probabilistic metric space

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Defines	distance distribution function
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Recall that a metric space is a set X equipped with a *distance function* $d : X \times X \to [0, \infty)$, such that

1. $d(a, b) = 0$ iff $a = b$,
2. $d(a, b) = d(b, a)$, and
3. $d(a, c) \leq d(a, b) + d(b, c)$.

In some real life situations, distance between two points may not be definite. When this happens, the distance function d may be replaced by a more general function F which takes any pair of points (a, b) to a distribution function $F_{(a,b)}$. Before precisely describing how this works, we first look at the properties of these $F_{(a,b)}$ should have, and how one translates the triangle inequality in this more general setting.

distance distribution functions. Since we are dealing with the distance between a and b, the distribution function $F_{(a,b)}$ must have the property that $F_{(a,b)}(0) = 0$. Any distribution function F such that $F(0) = 0$ is called a *distance distribution function*. The set of all distance distribution functions is denoted by Δ^+. For example, for any $r \geq 0$, the step functions defined by

$$e_r(x) = \begin{cases} 0 & \text{when } x \leq r, \\ 1 & \text{otherwise} \end{cases}$$

are distance distribution functions.

In addition to $F_{(a,b)}$ being a distance distribution function, we need that $F_{(a,b)} = e_0$ iff $a = b$ and $F_{(a,b)} = F_{(b,a)}$. These two conditions correspond to the first two conditions on d.

triangle functions. Finally, we need to generalize the binary operation $+$ so it works on the set of distance distribution functions. Clearly, ordinary addition won't work as the sum of two distribution functions is no longer a distribution function. Šerstnev developed what is called a *triangle function* that will do the trick.

First, partial order Δ^+ by $F \leq G$ iff $F(x) \leq G(x)$ for all $x \in \mathbb{R}$. It is not hard to see that $e_x \leq e_y$ iff $y \leq x$ and that e_0 is the top element of Δ^+. From the poset Δ^+, call a binary operator τ on Δ^+ a *triangle function* if τ turns Δ^+ into a `http://planetmath.org/PartiallyOrderedGroup`partially ordered commutative monoid with e_0 serving as the identity element. Spelling this out, for any $F, G, H \in \Delta^+$, we have

- $F\tau G = G\tau F$,
- $(F\tau G)\tau H = F\tau(G\tau H)$,
- $F\tau e_0 = e_0\tau F = F$, and
- if $G \leq H$, then $F\tau G \leq F\tau H$,

where $F\tau G$ means $\tau(F, G)$. For example, $F\tau G = F \cdot G$, $F\tau G = \min(F, G)$ are two triangle functions. In fact, since $F\tau G \leq F\tau e_0 = F$ and $F\tau G \leq G$ similarly, we have $F\tau G \leq \min(F, G)$ for any triangle function τ.

With this, we are ready for our main definition:

Definition. A *probabilistic metric space* is a (non-empty) set X, equipped with a function $F : X \times X \to \Delta^+$, where Δ^+ is the set of distance distribution functions on which a triangle function τ is defined, such that

1. $F_{(a,b)} = e_0$ iff $a = b$, where $F_{(a,b)} := F(a, b)$,
2. $F_{(a,b)} = F_{(b,a)}$, and
3. $F_{(a,c)} \geq F_{(a,b)}\tau F_{(b,c)}$.

Given a metric space (X, d), if we can find a triangle function τ such that $e_x\tau e_y = e_{x+y}$, then (X, F) with $F_{(a,b)} := e_{d(a,b)}$ is a probabilistic metric space.

References

[1] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, Elsevier Science Publishing Company, (1983).

[2] A. N. Šerstnev, *Random normed spaces: problems of completeness*, Kazan. Gos. Univ. Učen. Zap. 122, 3-20, (1962).