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every subspace of a normed space of finite dimension is closed

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Let $(V, \|\cdot\|)$ be a normed vector space, and $S \subset V$ a finite dimensional subspace. Then S is closed.

Proof

Let $a \in \overline{S}$ and choose a sequence $\{a_n\}$ with $a_n \in S$ such that a_n converges to a. Then $\{a_n\}$ is a Cauchy sequence in V and is also a Cauchy sequence in S. Since a finite dimensional normed space is a Banach space, S is complete, so $\{a_n\}$ converges to an element of S. Since limits in a normed space are unique, that limit must be a, so $a \in S$.

Example

The result depends on the field being the real or complex numbers. Suppose the $V = Q \times R$, viewed as a vector space over Q and $S = Q \times Q$ is the finite dimensional subspace. Then clearly $(1, \sqrt{2})$ is in V and is a limit point of S which is not in S. So S is not closed.

Example

On the other hand, there is an example where Q is the underlying field and we can still show a finite dimensional subspace is closed. Suppose that $V = Q^n$, the set of n-tuples of rational numbers, viewed as vector space over Q. Then if S is a finite dimensional subspace it must be that $S = \{x | Ax = 0\}$ for some matrix A. That is, S is the inverse image of the closed set $\{0\}$. Since the map $x \to Ax$ is continuous, it follows that S is a closed set.