near operators

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Contents

1 Perturbations and small perturbations: definitions and some results

We start our discussion on the Campanato theory of near operators with some preliminary tools.

Let X, Y be two sets and let a metric d be defined on Y. If $F\colon X \to Y$ is an injective map, we can define a metric on X by putting

$$d_F(x', x'') = d(F(x'), F(x'')).$$

Indeed, d_F is zero if and only if $x' = x''$ (since F is injective); d_F is obviously symmetric and the triangle inequality follows from the triangle inequality of d.

Moreover, if $F(X)$ is a complete subspace of Y, then X is complete with respect to the metric d_F.

Indeed, let (u_n) be a Cauchy sequence in X. By definition of d, then $(F(u_n))$ is a Cauchy sequence in Y, and in particular in $F(X)$, which is complete. Thus, there exists $y_0 = F(x_0) \in F(X)$ which is limit of the sequence $(F(u_n))$. x_0 is the limit of (x_n) in (X, d_F), which completes the proof.

A particular case of the previous statement is when F is onto (and thus a bijection) and (Y, d) is complete.

Similarly, if $F(X)$ is compact in Y, then X is compact with the metric d_F.

Definition 1.1 Let X be a set and Y be a metric space. Let F, G be two maps from X to Y. We say that G is a perturbation of F if there exist a constant $k > 0$ such that for each $x', x'' \in X$ one has:

$$d(G(x'), G(x'')) \leq k d(F(x'), F(x''))$$

Remark 1.2 In particular, if F is injective then G is a perturbation of F if G is uniformly continuous with respect to the metric induced on X by F.

Definition 1.3 In the same hypothesis as in the previous definition, we say that G is a small perturbation of F if it is a perturbation of constant $k < 1$.

We can now prove this generalization of the Banach-Caccioppoli fixed point theorem:

Theorem 1.4 *Let X be a set and (Y, d) be a complete metric space. Let F, G be two mappings from X to Y such that:*

1. *F is bijective;*

2. *G is a small perturbation of F.*

Then, there exists a unique $u \in X$ such that $G(u) = F(u)$

Proof. The hypothesis (**??**) ensures that the metric space (X, d_F) is complete. If we now consider the function $T : X \to X$ defined by

$$T(x) = F^{-1}(G(x))$$

we note that, by (**??**), we have

$$d(G(x'), G(x'')) \leq kd(F(x'), F(x''))$$

where $k \in (0, 1)$ is the constant of the small perturbation; note that, by the definition of d_F and applying $F \circ F^{-1}$ to the first side, the last equation can be rewritten as

$$d_F(T(x'), T(x'')) \leq kd_F(x', x'');$$

in other words, since $k < 1$, T is a contraction in the complete metric space (X, d_F); therefore (by the classical Banach-Caccioppoli fixed point theorem) T has a unique fixed point: there exist $u \in X$ such that $T(u) = u$; by definition of T this is equivalent to $G(u) = F(u)$, and the proof is hence complete. □

Remark 1.5 The hypothesis of the theorem can be generalized as such: let X be a set and Y a metric space (not necessarily complete); let F, G be two mappings from X to Y such that F is injective, $F(X)$ is complete and $G(X) \subseteq F(X)$; then there exists $u \in X$ such that $G(u) = F(u)$.

(Apply the theorem using $F(X)$ instead of Y as target space.)

Remark 1.6 The Banach-Caccioppoli fixed point theorem is obtained when $X = Y$ and F is the identity.

We can use theorem **??** to prove a result that applies to perturbations which are not necessarily small (i.e. for which the constant k can be greater than one). To prove it, we must assume some supplemental structure on the metric of Y: in particular, we have to assume that the metric d is invariant by dilations, that is that $d(\alpha y', \alpha y'') = \alpha d(y', y'')$ for each $y', y'' \in Y$. The most common case of such a metric is when the metric is deduced from a norm (i.e. when Y is a normed space, and in particular a Banach space). The result follows immediately:

Corollary 1.7 *Let X be a set and (Y, d) be a complete metric space with a metric d invariant by dilations. Let F, G be two mappings from X to Y such that F is bijective and G is a perturbation of F, with constant $K > 0$.*

Then, for each $M > K$ there exists a unique $u_M \in X$ such that $G(u) = MF(u)$

Proof. The proof is an immediate consequence of theorem **??** given that the map $\tilde{G}(u) = G(u)/M$ is a small perturbation of F (a property which is ensured by the dilation invariance of the metric d). □

We also have the following

Corollary 1.8 *Let X be a set and (Y, d) be a complete, compact metric space with a metric d invariant by dilations. Let F, G be two mappings from X to Y such that F is bijective and G is a perturbation of F, with constant $K > 0$.*

Then there exists at least one $u_K \in X$ such that $G(u_\infty) = KF(u_\infty)$

