Polish spaces up to Borel isomorphism

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Two topological spaces X and Y are http://planetmath.org/BorelIsomorphismBorel isomorphic if there is a Borel measurable function $f\colon X \to Y$ with Borel inverse. Such a function is said to be a Borel isomorphism. The following result classifies all Polish spaces up to Borel isomorphism.

Theorem. *Every uncountable Polish space is Borel isomorphic to* $\mathbb{R}$ *with the standard topology.*

As the Borel σ-algebra on any countable metric space is just its power set, this shows that every Polish space is Borel isomorphic to one and only one of the following.

1. $\{1, 2, \ldots, n\}$ for some $n \geq 0$, with the discrete topology.

2. $\mathbb{N} = \{1, 2, \ldots\}$ with the discrete topology.

3. $\mathbb{R}$ with the standard topology.

In particular, two Polish spaces are Borel isomorphic if and only if they have the same cardinality, and any uncountable Polish space has the cardinality of the continuum.