Tychonoff's theorem implies AC

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In this entry, we prove that Tychonoff's theorem implies that product of non-empty set of non-empty sets is non-empty, which is equivalent to the axiom of choice (AC). This fact, together with the fact that AC implies Tychonoff's theorem, shows that Tychonoff's theorem is equivalent to AC (under ZF). The proof was first discovered by John Kelley in 1950, and is now an exercise in axiomatic set theory.

Proof. Let C be a non-empty collection of non-empty sets. Let Y be the generalized cartesian product of all the elements in C. Our objective is show that Y is non-empty.

First, some notations: for each $A \in C$, set $X_A := A \cup \{A\}$, $D := \{X_A \mid A \in C\}$, X the generalized cartesian product of all the X_A's, and p_A the projection from X onto X_A.

We break down the proof into several steps:

1. Y is equipollent to $Z := \bigcap\{p_A^{-1}(A) \mid A \in C\}$.

   An element of X is a function $f : D \to \bigcup D$, such that $f(X_A) \in X_A$ for each $A \in C$. In other words, either $f(X_A) \in A$, or $f(X_A) = A$. An element of Y is a function $g : C \to \bigcup C$ such that $g(A) \in A$ for each $A \in C$. Finally, $h \in p_A^{-1}(A)$ iff $h(X_A) \in A$.

   Given $g \in Y$, define $g^* \in X$ by $g^*(X_A) := g(A) \in A$. Since A is arbitrary, $g^* \in Z$. Conversely, given $h \in Z$, define $h' \in Y$ by $h'(A) := h(X_A)$, which is well-defined, since $h(X_A) \in A$. Now, it is easy to see that the function $\phi : Y \to Z$ given by $\phi(g) = g^*$ is a bijection, whose inverse $\phi^{-1} : Z \to Y$ is given by $\phi^{-1}(h) = h'$. This shows that Y and Z are equipollent.

2. Next, we topologize each X_A in such a way that X_A is compact.

   Let $\mathcal{T}_A$ be the coarsest topology containing the cofinite topology on X_A and the singleton $\{A\}$. A typical open set of X_A is either the empty set, or has the form $S \cup \{A\}$, where S is cofinite in A.

   To show that X_A is compact under $\mathcal{T}_A$, let $\mathcal{D}$ be an open cover for X_A. We want to show that there is a finite subset of $\mathcal{D}$ covering X_A. If $X_A \in \mathcal{D}$, then we are done. Otherwise, pick a non-empty element $S \cup \{A\}$ in $\mathcal{D}$, so that $A - S \neq \varnothing$, and is finite. By assumption, each element in $A - S$ belongs to some open set in $\mathcal{D}$. So to cover $A - S$, only a finite number of open sets in $\mathcal{D}$ are needed. These open sets, together with $S \cup \{A\}$, cover X_A. Hence X_A is compact.

3. Finally, we prove that Z, and therefore Y, is non-empty.

   Apply Tychonoff's theorem, X is compact under the product topology. Furthermore, π_A is continuous for each $A \in C$. Since $\{A\}$ is open in X_A, and $A = X_A - \{A\}$, A is closed in X_A, and thus so is $p_A^{-1}(A)$ closed in X.

   To show that Z is non-empty, we employ a characterization of compact space: `http://planetmath.org/ASpaceIsCompactIfAndOnlyIfTheSpaceHasTheFiniteInt` is compact iff every collection of closed sets in X having FIP has non-empty intersection. Let us look at the collection $\mathcal{S} := \{p_A^{-1}(A) \mid A \in C\}$. Given $A_1, \ldots, A_n \in C$, pick an element $a_i \in A_i$, since $A_i \neq \varnothing$ by assumption. Note that this is possible, since there are only a finite number of sets. Define $f : D \to \bigcup D$ as follows:

$$f(X_A) := \begin{cases} a_i & \text{if } A = A_i \text{ for some } i = 1, \ldots, n, \\ A & \text{otherwise.} \end{cases}$$

   Since $f(X_{A_i}) = a_i \in A_i$, $f \in p_{A_i}^{-1}(A_i)$ for each $i = 1, \ldots, n$. Therefore,

$$f \in p_{A_1}^{-1}(A_1) \cap \cdots \cap p_{A_n}^{-1}(A_n).$$

   Since $p_{A_1}^{-1}(A_1), \ldots, p_{A_n}^{-1}(A_n)$ are arbitrarily picked from $\mathcal{S}$, the collection $\mathcal{S}$ has finite intersection property, and since X is compact, $Z = \bigcap S$ must be non-empty.

This completes the proof. □

Remark. In the proof, we see that the trick is to adjoin the set $\{A\}$ to each set $A \in C$. Instead of $\{A\}$, we could have picked some arbitrary, but fixed singleton $\{B\}$, as long as $B \notin A$ for each $A \in C$, and the proof follows essentially the same way.

References

[1] T. J. Jech, *The Axiom of Choice*. North-Holland Pub. Co., Amsterdam, 1973.

[2] J. L. Kelley, *The Tychonoff's product theorem implies the axiom of choice*. Fund. Math. 37, pp. 75-76, 1950.