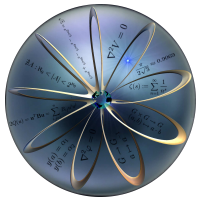

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uncountable Polish spaces contain Cantor space

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Cantor space is an example of a compact and uncountable Polish space. In fact, every uncountable Polish space contains Cantor space, as stated by the following theorem.

Theorem. *Let X be an uncountable Polish space. Then, it contains a subset S which is homeomorphic to Cantor space.*

For example, the set $\mathbb{R}$ of real numbers contains the `http://planetmath.org/CantorSet`Cantor middle thirds set. Note that, being homeomorphic to Cantor space, S must be a compact and hence closed subset of X. The result is trivial in the case of Baire space $\mathcal{N}$, in which case we may take S to be the set of all $s \in \mathcal{N}$ satisfying $s_n \in \{1, 2\}$ for all n. Then, for any uncountable Polish space X there exists a continuous and one-to-one function $f\colon \mathcal{N} \to X$ (see `http://planetmath.org/InjectiveImagesOfBaireSpace`here). Then f gives a continuous bijection from S to $f(S)$. The `http://planetmath.org/InverseFunctionThe` function theorem implies that f is a homeomorphism between S and $f(S)$ and, therefore, $f(S)$ is homeomorphic to Cantor space.