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proximal neighborhood

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Let X be a set and $P(X)$ its power set. Let $\ll$ be a binary relation on $P(X)$ satisfying the

following conditions, for any $A, B \subseteq X$:

1. $X \ll X$,
2. $A \ll B$ implies $A \subseteq B$,
3. $A \ll B$ and $C \ll D$ imply $A \cap C \ll B \cap D$,
4. $A \ll B$ implies $B' \ll A'$ ($'$ is the complement operator)
5. $A \subseteq B \ll C \subseteq D$, then $A \ll D$, and
6. if $A \ll B$, then there is $C \subseteq X$, such that $A \ll C \ll B$.

By 1 and 4, it is easy to see that $\varnothing \ll \varnothing$. Also, 3 and 4 show that $A \cup C \ll B \cup D$ whenever $A \ll B$ and $C \ll D$. So $\ll$ is a topogenous order, which means $\ll$ is transitive and anti-symmetric. Under this order relation, we say that B is a *proximal neighborhood* of A if $A \ll B$.

The reason why we call B a "proximal" neighborhood is due to the following:

Theorem 1. *Let X be a set. The following are true.*

- *Let $\ll$ be defined as above. Define a new relation δ on $P(X)$: $A\delta' B'$ iff $A \ll B$. Then δ so defined is a proximity relation, turning X into a proximity space.*
- *Conversely, let (X, δ) is a proximity space. Define a new relation $\ll$ on $P(X)$: $A \ll B$ iff $A\delta' B'$. Then $\ll$ satisfies the six properties above.*

Proof. Suppose first that X and $\ll$ are defined as above. We will verify the individual nearness relation axioms of δ by proving their contrapositives in each case, except the last axiom:

1. if $A\delta' B$, then $A \ll B'$, or $A \subseteq B'$, so $A \cap B = \varnothing$;
2. suppose either $A = \varnothing$ or $B = \varnothing$. In either case, $A \ll B'$, which means $A\delta' B$;
3. if $A\delta' B$, then $A \ll B'$, so $B'' \ll A'$, or $B \ll A'$, or $B\delta' A$;

4. if $A_1\delta' B$ and $A_2\delta' B$, then $A_1 \ll B$ and $A_2 \ll B$, so $(A_1 \cup A_2) \ll B$, or $(A_1 \cup A_2)\delta' B$;

5. if $A\delta' B$, then $A \ll B'$. So there is $D \subseteq X$ with $A \ll D$ and $D \ll B'$. Let $C = D'$. Then $A \ll C'$ and $C' \ll B'$, or $A\delta' C$ and $C'\delta' B$.

Next, suppose (X, δ) is a proximity space. We now verify the six properties of $\ll$ above.

1. since $X\delta'\varnothing$, $X \ll \varnothing'$, or $X \ll X$;

2. suppose $A\delta' B'$, then if $x \in A$, we have $x\delta' B'$, implying $x \cap B' = \varnothing$, or $x \in B$;

3. if $A \ll B$ and $C \ll D$, then $A\delta' B'$ and $C\delta' D'$, which means $A\delta'(B' \cup D')$ and $C\delta'(B' \cup D')$, which together imply $(A \cap C)\delta'(B' \cup D')$, or $(A \cap C)\delta(B \cap D)'$, or $A \cap C \ll B \cap D$;

4. if $A \ll B$, then $A\delta' B'$, so $B'\delta' A$ (as δ is symmetric, so is its complement), which is the same as $B'\delta' A''$, or $B' \ll A'$;

5. if $A\delta D'$, then $B\delta C'$ (since $A \subseteq B$ and $D' \subseteq C'$), so $B \ll' C$, a contradiction;

6. if $A \ll B$, then $A\delta' B'$, so there is $D \subseteq X$ with $A\delta' D$ and $D'\delta' B'$. Define $C = D'$, then $A \ll C$ and $C \ll B$, as desired.

This completes the proof. $\square$

Because of the above, we see that a proximity space can be equivalently defined using the proximal neighborhood concept. To emphasize its relationship with δ, a proximal neighborhood is also called a δ-neighbhorhood.

Furthermore, we have

Theorem 2. *if B is a proximal neighborhood of A in a proximity space (X, δ), then B is a (topological) neighborhood of A under the topology $\tau(\delta)$ induced by the proximity relation δ. In other words, if $A \ll B$, then $A \subseteq B^\circ$ and $A^c \subseteq B$, where $^\circ$ and ^{c} denote the interior and closure operators.*

Proof. Since $A\delta' B'$, then $x\delta' B'$ whenever $x \in A$, which is the contrapositive of the statement: $x \in A'$ whenever $x\delta B'$, which is equivalent to $B'^c \subseteq A'$, or $A \subseteq B^\circ$. Furthermore, if $x \notin B$, then $x \in B'$. But $A\delta' B'$ b assumption. This implies $x\delta' A$, which means $x \notin A^c$. Therefore $A^c \subseteq B$. $\square$

Remark. However, not every $\tau(\delta)$-neighborhood is a δ-neighborhood.