compactness and accumulation points of nets

Canonical name	CompactnessAndAccumulationPointsOfNets
Date of creation	2013-03-22 18:37:50
Last modified on	2013-03-22 18:37:50
Owner	azdbacks4234 (14155)
Last modified by	azdbacks4234 (14155)
Numerical id	7
Author	azdbacks4234 (14155)
Entry type	Theorem
Classification	msc 54A20
Related topic	Net
Related topic	Compact
Related topic	AccumulationPointsAndConvergentSubnets

Theorem. *A topological space X is compact if and only if every net in X has an accumulation point.*

Proof. Suppose X is compact and let $(x_\alpha)_{\alpha\in A}$ be a net in X. For each $\alpha \in A$, put $E_\alpha = \{x_\beta : \beta \geq \alpha\}$; the collection $\{\overline{E_\alpha} : \alpha \in A\}$ of closed subsets of X has the finite intersection property, for given $\alpha_1, \ldots, \alpha_n \in A$, because A is directed, there exists $\beta \in A$ satisfying $\beta \geq \alpha_i$ for each $i \in \{1, \ldots, n\}$, so that $x_\beta \in \bigcap_{i=1}^n \overline{E_{\alpha_i}}$. Therefore, by compactness, $\bigcap_{\alpha\in A} \overline{E_\alpha} \neq \emptyset$; let x be a point of this intersection. If U is any open subset of X and $\alpha \in A$, then because $x \in \overline{E_\alpha}$, $E_\alpha \cap U \neq \emptyset$, and thus there exists $\beta \geq \alpha \in A$ for which $x_\beta \in U$. It follows that x is an accumulation point of (x_α). For the converse, assume that X fails to be compact, and let $\{U_i : i \in I\}$ be an open cover of X with no finite subcover. If B is the set of finite subsets of I, then B is directed by inclusion. For each set $S \in B$, let x_S be a point in the complement of $\bigcup_{i\in S} U_i$. We contend that the net $(x_S)_{S\in B}$ has no accumulation points; indeed, given $x \in X$, we may select $i_0 \in I$ such that $x \in U_{i_0}$; if $S \in B$ is such that $i_0 \in S$, that is, if $S \geq \{i_0\}$, then by construction, $x_S \notin U_{i_0}$, establishing our contention. □

Corollary. *The following conditions on a topological space X are equivalent:*

1. *X is compact;*
2. *every net in X has an accumulation point;*
3. *every net in X has a convergent subnet;*

Proof. The preceding theorem establishes the equivalence of (1) and (2), while that of (2) and (3) is established in the entry on accumulation points and convergent subnets. □