first countable implies compactly generated

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Proposition 1. *Any first countable topological space is compactly generated.*

Proof. Suppose X is first countable, and $A \subseteq X$ has the property that, if C is any compact set in X, the set $A \cap C$ is closed in C. We want to show tht A is closed in X. Since X is first countable, this is equivalent to showing that any sequence (x_i) in A converging to x implies that $x \in A$. Let $C = \{x_i \mid i = 1, 2, \ldots\} \cup \{x\}$.

Lemma 1. *C is compact.*

Proof. Let $\{U_j \mid j \in J\}$ be a collection of open sets covering C. So $x \in U_j$ for some j. Since U_j is open, there is a positive integer k such that $x_i \in U_j$ for all $i \geq k$. Now, each $x_i \in U_{d(i)}$ for $i = 1, \ldots, k$. So C is covered by $U_{d(1)}, \ldots, U_{d(k)}$, and U_j, showing that C is compact. □

In addition, as a subspace of X, C is also first countable. By assumption, $A \cap C$ is closed in C. Since $x_i \in A \cap C$ for all $i \geq 1$, we see that $x \in A \cap C$ as well, since C is first countable. Hence $x \in A$, and A is closed in X. □