# uniform proximity is a proximity

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In this entry, we want to show that a uniform proximity is, as expected, a proximity.

First, the following equivalent characterizations of a uniform proximity is useful:

**Lemma 1.** *Let $X$ be a uniform space with uniformity $\mathcal{U}$, and $A, B$ are subsets of $X$. Denote $U[A]$ the image of $A$ under $U \in \mathcal{U}$:*

$$\{b \in X \mid (a,b) \in U \text{ for some } a \in A\}.$$

*The following are equivalent:*

1. $(A \times B) \cap U \neq \varnothing$ *for all* $U \in \mathcal{U}$
2. $U[A] \cap U[B] \neq \varnothing$ *for all* $U \in \mathcal{U}$
3. $U[A] \cap B \neq \varnothing$ *for all* $U \in \mathcal{U}$

If we define $A\delta B$ iff the pair $A, B$ satisfy any one of the above conditions for all $U \in \mathcal{U}$, we call $\delta$ the uniform proximity.

*Proof.* ($1 \Rightarrow 2$) Suppose $(a,b) \in (A \times B) \cap U$. Then $b \in U[A]$. Since $U$ is reflexive, $(b,b) \in U$, or $b \in U[B]$. This means $b \in U[A] \cap U[B]$.

($2 \Rightarrow 3$) For any $U \in \mathcal{U}$, we can find $V \in \mathcal{U}$ such that $V \circ V \subseteq U$. So $V = V \circ \Delta \subseteq V \circ V \subseteq W$, where $\Delta$ is the diagonal relation (since $V$ is reflexive). Set $W = V \cap V^{-1}$. By assumption, there is $c \in W[A] \cap W[B]$ (and hence $c \in U[A] \cap U[B]$ as well). This means $(a,c), (b,c) \in W$ for some $a \in A$ and $b \in B$. Since $W$ is symmetric, $(c,b) \in W \subseteq V$, so that $(a,b) = (a,c) \circ (c,b) \in V \subseteq U$. This means that $b \in U[A]$. As a result, $U[A] \cap B \neq \varnothing$.

($3 \Rightarrow 1$) If $b \in U[A] \cap B$, then there is $a \in A$ such that $(a,b) \in U$, or $(A \times B) \cap U \neq \varnothing$. □

We want to prove the following:

**Proposition 1.** *The binary relation $\delta$ on $P(X)$ defined by*

$$A\delta B \quad \textit{iff} \quad (A \times B) \cap U \neq \varnothing \text{ for all } U \in \mathcal{U}$$

*is a proximity on $X$.*

*Proof.* We verify each of the axioms of a proximity relation:

1. if $A \cap B \neq \varnothing$, then $A\delta B$:

   pick $c \in A \cap B$, then $(c, c) \in U$ since the diagonal relation $\Delta \subseteq U$ for all $U \in \mathcal{U}$.

2. if $A\delta B$, then $A \neq \varnothing$ and $B \neq \varnothing$:

   If $A\delta B$, then $(A \times B) \cap U \neq \varnothing$ for every $U \in \mathcal{U}$, since no $U$ is empty, there is $(a, b) \in U$ such that $(a, b) \in A \times B$, or $A \neq \varnothing$ and $B \neq \varnothing$.

3. (symmetry) if $A\delta B$, then $B\delta A$:

   If $A\delta B$, then there is $(a_U, b_U) \in (A \times B) \cap U^{-1}$ for every $U \in \mathcal{U}$, so $(b_U, a_U) \in U$, which implies $(B \times A) \cap U \neq \varnothing$, or $B\delta A$.

4. $(A_1 \cup A_2)\delta B$ iff $A_1\delta B$ or $A_2\delta B$:

   Since $(A_1 \cup A_2) \times B = (A_1 \times B) \cup (A_2 \times B)$,

   $$\begin{aligned} & (a, b) \in \big((A_1 \cup A_2) \times B\big) \cap U \\ \text{iff} \quad & (a, b) \in \big((A_1 \times B) \cup (A_2 \times B)\big) \cap U = \big((A_1 \times B) \cap U\big) \cup \big((A_2 \times B) \cap U\big) \\ \text{iff} \quad & (a, b) \in (A_1 \times B) \cap U \text{ or } (a, b) \in (A_2 \times B) \cap U. \end{aligned}$$

5. $A\delta' B$ implies the existence of $C \in P(X)$ with $A\delta' C$ and $(X - C)\delta' B$, where $A\delta' B$ means $(A, B) \notin \delta$.

   First note that $\delta'$ is symmetric because $\delta$ is. By assumption, there is $U \in \mathcal{U}$ such that $U[A] \cap U[B] = \varnothing$ (second equivalent characterization of uniform proximity from lemma above). Set $C = U[B]$. Then $U[A] \cap C = \varnothing$. By the third equivalent condition of uniform proximity, $A\delta' C$. Likewise, $U[B] \cap (X - C) = U[B] \cap (X - U[B]) = \varnothing$, so $B\delta'(X - C)$, or $(X - C)\delta' B$.

This shows that $\delta$ is a proximity on $X$. $\square$