a space is compact iff any family of closed sets having fip has non-empty intersection

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Theorem. A topological space is compact if and only if any collection of its closed sets having the finite intersection property has non-empty intersection.

The above theorem is essentially the definition of a compact space rewritten using de Morgan's laws. The usual definition of a compact space is based on open sets and unions. The above characterization, on the other hand, is written using closed sets and intersections.

Proof. Suppose X is compact, i.e., any collection of open subsets that cover X has a finite collection that also cover X. Further, suppose $\{F_i\}_{i\in I}$ is an arbitrary collection of closed subsets with the finite intersection property. We claim that $\cap_{i\in I} F_i$ is non-empty. Suppose otherwise, i.e., suppose $\cap_{i\in I} F_i = \emptyset$. Then,

$$\begin{aligned} X &= \left(\bigcap_{i\in I} F_i\right)^c \\ &= \bigcup_{i\in I} F_i^c. \end{aligned}$$

(Here, the complement of a set A in X is written as A^c.) Since each F_i is closed, the collection $\{F_i^c\}_{i\in I}$ is an open cover for X. By compactness, there is a finite subset $J \subset I$ such that $X = \cup_{i\in J} F_i^c$. But then $X = (\cap_{i\in J} F_i)^c$, so $\cap_{i\in J} F_i = \emptyset$, which contradicts the finite intersection property of $\{F_i\}_{i\in I}$.

The proof in the other direction is analogous. Suppose X has the finite intersection property. To prove that X is compact, let $\{F_i\}_{i\in I}$ be a collection of open sets in X that cover X. We claim that this collection contains a finite subcollection of sets that also cover X. The proof is by contradiction. Suppose that $X \neq \cup_{i\in J} F_i$ holds for all finite $J \subset I$. Let us first show that the collection of closed subsets $\{F_i^c\}_{i\in I}$ has the finite intersection property. If J is a finite subset of I, then

$$\bigcap_{i\in J} F_i^c = \left(\bigcup_{i\in J} F_i\right)^c \neq \emptyset,$$

where the last assertion follows since J was finite. Then, since X has the finite intersection property,

$$\emptyset \neq \bigcap_{i\in I} F_i^c = \left(\bigcup_{i\in I} F_i\right)^c.$$

This contradicts the assumption that $\{F_i\}_{i\in I}$ is a cover for X. $\square$

References

[1] R.E. Edwards, *Functional Analysis: Theory and Applications*, Dover Publications, 1995.