product topology and subspace topology

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Let X_α with $\alpha \in A$ be a collection of topological spaces, and let $Z_\alpha \subseteq X_\alpha$ be subsets. Let

$$X = \prod_\alpha X_\alpha$$

and

$$Z = \prod_\alpha Z_\alpha.$$

In other words, $z \in Z$ means that z is a function $z\colon A \to \cup_\alpha Z_\alpha$ such that $z(\alpha) \in Z_\alpha$ for each α. Thus, $z \in X$ and we have

$$Z \subseteq X$$

as sets.

Theorem 1. *The product topology of Z coincides with the subspace topology induced by X.*

Proof. Let us denote by τ_X and τ_Z the product topologies for X and Z, respectively. Also, let

$$\pi_{X,\alpha}\colon X \to X_\alpha, \quad \pi_{Z,\alpha}\colon Z \to Z_\alpha$$

be the canonical projections defined for X and Z. The `http://planetmath.org/Subbasis`subbases for X and Z are given by

$$\begin{aligned} \beta_X &= \{\pi^{-1}_{X,\alpha}(U) : \alpha \in A, U \in \tau(X_\alpha)\}, \\ \beta_Z &= \{\pi^{-1}_{Z,\alpha}(U) : \alpha \in A, U \in \tau(Z_\alpha)\}, \end{aligned}$$

where $\tau(X_\alpha)$ is the topology of X_α and $\tau(Z_\alpha)$ is the subspace topology of $Z_\alpha \subseteq X_\alpha$. The claim follows as

$$\beta_Z = \{B \cap Z : B \in \beta_X\}.$$

$\square$