subspace topology in a metric space

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Theorem 1. *Suppose X is a topological space whose topology is induced by a metric d, and suppose $Y \subseteq X$ is a subset. Then the subspace topology in Y is the same as the metric topology when by d restricted to Y.*

Let $d' \colon Y : Y \to \mathbb{R}$ be the restriction of d to Y, and let

$$\begin{aligned} B_r(x) &= \{z \in X : d'(z,x) < r\}, \\ B'_r(x) &= \{z \in Y : d'(z,x) < r\}. \end{aligned}$$

The proof rests on the identity

$$B'_r(x) = Y \cap B_r(x), \quad x \in Y, r > 0.$$

Suppose $A \subseteq Y$ is open in the subspace topology of Y, then $A = Y \cap V$ for some open $V \subseteq X$. Since V is open in X,

$$V = \cup\{B_{r_i}(x_i) : i = 1, 2, \ldots\}$$

for some $r_i > 0$, $x_i \in X$, and

$$\begin{aligned} A &= \cup\{Y \cap B_{r_i}(x_i) : i = 1, 2, \ldots\} \\ &= \cup\{B'_{r_i}(x_i) : i = 1, 2, \ldots\}. \end{aligned}$$

Thus A is open also in the metric topology of d'. The converse direction is proven similarly.