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open set

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Defines	Hausdorff axioms

In a metric space M a set O is called an *open subset* of M or just *open*, if for every $x \in O$ there is an open ball S around x such that $S \subset O$. If $d(x, y)$ is the distance from x to y then the open ball B_r with radius $r > 0$ around x is given as:

$$B_r = \{y \in M | d(x, y) < r\}.$$

Using the idea of an open ball one can define a neighborhood of a point x. A set containing x is called a neighborhood of x if there is an open ball around x which is a subset of the neighborhood.

These neighborhoods have some properties, which can be used to define a topological space using the Hausdorff axioms for neighborhoods, by which again an open set within a topological space can be defined. In this way we drop the metric and get the more general topological space. We can define a topological space X with a set of neighborhoods of x called U_x for every $x \in X$, which satisfy

1. $x \in U$ for every $U \in U_x$
2. If $U \in U_x$ and $V \subset X$ and $U \subset V$ then $V \in U_x$ (every set containing a neighborhood of x is a neighborhood of x itself).
3. If $U, V \in U_x$ then $U \cap V \in U_x$.
4. For every $U \in U_x$ there is a $V \in U_x$, such that $V \subset U$ and $V \in U_p$ for every $p \in V$.

The last point leads us back to open sets, indeed a set O is called open if it is a neighborhood of every of its points. Using the properties of these open sets we arrive at the usual definition of a topological space using open sets, which is equivalent to the above definition. In this definition we look at a set X and a set of subsets of X, which we call open sets, called $\mathcal{O}$, having the following properties:

1. $\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$.
2. Any union of open sets is open.
3. intersections of open sets are open.

Note that a topological space is more general than a metric space, i.e. on every metric space a topology can be defined using the open sets from the metric, yet we cannot always define a metric on a topological space such that all open sets remain open.

Examples:

- On the real axis the interval $I = (0, 1)$ is open because for every $a \in I$ the open ball with radius $\min(a, 1 - a)$ is always a subset of I. (Using the standard metric $d(x, y) = |x - y|$.)

- The open ball B_r around x is open. Indeed, for every $y \in B_r$ the open ball with radius $r - d(x, y)$ around y is a subset of B_r, because for every z within this ball we have:

  $$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r.$$

  So $d(x, z) < r$ and thus z is in B_r. This holds for every z in the ball around y and therefore it is a subset of B_r

- A non-metric topology would be the finite complement topology on infinite sets, in which a set is called open, if its complement is finite.