testing for continuity via nets

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Proposition 1. *Let X, Y be topological spaces and $f : X \to Y$. Then the following are equivalent:*

1. *f is continuous;*
2. *If (x_i) is a net in X converging to x, then $(f(x_i))$ is a net in Y converging to $f(x)$.*
3. *Whenever two nets (x_i) and (y_j) in X converge to the same point, then $(f(x_i))$ and $(f(y_j))$ converge to the same point in Y.*

Proof. (1) $\Leftrightarrow$ (2). Let A be the (directed) index set for i. Suppose $f(x) \in U$ is open in Y. Then $x \in f^{-1}(U)$ is open in X since f is continuous. By assumption, (x_i) is a net, so there is $b \in A$ such that $x_j \in f^{-1}(U)$ for all $j \geq b$. This means that $f(x_j) \in U$ for all $i \geq b$, so $(f(x_i))$ is a net too.

Conversely, suppose f is not continuous, say, at a point $x \in X$. Then there is an open set V containing $f(x)$ such that $f^{-1}(V)$ does not contain any open set containing x. Let A be the set of all open sets containing x. Then under reverse inclusion, A is a directed set (if $U_1, U_2 \in A$, then $U_1 \cap U_2 \in A$). Define a relation $R \subseteq A \times X$ as follows:

$$(U, x) \in R \qquad \text{iff} \qquad x \in U - f^{-1}(V).$$

Then for each $U \in A$, there is an $x \in X$ such that $(U, x) \in R$, since $U \not\subseteq f^{-1}(V)$. By the axiom of choice, we get a function $d \subseteq R$ from A to X. Write $d(U) := x_U$. Since A is directed, (x_U) is a net. In addition, (x_U) converges to x (just pick any $U \in A$, then for any $W \geq U$, we have $x \in W$ by the definition of A). However, $(f(x_U))$ does not converge to $f(x)$, since $x_U \notin f^{-1}(V)$ for any $U \in A$.

(2) $\Leftrightarrow$ (3). Suppose nets (x_i) and (y_j) both converge to $z \in X$. Then, by assumption, $(f(x_i))$ and $(f(y_j))$ are nets converging to $f(z) \in Y$.

Conversely, suppose (x_i) converges to x, and i is indexed by a directed set A. Define a net (y_i) such that $y_i = x$ for all $i \in A$. Then $(y_i) = (x)$ clearly converges to x. Hence both $(f(x_i))$ and $(f(y_i))$ converge to the same point in Y. But $(f(y_i)) = (f(x))$ converges to $f(x)$, we see that $(f(x_i))$ converges to $f(x)$ as well. $\square$

Remark. In particular, if X, Y are first countable, we may replace nets by sequences in the proposition. In other words, f is continuous iff it preserves converging sequences.