proof that products of connected spaces are connected

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Let $\{X_\alpha \text{ for } \alpha \in A\}$ be topological spaces, and let $X = \prod X_\alpha$ be the product, with projection maps π_α.

Using the Axiom of Choice, one can straightforwardly show that each π_α is surjective; they are continuous by definition, and the continuous image of a connected space is connected, so if X is connected, then all X_α are.

Let $\{X_\alpha \text{ for } \alpha \in A\}$ be connected topological spaces, and let $X = \prod X_\alpha$ be the product, with projection maps π_α.

First note that each π_α is an open map: If U is open, then it is the union of open sets of the form $\bigcap_{\beta \in F} \pi_\beta^{-1} U_\beta$ where F is a finite subset of A and U_β is an open set in X_β. But $\pi_\alpha(U_\beta)$ is always open, and the image of a union is the union of the images.

Suppose the product is the disjoint union of open sets U and V, and suppose U and V are nonempty. Then there is an $\alpha \in A$ and an element $u \in U$ and an element $v \in V$ that differ only in the α place. To see this, observe that for all but finitely many places γ, both $\pi_\beta(U)$ and $\pi_\beta(V)$ must be X_γ, so there are elements u and v that differ in finitely many places. But then since U and V are supposed to cover X, if $\pi_\beta(u) \neq \pi_\beta(v)$, changing u in the β place lands us in either U or V. If it lands us in V, we have elements that differ in only one place. Otherwise, we can make a $u' \in U$ such that $\pi_\beta(u') = \pi_\beta(v)$ and which otherwise agrees with u. Then by induction we can obtain elements $u \in U$ and $v \in V$ that differ in only one place. Call that place α.

We then have a map $\rho : X_\alpha \to X$ such that $\pi_\alpha \circ \rho$ is the identity map on X_α, and $(\rho \circ \pi)(u) = u$. Observe that since π_α is open, ρ is continuous. But $\rho^{-1}(U)$ and $\rho^{-1}(V)$ are disjoint nonempty open sets that cover X_α, which is impossible.

Note that if we do not assume the Axiom of Choice, the product may be empty, and hence connected, whether or not the X_α are connected; by taking the discrete topology on some X_α we get a counterexample to one direction of the theorem: we have a connected (empty!) space that is the product of non-connected spaces. For the other direction, if the product is empty, it is connected; if it is not empty, then the argument below works unchanged. So without the Axiom of Choice, this theorem becomes “If all X_α are connected, then X is.”