product of path connected spaces is path connected

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Proposition. Let X and Y be topological spaces. Then $X \times Y$ is path connected if and only if both X and Y are path connected.

Proof. "$\Leftarrow$" Assume that X and Y are path connected and let $(x_1, y_1), (x_2, y_2) \in X \times Y$ be arbitrary points. Since X is path connected, then there exists a continous map

$$\sigma : \mathrm{I} \to X$$

such that

$$\sigma(0) = x_1 \quad \text{and} \quad \sigma(1) = x_2.$$

Analogously there exists a continous map

$$\tau : \mathrm{I} \to Y$$

such that

$$\tau(0) = y_1 \quad \text{and} \quad \tau(1) = y_2.$$

Then we have an induced map

$$\sigma \times \tau : \mathrm{I} \to X \times Y$$

defined by the formula:

$$(\sigma \times \tau)(t) = (\sigma(t), \tau(t)),$$

which is continous path from (x_1, y_1) to (x_2, y_2).

"$\Rightarrow$" On the other hand assume that $X \times Y$ is path connected. Let $x_1, x_2 \in X$ and $y_0 \in Y$. Then there exists a path

$$\sigma : \mathrm{I} \to X \times Y$$

such that

$$\sigma(0) = (x_1, y_0) \quad \text{and} \quad \sigma(1) = (x_1, y_0).$$

We also have the projection map $\pi : X \times Y \to X$ such that $\pi(x, y) = x$. Thus we have a map

$$\tau : \mathrm{I} \to X$$

defined by the formula

$$\tau(t) = \pi(\sigma(t)).$$

This is a continous path from x_1 to x_2, therefore X is path connected. Analogously Y is path connected. $\square$