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ring of continuous functions

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Let X be a topological space and $C(X)$ be the function space consisting of all continuous functions from X into $\mathbb{R}$, the reals (with the usual metric topology).

Ring Structure on $C(X)$

To formally define $C(X)$ as a ring, we take a step backward, and look at $\mathbb{R}^X$, the set of all functions from X to $\mathbb{R}$. We will define a ring structure on $\mathbb{R}^X$ so that $C(X)$ inherits that structure and forms a ring itself.

For any $f, g \in \mathbb{R}^X$ and any $r \in \mathbb{R}$, we define the following operations:

1. (addition) $(f+g)(x) := f(x) + g(x)$,
2. (multiplication) $(fg)(x) := f(x)g(x)$,
3. (identities) Define $r(x) := r$ for all $x \in X$. These are the constant functions. The special constant functions $1(x)$ and $0(x)$ are the *multiplicative* and *additive identities* in $\mathbb{R}^X$.
4. (additive inverse) $(-f)(x) := -(f(x))$,
5. (multiplicative inverse) if $f(x) \neq 0$ for all $x \in X$, then we may define the multiplicative inverse of f, written f^{-1} by

   $$f^{-1}(x) := \frac{1}{f(x)}.$$

   This is not to be confused with the functional inverse of f.

All the ring axioms are easily verified. So $\mathbb{R}^X$ is a ring, and actually a commutative ring. It is immediate that any constant function other than the additive identity is invertible.

Since $C(X)$ is closed under all of the above operations, and that $0, 1 \in C(X)$, $C(X)$ is a subring of $\mathbb{R}^X$, and is called *the ring of continuous functions* over X.

Additional Structures on $C(X)$

$\mathbb{R}^X$ becomes an $\mathbb{R}$-algebra if we define scalar multiplication by $(rf)(x) := r(f(x))$. As a result, $C(X)$ is a subalgebra of $\mathbb{R}^X$.

In addition to having a ring structure, $\mathbb{R}^X$ also has a natural order structure, with the partial order defined by $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in X$. The positive cone is the set $\{f \mid 0 \leq f\}$. The absolute value, given by $|f|(x) := |f(x)|$, is an operator mapping $\mathbb{R}^X$ onto its positive cone. With the absolute value operator defined, we can put a `http://planetmath.org/Lattice`lattice structure on $\mathbb{R}^X$ as well:

- (meet) $f \vee g := 2^{-1}(f + g + |f - g|)$. Here, 2^{-1} is the constant function valued at $\frac{1}{2}$ (also as the multiplicative inverse of the constant function 2).
- (join) $f \wedge g := f + g - (f \vee g)$.

Since taking the absolute value of a continuous function is again continuous, $C(X)$ is a sublattice of $\mathbb{R}^X$. As a result, we may consider $C(X)$ as a lattice-ordered ring of continuous functions.

Remarks. Any subring of $C(X)$ is called a *ring of continuous functions* over X. This subring may or may not be a sublattice of $C(X)$. Other than $C(X)$, the two commonly used lattice-ordered subrings of $C(X)$ are

- $C^*(X)$, the subset of $C(X)$ consisting of all bounded continuous functions. It is easy to see that $C^*(X)$ is closed under all of the algebraic operations (ring-theoretic or lattice-theoretic). So $C^*(X)$ is a lattice-ordered subring of $C(X)$. When X is pseudocompact, and in particular, when X is compact, $C^*(X) = C(X)$.

  In this subring, there is a natural norm that can be defined:

  $$\|f\| := \sup_{x \in X} |f(x)| = \inf\{r \in \mathbb{R} \mid |f| \leq r\}.$$

  Routine verifications show that $\|fg\| \leq \|f\|\|g\|$, so that $C^*(X)$ becomes a normed ring.

- The subset of $C^*(X)$ consisting of all constant functions. This is isomorphic to $\mathbb{R}$, and is often identified as such, so that $\mathbb{R}$ is considered as a lattice-ordered subring of $C(X)$.

References

[1] L. Gillman, M. Jerison: *Rings of Continuous Functions*, Van Nostrand, (1960).