proof that a compact set in a Hausdorff space is closed

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Let X be a Hausdorff space, and $C \subseteq X$ a compact subset. We are to show that C is closed. We will do so, by showing that the complement $U = X \setminus C$ is open. To prove that U is open, it suffices to demonstrate that, for each $x \in U$, there exists an open set V with $x \in V$ and $V \subseteq U$.

Fix $x \in U$. For each $y \in C$, using the Hausdorff assumption, choose disjoint open sets A_y and B_y with $x \in A_y$ and $y \in B_y$.

Since every $y \in C$ is an element of B_y, the collection $\{B_y \mid y \in C\}$ is an open covering of C. Since C is compact, this open cover admits a finite subcover. So choose $y_1, \ldots, y_n \in C$ such that $C \subseteq B_{y_1} \cup \cdots \cup B_{y_n}$.

Notice that $A_{y_1} \cap \cdots \cap A_{y_n}$, being a finite intersection of open sets, is open, and contains x. Call this neighborhood of x by the name V. All we need to do is show that $V \subseteq U$.

For any point $z \in C$, we have $z \in B_{y_1} \cup \cdots \cup B_{y_n}$, and therefore $z \in B_{y_k}$ for some k. Since A_{y_k} and B_{y_k} are disjoint, $z \notin A_{y_k}$, and therefore $z \notin A_{y_1} \cap \cdots \cap A_{y_n} = V$. Thus C is disjoint from V, and V is contained in U.