proof of Borsuk-Ulam theorem

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Proof of the Borsuk-Ulam theorem: I'm going to prove a stronger statement than the one given in the statement of the Borsak-Ulam theorem here, which is:

Every odd (that is, antipode-preserving) map $f : S^n \to S^n$ has odd degree.

Proof: We go by induction on n. Consider the pair (S^n, A) where A is the equatorial sphere. f defines a map

$$\tilde{f} : \mathbb{R}P^n \to \mathbb{R}P^n$$

. By cellular approximation, this may be assumed to take the hyperplane at infinity (the $n-1$-cell of the standard cell structure on $\mathbb{R}P^n$) to itself. Since whether a map lifts to a covering depends only on its homotopy class, f is homotopic to an odd map taking A to itself. We may assume that f is such a map.

The map f gives us a morphism of the long exact sequences:

$$\begin{array}{ccccccccc}
H_n(A;\mathbb{Z}_2) & \xrightarrow{i} & H_n(S^n;\mathbb{Z}_2) & \xrightarrow{j} & H_n(S^n,A;\mathbb{Z}_2) & \xrightarrow{\partial} & H_{n-1}(A;\mathbb{Z}_2) & \xrightarrow{i} & H_{n-1}(S^n,A;\mathbb{Z}_2) \\
{\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow \\
H_n(A;\mathbb{Z}_2) & \xrightarrow{i} & H_n(S^n;\mathbb{Z}_2) & \xrightarrow{j} & H_n(S^n,A;\mathbb{Z}_2) & \xrightarrow{\partial} & H_{n-1}(A;\mathbb{Z}_2) & \xrightarrow{i} & H_{n-1}(S^n,A;\mathbb{Z}_2)
\end{array}$$

Clearly, the map $f|_A$ is odd, so by the induction hypothesis, $f|_A$ has odd degree. Note that a map has odd degree if and only if $f^* : H_n(S^n;\mathbb{Z}_2) \to H_n(S^n,\mathbb{Z}_2)$ is an isomorphism. Thus

$$f^* : H_{n-1}(A;\mathbb{Z}_2) \to H_{n-1}(A;\mathbb{Z}_2)$$

is an isomorphism. By the commutativity of the diagram, the map

$$f^* : H_n(S^n,A;\mathbb{Z}_2) \to H_n(S^n,A;\mathbb{Z}_2)$$

is not trivial. I claim it is an isomorphism. $H_n(S^n,A;\mathbb{Z}_2)$ is generated by cycles $[R^+]$ and $[R^-]$ which are the fundamental classes of the upper and lower hemispheres, and the antipodal map exchanges these. Both of these map to the fundamental class of A, $[A] \in H_{n-1}(A;\mathbb{Z}_2)$. By the commutativity of the diagram, $\partial(f^*([R^\pm])) = f^*(\partial([R^\pm])) = f^*([A]) = [A]$. Thus $f^*([R^+]) = [R^\pm]$ and $f^*([R^-]) = [R^\mp]$ since f commutes with the antipodal map. Thus f^* is an isomorphism on $H_n(S^n,A;\mathbb{Z}_2)$. Since $H_n(A,\mathbb{Z}_2) = 0$, by the exactness of the sequence $i : H_n(S^n;\mathbb{Z}_2) \to H_n(S^n,A;\mathbb{Z}_2)$ is injective, and so by the commutativity of the diagram (or equivalently by the 5-lemma) $f^* : H_n(S^n; \mathbb{Z}_2) \to H_n(S^n; \mathbb{Z}_2)$ is an isomorphism. Thus f has odd degree.

The other statement of the Borsuk-Ulam theorem is:

There is no odd map $S^n \to S^{n-1}$.

Proof: If f where such a map, consider f restricted to the equator A of S^n. This is an odd map from S^{n-1} to S^{n-1} and thus has odd degree. But the map

$$f^* H_{n-1}(A) \to H_{n-1}(S^{n-1})$$

factors through $H_{n-1}(S^n) = 0$, and so must be zero. Thus $f|_A$ has degree 0, a contradiction.