rational numbers are real numbers

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Let us first show that the natural numbers $0, 1, 2, \ldots$ are contained in the real numbers as constructed above. Heuristically, this should be clear. We start with 0. By adding 1 repeatedly we obtain the natural numbers

$$0, \quad 0+1, \quad (0+1)+1, \quad ((0+1)+1)+1, \ldots,$$

To make this precise, let $\mathbb{N}$ be the natural numbers. (We assume that these exist. For example, all the usual constructions of $\mathbb{R}$ rely on the existence of the natural numbers.) Then we can define a map $f\colon \mathbb{N} \to \mathbb{R}$ as

1. $f(0) = 0$, or more precisely, $f(0_{\mathbb{N}}) = 0_{\mathbb{R}}$,
2. $f(a+1) = f(a) + 1$ for $a \in \mathbb{N}$.

By induction on a one can prove that

$$\begin{aligned} f(a+b) &= f(a) + f(b), \\ f(ab) &= f(a)f(b), \quad a, b \in \mathbb{N} \end{aligned}$$

and

$$f(a) \geq 0, \; a \in \mathbb{N} \text{ with equality only when } a = 0.$$

The last claim follows since $f(a) > 0$ for $a = 1, 2, \ldots$ (by induction), and $f(0) = 0$. It follows that f is an injection: If $a \leq b$, then $f(a) = f(b)$ implies that $f(a) = f(a) + f(b-a)$, so $a = b$.

To conclude, let us show that $f(\mathbb{N}) \subset \mathbb{R}$ satisfies the Peano axioms with zero element $f(0)$ and sucessor operator

$$\begin{aligned} S\colon f(\mathbb{N}) &\to f(\mathbb{N}) \\ k &\mapsto f(f^{-1}(k) + 1) \end{aligned}$$

First, as f is a bijection, $x = y$ if and only if $S(x) = S(y)$ is clear. Second, if $S(k) = 0$ for some $k = f(a) \in f(\mathbb{N})$, then $a + 1 = 0$; a contradiction. Lastly, the axiom of induction follows since $\mathbb{N}$ satisfies this axiom. We have shown that $f(\mathbb{N})$ are a subset of the real numbers that behave as the natural numbers.

From the natural numbers, the integers and rationals can be defined as

$$\begin{aligned} \mathbb{Z} &= \mathbb{N} \cup \{-z \in \mathbb{R} : z \in \mathbb{N}\}, \\ \mathbb{Q} &= \left\{ \frac{a}{b} : a \in \mathbb{Z}, b \in \mathbb{N} \setminus \{0\} \right\}. \end{aligned}$$

Mathematically, $\mathbb{Z}$ and $\mathbb{Q}$ are subrings of $\mathbb{R}$ that are ring isomorphic to the integers and rationals, respectively.

Other constructions

The above construction follows [**?**]. However, there are also other constructions. For example, in [**?**], natural numbers in $\mathbb{R}$ are defined as follows. First, a set $L \subseteq \mathbb{R}$ is *inductive* if

1. $0 \in L$,
2. if $a \in L$, then $a + 1 \in L$.

Then the natural numbers are defined as real numbers that are contained in all inductive sets. A third approach is to explicitly exhibit the natural numbers when constructing the real numbers. For example, in [**?**], it is shown that the rational numbers form a subfield of $\mathbb{R}$ using explicit Dedekind cuts.

References

[1] H.L. Royden, *Real analysis*, Prentice Hall, 1988.

[2] M. Spivak, *Calculus*, Publish or Perish.

[3] W. Rudin, *Principles of mathematical analysis*, McGraw-Hill, 1976.