topic entry on real numbers

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Introduction

The real number system may be conceived as an attempt to fill in the gaps in the rational number system. These gaps first became apparent in connection with the Pythagorean theorem, which requires one to extract a square root in order to find the third side of a right triangle two of whose sides are known. Hypossos, a student of Pythagoras, showed that there is no rational number whose square is exactly 2. In particular, this means that there is no rational number which may be used to describe the length of the diagonal of a square the length of whose sides is rational. This result ruined the philosophical program of Pythagoras, which was to describe everything in terms of whole numbers (or ratios of whole numbers) and, according to legend, resulted in Hypossos drowning himself. Eventually, geometers reconciled themselves to the existence of irrational magnitudes and Eudoxos devised his method of exhaustion which allowed one to prove results about irrational magnitudes by considerations of rational magnitudes which are smaller and larger than the the irrational magnitude in question.

Centuries later, Descartes showed how it is systematically possible to reduce questions of geometry to algebra. This brought up the issue of irrational numbers again — if one is going to reformulate everything in terms of algebra, then one cannot have recourse to defining magnitudes geometrically, but have to find some sort of number which can adequately represent things like the hypotenuse of a square with rational sides. At first, such problems of logical consistency were swept under the rug, but eventually mathematicians realized that their subject needed to be put on a firm logical foundation. In particular, Dedekind solved this difficulty by defining the real numbers as a certain type of partition of the set of rational numbers which he termed a cut and defining operations on these numbers, such as addition, subtraction, multiplication, and division in terms of operations on these partitions.

Index of entries on real numbers

The below list presents entries on real numbers in an order suitable for studying the subject.

1. Rational numbers

2. Axiomatic definition of the real numbers.

3. Constructions of real numbers (advanced):

   (a) Dedekind cuts
   (b) http://planetmath.org/RealNumberCauchy sequences
   (c) http://planetmath.org/EveryOrderedFieldWithTheLeastUpperBoundPropertyIsIs of real numbers
   (d) http://planetmath.org/NonIsomorphicCompletionsOfMathbbQReals not isomorphic to p-adic numbers

4. commensurable numbers
5. positive
6. http://planetmath.org/InequalityForRealNumbersInequalities for real numbers
7. index of inequalities
8. rational numbers are real numbers
9. interval
10. nested interval theorem
11. http://planetmath.org/CantorsDiagonalArgumentReal numbers are uncountable
12. Archimedean property
13. Operations for real numbers
    (a) infimum and supremum for real numbers
    (b) minimal and maximal number
    (c) absolute value
    (d) square root
    (e) fraction power
14. http://planetmath.org/TheoryOfAlgebraicNumbersTopic entry on algebraic and transcendental numbers
    (a) http://planetmath.org/IrrationalIrrational number

(b) Transcendental number

(c) http://planetmath.org/AlgebraicNumberAlgebraic number

15. Particular real numbers

    (a) natural log base

    (b) pi

    (c) Mascheroni constant

    (d) golden ratio

Generalizations

There are many generalizations of real numbers. These include the complex numbers, quaternions, extended real numbers, http://planetmath.org/Hyperrealhyperreal numbers, and surreal numbers. Of course the field $\mathbb{R}$ has many other field extensions, e.g. the field $\mathbb{R}(x)$ of the rational functions.