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product topology preserves the Hausdorff property

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Theorem Suppose $\{X_\alpha\}_{\alpha\in A}$ is a collection of Hausdorff spaces. Then the generalized Cartesian product $\prod_{\alpha\in A} X_\alpha$ equipped with the product topology is a Hausdorff space.

Proof. Let $Y=\prod_{\alpha\in A} X_\alpha$, and let x, y be distinct points in Y. Then there is an index $\beta\in A$ such that $x(\beta)$ and $y(\beta)$ are distinct points in the Hausdorff space X_β. It follows that there are open sets U and V in X_β such that $x(\beta)\in U$, $y(\beta)\in V$, and $U\cap V=\emptyset$. Let π_β be the projection operator $Y\to X_\beta$ defined `http://planetmath.org/GeneralizedCartesianProduct`here. By the definition of the product topology, π_β is continuous, so $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are open sets in Y. Also, since the `http://planetmath.org/InverseImageCommutesWithSe` commutes with set operations, we have that

$$\begin{aligned}\pi_\beta^{-1}(U)\cap\pi_\beta^{-1}(V) &= \pi_\beta^{-1}\big(U\cap V\big)\\ &= \emptyset.\end{aligned}$$

Finally, since $x(\beta)\in U$, i.e., $\pi_\beta(x)\in U$, it follows that $x\in\pi_\beta^{-1}(U)$. Similarly, $y\in\pi_\beta^{-1}(V)$. We have shown that U and V are open disjoint neighborhoods of x respectively y. In other words, Y is a Hausdorff space. □