characterization of subspace topology

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Theorem. *Let X be a topological space and $Y \subset X$ any subset. The subspace topology on Y is the weakest topology making the inclusion map continuous.*

Proof. Let $\mathcal{S}$ denote the subspace topology on Y and $j\colon Y \hookrightarrow X$ denote the inclusion map.

Suppose $\{\mathcal{T}_\alpha \mid \alpha \in J\}$ is a family of topologies on Y such that each inclusion map $j_\alpha\colon (Y, \mathcal{T}_\alpha) \hookrightarrow X$ is continuous. Let $\mathcal{T}$ be the intersection $\bigcap_{\alpha\in J} \mathcal{T}_\alpha$. Observe that $\mathcal{T}$ is also a topology on Y. Let U be open in X. By continuity of j_α, the set $j_\alpha^{-1}(U) = j^{-1}(U)$ is open in each $\mathcal{T}_\alpha$; consequently, $j^{-1}(U)$ is also in $\mathcal{T}$. This shows that there is a weakest topology on Y making inclusion continuous.

We claim that any topology strictly weaker than $\mathcal{S}$ fails to make the inclusion map continuous. To see this, suppose $\mathcal{S}_0 \subsetneq \mathcal{S}$ is a topology on Y. Let V be a set open in $\mathcal{S}$ but not in $\mathcal{S}_0$. By the definition of subspace topology, $V = U \cap Y$ for some open set U in X. But $j^{-1}(U) = V$, which was specifically chosen not to be in $\mathcal{S}_\prime$. Hence $\mathcal{S}_\prime$ does not make the inclusion map continuous. This completes the proof. $\square$