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example of a connected space that is not path-connected

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This standard example shows that a connected topological space need not be path-connected (the converse is true, however).

Consider the topological spaces

$$\begin{aligned} X_1 &= \{(0, y) \mid y \in [-1, 1]\} \\ X_2 &= \left\{(x, \sin \tfrac{1}{x}) \mid x > 0\right\} \\ X &= X_1 \cup X_2 \end{aligned}$$

with the topology induced from $\mathbb{R}^2$.

X_2 is often called the "*topologist's sine curve*", and X is its closure.

X is not path-connected. Indeed, assume to the contrary that there exists a `http://planetmath.org/PathConnected`path $\gamma\colon [0, 1] \to X$ with $\gamma(0) = (\frac{1}{\pi}, 0)$ and $\gamma(1) = (0, 0)$. Let

$$c = \inf \{t \in [0, 1] \mid \gamma(t) \in X_1\}.$$

Then $\gamma([0, c])$ contains at most one point of X_1, while $\overline{\gamma([0, c])}$ contains all of X_1. So $\gamma([0, c])$ is not closed, and therefore not compact. But γ is continuous and $[0, c]$ is compact, so $\gamma([0, c])$ must be compact (as a continuous image of a compact set is compact), which is a contradiction.

But X is connected. Since both "parts" of the topologist's sine curve are themselves connected, neither can be partitioned into two open sets. And any open set which contains points of the line segment X_1 must contain points of X_2. So X is not the disjoint union of two nonempty open sets, and is therefore connected.