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proof of Lebesgue number lemma

Canonical name	ProofOfLebesgueNumberLemma
Date of creation	2013-03-22 13:09:20
Last modified on	2013-03-22 13:09:20
Owner	scanez (1021)
Last modified by	scanez (1021)
Numerical id	7
Author	scanez (1021)
Entry type	Proof
Classification	msc 54E45

By way of contradiction, suppose that no Lebesgue number existed. Then there exists an open cover $\mathcal{U}$ of X such that for all $\delta > 0$ there exists an $x \in X$ such that no $U \in \mathcal{U}$ contains $B_\delta(x)$ (the open ball of radius δ around x). Specifically, for each $n \in \mathbb{N}$, since $1/n > 0$ we can choose an $x_n \in X$ such that no $U \in \mathcal{U}$ contains $B_{1/n}(x_n)$. Now, X is compact so there exists a subsequence (x_{n_k}) of the sequence of points (x_n) that converges to some $y \in X$. Also, $\mathcal{U}$ being an open cover of X implies that there exists $\lambda > 0$ and $U \in \mathcal{U}$ such that $B_\lambda(y) \subseteq U$. Since the sequence (x_{n_k}) converges to y, for k large enough it is true that $d(x_{n_k}, y) < \lambda/2$ (d is the metric on X) and $1/n_k < \lambda/2$. Thus after an application of the triangle inequality, it follows that

$$B_{1/n_k}(x_{n_k}) \subseteq B_\lambda(y) \subseteq U,$$

contradicting the assumption that no $U \in \mathcal{U}$ contains $B_{1/n}(x_n)$. Hence a Lebesgue number for $\mathcal{U}$ does exist.