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completely separated

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Proposition 1. *Let A, B be two subsets of a topological space X . The following are equivalent:*

1. *There is a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$,*
2. *There is a continuous function $g : X \rightarrow \mathbb{R}$ such that $g(A) \leq r < s \leq g(B)$, $r, s \in \mathbb{R}$.*

Proof. Clearly 1 implies 2 (by setting $r = 0$ and $s = 1$). To see that 2 implies 1, first take the transformation

$$h(x) = \frac{g(x) - r}{s - r}$$

so that $h(A) \leq 0 < 1 \leq h(B)$. Then take the transformation $f(x) = (h(x) \vee 0) \wedge 1$, where $0(x) = 0$ and $1(x) = 1$ for all $x \in X$. Then $f(A) = (h(A) \vee 0) \wedge 1 = 0 \wedge 1 = 0$ and $f(B) = (h(B) \vee 0) \wedge 1 = h(B) \wedge 1 = 1$. Here, \vee and \wedge denote the binary operations of taking the maximum and minimum of two given real numbers (see ring of continuous functions for more detail). Since during each transformation, the resulting function remains continuous, the first assertion is proved. \square

Definition. Any two sets A, B in a topological space X satisfying the above equivalent conditions are said to be *completely separated*. When A and B are completely separated, we also say that $\{A, B\}$ is completely separated.

Clearly, two sets that are completely separated are disjoint, and in fact separated.

Remark. A T1 topological space in which every pair of disjoint closed sets are completely separated is a normal space. A T0 topological space in which every pair consisting of a closed set and a singleton is completely separated is a completely regular space.