applications of Urysohn's Lemma to locally compact Hausdorff spaces

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Let X be a locally compact Hausdorff space (LCH space) and X^* its one-point compactification. We employ the following notation:

- $C(X)$ denotes the set of continuous complex functions on X;
- $C_b(X)$ denotes the set of continuous and bounded complex functions on X;
- $C_0(X)$ denotes the set of continuous complex functions on X which vanish at infinity;
- $C_c(X)$ denotes the set of continuous complex functions on X with compact support

Note that we have $C_c(X) \subseteq C_0(X) \subseteq C_b(X) \subseteq C(X)$, and that when we replace X with X^* (in general, when X is compact), these four classes of functions coincide.

Now, while Urysohn's Lemma does not directly apply to X (since X need not in general be normal), it does apply to X^*, for being compact Hausdorff, X^* *is* necessarily normal. One may therefore indirectly apply Urysohn's Lemma to X by way of X^* to obtain various results asserting the existence of certain continuous functions on X with prescribed properties. The following results and their proofs illustrate this technique and are frequently useful in analysis.

Proposition 1. *If $K \subseteq U \subseteq X$ with K compact and U open, then there exists an open subset V of X with compact closure such that $K \subseteq V \subseteq \overline{V} \subseteq U$.*

Proof. Since K is a compact subset of the Hausdorff space X^*, it is closed, and because X is open in X^*, U is as well. Therefore, by normality, there exists an open subset V of X^* such that $K \subseteq V \subseteq \overline{V} \subseteq U$ (note that the closure of V in X^* coincides with that of V in X, since the former set is contained in X and the latter set is equal to the former intersected with X). As $\overline{V}$ is closed in X^*, it is compact, and because V is open in X^* and $V \subseteq X$, V is open in X. Thus V possesses the desired properties. □

Corollary 1. *For each $x \in X$ and each open subset U of X containing x, there exists an open subset V of X with compact closure such that $x \in V$ and $\overline{V} \subseteq U$.*

Proof. Take $K = \{x\}$ in the preceding proposition. $\square$

Theorem 1. *(Urysohn's Lemma for LCH Spaces) If $K \subseteq U \subseteq X$ with K compact and U open, then there exists $f \in C_c(X)$ such that $0 \leq f \leq 1$, $f|_K \equiv 1$, and* $\operatorname{supp} f \subseteq U$.

Proof. By the first Proposition, there exists an open subset V of X with compact closure such that $K \subseteq V \subseteq \overline{V} \subseteq U$; since K and $X^* - V$ are disjoint closed subsets of the normal space X^*, Urysohn's Lemma furnishes $g \in C(X^*)$ such that $0 \leq g \leq 1$, $g|_K \equiv 1$, and $g|_{X^*-V} \equiv 0$. Put $f = g|_X$. Then $f \in C(X)$, $0 \leq f \leq 1$, and $f|_K \equiv 1$. Moreover, f vanishes outside $\overline{V}$ because g does, so $\{x \in X : f(x) \neq 0\} \subseteq \overline{V} \subseteq U$; since $\overline{V}$ is compact, and consequently closed, the last inclusion gives $\operatorname{supp} f \subseteq \overline{V} \subseteq U$ and $f \in C_c(X)$. $\square$

Theorem 2. *(Tietze Extension Theorem for LCH Spaces) If $K \subseteq X$ is compact and $f \in C(K)$ is real, then there exists a real $g \in C_c(X)$ extending f.*

Corollary 2. *$C_0(X)$ is the uniform closure of $C_c(X)$ in $C_b(X)$.*

Proof. We first show that $C_0(X)$ is closed in $C_b(X)$. To this end, assume that $(f_n)_{n=1}^{\infty}$ is a uniformly convergent sequence of functions in $C_0(X)$ with limit f and let $\epsilon > 0$ be given. Select $N \in \mathbb{Z}^+$ such that $\|f - f_N\|_\infty < \epsilon/2$, and select a compact subset K of X such that $|f_N| < \epsilon/2$ for $x \in X - K$. We then have, for all such x,

$$|f(x)| = |f(x) - f_N(x) + f_N(x)| \leq |f(x) - f_N(x)| + |f_N(x)| \leq \|f - f_N\|_\infty + |f_N(x)| < \epsilon.$$

Thus f vanishes at infinity; since the uniform limit of continuous functions is continuous, we obtain $f \in C_0(X)$, whence $C_0(X)$ is closed. It remains to establish the density of $C_c(X)$ in $C_0(X)$. Given $f \in C_0(X)$ and $\epsilon > 0$, select a compact subset K of X such that $|f(x)| < \epsilon/2$ for $x \in X - K$. By Theorem 1, there exists $g \in C_c(X)$ with range in $[0, 1]$ satisfying $g|_K \equiv 1$. The function $h = fg$ is continuous and supported inside $\operatorname{supp} g$, hence lies in $C_c(X)$; moreover, if $x \in K$, then we have $|f(x) - h(x)| = |f(x) - f(x)| = 0$, while if $x \notin K$, then

$$|f(x) - h(x)| = |f(x) - f(x)g(x)| = |f(x)||1 - g(x)| \leq |f(x)| < \frac{\epsilon}{2}.$$

It follows that $\|f - h\|_\infty < \epsilon$, hence that $f \in \overline{C_c(X)}$, completing the proof. $\square$