proof of every filter is contained in an ultrafilter (alternate proof)

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Let $\mathfrak{U}$ be the family of filters over X which are finer than $\mathcal{F}$, under the partial order of inclusion.

Claim 1. *Every chain in* $\mathfrak{U}$ *has an upper bound also in* $\mathfrak{U}$.

Proof. Take any chain $\mathfrak{C}$ in $\mathfrak{U}$, and consider the set $\mathcal{C} = \cup\mathfrak{C}$. Then $\mathcal{C}$ is also a filter: it cannot contain the empty set, since no filter in the chain does; the intersection of two sets in $\mathcal{C}$ must be present in the filters of $\mathfrak{C}$; and $\mathcal{C}$ is closed under supersets because every filter in $\mathfrak{C}$ is. Obviously $\mathcal{C}$ is finer than $\mathcal{F}$. □

So we conclude, by Zorn's lemma, that $\mathfrak{U}$ must have a maximal filter say $\mathcal{U}$, which must contain $\mathcal{F}$. All we need to show is that $\mathcal{U}$ is an ultrafilter. Now, for any filter $\mathcal{U}$, and any set $Y \subseteq X$, we must have:

Claim 2. *Either* $\mathcal{U}_1 = \{Z \cap Y : Z \in \mathcal{U}\}$ *or* $\mathcal{U}_2 = \{Z \cap (X \backslash Y) : Z \in \mathcal{U}\}$ *(or both) are a filter subbasis.*

Proof. We prove by contradiction that at least one of $\mathcal{U}_1$ or $\mathcal{U}_2$ must have the finite intersection property. If neither has the finite intersection property, then for some $Z_1, \dots Z_k$ we must have

$$\varnothing = \bigcap_{1 \leq i \leq k} Z_i \cap Y = \bigcap_{1 \leq i \leq k} Z_i \cap (X \backslash Y).$$

But then

$$\varnothing = \left(\bigcap_{1 \leq i \leq k} Z_i \cap Y \right) \cup \left(\bigcap_{1 \leq i \leq k} Z_i \cap (X \backslash Y) \right) = \bigcap_{1 \leq i \leq k} Z_i,$$

and so $\mathcal{U}$ does not have the finite intersection property either. This cannot be, since $\mathcal{U}$ is a filter. □

Now, by Claim 2, if $\mathcal{U}$ were not an ultrafilter, i.e., if for some Y subset of X we would have neither Y nor $X \backslash Y$ in $\mathcal{U}$, then the filter generated $\mathcal{U}_1$ or $\mathcal{U}_2$ would be finer than $\mathcal{U}$, and then $\mathcal{U}$ would not be maximal.

So $\mathcal{U}$ is an ultrafilter containing $\mathcal{F}$, as intended.