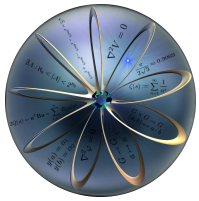

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ultrafilter

Canonical name	Ultrafilter
Date of creation	2013-03-22 12:13:59
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Entry type	Definition
Classification	msc 54A20
Related topic	Filter
Related topic	Ultranet
Related topic	EveryBoundedSequenceHasLimitAlongAnUltrafilter
Related topic	LatticeFilter
Defines	fixed ultrafilter
Defines	principal ultrafilter
Defines	trivial ultrafilter
Defines	free ultrafilter
Defines	non-principal ultrafilter
Defines	nonprincipal ultrafilter
Defines	uniform ultrafilter

Let X be a set.

Definitions

A collection $\mathcal{U}$ of subsets of X is an *ultrafilter* if $\mathcal{U}$ is a filter, and whenever $A \subseteq X$ then either $A \in \mathcal{U}$ or $X \setminus A \in \mathcal{U}$.

Equivalently, an ultrafilter on X is a `http://planetmath.org/MaximalElement`maximal filter on X.

More generally, an `http://planetmath.org/LatticeFilter`ultrafilter of a lattice is a maximal proper filter of the lattice. This is indeed a generalization, as an ultrafilter on X can then be defined as an ultrafilter of the power set $\mathcal{P}(X)$.

Types of ultrafilter

For any $x \in X$ the set $\{A \subseteq X \mid x \in A\}$ is an ultrafilter on X. An ultrafilter formed in this way is called a *fixed ultrafilter*, or a *principal ultrafilter*, or a *trivial ultrafilter*. Any other ultrafilter on X is called a *free ultrafilter*, or a *non-principal ultrafilter*. An ultrafilter on a finite set is necessarily fixed. On any infinite set there are free ultrafilters (`http://planetmath.org/NumberOfUltrafilters`in great abundance), but their existence depends on the Axiom of Choice, and so none can be explicitly constructed.

An ultrafilter $\mathcal{U}$ on X is called a *uniform ultrafilter* if every member of $\mathcal{U}$ has the same cardinality. (An ultrafilter on a singleton is uniform, but this is a degenerate case and is often excluded. All other uniform ultrafilters are free.)