equicontinuous

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1 Definition

Let X be a topological space, (Y, d) a metric space and $C(X, Y)$ the set of continuous functions $X \to Y$.

Let $\mathcal{F}$ be a subset of $C(X, Y)$. A function $f \in \mathcal{F}$ is continuous at a point x_0 when given $\epsilon > 0$ there is a neighbourhood U of x_0 such that $d(f(x), f(x_0)) < \epsilon$ for every $x \in U$. When the same neighbourhood U can be chosen for all functions $f \in \mathcal{F}$, the family $\mathcal{F}$ is said to be *equicontinuous*. More precisely:

Definition - Let $\mathcal{F}$ be a subset of $C(X, Y)$. The set of functions $\mathcal{F}$ is said to be **equicontinuous at** $x_0 \in X$ if for every $\epsilon > 0$ there is a neighbourhood U of x_0 such that for every $x \in U$ and every $f \in \mathcal{F}$ we have

$$d(f(x), f(x_0)) < \epsilon$$

The set $\mathcal{F}$ is said to be **equicontinuous** if it is equicontinuous at every point $x \in X$.

2 Examples

- A finite set of functions in $C(X, Y)$ is always equicontinuous.
- When X is also a metric space, a family of functions in $C(X, Y)$ that share the same Lipschitz constant is equicontinuous.
- The family of functions $\{f_n\}_{n \in \mathbb{N}}$, where $f_n : \mathbb{R} \to \mathbb{R}$ is given by $f_n(x) := \arctan(nx)$ is not equicontinuous at 0.

3 Properties

- If a subset $\mathcal{F} \subseteq C(X, Y)$ is totally bounded under the uniform metric, then $\mathcal{F}$ is equicontinuous.
- Suppose X is compact. If a sequence of functions $\{f_n\}$ in $C(X, \mathbb{R}^k)$ is equibounded and equicontinuous, then the sequence $\{f_n\}$ has a uniformly convergent subsequence. (`http://planetmath.org/AscoliArzelaTheorem`Arzel's theorem)

- Let $\{f_n\}$ be a sequence of functions in $C(X, Y)$. If $\{f_n\}$ is equicontinuous and converges pointwise to a function $f : X \to Y$, then f is continuous and $\{f_n\}$ converges to f in the compact-open topology.

References

[1] J. Munkres, *Topology* (2nd edition), Prentice Hall, 1999.