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identification topology

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Let f be a function from a topological space X to a set Y. The *identification topology* on Y with respect to f is defined to be the finest topology on Y such that the function f is continuous.

Theorem 1. *Let $f : X \to Y$ be defined as above. The following are equivalent:*

1. *$\mathcal{T}$ is the identification topology on Y.*
2. *$U \subseteq Y$ is open under $\mathcal{T}$ iff $f^{-1}(U)$ is open in X.*

Proof. (1. $\Rightarrow$ 2.) If U is open under $\mathcal{T}$, then $f^{-1}(U)$ is open in X as f is continuous under $\mathcal{T}$. Now, suppose U is not open under $\mathcal{T}$ and $f^{-1}(U)$ is open in X. Let $\mathcal{B}$ be a subbase of $\mathcal{T}$. Define $\mathcal{B}' := \mathcal{B} \cup \{U\}$. Then the topology $\mathcal{T}'$ generated by $\mathcal{B}'$ is a strictly finer topology than $\mathcal{T}$ making f continuous, a contradiction.

(2. $\Rightarrow$ 1.) Let $\mathcal{T}$ be the topology defined by 2. Then f is continuous. Suppose $\mathcal{T}'$ is another topology on Y making f continuous. Let U be $\mathcal{T}'$-open. Then $f^{-1}(U)$ is open in X, which implies U is $\mathcal{T}$-open. Thus $\mathcal{T}' \subseteq \mathcal{T}$ and $\mathcal{T}$ is finer than $\mathcal{T}'$. $\square$

Remarks.

- $\mathcal{S} = \{f(V) \mid V \text{ is open in } X\}$ is a subbasis for $f(X)$, using the subspace topology on $f(X)$ of the identification topology on Y.
- More generally, let X_i be a family of topological spaces and $f_i : X_i \to Y$ be a family of functions from X_i into Y. The *identification topology* on Y with respect to the family f_i is the finest topology on Y making each f_i a continuous function. In literature, this topology is also called the *final topology.*
- The dual concept of this is the initial topology.
- Let $f : X \to Y$ be defined as above. Define binary relation $\sim$ on X so that $x \sim y$ iff $f(x) = f(y)$. Clearly $\sim$ is an equivalence relation. Let X^* be the quotient $X/\sim$. Then f induces an injective map $f^* : X^* \to Y$ given by $f^*([x]) = f(x)$. Let Y be given the identification topology and X^* the quotient topology (induced by $\sim$), then f^* is continuous. Indeed, for if $V \subseteq Y$ is open, then $f^{-1}(V)$ is open in X. But then $f^{-1}(V) = \bigcup f^{*-1}(V)$, which implies $f^{*-1}(V)$ is open in X^*.

Furthermore, the argument is reversible, so that if U is open in X^*, then so is $f^*(U)$ open in Y. Finally, if f is surjective, so is f^*, so that f^* is a homeomorphism.