subspace of a subspace

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Theorem 1. *Suppose $X \subseteq Y \subseteq Z$ are sets and Z is a topological space with topology τ_Z. Let $\tau_{Y,Z}$ be the subspace topology in Y given by τ_Z, and let $\tau_{X,Y,Z}$ be the subspace topology in X given by $\tau_{Y,Z}$, and let $\tau_{X,Z}$ be the subspace topology in X given by τ_Z. Then $\tau_{X,Z} = \tau_{X,Y,Z}$.*

Proof. Let $U_X \in \tau_{X,Z}$, then there is by the definition of the subspace topology an open set $U_Z \in \tau_Z$ such that $U_X = U_Z \cap X$. Now $U_Z \cap Y \in \tau_{Y,Z}$ and therefore $U_Z \cap Y \cap X \in \tau_{X,Y,Z}$. But since $X \subseteq Y$, we have $U_Z \cap Y \cap X = U_Z \cap X = U_X$, so $U_X \in \tau_{X,Y,Z}$ and thus $\tau_{X,Z} \subseteq \tau_{X,Y,Z}$.

To show the reverse inclusion, take an open set $U_X \in \tau_{X,Y,Z}$. Then there is an open set $U_Y \in \tau_{Y,Z}$ such that $U_X = U_Y \cap X$. Furthermore, there is an open set $U_Z \in \tau_Z$ such that $U_Y = U_Z \cap Y$. Since $X \subseteq Y$, we have

$$U_Z \cap X = U_Z \cap Y \cap X = U_Y \cap X = U_X,$$

so $U_X \in \tau_{X,Z}$ and thus $\tau_{X,Y,Z} \subseteq \tau_{X,Z}$.

Together, both inclusions yield the equality $\tau_{X,Z} = \tau_{X,Y,Z}$. $\square$