the sphere is indecomposable as a topological space

Canonical name	TheSphereIsIndecomposableAsATopologicalSpace
Date of creation	2013-03-22 18:31:45
Last modified on	2013-03-22 18:31:45
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	8
Author	joking (16130)
Entry type	Theorem
Classification	msc 54F99

Proposition. If for any topological spaces X and Y the n-dimensional sphere $\mathbb{S}^n$ is homeomorphic to $X \times Y$, then either X has exactly one point or Y has exactly one point.

Proof. Recall that the homotopy group functor is additive, i.e. $\pi_n(X \times Y) \simeq \pi_n(X) \oplus \pi_n(Y)$. Assume that $\mathbb{S}^n$ is homeomorphic to $X \times Y$. Now $\pi_n(\mathbb{S}^n) \simeq \mathbb{Z}$ and thus we have:

$$\mathbb{Z} \simeq \pi_n(\mathbb{S}^n) \simeq \pi_n(X \times Y) \simeq \pi_n(X) \oplus \pi_n(Y).$$

Since $\mathbb{Z}$ is an indecomposable group, then either $\pi_n(X) \simeq 0$ or $\pi_n(Y) \simeq 0$.

Assume that $\pi_n(Y) \simeq 0$. Consider the map $p : X \times Y \to Y$ such that $p(x, y) = y$. Since $X \times Y$ is homeomorphic to $\mathbb{S}^n$ and $\pi_n(Y) \simeq 0$, then p is homotopic to some constant map. Let $y_0 \in Y$ and $H : \mathrm{I} \times X \times Y \to Y$ be such that

$$H(0, x, y) = p(x, y) = y;$$

$$H(1, x, y) = y_0.$$

Consider the map $F : \mathrm{I} \times X \times Y \to X \times Y$ defined by the formula

$$F(t, x, y) = (x, H(t, x, y)).$$

Note that $F(0, x, y) = (x, y)$ and $F(1, x, y) = (x, y_0)$ and thus $X \times \{y_0\}$ is a deformation retract of $X \times Y$. But $X \times Y$ is a sphere and spheres do not have proper deformation retracts (please see `http://planetmath.org/EveryMapIntoSphereWhichIsNot` entry for more details). Therefore $X \times \{y_0\} = X \times Y$, so $Y = \{y_0\}$ has exactly one point. $\square$