# dual of Stone representation theorem


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The Stone representation theorem characterizes a Boolean algebra as a field of sets in a topological space. There is also a dual to this famous theorem that characterizes a Boolean space as a topological space constructed from a Boolean algebra.

**Theorem 1.** *Let $X$ be a Boolean space. Then there is a Boolean algebra $B$ such that $X$ is homeomorphic to $B^*$, the* `http://planetmath.org/DualSpaceOfABooleanAlgebra` *space of $B$.*

*Proof.* The choice for $B$ is clear: it is the set of clopen sets in $X$ which, via the set theoretic operations of intersection, union, and complement, is a Boolean algebra.

Next, define a function $f : X \to B^*$ by

$$f(x) := \{U \in B \mid x \notin U\}.$$

Our ultimate goal is to prove that $f$ is the desired homeomorphism. We break down the proof of this into several stages:

**Lemma 1.** *$f$ is well-defined.*

*Proof.* The key is to show that $f(x)$ is a prime ideal in $B^*$ for any $x \in X$. To see this, first note that if $U, V \in f(x)$, then so is $U \cup V \in f(x)$, and if $W$ is any clopen set of $X$, then $U \cap W \in f(x)$ too. Finally, suppose that $U \cap V \in f(x)$. Then $x \in X - (U \cap V) = (X - U) \cup (X - V)$, which means that $x \notin U$ or $x \notin V$, which is the same as saying that $U \in f(x)$ or $V \in f(x)$. Hence $f(x)$ is a prime ideal, or a maximal ideal, since $B$ is Boolean. □

**Lemma 2.** *$f$ is injective.*

*Proof.* Suppose $x \neq y$, we want to show that $f(x) \neq f(y)$. Since $X$ is Hausdorff, there are disjoint open sets $U, V$ such that $x \in U$ and $y \in V$. Since $X$ is also totally disconnected, $U$ and $V$ are unions of clopen sets. Hence we may as well assume that $U, V$ clopen. This then implies that $U \in f(y)$ and $V \in f(x)$. Since $U \neq V$, $f(x) \neq f(y)$. □

**Lemma 3.** *$f$ is surjective.*

*Proof.* Pick any maximal ideal $I$ of $B^*$. We want to find an $x \in X$ such that $f(x) = I$. If no such $x$ exists, then for every $x \in X$, there is some clopen set $U \in I$ such that $x \in U$. This implies that $\bigcup I = X$. Since $X$ is compact,

$X = \bigcup J$ for some finite set $J \subseteq I$. Since $I$ is an ideal, and $X$ is a finite join of elements of $I$, we see that $X \in I$. But this would mean that $I = B^*$, contradicting the fact that $I$ is a maximal, hence a proper ideal of $B^*$. $\square$

**Lemma 4.** *$f$ and $f^{-1}$ are continuous.*

*Proof.* We use a fact about continuous functions between two Boolean spaces:

> a bijection is a homeomorphism iff it maps clopen sets to clopen sets (proof `http://planetmath.org/HomeomorphismBetweenBooleanSpaces`here).

So suppose that $U$ is clopen in $X$, we want to prove that $f(U)$ is clopen in $B^*$. In other words, there is an element $V \in B$ (so that $V$ is clopen in $X$) such that

$$f(U) = M(V) = \{M \in B^* \mid V \notin M\}.$$

This is because every clopen set in $B^*$ has the form $M(V)$ for some $V \in B^*$ (see the lemma in `http://planetmath.org/StoneRepresentationTheorem`this entry). Now, $f(U) = \{f(x) \mid x \in U\} = \{f(x) \mid U \notin f(x)\} = \{M \mid U \notin M\}$, the last equality is based on the fact that $f$ is a bijection. Thus by setting $V = U$ completes the proof of the lemma. $\square$

Therefore, $f$ is a homemorphism, and the proof of theorem is complete. $\square$