neighborhood system on a set

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In point-set topology, a neighborhood system is defined as the set of neighborhoods of some point in the topological space.

However, one can start out with the definition of a "abstract neighborhood system" $\mathfrak{N}$ on an arbitrary set X and define a topology T on X based on this system $\mathfrak{N}$ so that $\mathfrak{N}$ is the neighborhood system of T. This is done as follows:

> Let X be a set and $\mathfrak{N}$ be a subset of $X \times P(X)$, where $P(X)$ is the power set of X. Then $\mathfrak{N}$ is said to be a *abstract neighborhood system* of X if the following conditions are satisfied:
>
> 1. if $(x, U) \in \mathfrak{N}$, then $x \in U$,
> 2. for every $x \in X$, there is a $U \subseteq X$ such that $(x, U) \in \mathfrak{N}$,
> 3. if $(x, U) \in \mathfrak{N}$ and $U \subseteq V \subseteq X$, then $(x, V) \in \mathfrak{N}$,
> 4. if $(x, U), (x, V) \in \mathfrak{N}$, then $(x, U \cap V) \in \mathfrak{N}$,
> 5. if $(x, U) \in \mathfrak{N}$, then there is a $V \subseteq X$ such that
>   - $(x, V) \in \mathfrak{N}$, and
>   - $(y, U) \in \mathfrak{N}$ for all $y \in V$.
>
> In addition, given this $\mathfrak{N}$, define the *abstract neighborhood system around* $x \in X$ to be the subset $\mathfrak{N}_x$ of $\mathfrak{N}$ consisting of all those elements whose first coordinate is x. Evidently, $\mathfrak{N}$ is the disjoint union of $\mathfrak{N}_x$ for all $x \in X$. Finally, let
>
> $$\begin{aligned} T &= \{U \subseteq X \mid \text{for every } x \in U,\ (x, U) \in \mathfrak{N}\} \\ &= \{U \subseteq X \mid \text{for every } x \in U, \text{ there is a } V \subseteq U, \text{ such that } (x, V) \in \mathfrak{N}\}. \end{aligned}$$

The two definitions are the same by condition 3. We assert that T defined above is a topology on X. Furthermore, $T_x := \{U \mid (x, U) \in \mathfrak{N}_x\}$ is the set of neighborhoods of x under T.

Proof. We first show that T is a topology. For every $x \in X$, some $U \subseteq X$, we have $(x, U) \in \mathfrak{N}$ by condition 2. Hence $(x, X) \in \mathfrak{N}$ by condition 3. So $X \in T$. Also, $\varnothing \in T$ is vacuously satisfied, for no $x \in \varnothing$. If $U, V \in T$, then $U \cap V \in T$ by condition 4. Let $\{U_i\}$ be a subset of T whose elements are indexed by I $(i \in I)$. Let $U = \bigcup U_i$. Pick any $x \in U$, then $x \in U_i$ for some

$i \in I$. Since $U_i \in T$, $(x, U_i) \in \mathfrak{N}$. Since $U_i \subseteq U$, $(x, U) \in \mathfrak{N}$ by condition 3, so $U \in T$.

Next, suppose $\mathcal{N}$ is the set of neighborhoods of x under T. We need to show $\mathcal{N} = T_x$:

1. ($\mathcal{N} \subseteq T_x$). If $N \in \mathcal{N}$, then there is $U \in T$ with $x \in U \subseteq N$. But $(x, U) \in \mathfrak{N}$, so by condition 3, $(x, N) \in \mathfrak{N}$, or $(x, N) \in \mathfrak{N}_x$, or $N \in T_x$.

2. ($T_x \subseteq \mathcal{N}$). Pick any $U \in T_x$ and set $W = \{z \mid U \in T_z\}$. Then $x \in W \subseteq U$ by condition 1. We show W is open. This means we need to find, for each $z \in W$, a $V \subseteq W$ such that $(z, V) \in \mathfrak{N}$. If $z \in W$, then $(z, U) \in \mathfrak{N}$. By condition 5, there is $V \in \mathfrak{N}$ such that $(z, V) \in \mathfrak{N}$, and for any $y \in V$, $(y, U) \in \mathfrak{N}$, or $y \in U$ by condition 1. So $y \in W$ by the definition of W, or $V \subseteq W$. Thus W is open and $U \in \mathcal{N}$.

This completes the proof. By the way, W defined above is none other than the interior of U: $W = U^\circ$. $\square$

Remark. Conversely, if T is a topology on X, we can define $\mathfrak{N}_x$ to be the set consisting of (x, U) such that U is a neighborhood of x. The the union $\mathfrak{N}$ of $\mathfrak{N}_x$ for each $x \in X$ satisfies conditions 1 through 5 above:

1. (condition 1): clear

2. (condition 2): because $(x, X) \in \mathfrak{N}$ for each $x \in X$

3. (condition 3): if U is a neighborhood of x and V a supserset of U, then V is also a neighborhood of x

4. (condition 4): if U and V are neighborhoods of x, there are open A, B with $x \in A \subseteq U$ and $x \in B \subseteq V$, so $x \in A \cap B \subseteq U \cap V$, which means $U \cap V$ is a neighborhood of x

5. (condition 5): if U is a neighborhood of x, there is open A with $x \in A \subseteq U$; clearly A is a neighborhood of x and any $y \in A$ has U as neighborhood.

So the definition of a neighborhood system on an arbitrary set gives an alternative way of defining a topology on the set. There is a one-to-one correspondence between the set of topologies on a set and the set of abstract neighborhood systems on the set.