proof of characterization of connected compact metric spaces.

Canonical name	ProofOfCharacterizationOfConnectedCompactMetricSpaces
Date of creation	2013-03-22 14:17:06
Last modified on	2013-03-22 14:17:06
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Last modified by	paolini (1187)
Numerical id	4
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Entry type	Proof
Classification	msc 54A05

First we prove the right-hand arrow: if X is connected then the property stated in the Theorem holds. This implication is true in every metric space X, without additional conditions.

Let us denote with A_ε the set of all points $z \in X$ which can be joined to x with a sequence of points $p_1, \dots, p_n$ with $p_1 = x$, $p_n = z$ and $d(p_i, p_{i+1}) < \varepsilon$. If $z \in A_\varepsilon$ then also $B_\varepsilon(z) \subset A_\varepsilon$ since given $w \in B_\varepsilon(z)$ we can simply add the point $p_{n+1} = w$ to the sequence $p_1, \dots, p_n$. This immediately shows that A_ε is an open subset of X. On the other hand we can show that A_ε is also closed. In fact suppose that $x_n \in A_\varepsilon$ and $x_n \to \bar{x} \in X$. Then there exists k such that $\bar{x} \in B_\varepsilon(x_k)$ and hence $\bar{x} \in A_\varepsilon$ by the property stated above. Since both A_ε and its complementary set are open then, being X connected, we conclude that A_ε is either empty or its complementary set is empty. Clearly $x \in A_\varepsilon$ so we conclude that $A_\varepsilon = X$. Since this is true for all $\varepsilon > 0$ the first implication is proven.

Let us prove the reverse implication. Suppose by contradiction that X is not connected. This means that two non-empty open sets A, B exist such that $A \cup B = X$ and $A \cap B = \emptyset$. Since A is the complementary set of B and vice-versa, we know that A and B are closed too. Being X compact we conclude that both A and B are compact sets. We now claim that

$$\delta := \inf_{a \in A, b \in B} d(a, b) > 0.$$

Suppose by contradiction that $\delta = 0$. In this case by definition of infimum, there exist two sequences $a_k \in A$ and $b_k \in B$ such that $d(a_k, b_k) \to 0$. Since A and B are compact, up to a subsequence we may and shall suppose that $a_k \to a \in A$ and $b_k \to b \in B$. By the continuity of the distance function we conclude that $d(a, b) = 0$ i.e. $a = b$ which is in contradiction with the condition $A \cap B = \emptyset$. So the claim is proven.

As a consequence, given $\varepsilon < \delta$ it is not possible to join a point of A with a point of B. In fact in the sequence $p_1, \dots, p_n$ there should exists two consecutive points p_i and p_{i+1} with $p_i \in A$ and $p_{i+1} \in B$. By the previous observation we would conclude that $d(p_i, p_{i+1}) \geq \delta > \varepsilon$.