Kuratowski closure-complement theorem

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Problem. Let X be a topological space and A a subset of X. How many (distinct) sets can be obtained by iteratively applying the closure and complement operations to A?

Kuratowski studied this problem, and showed that at most 14 sets that can be generated from a given set in an arbitrary topological space. This is known as the *Kuratowski closure-complement theorem*.

Let us examine this problem more closely. For convenience, let us denote $^- : X \to X$ be the closure operator:

$$A \mapsto A^-,$$

and $^c : X \to X$ the complementation operator:

$$A \mapsto A^c.$$

A set that can be obtained from A by iteratively applying $^-$ and ^{c} has the form A^σ, where σ is an operator on X that is the composition of finitely many $^-$ and ^{c}. In other words, σ is a word on the alphabet $\{^-,^c\}$.

First, notice that $A^{--} = A^-$ and $A^{cc} = A$. This means that σ can be reduced (or simplified) to a form such that $^-$ and ^{c} occurs alternately.

In addition, we have the following:

Proposition 1. $A^{-c-c-c-} = A^{-c-}$.

Proof. For any set A in a topological space X, A^- is closed, so that A^{-c-} is regular closed. This means that $A^{-c-} = A^{-c-c-c-}$. $\square$

This means that σ can be reduced to one of the following cases:

$$1, ^-, ^{-c}, ^{-c-}, ^{-c-c}, ^{-c-c-}, ^{-c-c-c}, ^c, ^{c-}, ^{c-c}, ^{c-c-}, ^{c-c-c}, ^{c-c-c-}, ^{c-c-c-c},$$

where $1 = {}^{cc}$ is the identity operator. As there are a total of 14 combinations, proving the closure-complement theorem is to exhibit an example. To do this, pick $X = \mathbb{R}$, the real line. Let $A = (0,1) \cup \{2\} \cup ((3,4) \cap \mathbb{Q}) \cup ((5,7) - \{6\})$. In other words, A is the union of a real interval, a point, a rational interval, and a real interval with a point deleted. Then

1. $A^- = [0,1] \cup \{2\} \cup [3,4] \cup [5,7]$,
2. $A^{-c} = (-\infty,0) \cup (1,2) \cup (2,3) \cup (4,5) \cup (7,\infty)$,
3. $A^{-c-} = (-\infty,0] \cup [1,3] \cup [4,5] \cup [7,\infty)$,

4. $A^{-c-c} = (0,1) \cup (3,4) \cup (5,7)$,
5. $A^{-c-c-} = [0,1] \cup [3,4] \cup [5,7]$,
6. $A^{-c-c-c} = (-\infty,0) \cup (1,3) \cup (4,5) \cup (7,\infty)$,
7. $A^{c} = (-\infty,0] \cup [1,2) \cup (2,3] \cup ((3,4) - \mathbb{Q}) \cup [4,5] \cup \{6\} \cup [7,\infty)$,
8. $A^{c-} = (-\infty,0] \cup [1,5] \cup \{6\} \cup [7,\infty)$,
9. $A^{c-c} = (0,1) \cup (5,6) \cup (6,7)$,
10. $A^{c-c-} = [0,1] \cup [5,7]$,
11. $A^{c-c-c} = (-\infty,0) \cup (1,5) \cup (7,\infty)$,
12. $A^{c-c-c-} = (-\infty,0] \cup [1,5] \cup [7,\infty)$,
13. $A^{c-c-c-c} = (0,1) \cup (5,7)$,

together with A, are 14 pairwise distinct sets that can be generated by $^{-}$ and ^{c}.