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invariant forms on representations of compact groups

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Let G be a real Lie group. TFAE:

1. Every real representation of G has an invariant positive definite form, and G has at least one faithful representation.
2. One faithful representation of G has an invariant positive definite form.
3. G is compact.

Also, any group satisfying these criteria is reductive, and its Lie algebra is the direct sum of simple algebras and an abelian algebra (such an algebra is often called reductive).

Proof. (1) $\Rightarrow$ (2): Obvious.

(2) $\Rightarrow$ (3): Let Ω be the invariant form on a faithful representation V. Let then representation gives an embedding $\rho : G \to \mathrm{SO}(V, \Omega)$, the group of automorphisms of V preserving Ω. Thus, G is homeomorphic to a closed subgroup of $\mathrm{SO}(V, \Omega)$. Since this group is compact, G must be compact as well.

(Proof that $\mathrm{SO}(V, \Omega)$ is compact: By induction on $\dim V$. Let $v \in V$ be an arbitrary vector. Then there is a map, evaluation on v, from $\mathrm{SO}(V, \Omega) \to S^{\dim V - 1} \subset V$ (this is topologically a sphere, since (V, ω) is isometric to $\mathbb{R}^{\dim V}$ with the standard norm). This is a a fiber bundle, and the fiber over any point is a copy of $\mathrm{SO}(v^{\perp}, \Omega)$, which is compact by the inductive hypothesis. Any fiber bundle over a compact base with compact fiber has compact total space. Thus $\mathrm{SO}(V, \Omega)$ is compact).

(3) $\Rightarrow$ (1): Let V be an arbitrary representation of G. Choose an arbitrary positive definite form Ω on V. Then define

$$\tilde{\Omega}(v, w) = \int_G \Omega(gv, gw)dg,$$

where dg is Haar measure (normalized so that $\int_G dg = 1$). Since K is compact, this gives a well defined form. It is obviously bilinear, b$\mathrm{SO}(V, \Omega)$y the linearity of integration, and positive definite since

$$\tilde{\Omega}(gv, gv) = \int_G \Omega(gv, gv)dg \geq \inf_{g \in G} \Omega(gv, gv) > 0.$$

Furthermore, $\tilde{\Omega}$ is invariant, since

$$\tilde{\Omega}(hv, hw) = \int_G \Omega(ghv, ghw)dg = \int_G \Omega(ghv, ghw)d(gh) = \tilde{\Omega}(v, w).$$

For representation $\rho : T \to \mathrm{GL}(V)$ of the maximal torus $T \subset K$, there exists a representation ρ' of K, with ρ a T-subrepresentation of ρ'. Also, since every conjugacy class of K intersects any maximal torus, a representation of K is faithful if and only if it restricts to a faithful representation of T. Since any torus has a faithful representation, K must have one as well.

Given that these criteria hold, let V be a representation of G, Ω is positive definite real form, and W a subrepresentation. Now consider

$$W^{\perp} = \{v \in V | \Omega(v, w) = 0 \, \forall w \in W\}.$$

By the positive definiteness of Ω, $V = W \oplus W^{\perp}$. By induction, V is completely reducible.

Applying this to the adjoint representation of G on $\mathfrak{g}$, its Lie algebra, we find that $\mathfrak{g}$ in the direct sum of simple algebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$, in the sense that $\mathfrak{g}_i$ has no proper nontrivial ideals, meaning that $\mathfrak{g}_i$ is simple in the usual sense or it is abelian. $\square$