proof for one equivalent statement of Baire category theorem

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First, let's assume Baire's category theorem and prove the alternative statement.
We have $B = \bigcup_{n=1}^{\infty} B_n$, with $\mathrm{int}(\overline{B_k}) = \emptyset \ \forall k \in \mathbf{N}$.
Then

$$X = X - \mathrm{int}(\overline{B_k}) = \overline{X - \overline{B_k}} \ \forall k \in \mathbf{N}$$

Then $X - \overline{B_k}$ is dense in X for every k. Besides, $X - \overline{B_k}$ is open because X is open and $\overline{B_k}$ closed. So, by Baire's Category Theorem, we have that

$$\bigcap_{n=1}^{\infty}(X - \overline{B_n}) = X - \bigcup_{n=1}^{\infty} \overline{B_n}$$

is dense in X. But $B \subset \bigcup_{n=1}^{\infty} \overline{B_n} \implies X - \bigcup_{n=1}^{\infty} \overline{B_n} \subset X - B$, and then $X = \overline{X - \bigcup_{n=1}^{\infty} \overline{B_n}} \subset \overline{X - B} = X - \mathrm{int}(B) \implies \mathrm{int}(B) = \emptyset$.

Now, let's assume our alternative statement as the hypothesis, and let $(B_k)_{k \in N}$ be a collection of open dense sets in a complete metric space X. Then $\mathrm{int}(\overline{X - B_k}) = \mathrm{int}(X - \mathrm{int}(B_k)) = \mathrm{int}(X - B_k) = X - \overline{B_k} = \emptyset$ and so $X - B_k$ is nowhere dense for every k.

Then $X - \overline{\bigcap_{n=1}^{\infty} B_n} = \mathrm{int}(X - \bigcap_{n=1}^{\infty} B_n) = \mathrm{int}(\bigcup_{n=1}^{\infty} X - B_n) = \emptyset$ due to our hypothesis. Hence Baire's category theorem holds.
QED