homotopy groups

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The homotopy groups are an infinite series of (covariant) functors π_n indexed by non-negative integers from based topological spaces to groups for $n > 0$ and sets for $n = 0$. $\pi_n(X, x_0)$ as a set is the set of all homotopy classes of maps of pairs $(D^n, \partial D^n) \to (X, x_0)$, that is, maps of the disk into X, taking the boundary to the point x_0. Alternatively, these can be thought of as maps from the sphere S^n into X, taking a basepoint on the sphere to x_0. These sets are given a group structure by declaring the product of 2 maps f, g to simply attaching two disks D_1, D_2 with the right orientation along part of their boundaries to get a new disk $D_1 \cup D_2$, and mapping D_1 by f and D_2 by g, to get a map of $D_1 \cup D_2$. This is continuous because we required that the boundary go to a , and well defined up to homotopy.

If $f : X \to Y$ satisfies $f(x_0) = y_0$, then we get a homomorphism of homotopy groups $f^* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ by simply composing with f. If g is a map $D^n \to X$, then $f^*([g]) = [f \circ g]$.

More algebraically, we can define homotopy groups inductively by $\pi_n(X, x_0) \cong \pi_{n-1}(\Omega X, y_0)$, where ΩX is the loop space of X, and y_0 is the constant path sitting at x_0.

If $n > 1$, the groups we get are abelian.

Homotopy groups are invariant under homotopy equivalence, and higher homotopy groups ($n > 1$) are not changed by the taking of covering spaces.

Some examples are:

$\pi_n(S^n) = \mathbb{Z}$.

$\pi_m(S^n) = 0$ if $m < n$.

$\pi_n(S^1) = 0$ if $n > 1$.

$\pi_n(M) = 0$ for $n > 1$ where M is any surface of nonpositive Euler characteristic (not a sphere or projective plane).