proof of properties of the closure operator

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Recall that the closure of a set A in a topological space X is defined to be the intersection of all closed sets containing it.

$A \subset \overline{A}$: By definition

$$\overline{A} = \bigcap_{C \supseteq A,\ C \text{ closed}} C,$$

but since for every C we have $A \subseteq C$, we immediately find

$$A \subseteq \bigcap_{C \supseteq A,\ C \text{ closed}} C.$$

$\overline{A}$ **is closed** : Recall that the intersection of any number of closed sets is closed, so the closure is itself closed.

$\overline{\emptyset} = \emptyset$**,** $\overline{X} = X$**, and** $\overline{\overline{A}} = \overline{A}$: If C is any closed set, then

$$\overline{C} = \bigcap_{C' \supseteq C,\ C' \text{ closed}} C' = C \cap \bigcap_{C' \supsetneq C,\ C' \text{ closed}} C' = C.$$

$\overline{A \cup B} = \overline{A} \cup \overline{B}$: First write down the definition:

$$\overline{A} \cup \overline{B} = \bigcap_{C \supseteq A,\ C \text{ closed}} C \cup \bigcap_{D \supseteq B,\ D \text{ closed}} D,$$

then apply DeMorgan's law to get

$$= \bigcap_{C \supseteq A, D \supseteq B,\ C, D \text{ closed}} (C \cup D),$$

but for every such pair C, D, we have that $E = C \cup D$ is a closed set containing $A \cup B$. Conversely, every closed set E containing $A \cup B$ is obtained from such a pair — just take (E, E) to be the pair. Thus

$$\begin{aligned} &= \bigcap_{E \supseteq A \cup B,\ E \text{ closed}} (E) \\ &= \overline{A \cup B}. \end{aligned}$$

$\overline{A \cap B} \subset \overline{A} \cap \overline{B}$:

$$\begin{aligned}\overline{A} \cap \overline{B} &= \bigcap_{C \supseteq A, C \text{ closed}} C \cap \bigcap_{D \supseteq B,\ D \text{ closed}} D, \\ &= \bigcap_{C \supseteq A, D \supseteq B,\ C, D \text{ closed}} (C \cap D),\end{aligned}$$

but for every such pair C, D, we have that $E = C \cap D$ is a closed set containing $A \cap B$. However, some closed sets may not arise in this way, so we do not have equality. Thus

$$\begin{aligned}&\supseteq \bigcap_{E \supseteq A \cap B,\ E \text{ closed}} (E) \\ &= \overline{A \cap B}.\end{aligned}$$

so we have

$$\overline{A} \cap \overline{B} \supseteq \overline{A \cap B}.$$

$\overline{A} = A \cup A'$ **where** A' **is the set of all limit points of** A : Let a be a limit point of A, and let C be a closed set containing A. If a is not in C, then $X \setminus C$ is an open set containing a but not meeting C, which implies that $X \setminus C$ does not meet A, which contradicts the fact that a was a limit point of A. Conversely, suppose that a is not a limit point of A, and that a is not in A. Then there is some open neighborhood U of a which does not meet A. But then $X \setminus U$ is a closed set containing A but not containing a, so $a \notin \overline{A}$.