bijection between closed and open interval

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For mapping the end points of the closed unit interval $[0,\,1]$ and its inner points bijectively onto the corresponding open unit interval $(0,\,1)$, one has to discern suitable denumerable subsets in both sets:

$$\begin{aligned}
[0,\,1] \;&=\; \{0,\,1,\,1/2,\,1/3,\,1/4,\,\ldots\} \cup S,\\
(0,\,1) \;&=\; \{1/2,\,1/3,\,1/4,\,\ldots\} \cup S,
\end{aligned}$$

where

$$S \;:=\; [0,\,1] \smallsetminus \{0,\,1,\,1/2,\,1/3,\,1/4,\,\ldots\}.$$

Then the mapping f from $[0,\,1]$ to $(0,\,1)$ defined by

$$f(x) \;:=\; \begin{cases} 1/2 & \text{for} \quad x = 0, \\ 1/(n{+}2) & \text{for} \quad x = 1/n \quad (n = 1,\,2,\,3,\,\ldots), \\ x & \text{for} \quad x \in S \end{cases}$$

is apparently a bijection. This means the equicardinality of both intervals.

Note that the bijection is neither monotonic (e.g. $0 \mapsto \frac{1}{2},\ \frac{1}{2} \mapsto \frac{1}{4},\ 1 \mapsto \frac{1}{3}$) nor continuous. Generally, there does not exist any continuous surjective mapping $[0,\,1] \to (0,\,1)$, since by the intermediate value theorem a continuous function maps a closed interval to a closed interval.

References

[1] S. Lipschutz: *Set theory.* Schaum Publishing Co., New York (1964).