diagonal embedding

Canonical name	DiagonalEmbedding
Date of creation	2013-03-22 14:20:41
Last modified on	2013-03-22 14:20:41
Owner	waj (4416)
Last modified by	waj (4416)
Numerical id	8
Author	waj (4416)
Entry type	Definition
Classification	msc 54B10
Classification	msc 18A05
Synonym	diagonal map
Related topic	ASpaceMathnormalXIsHausdorffIfAndOnlyIfDeltaXIsClosed

Given a topological space X, the *diagonal embedding*, or *diagonal map* of X into $X \times X$ (with the product topology) is the map

$$x \overset{\Delta}{\longmapsto} (x, x).$$

X is homeomorphic to the image of Δ (which is why we use the word "embedding").

We can perform the same construction with objects other than topological spaces: for instance, there's a diagonal map $\Delta \colon G \to G \times G$, from a group into its direct sum with itself, given by the same . It's sensible to call this an embedding, too, since Δ is a monomorphism.

We could also imagine a diagonal map into an n-fold product given by

$$x \overset{\Delta_n}{\longmapsto} (x, x, \ldots, x).$$

Why call it the diagonal map?

Picture $\mathbb{R}$. Its diagonal embedding into the Cartesian plane $\mathbb{R} \times \mathbb{R}$ is the diagonal line $y = x$.

What's it good for?

Sometimes we can use information about the product space $X \times X$ together with the diagonal embedding to get back information about X. For instance, X is Hausdorff if and only if the image of Δ is closed in $X \times X$ [`http://planetmath.org/ASpaceMathnormalXIsHausdorffIfAndOnlyIfDeltaXIsClosed`proof]. If we know more about the product space than we do about X, it might be easier to check if $\operatorname{Im} \Delta$ is closed than to verify the Hausdorff condition directly.

When studying algebraic topology, the fact that we have a diagonal embedding for any space X lets us define a bit of extra structure in cohomology, called the cup product. This makes cohomology into a ring, so that we can bring additional algebraic muscle to bear on topological questions.

Another application from algebraic topology: there is something called an H-space, which is essentially a topological space in which you can multiply two points together. The diagonal embedding, together with the multiplication, lets us say that the cohomology of an H-space is a Hopf algebra; this structure lets us find out lots of things about H-spaces by analogy to what we know about compact Lie groups.