# Cantor-Bendixson derivative

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Let $A$ be a subset of a topological space $X$. Its *Cantor-Bendixson derivative* $A'$ is defined as the set of accumulation points of $A$. In other words

$$A' = \{x \in X \mid x \in \overline{A \setminus \{x\}}\}.$$

Through transfinite induction, the Cantor-Bendixson derivative can be defined to any order $\alpha$, where $\alpha$ is an arbitrary ordinal. Let $A^{(0)} = A$. If $\alpha$ is a successor ordinal, then $A^{(\alpha)} = \left(A^{(\alpha-1)}\right)'$. If $\lambda$ is a limit ordinal, then $A^{(\lambda)} = \bigcap_{\alpha<\lambda} A^{(\alpha)}$. The *Cantor-Bendixson rank* of the set $A$ is the least ordinal $\alpha$ such that $A^{(\alpha)} = A^{(\alpha+1)}$. Note that $A' = A$ implies that $A$ is a perfect set.

Some basic properties of the Cantor-Bendixson derivative include

1. $(A \cup B)' = A' \cup B'$,
2. $(\bigcup_{i\in I} A_i)' \supseteq \bigcup_{i\in I} A_i'$,
3. $(\bigcap_{i\in I} A_i)' \subseteq \bigcap_{i\in I} A_i'$,
4. $(A \setminus B)' \supseteq A' \setminus B'$,
5. $A \subseteq B \Rightarrow A' \subseteq B'$,
6. $\overline{A} = A \cup A'$,
7. $\overline{A'} = A'$.

The last property requires some justification. Obviously, $A' \subseteq \overline{A'}$. Suppose $a \in \overline{A'}$, then every neighborhood of $a$ contains some points of $A'$ distinct from $a$. But by definition of $A'$, each such neighborhood must also contain some points of $A$. This implies that $a$ is an accumulation point of $A$, that is $a \in A'$. Therefore $\overline{A'} \subseteq A'$ and we have $\overline{A'} = A'$.

Finally, from the definition of the Cantor-Bendixson rank and the above properties, if $A$ has Cantor-Bendixson rank $\alpha$, the sets

$$A^{(1)} \supset A^{(2)} \supset \cdots \supset A^{(\alpha)}$$

form a strictly decreasing chain of closed sets.