Proof. Let (a_n) be a decreasing sequence of real numbers greater than one, converging to one ($a_n \downarrow 1$) and let $M_n = a_n K$ for each $n \in \mathbb{N}$. We can apply corollary **??** to each M_n, obtaining a sequence u_n of elements of X for which one has

$$G(u_n) = M_n F(u_n). \tag{1}$$

Since (X, d_F) is compact, there exist a subsequence of u_n which converges to some u_∞; by continuity of G and F we can pass to the limit in (**??**), obtaining

$$G(u_\infty) = KF(u_\infty)$$

which completes the proof. □

Remark 1.9 For theorem **??** we cannot ensure uniqueness of u_∞, since in general the sequence u_n may change with the choice of a_n, and the limit might be different. So the corollary can only be applied as an existence theorem.

2 Near operators

We can now introduce the concept of near operators and discuss some of their properties.

A historical remark: Campanato initially introduced the concept in Hilbert spaces; subsequently, it was remarked that most of the theory could more generally be applied to Banach spaces; indeed, it was also proven that the basic definition can be generalized to make part of the theory available in the more general environment of metric vector spaces.

We will here discuss the theory in the case of Banach spaces, with only a couple of exceptions: to see some of the extra properties that are available in Hilbert spaces and to discuss a generalization of the Lax-Milgram theorem to metric vector spaces.

2.1 Basic definitions and properties

Definition 2.1 Let X be a set and Y a Banach space. Let A, B be two operators from X to Y. We say that A is near B if and only if there exist two constants $\alpha > 0$ and $k \in (0, 1)$ such that, for each $x', x'' \in X$ one has

$$\|B(x') - B(x'') - \alpha(A(x') - A(x''))\| \leq k\|B(x') - B(x'')\|$$

In other words, A is near B if $B - \alpha A$ is a small perturbation of B for an appropriate value of α.

Observe that in general the property is not symmetric: if A is near B, it is not necessarily true that B is near A; as we will briefly see, this can only be proven if $\alpha < 1/2$, or in the case that Y is a Hilbert space, by using an equivalent condition that will be discussed later on. Yet it is possible to define a topology with some interesting properties on the space of operators, by using the concept of nearness to form a base.

The core point of the nearness between operators is that it allows us to "transfer" many important properties from B to A; in other words, if B satisfies certain properties, and A is near B, then A satisfies the same properties. To prove this, and to enumerate some of these "nearness-invariant" properties, we will emerge a few important facts.

In what follows, unless differently specified, we will always assume that X is a set, Y is a Banach space and A, B are two operators from X to Y.

Lemma 2.2 *If A is near B then there exist two positive constants M_1, M_2 such that*

$$\|B(x') - B(x'')\| \leq M_1 \|A(x') - A(x'')\|$$
$$\|A(x') - A(x'')\| \leq M_2 \|B(x') - B(x'')\|$$

Proof. We have:

$$\begin{aligned}
&\|B(x') - B(x'')\| \leq \\
&\quad \leq \|B(x') - B(x'') - \alpha(A(x') - A(x''))\| + \alpha\|A(x') - A(x'')\| \leq \\
&\qquad \leq k\|B(x') - B(x'')\| + \alpha\|A(x') - A(x'')\|
\end{aligned}$$

and hence
$$\|B(x') - B(x'')\| \leq \frac{\alpha}{1-k}\|A(x') - A(x'')\|$$
which is the first inequality with $M_1 = \alpha/(1-k)$ (which is positive since $k < 1$).

But also

$$\begin{aligned}
&\|A(x') - A(x'')\| \leq \\
&\quad \leq \frac{1}{\alpha}\|B(x') - B(x'') - \alpha(A(x') - A(x''))\| + \frac{1}{\alpha}\|B(x') - B(x'')\| \leq \\
&\qquad \leq \frac{k}{\alpha}\|B(x') - B(x'')\| + \frac{1}{\alpha}\|B(x') - B(x'')\|
\end{aligned}$$

and hence
$$\|A(x') - A(x'')\| \leq \frac{1+k}{\alpha}\|B(x') - B(x'')\|$$
which is the second inequality with $M_2 = (1+k)/\alpha$. □

The most important corollary of the previous lemma is the following

Corollary 2.3 *If A is near B then two points of X have the same image under A if and only if the have the same image under B.*

We can express the previous concept in the following formal way: for each y in $B(X)$ there exist z in Y such that $A(B^{-1}(y)) = \{z\}$ and conversely. In yet other words: each fiber of A is a fiber (for a different point) of B, and conversely.

It is therefore possible to define a map $T_A : B(X) \to Y$ by putting $T_A(y) = z$; the range of T_A is $A(X)$. Conversely, it is possible to define $T_B : A(X) \to Y$, by putting $T_B(z) = y$; the range of T_B is $B(X)$. Both maps are injective and, if restricted to their respective ranges, one is the inverse of the other.

