compactly supported continuous functions are dense in L^p

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Let $(X, \mathcal{B}, \mu)$ be a measure space, where X is a locally compact Hausdorff space, $\mathcal{B}$ a http://planetmath.org/SigmaAlgebraσ-algebra that contains all compact subsets of X and μ a measure such that:

- $\mu(K) < \infty$ for all compact sets $K \subset X$.
- μ is inner regular, meaning $\mu(A) = \sup\{\mu(K) : K \subset A,\ K \text{ is compact}\}$
- μ is outer regular, meaning $\mu(A) = \inf\{\mu(U) : A \subset U,\ U \in \mathcal{B}\text{and } U \text{ is open}\}$

We denote by $C_c(X)$ the space of continuous functions $X \to \mathbb{C}$ with compact support.

Theroem - For every $1 \leq p < \infty$, $C_c(X)$ is dense in http://planetmath.org/LpSpace$L^p(X)$.

: It is clear that $C_c(X)$ is indeed contained in $L^p(X)$, where we identify each function in $C_c(X)$ with its class in $L^p(X)$.

We begin by proving that for each $A \in \mathcal{B}$ with finite measure, the characteristic function χ_A can be approximated, in the L^p norm, by functions in $C_c(X)$. Let $\epsilon > 0$. By of μ, we know there exist an open set U and a compact set K such that $K \subset A \subset U$ and

$$\mu(U \setminus K) = \mu(U) - \mu(K) < \epsilon$$

By the http://planetmath.org/ApplicationsOfUrysohnsLemmaToLocallyCompactHausdorf lemma for locally compact Hausdorff spaces, we know there is a function $f \in C_c(X)$ such that $0 \leq f \leq 1$, $f|_K = 1$ and $\operatorname{supp} f \subset U$. Hence,

$$\int_X |\chi_A - f|^p \, d\mu = \int_{U \setminus K} |\chi_A - f|^p \, d\mu < \epsilon$$

Thus, χ_A can be approximated in L^p by functions in $C_c(X)$.

Now, it follows easily that any simple function $\sum_{i=1}^n c_i \chi_{A_i}$, where each A_i has finite measure, can also be approximated by a compactly supported continuous function. Since this kind of simple functions are dense in $L^p(X)$ we see that $C_c(X)$ is also dense in $L^p(X)$. $\square$