accumulation points and convergent subnets

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Proposition. *Let X be a topological space and $(x_\alpha)_{\alpha\in A}$ a net in X. A point $x \in X$ is an accumulation point of (x_α) if and only if some subnet of (x_α) converges to x.*

Proof. Suppose first that $(x_{\alpha_\beta})_{\beta\in B}$ is a subnet of (x_α) converging to x. Given an open subset U of X containing x and $\alpha \in A$, we may select $\beta_1 \in B$ such that $x_{\alpha_\beta} \in U$ for $\beta \geq \beta_1$, as well as $\beta_2 \in B$ such that $\alpha_\beta \geq \alpha$ for $\beta \geq \beta_2$. Finally, because B is directed, there exists $\beta \in B$ such that $\beta \geq \beta_1$ and $\beta \geq \beta_2$; we then have $\alpha_\beta \geq \alpha$ and $x_{\alpha_\beta} \in U$, so that (x_α) is frequently in U, whence x is an accumulation point of (x_α). Conversely, suppose that x is an accumulation point of (x_α), let N be the set of open neighborhoods of x in X, directed by reverse inclusion, and let $B = A \times N$, directed in the natural way. For each pair $(\gamma, U) \in B$, select $\alpha_{(\gamma,U)} \in B$ such that $\alpha \geq \gamma$ and $x_{\alpha_{(\gamma,U)}} \in U$; $(x_{\alpha_{(\gamma,U)}})_{(\gamma,U)\in B}$ is then a subnet of (x_α) that converges to x, for given $U \in N$ and $\gamma \in A$, if $(\gamma', U') \geq (\gamma, U)$, then $\alpha_{(\gamma',U)} \geq \gamma' \geq \gamma$ and $x_{\alpha_{(\gamma',U')}} \in U' \subseteq U$. $\square$