infimum and supremum for real numbers

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Suppose A is a non-empty subset of $\mathbb{R}$. If A is bounded from above, then the axioms of the real numbers imply that there exists a *least upper bound* for A. That is, there exists an $m \in \mathbb{R}$ such that

1. m is an upper bound for A, that is, $a \leq m$ for all $a \in A$,
2. if M is another upper bound for A, then $m \leq M$.

Such a number m is called the *supremum* of A, and it is denoted by $\sup A$. It is easy to see that there can be only one least upper bound. If m_1 and m_2 are two least upper bounds for A. Then $m_1 \leq m_2$ and $m_2 \leq m_1$, and $m_1 = m_2$.

Next, let us consider a set A that is bounded from below. That is, for some $m \in \mathbb{R}$ we have $m \leq a$ for all $a \in A$. Then we say that $M \in \mathbb{R}$ is a a *greatest lower bound* for A if

1. M is an lower bound for A, that is, $M \leq a$ for all $a \in A$,
2. if m is another lower bound for A, then $m \leq M$.

Such a number M is called the *infimum* of A, and it is denoted by $\inf A$. Just as we proved that the supremum is unique, one can also show that the infimum is unique. The next lemma shows that the infimum exists.

Lemma 1. *Every non-empty set bounded from below has a greatest lower bound.*

Proof. Let $m \in \mathbb{R}$ be a lower bound for non-empty set A. In other words, $m \leq a$ for all $a \in A$. Let

$$-A = \{-a \in \mathbb{R} : a \in A\}.$$

Let us recall the following result from `http://planetmath.org/InequalityForRealNumbers`this page; if m is an upper(lower) bound for A, then $-m$ is a lower(upper) bound for $-A$.

Thus $-A$ is bounded from above by $-m$. It follows that $-A$ has a least upper bound $\sup(-A)$. Now $-\sup(-A)$ is a greatest lower bound for A. First, by the result, it is a lower bound for A. Second, if m is a lower bound for A, then $-m$ is a upper bound for $-A$, and $\sup(-A) \leq -m$, or $m \geq -\sup(-A)$. $\square$

The proof shows that if A is non-empty and bounded from below, then

$$\inf A = -\sup(-A).$$

In consequence, if A is bounded from above, then

$$\sup A = -\inf(-A).$$

In many respects, the supremum and infimum are similar to the maximum and minimum, or the largest and smallest element in a set. However, it is important to notice that the $\inf A$ and $\sup A$ do not need to belong to A. (See examples below.)

Examples

1. For example, consider the set of negative real numbers

   $$A = \{x \in \mathbb{R} : \ x < 0\}.$$

   Then $\sup A = 0$. Indeed. First, $a < 0$ for all $a \in A$, and if $a < b$ for all $a \in A$, then $0 \leq b$.

2. The sequence $-(1-\frac{1}{1}), 1-\frac{1}{2}, -(1-\frac{1}{3}), 1-\frac{1}{4}, -(1-\frac{1}{5}), \ldots$ is not convergent. The set $A = \{(-1)^n(1-\frac{1}{n}) : \ n \in \mathbb{Z}_+\}$ formed by its members has the infimum -1 and the supremum 1.