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tube lemma

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Tube lemma - Let X and Y be topological spaces such that Y is compact. If N is an open set of $X \times Y$ containing a "slice" $x_0 \times Y$, then N contains some "tube" $W \times Y$, where W is a neighborhood of x_0 in X.

Proof : N is a union of basis elements $U \times V$, with U and V open sets in X and Y respect. Since $x_0 \times Y$ is compact (it is homeomorphic to Y), only a finite number $U_1 \times V_1, \ldots, U_n \times V_n$ of such basis elements cover $x_0 \times Y$.

We may assume that each of the basis elements $U_i \times V_i$ actually intersects $x_0 \times Y$, since otherwise we could discard it from the finite collection and still have a covering of $x_0 \times Y$.

Define $W := U_1 \cap \cdots \cap U_n$. The set W is open and contains x_0 because each $U_i \times V_i$ intersects $x_0 \times Y$ by the previous remark.

We now claim that $W \times Y \subseteq N$. Let (x, y) be a point in $W \times Y$. The point (x_0, y) is in some $U_i \times V_i$ and so $y \in V_i$. We also know that $x \in W = U_1 \cap \cdots \cap U_n \subseteq U_i$.

Therefore $(x, y) \in U_i \times V_i \subseteq N$ as desired. $\square$

References

[1] J. Munkres, *Topology* (2nd edition), Prentice Hall, 1999.