KKM lemma

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1 Preliminaries

We start by introducing some standard notation. $\mathbb{R}^{n+1}$ is the $(n+1)$-dimensional real space with Euclidean norm and metric. For a subset $A \subset \mathbb{R}^{n+1}$ we denote by $\mathrm{diam}(A)$ the diameter of A.

The n-dimensional simplex $\mathcal{S}_n$ is the following subset of $\mathbb{R}^{n+1}$

$$\left\{ (\alpha_1, \alpha_2, \ldots, \alpha_{n+1}) \,\Big|\, \sum_{i=1}^{n+1} \alpha_i = 1, \quad \alpha_i \geq 0 \quad \forall i = 1, \ldots, n+1 \right\}$$

More generally, if $V = \{v_1, v_2, \ldots, v_k\}$ is a set of vectors, then $S(V)$ is the simplex spanned by V:

$$S(V) = \left\{ \sum_{i=1}^{k} \alpha_i v_i \;, \Big|\, \sum_{i=1}^{k} \alpha_i = 1, \quad \alpha_i \geq 0 \quad \forall i = 1, \ldots, k \right\}$$

Let $\mathcal{E} = \{e_1, e_1, \ldots, e_{n+1}\}$ be the standard orthonormal basis of $\mathbb{R}^{n+1}$. So, $\mathcal{S}_n$ is the simplex spanned by $\mathcal{E}$. Any element v of $S(V)$ is represented by a vector of coordinates $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that $v = \sum_i \alpha_i v_i$; these are called a barycentric coordinates of v. If the set V is in general position then the above representation is unique and we say that V is a basis for $S(V)$. If we write $S(V)$ then V is always a basis.

Let v be in $S(V)$, $V = \{v_1, v_2, \ldots, v_k\}$ a basis and v represented (uniquely) by barycentric coordinates $(\alpha_1, \alpha_2, \ldots, \alpha_k)$. We denote by $F_V(v)$ the subset $\{j \,|\, \alpha_j \neq 0\}$ of $\{1, 2, \ldots, k\}$ (i.e., the set of non-null coordinates). Let $I \subset \{1, 2, \ldots, k\}$, the I-th face of $S(V)$ is the set $\{v \in S(V) | F_V(v) \subseteq I\}$. A *face* of $S(V)$ is an I-face for some I (note that this is independent of the choice of basis).

2 KKM Lemma

The main result we prove is the following:

Theorem 1 (Knaster-Kuratowski-Mazurkiewicz Lemma [**?**])**.** *Let $\mathcal{S}_n$ be the standard simplex spanned by $\mathcal{E}$ the standard orthonormal basis for $\mathbb{R}^{n+1}$. Assume we have $n+1$ closed subsets $C_1, \ldots, C_{n+1}$ of $\mathcal{S}_n$ with the property that for every subset I of $\{1, 2, \ldots, n+1\}$ the following holds: the I-th face of $\mathcal{S}_n$ is a subset of $\cup_{i \in I} C_i$. Then, the intersection of the sets $C_1, C_2, \ldots, C_{n+1}$ is non-empty.*

We prove the KKM Lemma by using Sperner's Lemma; Sperner's Lemma is based on the notion of simplicial subdivision and coloring.

Definition 2 (Simplicial subdivision of $\mathcal{S}_n$)**.** *A* simplicial subdivision *of $\mathcal{S}_n$ is a couple $D = (V, \mathcal{B})$; V are the vertices, a finite subset of $\mathcal{S}_n$; $\mathcal{B}$ is a set of simplexes $S(V_1), S(V_2), \ldots, S(V_k)$ where each V_i is a subset of V of size $n + 1$. D has the following properties:*

1. *The union of the simplexes in $\mathcal{B}$ is $\mathcal{S}_n$.*
2. *If $S(V_i)$ and $S(V_j)$ intersect then the intersection is a face of both $S(V_i)$ and $S(V_j)$.*

The norm *of D, denoted by $|D|$, is the diameter of the largest simplex in $\mathcal{B}$.*

An $(n + 1)$-*coloring* of a subdivision $D = (V, \mathcal{B})$ of $\mathcal{S}_n$ is a function

$$C : V \to \{1, 2, \ldots, n + 1\}$$

A *Sperner Coloring* of D is an $(n + 1)$-coloring C such that $C(v) \in F_{\mathcal{E}}(v)$ for every $v \in V$, that is, if v is on the I-th face then its color is from I. For example, if $D = (V, \mathcal{B})$ is a subdivision of the standard simplex $\mathcal{S}_n$ then the standard basis $\mathcal{E}$ is a subset of V and $F_{\mathcal{E}}(e_i) = i$. Hence, If C is a Sperner Coloring of D then $C(e_i) = i$ for all $i = 1, 2, \ldots, n + 1$.

Theorem 3 (Sperner's Lemma)**.** *Let $D = (V, \mathcal{B})$ be a simplicial subdivision of $\mathcal{S}_n$ and $C : V \to \{1, 2, \ldots, n + 1\}$ a Sperner Coloring of D. Then, there is a simplex $S(V_i) \in \mathcal{B}$ such that $C(V_i) = \{1, 2, \ldots, n + 1\}$.*

It is a standard result, for example by barycentric subdivisions, that $\mathcal{S}_n$ has a sequence of simplicial subdivisions $D_1, D_2, \ldots$ such that $|D_i| \to 0$. We use this fact to prove the KKM Lemma:

Proof of KKM Lemma. Let $C_1, C_2, \ldots, C_{n+1}$ be closed subsets of $\mathcal{S}_n$ as given in the lemma. We define the following function $\gamma : \mathcal{S}_n \to \{1, 2, \ldots, n + 1\}$ as follows:

$$\gamma(v) = \min\{i | i \in F_{\mathcal{E}}(v) \text{ and } v \in C_i\}$$

γ is well defined by the hypothesis of the lemma and $\gamma(v) \in F_{\mathcal{E}}(v)$. Also, if $\gamma(v) = i$ then $v \in C_i$. Let $D_1, D_2, \ldots$ be a sequence of simplicial subdivisions such that $|D_i| \to 0$. We set the color of every vertex v in D_i to be $\gamma(v)$. This is a Sperner Coloring since if v is in I-fact then $\gamma(v) \in F_{\mathcal{E}}(v) \subseteq I$.

Therefore, by Sperner's Lemma we have in each subdivision D_i a simplex $S(V_i)$ such that $\gamma(V_i) = \{1, 2, \ldots, n+1\}$. Moreover, $\mathrm{diam}(S(V_i)) \to 0$. By the properties of γ for every $i = 1, 2, \ldots$ and every $j \in \{1, 2, \ldots, n+1\}$ we have that $S(V_i) \cap C_j \neq \phi$. Let u_i be the arithmetic mean of the elements of V_i (this is an element of $S(V_i)$ and thus an element of $\mathcal{S}_n$). Since $\mathcal{S}_n$ is bounded and closed we get that u_i has a converging subsequence with a limit $L \in \mathcal{S}_n$. Now, every set C_j is closed, and for every $\epsilon > 0$ we have an element of C_j of ϵ-distance from L; thus L is in C_j.

Therefore, L is in the intersection of all the sets $C_1, C_2, \ldots, C_{n+1}$, and that proves the assertion. $\square$

References

[1] B. Knaster, C. Kuratowski, and S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe, *Fund. Math. 14 (1929) 132-137.*