conditions for a collection of subsets to be a basis for some topology

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Not just any collection of subsets of X can be a basis for a topology on X. For instance, if we took $\mathcal{C}$ to be all open intervals of length 1 in $\mathbb{R}$, $\mathcal{C}$ isn't the basis for any topology on $\mathbb{R}$: $(0,1)$ and $(.5,1.5)$ are unions of elements of $\mathcal{C}$, but their intersection $(.5,1)$ is not. The collection formed by arbitrary unions of members of $\mathcal{C}$ isn't closed under finite intersections and isn't a topology.

We'd like to know which collections $\mathcal{B}$ of subsets of X could be the basis for some topology on X. Here's the result:

Theorem. A collection $\mathcal{B}$ of subsets of X is a basis for some topology on X if and only if:

1. Every $x \in X$ is contained in some $B_x \in \mathcal{B}$, and
2. If B_1 and B_2 are two elements of $\mathcal{B}$ containing $x \in X$, then there's a third element B_3 of $\mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Proof. First, we'll show that if $\mathcal{B}$ is the basis for some topology $\mathcal{T}$ on X, then it satisfies the two conditions listed.

$\mathcal{T}$ is a topology on X, so $X \in \mathcal{T}$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, that means X can be written as a union of members of $\mathcal{B}$: since every $x \in X$ is in this union, every $x \in X$ is contained in some member of $\mathcal{B}$. That takes care of the first condition.

For the second condition: if B_1 and B_2 are elements of $\mathcal{B}$, they're also in $\mathcal{T}$. $\mathcal{T}$ is closed under intersection, so $B_1 \cap B_2$ is open in $\mathcal{T}$. Then $B_1 \cap B_2$ can be written as a union of members of $\mathcal{B}$, and any $x \in B_1 \cap B_2$ is contained by some basis element in this union.

Second, we'll show that if a collection $\mathcal{B}$ of subsets of X satisfies the two conditions, then the collection $\mathcal{T}$ of unions of members of $\mathcal{B}$ is a topology on X.

- $\emptyset \in \mathcal{T}$: $\emptyset$ is the null union of zero elements of $\mathcal{B}$.
- $X \in \mathcal{T}$: by the first condition, every X is contained in some member of $\mathcal{B}$. The union of all the members of $\mathcal{B}$ is then all of X.
- $\mathcal{T}$ is closed under arbitrary unions: Say we have a union of sets $T_\alpha \in \mathcal{T}$...

$$\bigcup_{\alpha \in I} T_\alpha = \bigcup_{\alpha \in I} \bigcup_{\beta \in J_\alpha} B_\beta$$

(since each T_α is a union of sets in $\mathcal{B}$)

$$= \bigcup_{\beta \in \bigcup_{\alpha \in I} J_\alpha} B_\beta$$

Since that's a union of elements of $\mathcal{B}$, it's also a member of $\mathcal{T}$.

- $\mathcal{T}$ is closed under finite intersections: since a collection of sets is closed under finite intersections if and only if it is closed under pairwise intersections, we need only check that the intersection of two members T_1, T_2 of $\mathcal{T}$ is in $\mathcal{T}$.

  Any $x \in T_1 \cap T_2$ is contained in some $B_x^1 \subset T_1$ and $B_x^2 \subset T_2$. By the second condition, $x \in B_x^1 \cap B_x^2$ gets us a B_x^3 with $x \in B_x^3 \subset B_x^1 \cap B_x^2 \subset T_1 \cap T_2$. Then

  $$T_1 \cap T_2 = \bigcup_{x \in T_1 \cap T_2} B_x^3$$

  which is in $\mathcal{T}$.

$\square$