uniformities on a set form a complete lattice

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Theorem. *The collection of uniformities on a given set ordered by set inclusion forms a complete lattice.*

Proof. Let X be a set. Let $\mathfrak{U}(X)$ denote the collection of uniformities on X. The coarsest uniformity on X is $\{X \times X\}$, and the finest is the *discrete uniformity*:

$$\{S \subset X \times X \colon \Delta(X) \subseteq S\}.$$

Hence $\mathfrak{U}(X)$ is bounded. To show that $\mathfrak{U}(X)$ is complete, we must prove that it has the least upper bound property.

Suppose $\{\mathcal{U}_\alpha\}_{\alpha \in I}$ is a nonempty family of uniformities on X. Let $\mathcal{B}$ consist of all finite intersections of elements of the $\mathcal{U}_\alpha$. Let us check that $\mathcal{B}$ is a fundamental system of entourages for a uniformity on X.

(B1) Let S, $T \in \mathcal{B}$. Each of S and T is a finite intersection of elements of the $\mathcal{U}_\alpha$, so their intersection is as well. Hence $S \cap T \in \mathcal{B}$.

(B2) Every element of $\mathcal{B}$ is a finite intersection of subsets of $X \times X$ containing $\Delta(X)$. So every element of $\mathcal{B}$ contains the diagonal.

(B3) Let $S \in \mathcal{B}$. Without loss of generality, $S = S_\alpha \cap S_\beta$, where $S_\alpha \in \mathcal{U}_\alpha$ and $S_\beta \in \mathcal{U}_\beta$. Since $S_\alpha \in \mathcal{U}_\alpha$, $S_\alpha^{-1} \in \mathcal{U}_\alpha$. Similarly, $S_\beta^{-1} \in \mathcal{U}_\beta$. Since the process of taking the inverse of a relation commutes with taking finite intersections, $(S_\alpha \cap S_\beta)^{-1} \in \mathcal{B}$.

(B4) Let $S \in \mathcal{B}$. Again suppose $S = S_\alpha \cap S_\beta$ with $S_\alpha \in \mathcal{U}_\alpha$ and $S_\beta \in \mathcal{U}_\beta$. Then there exist $T_\alpha \in \mathcal{U}_\alpha$ and $T_\beta \in \mathcal{U}_\beta$ such that $T_\alpha \circ T_\alpha \subseteq S_\alpha$ and $T_\beta \circ T_\beta \subseteq S_\beta$. The set $T = T_\alpha \cap T_\beta$ is in $\mathcal{U}$, and since $T \circ T$ is a subset of both S_α and S_β, it is a subset of S.

The fundamental system $\mathcal{B}$ generates a uniformity $\mathcal{U}$. By construction, $\mathcal{U}$ is an upper bound of the $\mathcal{U}_\alpha$. But any upper bound of the $\mathcal{U}_\alpha$ would have to contain all finite intersections of elements of the $\mathcal{U}_\alpha$. So $\mathcal{U} = \sup_{\alpha \in I} \mathcal{U}_\alpha$. $\square$

This theorem is useful because it allows us to assert the existence of the coarsest uniform space satisfying a particular property.

Corollary. *Let X be a set and let $\{Y_\alpha\}_{\alpha \in I}$ be a family of uniform spaces. Then for any family of functions $\{f_\alpha \colon X \to Y_\alpha\}$, there is a coarsest uniformity on X making all the f_α uniformly continous.*

The coarsest uniformity making a family of functions uniformly continuous is called the *initial uniformity* or *weak uniformity*.

References

[1] Nicolas Bourbaki, *Elements of Mathematics: General Topology: Part 1*, Hermann, 1966.