nets and closures of subspaces

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Theorem. *A point of a topological space is in the closure of a subspace if and only if there is a net of points of the subspace converging to the point.*

Proof. Let X be a topological space, x a point of X, and A a subspace of X. Suppose first that $x \in \bar{A}$, and let $\mathcal{U}$ be the collection of neighborhoods of x, `http://planetmath.org/node/123`partially ordered by reverse . For each $U \in \mathcal{U}$, select a point $x_U \in U \cap A$ (such a point is guaranteed to exist because $x \in \bar{A}$); then $(x_U)_{U \in \mathcal{U}}$ is a net of points in A, and we claim that $x_U \to x$. To see this, let V be a neighborhood of x in X, and note that, by construction, $x_V \in V$; furthermore, if $U \in \mathcal{U}$ satisfies $V \supset U$, then because $x_U \in U$, $x_U \in V$. It follows that $x_U \to x$. Conversely, suppose there exists a net $(x_\alpha)_{\alpha \in J}$ of points of A converging to x, and let $U \subset X$ be a neighborhood of x. Since $x_\alpha \to x$, there exists $\beta \in J$ such that $x_\alpha \in U$ whenever $\beta \preceq \alpha$. Because $x_\alpha \in A$ for each $\alpha \in J$ by hypothesis, we may conclude that $U \cap A \neq \emptyset$, hence that $x \in \bar{A}$. $\square$

The forward implication of the preceding is a generalization of the result that a point of a topological space is in the closure of a subspace if there is a *sequence* of points of the subspace converging to the point, as a sequence is just a net with the positive integers as its `http://planetmath.org/node/360`domain; however, the converse (if a point is in the closure of a subspace then there exists a sequence of points of the subspace converging to the point) requires the additional condition that the ambient topological space be first countable.