the groups of real numbers

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Proposition 1. *The additive group of real number $\langle \mathbb{R}, + \rangle$ is isomorphic to the multiplicative group of positive real numbers $\langle \mathbb{R}^+, \times \rangle$.*

Proof. Let $f(x) = e^x$. This maps the group $\langle \mathbb{R}, + \rangle$ to the group $\langle \mathbb{R}^+, \times \rangle$. As f has an inverse $f^{-1}(x) = \ln x$ we observe f is invertible. Furthermore, $f(x + y) = e^{x+y} = e^x e^y = f(x)f(y)$ so f is a homomorphism. Thus f is an isomorphism. $\square$

Corollary 2. *The multiplicative group of non-zero real number $\mathbb{R}^\times$ is isomorphic to $\mathbb{Z}_2 \oplus \langle \mathbb{R}, + \rangle$.*

Proof. Use the map $f : \mathbb{Z}_2 \oplus \langle \mathbb{R}, + \rangle \to \mathbb{R}^\times$ defined by $f(s, r) = (-1)^s e^r$.[1] Then

$$f((s_1, r_1)+(s_2, r_2)) = f(s_1+s_2, r_1+r_2) = (-1)^{s_1+s_2} e^{r_1+r_2} = (-1)^{s_1} e^{r_1} (-1)^{s_2} e^{r_2} = f(s_1, r_1) f(s_2, r_2)$$

so that f is a homomorphism. Furthermore, $f^{-1}(r) = (\operatorname{sign} r, \ln |r|)$ is the inverse of f so that f is bijective and thus an isomorphism of groups. $\square$

[1] We write $(-1)^s$ to mean $(-1)^{s'}$ for any integer s' representative of the equivalence class of s in $\mathbb{Z}_2$.