pseudometric topology

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Let (X,d) be a pseudometric space. As in a metric space, we define

$$B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}.$$

for $x \in X$, $\varepsilon > 0$.

In the below, we show that the collection of sets

$$\mathscr{B} = \{B_\varepsilon(x) \mid \varepsilon > 0, x \in X\}$$

form a base for a topology for X. We call this topology the on X induced by d. Also, a topological space X is a *pseudometrizable topological space* if there exists a pseudometric d on X whose pseudometric topology coincides with the given topology for X [**?**, **?**].

Proposition 1. *$\mathscr{B}$ is a base for a topology.*

Proof. We shall use the `http://planetmath.org/node/5845`this result to prove that $\mathscr{B}$ is a base.

First, as $d(x,x) = 0$ for all $x \in X$, it follows that $\mathscr{B}$ is a cover. Second, suppose $B_1, B_2 \in \mathscr{B}$ and $z \in B_1 \cap B_2$. We claim that there exists a $B_3 \in \mathscr{B}$ such that

$$z \in B_3 \subseteq B_1 \cap B_2. \tag{1}$$

By definition, $B_1 = B_{\varepsilon_1}(x_1)$ and $B_2 = B_{\varepsilon_2}(x_2)$ for some $x_1, x_2 \in X$ and $\varepsilon_1, \varepsilon_2 > 0$. Then

$$d(x_1, z) < \varepsilon_1, \quad d(x_2, z) < \varepsilon_2.$$

Now we can define $\delta = \min\{\varepsilon_1 - d(x_1,z), \varepsilon_2 - d(x_2,z)\} > 0$, and put

$$B_3 = B_\delta(z).$$

If $y \in B_3$, then for $k = 1,2$, we have by the triangle inequality

$$\begin{aligned} d(x_k, y) &\leq d(x_k, z) + d(z, y) \\ &< d(x_k, z) + \delta \\ &\leq \varepsilon_k, \end{aligned}$$

so $B_3 \subseteq B_k$ and condition **??** holds. □

Remark

In the proof, we have not used the fact that d is symmetric. Therefore, we have, in fact, also shown that any quasimetric induces a topology.

References

[1] J.L. Kelley, *General Topology*, D. van Nostrand Company, Inc., 1955.

[2] S. Willard, *General Topology*, Addison-Wesley, Publishing Company, 1970.