induced partial order on an Alexandroff space

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Let X be a T_0, Alexandroff space. For $A \subseteq X$ denote by A^o the intersection of all open neighbourhoods of A. Define a relation $\leq$ on X as follows: for any $x, y \in X$ we have $x \leq y$ if and only if $x \in \{y\}^o$. This relation will be called the *induced partial order on* X.

Proposition 1. $(X, \leq)$ is a poset.

Proof. Of course $x \in \{x\}^o$ for any $x \in X$. Thus $\leq$ is reflexive.

Assume now that $x \leq y$ and $y \leq x$ for some $x, y \in X$. Assume that $x \neq y$. Then, since X is a T_0 space, there is an open set U such that $x \in U$ and $y \notin U$ or there is an open set V such that $y \in V$ and $x \notin V$. Both cases lead to contradiction, because we assumed that $x \in \{y\}^o$ and $y \in \{x\}^o$. Thus every open neighbourhood of one element must also contain the other. Thus $\leq$ is antisymmetric.

Finally assume that $x \leq y$ and $y \leq z$ for some $x, y, z \in X$. Since $y \in \{z\}^o$, then $\{z\}^o$ is an open neighbourhood of y and thus $\{y\}^o \subseteq \{z\}^o$. Therefore $x \in \{z\}^o$, so $\leq$ is transitive, which completes the proof. $\square$

Proposition 2. Let X, Y be two, T_0, Alexandroff spaces and $f : X \to Y$ be a function. Then f is continuous if and only if f preserves the induced partial order.

Proof. „$\Rightarrow$" Assume that f is continuous and suppose that $x, y \in X$ are such that $x \leq y$. We wish to show that $f(x) \leq f(y)$, so assume this is not the case. Let $A = \{f(y)\}^o$. Then $f(x) \notin A$. But A is open, so $f^{-1}(A)$ is also open (because we assumed that f is continuous). Furthermore $y \in f^{-1}(A)$ and because $x \leq y$, then $x \in f^{-1}(A)$, but this implies that $f(x) \in A$. Contradiction.

„$\Leftarrow$" Assume that f preserves the induced partial order and let $U \subseteq Y$ be an open subset. Let $x \in U$. Then for any $y \leq x$ we have $f(y) \leq f(x)$ (because f preserves the induced partial order) and since $\{f(x)\}^o \subseteq U$ (because U is open and $\{f(x)\}^o$ is the smallest open neighbourhood of $f(x)$) we have that $f(y) \in U$. Thus

$$\{x\}^o = \{y \in X \mid y \leq x\} \subseteq f^{-1}(U)$$

which implies that $f^{-1}(U)$ is open because $f^{-1}(A)$ contains a small neighbourhood of each point. This completes the proof. $\square$