alternative definition of metric space

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1 Definition

The notion of *metric space* may be defined in a way which makes use only of rational numbers as opposed to real numbers. To avoid mention of real numbers we take a three place relation as our primitive term instead of the distance function. Thus, one can use this definition to define the set of real numbers as the completion of the set of rational numbers as a metric space. Let $\mathbb{Q}_+$ denote the set of strictly positive rational numbers

Definition 1. *A* metric space *consists of a set X and a relation $D \subset X \times X \times \mathbb{Q}_+$ which satisfies the following properties:*

1. *For all $x, y \in X$, it is the case that $D(x, y, r)$ for all $r \in \mathbb{Q}_+$ if and only if $x = y$.*
2. *For all $x, y \in X$ and all $r \in \mathbb{Q}_+$, it is the case that $D(x, y, r)$ if and only if $D(y, x, r)$.*
3. *For all $x, y \in X$ and all $r, s \in \mathbb{Q}_+$, if $r \leq s$ and $D(x, y, r)$ then $D(x, y, s)$.*
4. *For all $x, y, z \in X$ and all $r, s \in \mathbb{Q}_+$, if $D(x, y, r)$ and $D(y, z, s)$, then $D(x, z, r + s)$.*

2 Equivalence with usual definition

The definition presented above is equivalent to the definition presented in `http://planetmath.org/MetricSpace`metric space. This equivalence is given by the following equations which relate the primitive term "D" used in this definition with the primitive term "d" of the previous definition:

$$d(x, y) = \inf\{r \mid D(x, y, r)\}$$

$$D(x, y, r) \leftrightarrow d(x, y) \leq r$$