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metric spaces are Hausdorff

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Suppose we have a space X and a metric d on X. We'd like to show that the metric topology that d gives X is Hausdorff.

Say we've got distinct $x, y \in X$. Since d is a metric, $d(x, y) \neq 0$. Then the open balls $B_x = B(x, \frac{d(x,y)}{2})$ and $B_y = B(y, \frac{d(x,y)}{2})$ are open sets in the metric topology which contain x and y respectively. If we could show B_x and B_y are disjoint, we'd have shown that X is Hausdorff.

We'd like to show that an arbitrary point z can't be in both B_x and B_y. Suppose there is a z in both, and we'll derive a contradiction. Since z is in these open balls, $d(z, x) < \frac{d(x,y)}{2}$ and $d(z, y) < \frac{d(x,y)}{2}$. But then $d(z, x) + d(z, y) < d(x, y)$, contradicting the triangle inequality.

So B_x and B_y are disjoint, and X is Hausdorff.$\square$