# hemicompact space

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A topological space $(X, \tau)$ is called a *hemicompact* space if there is an *admissible sequence* in $X$, i.e. there is a sequence of compact sets $(K_n)_{n\in\mathbb{N}}$ in $X$ such that for every $K \subset X$ compact there is an $n \in \mathbb{N}$ with $K \subset K_n$.

- The above conditions imply that if $X$ is hemicompact with admissible sequence $(K_n)_{n\in\mathbb{N}}$ then $X = \bigcup_{n\in\mathbb{N}} K_n$ because every point of $X$ is compact and lies in one of the $K_n$.
- A hemicompact space is clearly $\sigma$-compact. The converse is false in general. This follows from the fact that a first countable hemicompact space is locally compact (see below). Consider the set of rational numbers $\mathbb{Q}$ with the induced euclidean topology. $\mathbb{Q}$ is $\sigma$-compact but not hemicompact. Since $\mathbb{Q}$ satisfies the first axiom of countability it can't be hemicompact as this would imply local compactness.
- Not every locally compact space (like $\mathbb{R}$) is hemicompact. Take for example an uncountable discrete space. If we assume in addition $\sigma$-compactness we obtain a hemicompact space (see below).

**Proposition.** Let $(X, \tau)$ be a first countable hemicompact space. Then $X$ is locally compact.

*Proof.* Let $\cdots \subset K_n \subset K_{n+1} \subset \cdots$ be an admissible sequence of $X$. Assume for contradiction that there is an $x \in X$ without compact neighborhood. Let $U_n \supset U_{n+1} \supset \cdots$ be a countable basis for the neighbourhoods of $x$. For every $n \in \mathbb{N}$ choose a point $x_n \in U_n \setminus K_n$. The set $K := \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact but there is no $n \in \mathbb{N}$ with $K \subset K_n$. We have a contradiction. $\square$

**Proposition.** Let $(X, \tau)$ be a locally compact and $\sigma$-compact space. Then $X$ is hemicompact.

*Proof.* By local compactness we choose a cover $X \subset \bigcup_{i\in I} U_i$ of open sets with compact closure (take a compact neighborhood of every point). By $\sigma$-compactness there is a sequence $(K_n)_{n\in\mathbb{N}}$ of compacts such that $X = \bigcup_{n\in\mathbb{N}} K_n$. To each $K_n$ there is a finite subfamily of $(U_i)_{i\in I}$ which covers $K_n$. Denote the union of this finite family by $U_n$ for each $n \in \mathbb{N}$. Set $\tilde{K}_n := \overline{\bigcup_{k=1}^n U_k}$. Then $(\tilde{K}_n)_{n\in\mathbb{N}}$ is a sequence of compacts. Let $K \subset X$ be compact then there is a finite subfamily of $(U_i)_{i\in I}$ covering $K$. Therefore $K \subset K_n$ for some $n \in \mathbb{N}$. $\square$