closed Hausdorff neighbourhoods, a theorem on

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Theorem. If X is a topological space in which every point has a closed Hausdorff neighbourhood, then X is Hausdorff.

Note. In this theorem (and the proof that follows) neighbourhoods are not assumed to be open. That is, a neighbourhood of a point x is a set A such that x lies in the interior of A.

Proof of theorem. Let X be a topological space in which every point has a closed Hausdorff neighbourhood. Suppose $a, b \in X$ are distinct. It suffices to show that a and b have disjoint neighbourhoods. By assumption, there is a closed Hausdorff neighbourhood N of b. If $a \notin N$, then $X \setminus N$ and N are disjoint neighbourhoods of a and b (as N is closed).

So suppose $a \in N$. As N is Hausdorff, there are disjoint sets $U_0, V_0 \subseteq N$ that are open in N, such that $a \in U_0$ and $b \in V_0$. There are open sets U and V of X such that $U_0 = U \cap N$ and $V_0 = V \cap N$. Note that U is a neighbourhood of a, and V is a neighbourhood of b. As N is a neighbourhood of b, it follows that $V \cap N$ (that is, V_0) is a neighbourhood of b. We have $U \cap V_0 = U_0 \cap V_0 = \varnothing$. So U and V_0 are disjoint neighbourhoods of a and b. QED.