graph theorems for topological spaces

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We wish to show the relation between continuous maps and their graphs is closer that it may look. Recall, that if $f : X \to Y$ is a function between sets, then the set $\Gamma(f) = \{(x, f(x)) \in X \times Y\}$ is called *the graph of* f.

Proposition 1. If $f : X \to Y$ is a continuous map between topological spaces such that Y is Hausdorff, then the graph $\Gamma(f)$ is a closed subset of $X \times Y$ in product topology.

Proof. Indeed, we will show, that $Z = (X \times Y)\backslash\Gamma(f)$ is open. Let $(x, y) \in Z$. Then $f(x) \neq y$ and thus (since Y is Hausdorff) there exist open subsetes $V_1, V_2 \subseteq Y$ such that $f(x) \in V_1$, $y \in V_2$ and $V_1 \cap V_2 = \emptyset$. Since f is continuous, then $U = f^{-1}(V_1)$ is open in X.

Note, that the condition $V_1 \cap V_2 = \emptyset$ implies, that $f(U) \cap V_2 = \emptyset$. Therefore $U \times V_2$ is a subset of Z. On the other hand this subset is open (since it is a product of two open sets) in product topology and $(x, y) \in U \times V_2$. This shows, that every point in Z belongs to Z together with a small neighbourhood, which completes the proof. $\square$

Unfortunetly, the converse of this theorem is not true as we will see later. Nevertheless we can achieve similar result, if we assume a bit more about spaces:

Proposition 2. Let $f : X \to Y$ be a function, where X, Y are Hausdorff spaces with Y compact. If $\Gamma(f)$ is a closed subset of $X \times Y$ in product topology, then f is continuous.

Proof. Let $F \subseteq Y$ be a closed set. We will show that $f^{-1}(F)$ is also closed. Consider projections

$$\pi_Y : X \times Y \to Y; \quad \pi_X : X \times Y \to X.$$

They are both continuous and thus $\pi_Y^{-1}(F)$ is closed in $X \times Y$. Since $\Gamma(f)$ is also closed, then

$$Z = \pi_Y^{-1}(F) \cap \Gamma(f)$$

is closed in $X \times Y$. It is well known, that since Y is compact, then π_X is a closed map (this is easily seen to be equivalent to *the tube lemma*). Furthermore it is easy to see, that $\pi_X(Z) = f^{-1}(F)$ and the proof is complete. $\square$

Counterexample. Let $\mathbb{R}$ denote the set of reals (with standard topology). Consider function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1/x$ and $f(0) = 0$. It is obvious, that f is discontinuous at $x = 0$, but also it can be easily checked, that $\Gamma(f)$ is closed in $\mathbb{R}^2$. Note, that $\mathbb{R}$ is not compact.