closure map

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Defines	dual closure
Defines	fixed point

Let P be a poset. A function $c : P \to P$ is called a *closure map* if

- c is order preserving,
- $1_P \leq c$,
- c is idempotent: $c \circ c = c$.

If the second condition is changed to $c \leq 1_P$, then c is called a *dual closure map* on P.

For example, the real function f such that $f(r)$ is the least integer greater than or equal to r is a closure map (see Archimedean property). The rounding function $[\cdot]$ is an example of a dual closure map.

A *fixed point* of a closure map c on P is an element $x \in P$ such that $c(x) = x$. It is evident that every image point of c is a fixed point: for if $x = c(a)$ for some $a \in P$, then $c(x) = c(c(a)) = c(a) = x$.

In the example above, any integer is a fixed point of f.

Every closure map can be characterized by an interesting decomposition property: $c : P \to P$ is a closure map iff there is a set Q and a residuated function $f : P \to Q$ such that $c = f^+ \circ f$, where f^+ denotes the residual of f.

Again, in the example above, $f = g^+ \circ g$, where $g : \mathbb{R} \to \mathbb{Z}$ is the function taking any real number r to the largest integer smaller than r. g is residuated, and its residual is $g^+(x) = x + 1$.

Remark. Closure maps are generalizations to closure operator on a set (see the parent entry). Indeed, any closure operator on a set X takes a subset A of X to a subset A^c of X satisfying the closure axioms, where Axiom 2 corresponds to condition 2 above, and Axiom 3 says the operator is idempotent. To see that the operator is order preserving, suppose $A \subseteq B$. Then $B^c = (A \cup B)^c = A^c \cup B^c$ by Axiom 4, and hence $A^c \subseteq B^c$. Axiom 1 says that the empty set $\varnothing$ is a fixed point of the operator. However, in general, this is not the case, for P may not even have a minimal element, as indicated by the above example.

References

[1] T.S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer, New York (2005).