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properties of first countability

Canonical name	PropertiesOfFirstCountability
Date of creation	2013-03-22 19:09:26
Last modified on	2013-03-22 19:09:26
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	11
Author	CWoo (3771)
Entry type	Result
Classification	msc 54D99

Proposition 1. *Let X be a first countable topological space and $x \in X$. Then $x \in \overline{A}$ iff there is a sequence (x_i) in A that converges to x.*

Proof. One side is true for all topological spaces: if (x_i) is in A converging x, then for any open set U of x, there is some i such that $x_i \in U$, whence $U \cap A \neq \varnothing$. As a result, $x \in \overline{A}$.

Conversely, suppose $x \in \overline{A}$. Let $\{B_i \mid i = 1, 2, \ldots\}$ be a neighborhood base around x. We may as well assume each B_i open. Next, let

$$N_n := B_1 \cap B_2 \cap \cdots \cap B_n,$$

then we obtain a set of nested open sets containing x:

$$N_1 \supseteq N_2 \supseteq \cdots .$$

Since each N_i is open, its intersection with A is non-empty. So we may choose $x_i \in N_i \cap A$. We want to show that (x_i) converges to x. First notice tat for any fixed j, $x_i \in N_j$ for all $i \geq j$. Pick any open set U containing x. Then $N_j \subseteq B_j \subseteq U$. Hence $x_i \in U$ for all $i \geq j$. $\square$

From this, we can prove the following corollaries (assuming all spaces involved are first countable):

Corollary 1. *C is closed iff every sequence (x_i) in C that converges to x implies that $x \in C$.*

Proof. First, assume (x_i) is in a closed set C converging to x. Then $x \in \overline{C}$ by the proposition above. As C is closed, we have $x \in \overline{C} = C$.

Conversely, pick any $x \in \overline{C}$. By the proposition above, there is a sequence (x_i) in C converging to x. By assumption $x \in C$. So $\overline{C} \subseteq C$, which means that C is closed. $\square$

Corollary 2. *U is open iff every sequence (x_i) that converges to $x \in U$ is eventually in U.*

Proof. First, suppose U is open and (x_i) converges to $x \in U$. If none of x_i is in U, then all of x_i is in its complement $X - U$, which is closed. Then by the proposition, x must be in the closure of $X - U$, which is just $X - U$, contradicting the assumption that $x \in U$. Hence $x_i \in U$ for some i.

Conversely, assume the right hand side statement. Suppose $x \notin U^\circ = X - \overline{X - U}$. Then $x \in \overline{X - U}$. By the proposition, there is a sequence (x_i)

in $X - U$ converging to x. If $x \in U$, then by assumption, (x_i) is eventually in U, which means $x_i \in U$ for some i, contradicting the earlier statement that (x_i) is in $X - U$. Therefore, $x \notin U$, which implies that $U \subseteq U^\circ$, or U is open. $\square$

Corollary 3. *A function $f : X \to Y$ is continuous iff it preserves converging sequences.*

Proof. Suppose first that f is continuous, and (x_i) in X converging to x. We want to show that $(f(x_i))$ converges to $f(x)$. Let V be an open set containing $f(x)$. So $f^{-1}(V)$ is open containing x, which implies that there is some j such that $x_i \in f^{-1}(V)$ for all $i \geq j$, or $f(x_i) \in V$ for all $i \geq j$, which means that $(f(x_i)) \to f(x)$.

Conversely, suppose f preserves converging sequences and C a closed set in Y. We want to show that $D := f^{-1}(C)$ is closed. Suppose (x_i) is a sequence in D converging to x. Then $(f(x_i))$ converges to $f(x)$. Since $(f(x_i))$ is in C and C is closed, $f(x) \in C$ by the first corollary above. So $x \in f^{-1}(C) = D$ too. Hence D is closed, again by the same corollary. $\square$