha.$$

Therefore we obtain continuous, injective maps $f_\alpha : I_\alpha \to \mathbb{R}$ which are restrictions of f to I_α. By lemma we have that $f_\alpha(I_\alpha)$ is open and therefore

$$f(U) = f\left(\bigcup_\alpha I_\alpha\right) = \bigcup_\alpha f(I_\alpha) = \bigcup_\alpha f_\alpha(I_\alpha)$$

is open. This shows that f is a homeomorphism onto image. $\square$