compact subspace of a Hausdorff space is closed

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Let X be a Hausdorff space, and Y be a compact subspace of X. We prove that $X \setminus Y$ is open, by finding for every point $x \in X \setminus Y$ a neighborhood U_x disjoint from Y.

Let $y \in Y$. $x \neq y$, so by the definition of a Hausdorff space, there exist open neighborhoods $U_x^{(y)}$ of x and $V_x^{(y)}$ of y such that $U_x^{(y)} \cap V_x^{(y)} = \emptyset$. Clearly

$$Y \subseteq \bigcup_{y \in Y} V_x^{(y)}$$

but since Y is compact, we can select from these a finite subcover of Y

$$Y \subseteq V_x^{(y_1)} \cup \cdots \cup V_x^{(y_n)}$$

Now for every $y \in Y$ there exists $k \in 1...n$ such that $y \in V_x^{(y_k)}$. Since $U_x^{(y_k)}$ and $V_x^{(y_k)}$ are disjoint, $y \notin U_x^{(y_k)}$, therefore neither is it in the intersection

$$U_x = \bigcap_{j=1}^{n} U_x^{(y_j)}$$

A finite intersection of open sets is open, hence U_x is a neighborhood of x disjoint from Y.