fundamental system of entourages

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Let $(X,\mathcal{U})$ be a uniform space. A subset $\mathcal{B} \subseteq \mathcal{U}$ is a *fundamental system of entourages* for $\mathcal{U}$ provided that each entourage in $\mathcal{U}$ contains an element of $\mathcal{B}$.

To see that each uniform space $(X,\mathcal{U})$ has a fundamental system of entourages, define

$$\mathcal{B} = \{U \cap U^{-1} \colon U \in \mathcal{U}\},$$

where U^{-1} denotes the inverse relation of U. Since $\mathcal{U}$ is closed under taking relational inverses and binary intersections, $\mathcal{B} \subseteq \mathcal{U}$. By construction, each $U \in \mathcal{U}$ contains the element of $U \cap U^{-1} \in \mathcal{B}$.

There is a useful equivalent condition for being a fundamental system of entourages. Let $\mathcal{B}$ be a nonempty family of subsets of $X \times X$. Then $\mathcal{B}$ is a fundamental system of entourages of a uniformity on X if and only if it the following axioms.

- (B1) If S, $T \in \mathcal{B}$, then $S \cap T$ contains an element of $\mathcal{B}$.
- (B2) Each element of $\mathcal{B}$ contains the diagonal $\Delta(X)$.
- (B3) For any $S \in \mathcal{B}$, the inverse relation of S contains an element of $\mathcal{B}$.
- (B4) For any $S \in \mathcal{B}$, there is an element $T \in \mathcal{B}$ such that the relational composition $T \circ T$ is contained in S.

Suppose $\mathcal{B}$ is a fundamental system of entourages for uniformities $\mathcal{U}$ and $\mathcal{V}$. Then $\mathcal{U} \subset \mathcal{V}$. To see this, suppose $S \in \mathcal{U}$. Since $\mathcal{B}$ is a fundamental system of entourages for $\mathcal{U}$, there is some element $B \in \mathcal{B}$ such that $B \subset S$. But $\mathcal{B} \subset \mathcal{V}$, so $B \in \mathcal{V}$. Hence by applying the fact that $\mathcal{V}$ is closed under taking supersets we may conclude that $S \in \mathcal{V}$. So if $\mathcal{B}$ is a fundamental system of entourages, it is a fundamental system for a unique uniformity $\mathcal{U}$. Thus it makes sense to call $\mathcal{U}$ the *uniformity generated by the fundamental system* $\mathcal{B}$.

References

[1] Nicolas Bourbaki, *Elements of Mathematics: General Topology: Part 1*, Hermann, 1966.