basis (topology)

Canonical name	Basistopology
Date of creation	2013-03-22 12:05:03
Last modified on	2013-03-22 12:05:03
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	16
Author	rspuzio (6075)
Entry type	Definition
Classification	msc 54A99
Synonym	basis
Synonym	base
Synonym	topology generated by a basis
Related topic	Subbasis
Related topic	CompactMetricSpacesAreSecondCountable

Let $(X, \mathcal{T})$ be a topological space. A subset $\mathcal{B}$ of $\mathcal{T}$ is a *basis* for $\mathcal{T}$ if every member of $\mathcal{T}$ is a union of members of $\mathcal{B}$.

Equivalently, $\mathcal{B}$ is a basis if and only if whenever U is open and $x \in U$ then there is an open set $V \in \mathcal{B}$ such that $x \in V \subseteq U$.

The topology generated by a basis $\mathcal{B}$ consists of exactly the unions of the elements of $\mathcal{B}$.

We also have the following easy characterization: (for a proof, see the attachment)

Proposition. *A collection of subsets $\mathcal{B}$ of X is a basis for some topology on X if and only if each $x \in X$ is in some element $B \in \mathcal{B}$ and whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

0.0.1 Examples

1. A basis for the usual topology of the real line is given by the set of open intervals since every open set can be expressed as a union of open intervals. One may choose a smaller set as a basis. For instance, the set of all open intervals with rational endpoints and the set of all intervals whose length is a power of 1/2 are also bases. However, the set of all open intervals of length 1 is not a basis although it is a subbasis (since any interval of length less than 1 can be expressed as an intersection of two intervals of length 1).

2. More generally, the set of open balls forms a basis for the topology on a metric space.

3. The set of all subsets with one element forms a basis for the discrete topology on any set.