another proof of the non-existence of a continuous function that switches the rational and the irrational numbers

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Let $\mathbb{J} = \mathbb{R} \setminus \mathbb{Q}$ denote the irrationals. There is no continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subseteq \mathbb{J}$ and $f(\mathbb{J}) \subseteq \mathbb{Q}$.

Proof

Suppose f is such a function. Since $\mathbb{Q}$ is countable, $f(\mathbb{Q})$ and $f(\mathbb{J})$ are also countable. Therefore the image of f is countable. If f is not a constant function, then by the intermediate value theorem the image of f contains a nonempty interval, so the image of f is uncountable. We have just shown that this isn't the case, so there must be some c such that $f(x) = c$ for all $x \in \mathbb{R}$. Therefore $f(\mathbb{Q}) = \{c\} \subset \mathbb{J}$ and $f(\mathbb{J}) = \{c\} \subset \mathbb{Q}$. Obviously no number is both rational and irrational, so no such f exists.