proof of Cauchy condition for limit of function

Canonical name	ProofOfCauchyConditionForLimitOfFunction
Date of creation	2013-03-22 18:59:08
Last modified on	2013-03-22 18:59:08
Owner	puff (4175)
Last modified by	puff (4175)
Numerical id	8
Author	puff (4175)
Entry type	Proof
Classification	msc 54E35
Classification	msc 26A06
Classification	msc 26B12

The forward direction is . Assume that $\lim_{x\to x_0} f(x) = L$. Then given ϵ there is a δ such that

$$|f(u) - L| < \epsilon/2 \text{ when } 0 < |u - x_0| < \delta.$$

Now for $0 < |u - x_0| < \delta$ and $0 < |v - x_0| < \delta$ we have

$$|f(u) - L| < \epsilon/2 \text{ and } |f(v) - L| < \epsilon/2$$

and so

$$|f(u)-f(v)| = |f(u)-L-(f(v)-L)| \leq |f(u)-L|+|f(v)-L| < \epsilon/2+\epsilon/2 = \epsilon.$$

We prove the reverse by contradiction. Assume that the condition holds. Now suppose that $\lim_{x\to x_0} f(x)$ does not exist. This means that for any l and any ϵ sufficiently small then for any $\delta > 0$ there is x_l such that $0 < |xl - x_0| < \delta$ and $|f(x_l) - l| \geq \epsilon$. For any such ϵ choose u such that $0 < |u - x_0| < \delta$ and put $l = f(v)$ then substituting in the condition with $u = x_l$ we get $|f(x_l) - l| < \epsilon$. A contradiction.