limit points and closure for connected sets

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The below theorem shows that adding limit points to a connected set preserves connectedness.

Theorem 1. *Suppose A is a connected set in a topological space. If $A \subseteq B \subseteq \overline{A}$, then B is connected. In particular, $\overline{A}$ is connected.*

Thus, one way to prove that a space X is connected is to find a dense subspace in X which is connected.

Two touching closed balls in $\mathbb{R}^2$ shows that this theorem does not hold for the interior. Along the same lines, taking the closure does not preserve separatedness.

Proof. Let X be the ambient topological space. By assumption, if $U, V \subseteq A$ are open and $U \cup V = A$, then $U \cap V \neq \emptyset$. To prove that B is connected, let U, V be open sets in B such that $U \cup V = B$ and for a contradition, suppose that $U \cap V = \emptyset$. Then there are open sets $R, S \subseteq X$ such that

$$U = R \cap B, \quad V = S \cap B.$$

It follows that $(R \cup S) \cap B = B$ and $(R \cap S) \cap B = \emptyset$. Next, let $\tilde{U}, \tilde{V}$ be open sets in A defined as

$$\tilde{U} = R \cap A, \quad \tilde{V} = S \cap A.$$

Now

$$A = B \cap A \subseteq (R \cup S) \cap A \subseteq A$$

and as $(R \cup S) \cap A = \tilde{U} \cup \tilde{V}$, it follows that $\emptyset \neq \tilde{U} \cap \tilde{V} = (R \cap S) \cap A$. Then, by the properties of the closure operator,

$$\emptyset \neq \overline{(R \cap S) \cap A} \supseteq (R \cap S) \cap \overline{A} \supseteq (R \cap S) \cap B = \emptyset.$$

$\square$