examples of compact spaces

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Author	yark (2760)
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Here are some examples of `http://planetmath.org/Compact`compact spaces:

- The unit interval [0,1] is compact. This follows from the Heine-Borel Theorem. Proving that theorem is about as hard as proving directly that [0,1] is compact. The half-open interval (0,1] is not compact: the open cover $(1/n, 1]$ for $n = 1, 2, ...$ does not have a finite subcover.

- Again from the Heine-Borel Theorem, we see that the closed unit ball of any finite-dimensional normed vector space is compact. This is not true for infinite dimensions; in fact, a normed vector space is finite-dimensional if and only if its closed unit ball is compact.

- Any finite topological space is compact.

- Consider the set $2^{\mathbb{N}}$ of all infinite sequences with entries in $\{0, 1\}$. We can turn it into a metric space by defining $d((x_n), (y_n)) = 1/k$, where k is the smallest index such that $x_k \neq y_k$ (if there is no such index, then the two sequences are the same, and we define their distance to be zero). Then $2^{\mathbb{N}}$ is a compact space, a consequence of Tychonoff's theorem. In fact, $2^{\mathbb{N}}$ is homeomorphic to the Cantor set (which is compact by Heine-Borel). This construction can be performed for any finite set, not just {0,1}.

- Consider the set K of all functions $f : \mathbb{R} \to [0, 1]$ and defined a topology on K so that a sequence (f_n) in K converges towards $f \in K$ if and only if $(f_n(x))$ converges towards $f(x)$ for all $x \in \mathbb{R}$. (There is only one such topology; it is called the topology of pointwise convergence). Then K is a compact topological space, again a consequence of Tychonoff's theorem.

- Take any set X, and define the cofinite topology on X by declaring a subset of X to be open if and only if it is empty or its complement is finite. Then X is a compact topological space.

- The prime spectrum of any commutative ring with the Zariski topology is a compact space important in algebraic geometry. These prime spectra are almost never Hausdorff spaces.

- If H is a Hilbert space and $A : H \to H$ is a continuous linear operator, then the spectrum of A is a compact subset of $\mathbb{C}$. If H is infinite-dimensional, then any compact subset of $\mathbb{C}$ arises in this manner from some continuous linear operator A on H.

- If $\mathcal{A}$ is a complex C*-algebra which is commutative and contains a one, then the set X of all non-zero algebra homomorphisms $\phi : \mathcal{A} \to \mathbb{C}$ carries a natural topology (the weak-* topology) which turns it into a compact Hausdorff space. $\mathcal{A}$ is isomorphic to the C*-algebra of continuous complex-valued functions on X with the supremum norm.

- Any profinite group is compact Hausdorff: finite discrete spaces are compact Hausdorff, therefore their product is compact Hausdorff, and a profinite group is a closed subset of such a product.

- Any locally compact Hausdorff space can be turned into a compact space by adding a single point to it (`http://planetmath.org/AlexandrovOnePointCompacti` one-point compactification). The one-point compactification of $\mathbb{R}$ is homeomorphic to the circle S^1; the one-point compactification of $\mathbb{R}^2$ is homeomorphic to the sphere S^2. Using the one-point compactification, one can also easily construct compact spaces which are not Hausdorff, by starting with a non-Hausdorff space.

- Other non-Hausdorff compact spaces are given by the left order topology (or right order topology) on bounded totally ordered sets.