proof of Riesz' Lemma

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Let's consider $x \in E - S$ and let $r = d(x, S)$. Recall that r is the distance between x and S: $d(x, S) = \inf\{d(x, s)$ such that $s \in S\}$. Now, $r > 0$ because S is closed. Next, we consider $b \in S$ such that

$$\|x - b\| < \frac{r}{\alpha}$$

This vector b exists: as $0 < \alpha < 1$ then

$$\frac{r}{\alpha} > r$$

But then the definition of infimum implies there is $b \in S$ such that

$$\|x - b\| < \frac{r}{\alpha}$$

Now, define

$$x_\alpha = \frac{x - b}{\|x - b\|}$$

Trivially,

$$\|x_\alpha\| = 1$$

Notice that $x_\alpha \in E - S$, because if $x_\alpha \in S$ then $x - b \in S$, and so $(x - b) + b = x \in S$, an absurd. Plus, for every $s \in S$ we have

$$\|s - x_\alpha\| = \|s - \frac{x - b}{\|x - b\|}\| = \frac{1}{\|x - b\|} \cdot \|\|x - b\| \cdot s + b - x\| \geq \frac{r}{\|x - b\|}$$

because

$$\|x - b\| \cdot s + b \in S$$

But

$$\frac{r}{\|x - b\|} > \frac{\alpha}{r} \cdot r = \alpha$$

QED.