intervals are connected

Canonical name	IntervalsAreConnected
Date of creation	2013-03-22 18:32:49
Last modified on	2013-03-22 18:32:49
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	5
Author	joking (16130)
Entry type	Example
Classification	msc 54D05

We wish to show that intervals (with standard topology) are connected. In order to this, we will prove that the space of real numbers $\mathbb{R}$ is connected. First we need a lemma.

Let (X, d) be a metric space. Recall that for $x \in X$ and $r \in \mathbb{R}^+$ we have

$$B(x, r) = \{y \in X \mid d(x, y) < r\}.$$

Lemma. Let (X, d) be a metric space and $R \subset \mathbb{R}^+$ such that R is nonempty and bounded. Then for any $x \in X$ we have

$$\bigcup_{r \in R} B(x, r) = B(x, \sup(R)).$$

Proof. Assume that $y \in \bigcup_{r \in R} B(x, r)$. Then there is $r_0 \in R$ such that $d(x, y) < r_0$ and thus $d(x, y) < \sup(R)$, so $y \in B(x, \sup(R))$.

Now assume that $y \in B(x, \sup(R))$. Then $d(x, y) < \sup(R)$ and it follows (from the definition of supremum) that there is $r_0 \in R$ such that $d(x, y) < r_0$ and therefore $y \in B(x, r_0) \subset \bigcup_{r \in R} B(x, r)$, which completes the proof. $\square$

Proposition. The space of real numbers is connected.

Proof. Assume that $U, V \subseteq \mathbb{R}$ are open subsets of $\mathbb{R}$ such that $U \cap V = \emptyset$ and $U \cup V = \mathbb{R}$. Furthermore assume that $U \neq \emptyset$ and take any $x_0 \in U$. Then (since U is open) there is $r_0 \in \mathbb{R}$ such that the open ball

$$B(x_0, r_0) = \{x \in \mathbb{R} \mid |x - x_0| < r_0\}$$

is contained in U. Consider the set

$$R = \{r \in \mathbb{R}^+ \mid B(x_0, r) \subseteq U\}.$$

Thus R is nonempty.

Assume that R is bounded. Denote by $s = \sup(R) < \infty$. We can apply the lemma:

$$\bigcup_{r \in R} B(x_0, r) = B(x_0, s).$$

Thus (due to the definition of R) $B(x_0, s)$ is a maximal open ball (with the center in x_0) which is contained in U. Now

$$B(x_0, s) = (a, b)$$

for some $a, b \in \mathbb{R}$. Since (a, b) is maximal then $a \notin U$ or $b \notin U$. Indeed, if both $a \in U$ and $b \in U$, then (since U is open) small neighbourhoods of a and

b are also contained in U, so $(a-\epsilon, b+\epsilon)$ is contained in U (for some $\epsilon > 0$), but (a, b) was maximal. Contradiction.

Without loss of generality we can assume that $b \notin U$. Then $b \in V$, because $U \cup V = \mathbb{R}$. But then (since V is open) there is $c \in \mathbb{R}$ such that $a < c < b$ and $c \in V$. Thus $U \cap V \neq \emptyset$. Contradiction. Therefore R is unbounded.

Take any unbounded sequence $(a_n)_{n=1}^{\infty}$ from R. Then we have

$$\mathbb{R} = \bigcup_{n=1}^{\infty} B(x_0, a_n) \subseteq U$$

and thus $U = \mathbb{R}$, so $V = \emptyset$. This completes the proof. $\square$

Corollary. For any $a, b \in \mathbb{R}$ such that $a < b$ intervals (a, b), $[a, b)$, $(a, b]$ and $[a, b]$ are connected.

Proof. One can easily show that intervals are continous image of $\mathbb{R}$ and therefore intervals are connected.