ultrametric

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Any metric $d : X \times X \to \mathbb{R}$ on a set X must satisfy the triangle inequality:

$$(\forall x, y, z) \quad d(x, z) \leq d(x, y) + d(y, z)$$

An *ultrametric* must additionally satisfy a stronger version of the triangle inequality:

$$(\forall x, y, z) \quad d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

Here is an example of an ultrametric on a space with 5 points, labelled a, b, c, d, e:

	a	b	c	d	e
a	0	12	4	6	12
b		0	12	12	5
c			0	6	12
d				0	12
e					0

In the table above, an entry n in the for element x and the for element y indicates that $d(x, y) = n$, where d is the ultrametric. By symmetry of the ultrametric ($d(x, y) = d(y, x)$), the above table yields all values of $d(x, y)$ for all $x, y \in \{a, b, c, d, e\}$.

The ultrametric condition is equivalent to the ultrametric three point condition:

$$(\forall x, y, z) \quad x, y, z \text{ can be renamed such that } d(x, z) \leq d(x, y) = d(y, z)$$

Ultrametrics can be used to model bifurcating hierarchical systems. The distance between nodes in a weight-balanced binary tree is an ultrametric. Similarly, an ultrametric can be modelled by a weight-balanced binary tree, although the choice of tree is not necessarily unique. Tree models of ultrametrics are sometimes called *ultrametric trees.*

The metrics induced by non-Archimedean valuations are ultrametrics.