axiomatic definition of the real numbers

Canonical name	AxiomaticDefinitionOfTheRealNumbers
Date of creation	2013-03-22 15:39:29
Last modified on	2013-03-22 15:39:29
Owner	matte (1858)
Last modified by	matte (1858)
Numerical id	17
Author	matte (1858)
Entry type	Definition
Classification	msc 54C30
Classification	msc 26-00
Classification	msc 12D99
Related topic	RealNumber

Axiomatic definition of the real numbers

The real numbers consist of a set $\mathbb{R}$ together with mappings $+\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $\cdot\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and a relation $<\subseteq \mathbb{R}\times\mathbb{R}$ satisfying the following conditions:

1. $(\mathbb{R},+)$ is an Abelian group:

   (a) For $a,b,c\in\mathbb{R}$, we have

   $$\begin{aligned} a+b &= b+a,\\ (a+b)+c &= a+(b+c), \end{aligned}$$

   (b) there exists an element $0\in\mathbb{R}$ such that $a+0=a$ for all $a\in\mathbb{R}$,

   (c) every $a\in\mathbb{R}$ has an inverse $(-a)\in\mathbb{R}$ such that $a+(-a)=0$.

2. $(\mathbb{R}\setminus\{0\},\cdot)$ is an Abelian group:

   (a) For $a,b,c\in\mathbb{R}$, we have

   $$\begin{aligned} a\cdot b &= b\cdot a,\\ (a\cdot b)\cdot c &= a\cdot(b\cdot c), \end{aligned}$$

   (b) there exists an element $1\in\mathbb{R}\setminus\{0\}$ such that $a\cdot 1=a$ for all $a\in\mathbb{R}$,

   (c) every $a\in\mathbb{R}\setminus\{0\}$ has an inverse $a^{-1}\in\mathbb{R}$ such that $a^{-1}\cdot a=1$.

3. The operation $\cdot$ is distributive over $+$: If $a,b,c\in\mathbb{R}$, then

   $$\begin{aligned} a\cdot(b+c) &= a\cdot b+a\cdot c,\\ (b+c)\cdot a &= b\cdot a+c\cdot a. \end{aligned}$$

4. $(\mathbb{R},<)$ is a total order:

   (a) (transitivity) if $c\in\mathbb{R}$, $a<b$, and $b<c$, then $a<c$,

   (b) (trichotomy) precisely one of the below alternatives hold:

   $$a<b,\quad a=b,\quad b<a.$$

   For convenience we make the following notational definitions: $a>b$ means $b<a$, $a\le b$ means either $a<b$ or $a=b$, and $a\ge b$ means either $b<a$ or $a=b$.

5. The operations $+$ and $\cdot$ are compatible with the order $<$:
   (a) If a, b, $c \in \mathbb{R}$ and $a < b$, then $a + c < b + c$.
   (b) If a, b, $c \in \mathbb{R}$ with $a < b$ and $0 < c$, then $ac < bc$.
6. $\mathbb{R}$ has the least upper bound property: If $A \subset \mathbb{R}$, then an element $M \in \mathbb{R}$ is an for A if

$$a < M, \text{ for all } \ a \in A.$$

   If A is non-empty, we then say that A is bounded from above. That $\mathbb{R}$ has the least upper bound property means that if $A \subset \mathbb{R}$ is bounded from above, it has a least upper bound $m \in \mathbb{R}$. That is, A has an upper bound m such that if M is any upper bound from M, then $m \leq M$.

Here it should be emphasized that from the above we can not deduce that a set $\mathbb{R}$ with operations $+, \cdot, <$ exists. To settle this question such a set has to be explicitly constructed. However, this can be done in various ways, as discussed on `http://planetmath.org/RealNumber`this page. One can also show the above conditions uniquely determine the real numbers (up to an isomorphism). The proof of this can be found on `http://planetmath.org/EveryOrderedFieldWithT` page.

Basic properties

In condensed form, the above conditions state that $\mathbb{R}$ is an ordered field with the least upper bound property. In particular $(\mathbb{R}, +, \cdot)$ is a ring, and $(\mathbb{R} \setminus \{0\}, \cdot)$ is a group, and we have the following basic properties:

Lemma 1. *Suppose $a, b \in \mathbb{R}$.*

1. *The additive inverse $(-a)$ is unique* `http://planetmath.org/UniquenessOfAdditiveIdent`
2. *The additive identity 0 is unique* `http://planetmath.org/UniquenessOfAdditiveIdentit`
3. $(-1) \cdot a = (-a)$ `http://planetmath.org/1cdotAA`*(proof).*
4. $(-a) \cdot (-b) = a \cdot b$ `http://planetmath.org/XcdotYXcdotY`*(proof).*
5. $0 \cdot a = 0$ `http://planetmath.org/0cdotA0`*(proof)*

6. *The multiplicative inverse a^{-1} is unique* ***http://planetmath.org/UniquenessOfInverseFo***

7. *If a, b are non-zero, then $(ab)^{-1} = b^{-1}a^{-1}$* ***http://planetmath.org/InverseOfAProduct****(pro*

In view of property **??**, we can write simply $-a$ instead of $(-1) \cdot a$ and $(-a)$.

Because of the additive inverse of a real number is unique (by property 1 above), and $(-a) + a = a + (-a) = 0$, we see that the additive inverse of $-a$ is a, or that $-(-a) = a$. Similarly, if $a \neq 0$, then $a^{-1} \neq 0$ (or we'll end up with $1 = aa^{-1} = a0 = 0$), and therefore by Property 6 above, a^{-1} has a unique multiplicative inverse. Since $aa^{-1} = a^{-1}a = 1$, we see that a is the multiplicative inverse of a^{-1}. In other words, $(a^{-1})^{-1} = a$.

For $a, b \in \mathbb{R}$ let us also define $a - b = a + (-b)$, which is called the *difference* of a and b. By commutativity, $a - b = -b + a$. It is also common to leave out the multiplication symbol and simply write $ab = a \cdot b$. Suppose $a \in \mathbb{R}$ and $b \in \mathbb{R}$ is non-zero. Then b ***http://planetmath.org/Division****divided by* a is defined as

$$\frac{a}{b} = ab^{-1}.$$

In consequence, if $a, b, c, d \in \mathbb{R}$ and b, c, d are non-zero, then

- $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{bd}{ac}$,
- $\frac{ab}{b} = a$.

For example,

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{ab^{-1}}{cd^{-1}} = ab^{-1}(cd^{-1})^{-1} = ab^{-1}dc^{-1} = \frac{ad}{bc}.$$