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proximity space

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Synonym	proximity
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Defines	nearness relation
Defines	separated proximity space
Defines	discrete proximity
Defines	indiscrete proximity

Let X be a set. A binary relation δ on $P(X)$, the power set of X, is called a *nearness relation* on X if it satisfies the following conditions: for $A, B \in P(X)$,

1. if $A \cap B \neq \varnothing$, then $A\delta B$;
2. if $A\delta B$, then $A \neq \varnothing$ and $B \neq \varnothing$;
3. (symmetry) if $A\delta B$, then $B\delta A$;
4. $(A_1 \cup A_2)\delta B$ iff $A_1\delta B$ or $A_2\delta B$;
5. $A\delta' B$ implies the existence of $C \subseteq X$ with $A\delta' C$ and $(X-C)\delta' B$, where $A\delta' B$ means $(A, B) \notin \delta$.

If $x, y \in X$ and $A \subseteq X$, we write $x\delta A$ to mean $\{x\}\delta A$, and $x\delta y$ to mean $\{x\}\delta\{y\}$.

When $A\delta B$, we say that A is δ-*near*, or just *near* B. δ is also called a *proximity relation*, or *proximity* for short. Condition 1 is equivalent to saying if $A\delta' B$, then $A \cap B = \varnothing$. Condition 4 says that if A is near B, then any superset of A is near B. Conversely, if A is not near B, then no subset of A is near B. In particular, if $x \in A$ and $A\delta' B$, then $x\delta' B$.

Definition. A set X with a proximity as defined above is called a *proximity space*.

For any subset A of X, define $A^c = \{x \in X \mid x\delta A\}$. Then ^{c} is a closure operator on X:

Proof. Clearly $\varnothing^c = \varnothing$. Also $A \subseteq A^c$ for any $A \subseteq X$. To see $A^{cc} = A^c$, assume $x\delta A^c$, we want to show that $x\delta A$. If not, then there is $C \subseteq X$ such that $x\delta' C$ and $(X-C)\delta' A$. The second part says that if $y \in X-C$, then $y\delta' A$, which is equivalent to $A^c \subseteq C$. But $x\delta' C$, so $x\delta' A^c$. Finally, $x \in (A \cup B)^c$ iff $x\delta(A \cup B)$ iff $x\delta A$ or $x\delta B$ iff $x \in A^c$ or $x \in B^c$. $\square$

This turns X into a topological space. Thus any proximity space is a topological space induced by the closure operator defined above.

A proximity space is said to be *separated* if for any $x, y \in X$, $x\delta y$ implies $x = y$.

Examples.

- Let (X, d) be a pseudometric space. For any $x \in X$ and $A \subseteq X$, define $d(x, A) := \inf_{y \in A} d(x, y)$. Next, for $B \subseteq X$, define $d(A, B) := \inf_{x \in A} d(x, B)$. Finally, define $A\delta B$ iff $d(A, B) = 0$. Then δ is a proximity and (X, d) is a proximity space as a result.

- *discrete proximity*. Let X be a non-empty set. For $A, B \subseteq X$, define $A\delta B$ iff $A \cap B \neq \varnothing$. Then δ so defined is a proximity on X, and is called the *discrete proximity* on X.

- *indiscrete proximity*. Again, X is a non-empty set and $A, B \subseteq X$. Define $A\delta B$ iff $A \neq \varnothing$ and $B \neq \varnothing$. Then δ is also a proximity. It is called the *indiscrete proximity* on X.

References

[1] S. Willard, *General Topology*, Addison-Wesley, Publishing Company, 1970.

[2] S.A. Naimpally, B.D. Warrack, *Proximity Spaces*, Cambridge University Press, 1970.