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proof of injective images of Baire space

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We show that, for an uncountable Polish space X, there exists a continuous and one-to-one function $f\colon \mathcal{N} \to X$ such that $X \setminus f(\mathcal{N})$ is countable. Furthermore, the inverse of f defined on $f(\mathcal{N})$ is Borel measurable.

The construction of the function relies on the following result. We let d be a complete metric on X with respect to which the diameter of any subset is defined.

Lemma. *Let S be an uncountable subset of X which can be written as the difference of two closed sets, and choose any $\epsilon > 0$.*

Then, there exists a sequence $S_1, S_2, \ldots$ of pairwise disjoint and uncountable subsets of S with diameter no more than ϵ and such that,

1. *for each $n > 0$, $\bigcup_{k \leq n} S_k$ is closed.*
2. *$S \setminus \bigcup_n S_n$ is countable.*

Proof. As $S = A \setminus B$ is the difference of closed sets, it is a Polish space under the subspace topology. In fact, if B is nonempty, the topology is generated by the complete metric

$$d_S(x, y) = d(x, y) + \sup\left\{ |d(x, z)^{-1} - d(y, z)^{-1}| \colon z \in B \right\},$$

otherwise we may take $d_S = d$. In either case, $d_S \geq d$, so it is enough to choose the sets S_n to have diameter no more than 2^{-n} with respect to d_S. Note also that any bounded and closed set with respect to this metric is also closed as a subset of X.

Let S^c be the condensation points of S, which are the points whose neighborhoods all contain uncountably many points of S. Then, $S \setminus S^c$ is a union of countably many countable and open subsets of S, so is countable and open. Hence, S^c is uncountable and closed in S, and every open subset is uncountable. Choosing any $p \in S^c$ then $S^c \setminus p$ will not be a closed subset of X. So, replacing S by $S \setminus \{p\}$ if necessary, we may suppose that S^c is not closed as a subset of X and, therefore is not compact.

So, for some $\delta > 0$, S^c cannot be covered by finitely many sets with d_S-diameter no more than δ (see `http://planetmath.org/ProofThatAMetricSpaceIsCompactIfAndO`
By separability, there is a sequence $T_1, T_2, \ldots$ of open balls in S^c with diameter less than $\min(\epsilon, \delta)$, and covering S^c. Writing $\bar{T}_n$ for the d_S-closure of T_n and eliminating any terms such that $T_n \subseteq \bigcup_{k<n} \bar{T}_k$, then $S_n \equiv \bar{T}_n \setminus \bigcup_{k<n} \bar{T}_k$ have nonempty interior and hence are uncountable, and

$$S \setminus \bigcup_n S_n = S \setminus S^c$$

is countable, as required. $\square$

Note that $S_n = \bigcup_{k\leq n} S_k \setminus \bigcup_{k\leq n-1} S_k$ is also a difference of closed sets. So, the lemma allows us to inductively choose sets $C(n_1,\ldots,n_k)\subseteq X$ for integers $k\geq 0$ and $n_1,\ldots,n_k\geq 1$ such that $C()=X$ and the following are satisfied.

1. $C(n_1,\ldots,n_k,m)$ are uncountable, contained in $C(n_1,\ldots,n_k)$, and pairwise disjoint as m runs through the positive integers.

2. $\bigcup_{j\leq m} C(n_1,\ldots,n_k,j)$ is closed for all $m\geq 1$.

3. $C(n_1,\ldots,n_k)\setminus\bigcup_m C(n_1,\ldots,n_k,m)$ is countable and, for $k\geq 1$, has diameter no more than 2^{-k}.

For any $n\in\mathcal{N}$ we may choose a sequence $x_k\in C(n_1,\ldots,n_k)$. Since, for $k\geq 1$, this set has diameter no more than 2^{-k}, then $d(x_j,x_k)\leq 2^{-k}$ whenever $j\geq k\geq 1$. So, the sequence is `http://planetmath.org/CauchySequence`Cauchy and hence has a limit x. Furthermore, as x_j is contained in the closed set

$$\bigcup_{m\leq n_{k+1}} C(n_1,\ldots,n_k,m)$$

for $j>k$, then x must also be contained in it and hence is in $C(n_1,\ldots,n_k)$. So

$$C(n)\equiv\bigcap_{k=0}^{\infty} C(n_1,\ldots,n_k)$$

contains x and is nonempty. Furthermore, as it has zero diameter, it is a singleton. So $f\colon\mathcal{N}\to X$ is uniquely defined by $f(n)\in C(n)$.

Given any $m,n\in\mathcal{N}$ such that $m_j=n_j$ for $j\leq k$, then $f(m)$ and $f(n)$ are both in the set $C(m_1,\ldots,m_k)$, which has diameter no more than 2^{-k}. So, $d(f(m),f(n))\leq 2^{-k}$ and f is continuous.

If m and n are distinct elements of $\mathcal{N}$ and k is the smallest integer such that $m_k\neq n_k$, then $f(m)$ and $f(n)$ are in the disjoint sets $C(m_1,\ldots,m_k)$ and $C(n_1,\ldots,n_k)$ respectively. So $f(m)\neq f(n)$, and f is one-to-one.

Now let A be the countable set

$$A=\bigcup_k\bigcup_{n_1,\ldots,n_k}\left(C(n_1,\ldots,n_k)\setminus\bigcup_m C(n_1,\ldots,n_k,m)\right)\subseteq X.$$

Also define

$$\mathcal{N}(n_1,\dots,n_k) \equiv \{m \in \mathcal{N} \colon m_j = n_j \text{ for } j \le k\}\,.$$

These sets form a basis for the topology on $\mathcal{N}$. Clearly,

$$f\left(\mathcal{N}(n_1,\dots,n_k)\right) \subseteq C(n_1,\dots,n_k) \setminus A.$$

Choosing any $x \in C(n_1,\dots,n_k) \setminus A$ then we can inductively find n_j for $j > k$ such that $x \in C(n_1,\dots,n_j)$. Then, setting $n = (n_1, n_2, \dots)$ gives $f(n) = x$. This shows that

$$f\left(\mathcal{N}(n_1,\dots,n_k)\right) = C(n_1,\dots,n_k) \setminus A. \tag{1}$$

In particular, $f(\mathcal{N}) = X \setminus A$ and, therefore $X \setminus f(\mathcal{N})$ is countable. Finally, as $C(n_1,\dots,n_k)$ is a difference of closed sets, it is Borel, and equation (**??**) shows that the inverse of f is Borel measurable.