the continuous image of a compact space is compact

Canonical name	TheContinuousImageOfACompactSpaceIsCompact
Date of creation	2013-03-22 15:52:48
Last modified on	2013-03-22 15:52:48
Owner	cvalente (11260)
Last modified by	cvalente (11260)
Numerical id	7
Author	cvalente (11260)
Entry type	Proof
Classification	msc 54D30
Related topic	CompactnessIsPreservedUnderAContinuousMap

Consider $f : X \to Y$ a continuous and surjective function and X a compact set. We will prove that Y is also a compact set.

Let $\{V_a\}$ be an open covering of Y. By the continuity of f the pre-image by f of any http://planetmath.org/OpenSubsetopen subset of Y will also be an open subset of X. So we have an open covering $\{U_a\}$ of X where $U_a = f^{-1}(V_a)$.

To see this remember, since the continuity of f implies that each U_a is open, all we need to prove is that $\bigcup_a U_a \supset X$. Consider $x \in X$, we know that since $\{V_a\}$ is a covering of Y that there exists i such that $f(x) \in V_i$ but then by construction $x \in U_i$ and $\{U_a\}$ is indeed an open covering of X.

Since X is compact we can consider a finite set of indices $\{a_i\}$ such that $\{U_{a_i}\}$ is a finite open covering of X, but then $\{V_{a_i}\}$ will be a finite open covering of Y and it will thus be a compact set.

To see that $\{V_{a_i}\}$ is a covering of Y consider $y \in Y$. By the surjectivity of f there must exist (at least) one $x \in X$ such that $f(x) = y$ and since $\{U_{a_i}\}$ is a finite covering of X, there exists k such that $x \in U_{a_k}$. But then since $f(U_{a_k}) = V_{a_k}$, we must have that $y \in V_{a_k}$ and $\{V_{a_i}\}$ is indeed a finite open covering of Y.