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door space

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A topological space X is called a *door space* if every subset of X is either open or closed.

From the definition, it is immediately clear that any discrete space is door.

To find more examples, let us look at the singletons of a door space X. For each $x \in X$, either $\{x\}$ is open or closed. Call a point x in X open or closed according to whether $\{x\}$ is open or closed. Let A be the collection of open points in X. If $A = X$, then X is discrete. So suppose now that $A \neq X$. We look at the special case when $X - A = \{x\}$. It is now easy to see that the topology τ generated by all the open singletons makes X a door space:

Proof. If $B \subseteq X$ does not contain x, it is the union of elements in A, and therefore open. If $x \in B$, then its complement B^c does not, so is open, and therefore B is closed. $\square$

Since $\tau = P(A) \cup \{X\}$, the space X not discrete. In addition, X and $\varnothing$ are the only clopen sets in X.

When $X - A$ has more than one element, the situation is a little more complicated. We know that if X is door, then its topology $\mathcal{T}$ is strictly finer then the topology τ generated by all the open singletons. McCartan has shown that $\mathcal{T} = \tau \cup \mathcal{U}$ for some ultrafilter in X. In fact, McCartan showed $\mathcal{T}$, as well as the previous two examples, are the only types of possible topologies on a set making it a door space.

References

[1] J.L. Kelley, *General Topology*, D. van Nostrand Company, Inc., 1955.

[2] S.D. McCartan, *Door Spaces are identifiable*, Proc. Roy. Irish Acad. Sect. A, 87 (1) 1987, pp. 13-16.