proof of Baire category theorem

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Let (X, d) be a complete metric space, and U_k a countable collection of dense, open subsets. Let $x_0 \in X$ and $\epsilon_0 > 0$ be given. We must show that there exists a $x \in \bigcap_k U_k$ such that

$$d(x_0, x) < \epsilon_0.$$

Since U_1 is dense and open, we may choose an $\epsilon_1 > 0$ and an $x_1 \in U_1$ such that

$$d(x_0, x_1) < \frac{\epsilon_0}{2}, \quad \epsilon_1 < \frac{\epsilon_0}{2},$$

and such that the open ball of radius ϵ_1 about x_1 lies entirely in U_1. We then choose an $\epsilon_2 > 0$ and a $x_2 \in U_2$ such that

$$d(x_1, x_2) < \frac{\epsilon_1}{2}, \quad \epsilon_2 < \frac{\epsilon_1}{2},$$

and such that the open ball of radius ϵ_2 about x_2 lies entirely in U_2. We continue by induction, and construct a sequence of points $x_k \in U_k$ and positive ϵ_k such that

$$d(x_{k-1}, x_k) < \frac{\epsilon_{k-1}}{2}, \quad \epsilon_k < \frac{\epsilon_{k-1}}{2},$$

and such that the open ball of radius ϵ_k lies entirely in U_k.

By construction, for $0 \leq j < k$ we have

$$d(x_j, x_k) < \epsilon_j \left(\frac{1}{2} + \cdots + \frac{1}{2^{k-j}} \right) < \epsilon_j \leq \frac{\epsilon_0}{2^j}.$$

Hence the sequence $x_k,\ k = 1, 2, \ldots$ is Cauchy, and converges by hypothesis to some $x \in X$. It is clear that for every k we have

$$d(x, x_k) \leq \epsilon_k.$$

Moreover it follows that

$$d(x, x_k) \leq d(x, x_{k+1}) + d(x_k, x_{k+1}) < \epsilon_{k+1} + \frac{\epsilon_k}{2},$$

and hence a fortiori

$$d(x, x_k) < \epsilon_k$$

for every k. By construction then, $x \in U_k$ for all $k = 1, 2, \ldots$, as well. QED