planetmath.org

Math for the people, by the people.

proof of Banach fixed point theorem

Canonical name	ProofOfBanachFixedPointTheorem
Date of creation	2013-03-22 13:08:34
Last modified on	2013-03-22 13:08:34
Owner	asteroid (17536)
Last modified by	asteroid (17536)
Numerical id	5
Author	asteroid (17536)
Entry type	Proof
Classification	msc 54A20
Classification	msc 47H10
Classification	msc 54H25

Let (X,d) be a non-empty, complete metric space, and let T be a contraction mapping on (X,d) with constant q. Pick an arbitrary $x_0 \in X$, and define the sequence $(x_n)_{n=0}^{\infty}$ by $x_n := T^n x_0$. Let $a := d(Tx_0, x_0)$. We first show by induction that for any $n \geq 0$,

$$d(T^n x_0, x_0) \leq \frac{1-q^n}{1-q} a.$$

For $n = 0$, this is obvious. For any $n \geq 1$, suppose that $d(T^{n-1}x_0, x_0) \leq \frac{1-q^{n-1}}{1-q} a$. Then

$$\begin{aligned} d(T^n x_0, x_0) &\leq d(T^n x_0, T^{n-1} x_0) + d(x_0, T^{n-1} x_0) \\ &\leq q^{n-1} d(Tx_0, x_0) + \frac{1-q^{n-1}}{1-q} a \\ &= \frac{q^{n-1} - q^n}{1-q} a + \frac{1-q^{n-1}}{1-q} a \\ &= \frac{1-q^n}{1-q} a \end{aligned}$$

by the triangle inequality and repeated application of the property $d(Tx,Ty) \leq qd(x,y)$ of T. By induction, the inequality holds for all $n \geq 0$.

Given any $\epsilon > 0$, it is possible to choose a natural number N such that $\frac{q^n}{1-q} a < \epsilon$ for all $n \geq N$, because $\frac{q^n}{1-q} a \to 0$ as $n \to \infty$. Now, for any $m, n \geq N$ (we may assume that $m \geq n$),

$$\begin{aligned} d(x_m, x_n) &= d(T^m x_0, T^n x_0) \\ &\leq q^n d(T^{m-n} x_0, x_0) \\ &\leq q^n \frac{1-q^{m-n}}{1-q} a \\ &< \frac{q^n}{1-q} a < \epsilon, \end{aligned}$$

so the sequence (x_n) is a Cauchy sequence. Because (X,d) is complete, this implies that the sequence has a limit in (X,d); define x^* to be this limit. We now prove that x^* is a fixed point of T. Suppose it is not, then $\delta := d(Tx^*, x^*) > 0$. However, because (x_n) converges to x^*, there is a natural number N such that $d(x_n, x^*) < \delta/2$ for all $n \geq N$. Then

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx^*, x_{N+1}) + d(x^*, x_{N+1}) \\ &\leq qd(x^*, x_N) + d(x^*, x_{N+1}) \\ &< \delta/2 + \delta/2 = \delta, \end{aligned}$$

contradiction. So x^* is a fixed point of T. It is also unique. Suppose there is another fixed point x' of T; because $x' \neq x^*$, $d(x', x^*) > 0$. But then

$$d(x', x^*) = d(Tx', Tx^*) \leq qd(x', x^*) < d(x', x^*),$$

contradiction. Therefore, x^* is the unique fixed point of T.