Banach fixed point theorem

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Related topic	FixedPoint
Defines	contraction mapping
Defines	contraction operator

Let (X, d) be a complete metric space. A function $T : X \to X$ is said to be a *contraction mapping* if there is a constant q with $0 \leq q < 1$ such that

$$d(Tx, Ty) \leq q \cdot d(x, y)$$

for all $x, y \in X$. Contractions have an important property.

Theorem 1 (Banach Theorem)**.** *Every contraction has a unique* `http://planetmath.org/node/2` *point.*

There is an estimate to this fixed point that can be useful in applications. Let T be a contraction mapping on (X, d) with constant q and unique fixed point $x^* \in X$. For any $x_0 \in X$, define recursively the following sequence

$$\begin{aligned} x_1 &:= Tx_0 \\ x_2 &:= Tx_1 \\ &\vdots \\ x_{n+1} &:= Tx_n. \end{aligned}$$

The following inequality then holds:

$$d(x^*, x_n) \leq \frac{q^n}{1-q} d(x_1, x_0).$$

So the sequence (x_n) converges to x^*. This estimate is occasionally responsible for this result being known as *the method of successive approximations*.