completion

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Let (X, d) be a metric space. Let $\bar{X}$ be the set of all Cauchy sequences $\{x_n\}_{n \in \mathbb{N}}$ in X. Define an equivalence relation $\sim$ on $\bar{X}$ by setting $\{x_n\} \sim \{y_n\}$ if the interleave sequence of the sequences $\{x_n\}$ and $\{y_n\}$ is also a Cauchy sequence. The *completion* of X is defined to be the set $\hat{X}$ of equivalence classes of $\bar{X}$ modulo $\sim$.

The metric d on X extends to a metric on $\hat{X}$ in the following manner:

$$d(\{x_n\}, \{y_n\}) := \lim_{n \to \infty} d(x_n, y_n),$$

where $\{x_n\}$ and $\{y_n\}$ are representative Cauchy sequences of elements in $\hat{X}$. The definition of $\sim$ is tailored so that the limit in the above definition is well defined, and the fact that these sequences are Cauchy, together with the fact that $\mathbb{R}$ is complete, ensures that the limit exists. The space $\hat{X}$ with this metric is of course a complete metric space.

The original metric space X is isometric to the subset of $\hat{X}$ consisting of equivalence classes of constant sequences.

Note the similarity between the construction of $\hat{X}$ and the construction of $\mathbb{R}$ from $\mathbb{Q}$. The process used here is the same as that used to construct the real numbers $\mathbb{R}$, except for the minor detail that one can not use the terminology of metric spaces in the construction of $\mathbb{R}$ itself because it is necessary to construct $\mathbb{R}$ in the first place before one can define metric spaces.

1 Metric spaces with richer structure

If the metric space X has an algebraic structure, then in many cases this algebraic structure carries through unchanged to $\hat{X}$ simply by applying it one element at a time to sequences in X. We will not attempt to state this principle precisely, but we will mention the following important instances:

1. If $(X, \cdot)$ is a topological group, then $\hat{X}$ is also a topological group with multiplication defined by

$$\{x_n\} \cdot \{y_n\} = \{x_n \cdot y_n\}.$$

2. If X is a topological ring, then addition and multiplication extend to $\hat{X}$ and make the completion into a topological ring.

3. If F is a field with a valuation v, then the completion of F with respect to the metric imposed by v is a topological field, denoted F_v and called the completion of F at v.

2 Universal property of completions

The completion $\hat{X}$ of X satisfies the following universal property: for every uniformly continuous map $f : X \longrightarrow Y$ of X into a complete metric space Y, there exists a unique lifting of f to a continuous map $\hat{f} : \hat{X} \longrightarrow Y$ making the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr] & & Y \\
& \hat{X} \arrow[ur, swap, "\hat{f}"] &
\end{tikzcd}$$

commute. Up to isomorphism, the completion of X is the unique metric space satisfying this property. The ability to extend uniformly continuous functions from X to $\hat{X}$ is often the reason why algebraic structures on X extend to $\hat{X}$ as described in the previous section.