proof of alternative characterization of ultrafilter

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Proof that $A \coprod B = X$ implies $A \in \mathcal{U}$ or $B \in \mathcal{U}$ Once we show that $A \notin \mathcal{U}$ implies $B \notin \mathcal{U}$, this result will follow immediately.

On the one hand, suppose that $A \notin \mathcal{U}$ and that there exists a $C \in \mathcal{U}$ such that $A \cap C$ is empty. Then $C \subseteq B$. Since $\mathcal{U}$ is a filter and $C \in \mathcal{U}$, this implies that $B \in \mathcal{U}$.

On the other hand, suppose that $A \notin \mathcal{U}$ and that $A \cap C$ is not empty for any C in $\mathcal{U}$. Then $\{A\} \cup \mathcal{U}$ would be a filter subbasis. The filter which it would generate would be finer than $\mathcal{U}$. The fact that $\mathcal{U}$ is an ultrafilter means that there exists no filter finer than $\mathcal{U}$. This contradiction shows that, if $A \notin \mathcal{U}$, then there exists a C such that $A \cap C$ is empty. But this would imply that $C \subseteq B$ which, in turn would imply that $B \in \mathcal{U}$.

Proof that $\mathcal{U}$ is an ultrafilter. Assume that $\mathcal{U}$ is a filter, but not an ultrafilter and that $A \coprod B = X$ implies $A \in \mathcal{U}$ or $B \in \mathcal{U}$. Since $\mathcal{U}$ is not an ultrafilter, there must exist filter $\mathcal{U}'$ which is strictly finer. Hence there must exist $A \in \mathcal{U}'$ such that $A \notin \mathcal{U}$. Set $B = X \setminus A$. Since $A \coprod B = X$ and $A \notin \mathcal{U}$, it follows that $B \in \mathcal{U}$. Since $\mathcal{U} \subset \mathcal{U}'$, it is also the case that $B \in \mathcal{U}'$. But $A \in \mathcal{U}'$ as well; since $\mathcal{U}'$ is a filter, $A \cap B \in \mathcal{U}'$. This is impossible because $A \cap B \in \mathcal{U}'$ is empty. Therefore, no such filter $\mathcal{U}'$ can exist and $\mathcal{U}$ must be an ultrafilter.

Proof of generalization to $A \cup B = X$ On the one hand, since $A \cup B = X$ implies $A \coprod B = X$, the condition $A \cup B = X \Rightarrow A \in \mathcal{U} \vee B \in \mathcal{U}$ will also imply that $\mathcal{U}$ is an ultrafilter.

On the other hand, if $A \cup B = X$, there must exists $A' \subseteq A$ and $B' \subseteq B$ such that $A' \coprod B' = X$. If $\mathcal{U}$ is assumed to be a filter, $A' \in \mathcal{U}$ implies that $A \in \mathcal{U}$. Likewise, $B' \in \mathcal{U}$ implies that $B \in \mathcal{U}$. Hence, if $\mathcal{U}$ is a filter such that $A \cup B = X$ implies that either $A \in \mathcal{U}$ or $B \in \mathcal{U}$, then $\mathcal{U}$ is an ultrafilter.

Proof of first proposition regarding finite unions Let $B_j = \coprod_{i=1}^{j} A_i$ and let $C_j = \coprod_{i=j+1}^{n} A_i$. For each i between 1 and $n-1$, we have $B_i \coprod C_i = X$. Hence, either $B_i \in \mathcal{U}$ or $C_i \in \mathcal{U}$ for each i between 1 and $n-1$. Next, consider three possibilities:

1. $B_1 \in \mathcal{U}$: Since $B_1 = A_1$, it follows that $A_1 \in \mathcal{U}$.
2. $B_{n-1} \notin \mathcal{U}$: Since $B_{n-1} \coprod C_{n-1} = X$, it follows that $C_{n-1} \in \mathcal{U}$. Because $C_{n-1} = A_n$, it follows that $A_n \in \mathcal{U}$.

3. $B_1 \notin \mathcal{U}$ and $B_{n-1} \in \mathcal{U}$: There must exist an $i \in \{2, \ldots, n-1\}$ such that $B_{i-1} \notin \mathcal{U}$ and $B_i \in \mathcal{U}$. Since $B_{i-1} \notin \mathcal{U}$, $C_{i-1} \in \mathcal{U}$. Since $\mathcal{U}$ is a filter, $C_{i-1} \cap B_i \in \mathcal{U}$. But also $C_{i-1} \cap B_i = A_i$ which implies that $A_i \in \mathcal{U}$.

This examination of cases shows that if $\coprod_{i=1}^{n} A_i = X$, then there must exist an i such that $A_i \in \mathcal{U}$. It is also easy to see that this i is unique — If $A_i \in \mathcal{U}$ and $A_j \in \mathcal{U}$ and $i \neq j$, then $A_i \cap A_j = \emptyset$, but this cannot be the case since $\mathcal{U}$ is a filter.

Proof of second proposition regarding finite unions There exist sets A_i' such that $A_i' \subseteq A_i$ and $\coprod_{i=1}^{n} A_i = X$. By the result just proven, there exists an i such that $A_i' \in \mathcal{U}$. Since $\mathcal{U}$ is a filter, $A_i' \in \mathcal{U}$ implies $A_i \in \mathcal{U}$. Note that we can no longer assert that i is unique because the A_i's no longer are required to be pairwise disjoint.