ordered space

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Synonym	ordered topological space
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Synonym	partially ordered space
Synonym	partially ordered topological space
Related topic	OrderTopology
Defines	upper topology
Defines	lower topology
Defines	interval topology
Defines	upper semiclosed
Defines	lower semiclosed
Defines	semiclosed
Defines	pospace
Defines	totally ordered space

Definition. A set X that is both a topological space and a poset is variously called a *topological ordered space*, *ordered topological space*, or simply an *ordered space*. Note that there is no compatibility conditions imposed on X. In other words, the topology $\mathcal{T}$ and the partial ordering $\leq$ on X operate independently of one another.

If the partial order is a total order, then X is called a *totally ordered space*. In some literature, a totally ordered space is called an ordered space. In this entry, however, an ordered space is always a *partially* ordered space.

One can construct an ordered space from a set with fewer structures.

1. For example, any topological space is trivially an ordered space, with the partial order defined by $a \leq b$ iff $a = b$. But this is not so interesting. A more interesting example is to take a T_0 space X, and define $a \leq b$ iff $a \in \overline{\{b\}}$. The relation so defined turns out to be a partial order on X, called the specialization order, making X an ordered space.

2. On the other hand, given any poset P, we can arbitrarily assign a topology on it, making it an ordered space, so that every poset is trivially an ordered space. Again this is not very interesting.

3. A slightly more useful example is to take a poset P, and take

   $$\mathcal{L}(P) := \{P - \uparrow x \mid x \in P\},$$

   the family of all set complements of principal upper sets of P, as the subbasis for the topology $\omega(P)$ of P. The topology $\omega(P)$ so generated is called the *lower topology* on P.

4. Dually, if we take

   $$\mathcal{U}(P) := \{P - \downarrow x \mid x \in P\},$$

   as the subbasis, we get the *upper topology* on P, denoted by $\nu(P)$.

5. In the lower topology $\omega(P)$ of P, if $y \in P - \uparrow x$, then either $y < x$ (strict inequality) or $x \parallel y$ (incomparable with x). If x is an isolated element, then $P - \uparrow x = P - \{x\}$. This means that $\{x\}$ is a closed set. Similarly, $\{x\}$ is closed in the upper topology $\nu(P)$.

   If x is the top element of P, then $\{x\}$ is a closed set in $\omega(P)$, since $P - \uparrow x = P - \{x\}$ is open. Similarly $\{x\}$ is closed in $\nu(P)$ if x is the bottom element in P.

If P is totally ordered, there are no isolated elements. As a result, we may write $P- \uparrow x$ in a more familiar fashion: $(-\infty, x)$. Similarly, $P- \downarrow x$ may be written as (x, ∞).

6. Things get more interesting when we take the common refinement of $\omega(P)$ and $\nu(P)$. What we end up with is called the *interval topology* of P.

   When P is totally ordered, the interval topology on P has

   $$\mathcal{I}(P) := \{(x, y) \mid x, y \in P\}$$

   as a subbasis, where (x, y) denotes the *open* poset interval, consisting of elements $a \in P$ such that $x < a < y$. Since finite intersections of open poset intervals is a poset interval, an open set in P can be written as an (arbitrary) union of open poset intervals.

   As an example, the usual topology on $\mathbb{R}$ is precisely the interval topology generated by the linear order on $\mathbb{R}$.

Remark. It is a common practice in mathematics to impose special compatibility conditions on a structure having two inherent substructures so the substructures inter-relate, so that one can derive more interesting fruitful results. This is true also in the case of an ordered space. Let X be an ordered space. Below are some of the common conditions that can be imposed on X:

- X is said to be *upper semiclosed* if $\uparrow x$ is a closed set for every $x \in X$.
- Similarly, X is *lower semiclosed* if $\downarrow x$ is closed in X.
- X is *semiclosed* if it is both upper and lower semiclosed.
- If $\leq$, as a subset of $X \times X$, is closed in the product topology, then X is called a *pospace*.

Other structures, such as ordered topological vector spaces, `http://planetmath.org/Topological` lattices, and `http://planetmath.org/TopologicalVectorLattice`topological vector lattices are ordered spaces with algebraic structures satisfying certain additional compatibility conditions. Please click on the links for details.

References

[1] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, D. S. Scott, *Continuous Lattices and Domains*, Cambridge University Press, Cambridge (2003).