some properties of uncountable subsets of the real numbers

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Let S be an uncountable subset of $\mathbb{R}$. Let $\mathscr{A} := \{(x, y) : (x, y) \cap S \text{ is countable}\}$. For $\mathbb{R}$ is hereditarily Lindelöff, there is a countable subfamily $\mathscr{A}'$ of $\mathscr{A}$ such that $\bigcup \mathscr{A}' = \bigcup \mathscr{A}$. For the reason that each of members of $\mathscr{A}'$ has a countable intersection with S, we have that $(\bigcup \mathscr{A}') \cap S$ is countable. As the open set $\bigcup \mathscr{A}'$ can be expressed uniquely as the union of its components, and the components are countably many, we label the components as $\{(a_n, b_n) : n \in \mathbb{N}\}$.

See that $(\bigcup \mathscr{A}') \cap S$ is precisely the set of the elements of S that are NOT the condensation points of S.

Now we'd propose to show that $\{a_n, b_n : n \in \mathbb{N}\}$ is precisely the set of the points which are *unilateral* condensation points of S.

Let x be a unilateral (left, say) condensation point of S. So, there is some $r > 0$ with $(x, x + r) \cap S$ countable. So, there is some (a_n, b_n) such that $(x, x + r) \subseteq (a_n, b_n)$. See, if $x \in (a_n, b_n)$, then x is NOT a condensation point, for x has a neighbourhood (a_n, b_n) which has a countable intersection with S. But x is a condensation point; so, $x = a_n$. Similarly, if x is a right condensation point, then $x = b_n$.

Conversely, each $a_n(b_n$, resp) is a left (right, resp) condensation point. Because, for each $\epsilon \in (0, b_n - a_n)$, we have $(a_n, a_n + \epsilon) \cap S$ countable. And as no a_n, b_n is in $\bigcup \mathscr{A}'$, a_n, b_n are condensation points.

So, $\bigcup \mathscr{A}'$ is the set of non-condensation points - it is countable; and $\{a_n, b_n\}$ are precisely the unilateral condensation points. So, all the rest are bilateral condensation points. Now we see, all but a countable number of points of S are the bilateral condensation points of S.

Call T the set of all the bilateral condensation points that are IN S. Now, take two $x < y$ in T. As x is a bilateral condensation point of S, $(x, y) \cap S$ is uncountable; and as T misses atmost countably many points of S, $(x, y) \cap T$ is uncountable. So, T is a subset of S with in-between property.

We summarize the moral of the story: If S is an uncountable subset of $\mathbb{R}$, then

1. The points of S which are NOT condensation points of S, are at most countable.

2. The set of points in S which are unilateral condensation points of S, is, again, countable.

3. The bilateral condensation points of S, that are in S, are uncountable; even, all but countably many points of S are bilateral condensation

points of S.

4. The set $T \subseteq S$ of all the bilateral condensation points of S has got the property: if $\exists x < y \in T$, then there is also $z \in T$ with $x < z < y$.