spaces homeomorphic to Baire space

Canonical name	SpacesHomeomorphicToBaireSpace
Date of creation	2013-03-22 18:46:48
Last modified on	2013-03-22 18:46:48
Owner	gel (22282)
Last modified by	gel (22282)
Numerical id	7
Author	gel (22282)
Entry type	Theorem
Classification	msc 54E50
Related topic	PolishSpace
Related topic	InjectiveImagesOfBaireSpace

Baire space, $\mathcal{N} \equiv \mathbb{N}^{\mathbb{N}}$, is the set of all functions $x\colon \mathbb{N} \to \mathbb{N}$ together with the product topology. This is homeomorphic to the set of irrational numbers in the unit interval, with the homeomorphism $f\colon \mathcal{N} \to (0,1) \setminus \mathbb{Q}$ given by continued fraction expansion

$$f(x) = \cfrac{1}{x(1) + \cfrac{1}{x(2) + \cfrac{1}{\ddots}}}.$$

Theorem 1. *Let I be an open interval of the real numbers and S be a countable dense subset of I. Then, $I \setminus S$ is homeomorphic to Baire space.*

More generally, Baire space is uniquely characterized up to homeomorphism by the following properties.

Theorem 2. *A topological space X is homeomorphic to Baire space if and only if*

1. *It is a nonempty Polish space.*
2. *It is* `http://planetmath.org/ZeroDimensional`*zero dimensional.*
3. *No nonempty open subsets are compact.*

In particular, for an open interval I of the real numbers and countable dense subset $S \subseteq I$, then $I \setminus S$ is easily seen to satisfy these properties and Theorem **??** follows.