proof that uniformly continuous is proximity continuous

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Let $f: X \to Y$ be a uniformly continuous function from uniform spaces X to Y with uniformities $\mathcal{U}$ and $\mathcal{V}$ respectively. Let δ and ϵ be the http://planetmath.org/UniformPr generated by $\mathcal{U}$ and $\mathcal{V}$ respectively. It is known that X and Y are proximity spaces with proximities δ and ϵ respectively. Furthermore, we have the following:

Theorem 1. *$f: X \to Y$ is proximity continuous.*

Proof. Let A, B be any subsets of X with $A\delta B$. We want to show that $f(A)\epsilon f(B)$, or equivalently,

$$V[f(A)] \cap V[f(B)] \neq \varnothing,$$

for any $V \in \mathcal{V}$. Pick any $V \in \mathcal{V}$. Since f is uniformly continuous, there is $U \in \mathcal{U}$ such that

$$U[x] \subseteq f^{-1}(V[f(x)]),$$

for any $x \in X$. As a result,

$$U[A] \subseteq f^{-1}(V[f(A)]),$$

which implies that

$$f(U[A]) \subseteq V[f(A)].$$

Similarly $f(U[B]) \subseteq V[f(B)]$. Now, $A\delta B$ is equivalent to $U[A] \cap U[B] \neq \varnothing$, so we can pick

$$z \in U[A] \cap U[B].$$

Then

$$f(z) \in f(U[A]) \cap f(U[B]) \subseteq V[f(A)] \cap V[f(B)],$$

and therefore

$$V[f(A)] \cap V[f(B)] \neq \varnothing.$$

This shows that f is proximity continuous. $\square$