triangle inequality

Canonical name	TriangleInequality
Date of creation	2013-03-22 12:14:49
Last modified on	2013-03-22 12:14:49
Owner	drini (3)
Last modified by	drini (3)
Numerical id	12
Author	drini (3)
Entry type	Definition
Classification	msc 54-00
Classification	msc 54E35
Related topic	ProofOfLimitRuleOfProduct
Related topic	TriangleInequalityOfComplexNumbers
Defines	reverse triangle inequality

Let (X, d) be a metric space. The *triangle inequality* states that for any three points $x, y, z \in X$ we have

$$d(x, y) \leq d(x, z) + d(z, y).$$

The name comes from the special case of $\mathbb{R}^n$ with the standard topology, and geometrically meaning that in any triangle, the sum of the lengths of two sides is greater (or equal) than the third.

Actually, the triangle inequality is one of the properties that define a metric, so it holds in any metric space. Two important cases are $\mathbb{R}$ with $d(x, y) = |x - y|$ and $\mathbb{C}$ with $d(x, y) = \|x - y\|$ (here we are using complex modulus, not absolute value).

There is a second triangle inequality, sometimes called the *reverse triangle inequality*, which also holds in any metric space and is derived from the definition of metric:

$$d(x, y) \geq |d(x, z) - d(z, y)|.$$

In Euclidean geometry, this inequality is expressed by saying that each side of a triangle is greater than the difference of the other two.

The reverse triangle inequality can be proved from the first triangle inequality, as we now show.

Let $x, y, z \in X$ be given. For any $a, b, c \in X$, from the first triangle inequality we have:

$$d(a, b) \leq d(a, c) + d(c, b)$$

and thus (using $d(b, c) = d(c, b)$ for any $b, c \in X$):

$$d(a, c) \geq d(a, b) - d(b, c) \tag{1}$$

and writing (**??**) with $a = x, b = z, c = y$:

$$d(x, y) \geq d(x, z) - d(z, y) \tag{2}$$

while writing (**??**) with $a = y, b = z, c = x$ we get:

$$d(y, x) \geq d(y, z) - d(z, x)$$

or

$$d(x, y) \geq d(z, y) - d(x, z); \tag{3}$$

from (**??**) and (**??**), using the properties of the absolute value, it follows finally:

$$d(x, y) \geq |d(x, z) - d(z, y)|$$

which is the second triangle inequality.