universal nets in compact spaces are convergent

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Theorem - A universal net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in a compact space X is convergent.

Proof : Suppose by contradiction that $(x_\alpha)_{\alpha\in\mathcal{A}}$ was not convergent. Then for every $x \in X$ we would find neighborhoods U_x such that

$$\forall_{\alpha\in\mathcal{A}} \quad \exists_{\alpha\leq\alpha_0} \quad x_{\alpha_0} \notin U_x$$

The collection of all this neighborhoods cover X, and as X is compact, a finite number $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ also cover X.

The net $(x_\alpha)_{\alpha\in\mathcal{A}}$ is not eventually in U_{x_k} so it must be eventually in $X - U_{x_k}$ (because it is a net). Therefore we can find $\alpha_k \in \mathcal{A}$ such that

$$\forall_{\alpha_k\leq\alpha} \quad x_\alpha \in X - U_{x_k}$$

Because we have a finite number $\alpha_1, \alpha_2 \ldots, \alpha_n \in \mathcal{A}$ we can find $\gamma \in \mathcal{A}$ such that $\alpha_k \leq \gamma$ for each $1 \leq k \leq n$.

Then $x_\gamma \in X - U_{x_k}$ for all k, i.e. $x_\gamma \notin U_{x_k}$ for all k. But $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ cover X and thus we have a contradiction. $\square$