derivation of properties on interior operation

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Let X be a topological space and A a subset of X. Then

1. $\mathrm{int}(A) \subseteq A$.

   Proof. If $a \in \mathrm{int}(A)$, then $a \in U$ for some open set $U \subseteq A$. So $a \in A$. $\square$

2. $\mathrm{int}(A)$ is open.

   Proof. Since $\mathrm{int}(A)$ is a union of open sets, $\mathrm{int}(A)$ is open. $\square$

3. $\mathrm{int}(A)$ is the largest open set contained in A.

   Proof. If U is open set with $\mathrm{int}(A) \subseteq U \subseteq A$, then $U \subseteq \bigcup\{V \subseteq A \mid V \text{ open }\} = \mathrm{int}(A)$, so $U = \mathrm{int}(A)$. $\square$

4. A is open if and only if $A = \mathrm{int}(A)$.

   Proof. If A is open, then A is the largest open set contained in A, and so $\mathrm{int}(A) = A$ by property 3 above. On the other hand, if $\mathrm{int}(A) = A$, then A is open, since $\mathrm{int}(A)$ is, by property 2 above. $\square$

5. $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$.

   Proof. Since $\mathrm{int}(A)$ is open by property 2, $\mathrm{int}(A) = \mathrm{int}(\mathrm{int}(A))$ by property 4. $\square$

6. $\mathrm{int}(X) = X$ and $\mathrm{int}(\varnothing) = \varnothing$.

   Proof. This is so because both X and $\varnothing$ are open sets. $\square$

7. $\overline{A^{\complement}} = (\mathrm{int}(A))^{\complement}$.

Proof. (LHS $\subseteq$ RHS). If $a \in \overline{A^{\complement}}$, then $a \in B$ for every closed set B such that $A^{\complement} \subseteq B$. In particular, $a \in (\mathrm{int}(A))^{\complement}$, for $(\mathrm{int}(A))^{\complement}$ is the complement of an open set by property 2, and $A^{\complement} \subseteq (\mathrm{int}(A))^{\complement}$ by taking the complement of property 1.

(RHS $\subseteq$ LHS). If $a \in (\mathrm{int}(A))^{\complement}$, then $a \notin \mathrm{int}(A)$. If B is a closed set such that $A^{\complement} \subseteq B$, then $B^{\complement} \subseteq A$. Since $B^{\complement}$ is open, $B^{\complement} \subseteq \mathrm{int}(A)$ by property 3, so $a \notin B^{\complement}$, and thus $a \in B$. Since B is arbitrary, $a \in \overline{A^{\complement}}$ as desired. $\square$

8. $\overline{A}^{\complement} = \mathrm{int}(A^{\complement})$.

   Proof. Set $B = A^{\complement}$, and apply property 7. So $\overline{A}^{\complement} = \overline{B^{\complement}}^{\complement} = (\mathrm{int}(B))^{\complement\complement} = \mathrm{int}(B) = \mathrm{int}(A^{\complement})$. $\square$

9. $A \subseteq B$ implies that $\mathrm{int}(A) \subseteq \mathrm{int}(B)$.

   Proof. This is so because $\mathrm{int}(A)$ is open (property 2), contained in A (and therefore contained in B), so contained in $\mathrm{int}(B)$, as $\mathrm{int}(B)$ is the largest open set contained in B (property 3). $\square$

10. $\mathrm{int}(A) = A \setminus \partial A$, where ∂A is the boundary of A.

    Proof. Recall that $\partial A = \overline{A} \cap \overline{A^{\complement}}$. So $\partial A = \overline{A} \cap (\mathrm{int}(A))^{\complement}$ by property 7. By direct computation, we have $A \setminus \partial A = A \setminus (\overline{A} \cap (\mathrm{int}(A))^{\complement}) = (A \setminus \overline{A}) \cup (A \setminus (\mathrm{int}(A))^{\complement})$. Since $A \setminus \overline{A} = \varnothing$ and $A \setminus (\mathrm{int}(A))^{\complement} = A \cap (\mathrm{int}(A))^{\complement\complement} = A \cap \mathrm{int}(A)$, which is $\mathrm{int}(A)$ by property 2. $\square$

11. $\overline{A} = \mathrm{int}(A) \cup \partial A$.

    Proof. Again, by direct computation:

$$\begin{aligned}
\mathrm{int}(A) \cup \partial A &= \mathrm{int}(A) \cup (\overline{A} \cap (\mathrm{int}(A))^{\complement}) && \text{because } \partial A = \overline{A} \cap (\mathrm{int}(A))^{\complement} \\
&= (\mathrm{int}(A) \cup \overline{A}) \cap (\mathrm{int}(A) \cup (\mathrm{int}(A))^{\complement}) && \cap \text{ distributes over } \cup \\
&= \overline{A} \cap X = \overline{A}. && \mathrm{int}(A) \subseteq A \subseteq \overline{A}
\end{aligned}$$

$\square$

12. $X = \text{int}(A) \cup \partial A \cup \text{int}(A^{\complement})$.

    Proof. By property 11, $\text{int}(A) \cup \partial A \cup \text{int}(A^{\complement}) = \overline{A} \cup \text{int}(A^{\complement})$, which, by property 8, is $\overline{A} \cup \overline{A}^{\complement}$, and the last expression is just X. $\square$

13. $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$.

    Proof. (LHS $\subseteq$ RHS). Let $C = \text{int}(A \cap B)$. Since C is open and contained in both A and B, C is contained in both $\text{int}(A)$ and $\text{int}(B)$, since $\text{int}(A)$ and $\text{int}(B)$ are the largest open sets in A and B respectively. (RHS $\subseteq$ LHS). Let $D = \text{int}(A) \cap \text{int}(B)$. So D is open and is a subset of both A and B, hence a subset of $A \cap B$, and therefore a subset of $\text{int}(A \cap B)$, since it is the largest open set contained in $A \cap B$. $\square$

Remark. Using property 7, we see that an alternative definition of interior can be given:

$$\text{int}(A) = \overline{A^{\complement}}^{\complement}.$$