proof of Polish spaces up to Borel isomorphism

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We show that every uncountable Polish space X is Borel isomorphic to the real numbers. First, there exists a continuous one-to-one and injective function f from Baire space $\mathcal{N}$ to X such that $X \setminus f(\mathcal{N})$ is countable, and such that the inverse from $f(\mathcal{N})$ to $\mathcal{N}$ is Borel measurable (see `http://planetmath.org/InjectiveImagesOfBaireSpace`here). Letting S be any countably infinite subset of X, the same result can be applied to $X \setminus S$, which is also a Polish space. So, there is a continuous and one-to-one function $f\colon \mathcal{N} \to X \setminus S$ such that $S' \equiv X \setminus f(\mathcal{N})$ is countable and such that the inverse defined on $X \setminus S'$ is Borel. Then, S' contains S and is countably infinite. Hence, there is a invertible function g from $\mathbb{N} = \{1, 2, \ldots\}$ to S'. Under the discrete topology on $\mathbb{N}$ this is necessarily a continuous function with Borel measurable inverse. By combining the functions f and g, this gives a continuous, one-to-one and onto function from the `http://planetmath.org/TopologicalSum`disjoint union

$$u\colon \mathcal{N} \coprod \mathbb{N} \to X$$

with Borel measurable inverse. Similarly, the set of real numbers $\mathbb{R}$ with the standard topology is an uncountable Polish space and, therefore, there is a continuous function v from $\mathcal{N} \coprod \mathbb{N}$ to $\mathbb{R}$ with Borel inverse. So, $v \circ u^{-1}$ gives the desired Borel isomorphism from X to $\mathbb{R}$.