Also observe that T_B and T_A are continuous. This follows from the fact that for each $x \in X$ one has

$$T_A(B(x)) = A(x), \qquad T_B(A(x)) = B(x)$$

and that the lemma ensures that given a sequence (x_n) in X, the sequence $(B(x_n))$ converges to $B(x_0)$ if and only if $(A(x_n))$ converges to $A(x_0)$.

We can now list some invariant properties of operators with respect to nearness. The properties are given in the form "if and only if" because each operator is near itself (therefore ensuring the "only if" part).

1. a map is injective if and only if it is near an injective operator;
2. a map is surjective if and only if it is near a surjective operator;
3. a map is open if and only if it is near an open map;
4. a map has dense range if and only if it is near a map with dense range.

To prove (**??**) it is necessary to use theorem **??**.

Another important property that follows from the lemma is that if there exist $y \in Y$ such that $A^{-1}(y) \cap B^{-1}(y) \neq \varnothing$, then it is $A^{-1}(y) = B^{-1}(y)$: intersecting fibers are equal. (Campanato only stated this property for the case $y = 0$ and called it "'the kernel property"; I prefer to call it the "fiber persistence" property.)

2.1.1 A topology based on nearness

In this section we will show that the concept of nearness between operator can indeed be connected to a topological understanding of the set of maps from X to Y.

Let $\mathcal{M}$ be the set of maps between X and Y. For each $F \in \mathcal{M}$ and for each $k \in (0, 1)$ we let $U_k(F)$ the set of all maps $G \in \mathcal{M}$ such that $F - G$ is a small perturbation of F with constant k. In other words, $G \in U_k(F)$ if and only if G is near F with constants $1, k$.

The set $\mathcal{U}(F) = \{U_k(F) \mid 0 < k < 1\}$ satisfies the axioms of the set of fundamental neighbourhoods. Indeed:

1. F belongs to each $U_k(F)$;
2. $U_k(F) \subset U_h(F)$ if and only if $k < h$, and thus the intersection property of neighbourhoods is trivial;
3. for each $U_k(F)$ there exist $U_h(F)$ such that for each $G \in U_h(F)$ there exist $U_j(G) \subseteq U_k(F)$.

This last property (permanence of neighbourhoods) is somewhat less trivial, so we shall now prove it.

Proof. Let $U_k(F)$ be given.

Let $U_h(F)$ be another arbitrary neighbourhood of F and let G be an arbitrary element in it. We then have:

$$\|F(x') - F(x'') - (G(x') - G(x''))\| \leq h\|F(x') - F(x'')\|. \tag{2}$$

but also (lemma **??**)

$$\|(G(x') - G(x''))\| \leq (1 + h)\|F(x') - F(x'')\|. \tag{3}$$

Let also $U_j(G)$ be an arbitrary neighbourhood of G and H an arbitrary element in it. We then have:

$$\|G(x') - G(x'') - (H(x') - H(x''))\| \leq j\|G(x') - G(x'')\|. \tag{4}$$

The nearness between F and H is calculated as such:

$$\begin{gathered} \|F(x') - F(x'') - (H(x') - H(x''))\| \leq \\ \|F(x') - F(x'') - (G(x') - G(x''))\| + \|G(x') - G(x'') - (H(x') - H(x''))\| \leq \\ h\|F(x') - F(x'')\| + j\|G(x') - G(x'')\| \leq (h + j(1 + h))\|F(x') - F(x'')\|. \end{gathered} \tag{5}$$

We then want $h + j(1 + h) \leq k$, that is $j \leq (k - h)/(1 + h)$; the condition $0 < j < 1$ is always satisfied on the right side, and the left side gives us $h < k$. □

It is important to observe that the topology generated this way is not a Hausdorff topology: indeed, it is not possible to separate F and $F+y$ (where $F \in \mathcal{M}$ and y is a constant element of Y). On the other hand, the subset of all maps with with a fixed valued at a fixed point ($F(x_0) = y_0$) is a Hausdorff subspace.

Another important characteristic of the topology is that the set $\mathcal{H}$ of invertible operators from X to Y is open in $\mathcal{M}$ (because a map is invertible if and only if it is near an invertible map). This is not true in the topology of uniform convergence, as is easily seen by choosing $X = Y = \mathbb{R}$ and the sequence with generic element $F_n(x) = x^3 - x/n$: the sequence converges (in the uniform convergence topology) to $F(x) = x^3$, which is invertible, but none of the F_n is invertible. Hence F is an element of $\mathcal{H}$ which is not inside $\mathcal{H}$, and $\mathcal{H}$ is not open.

2.2 Some applications

As we mentioned in the introduction, the Campanato theory of near operators allows us to generalize some important theorems; we will now present some generalizations of the Lax-Milgram theorem, and a generalization of the Riesz representation theorem.

[TODO]