M. H. Stone's representation theorem

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Theorem 1. *Given a Boolean algebra B there exists a totally disconnected compact Hausdorff space X such that B is isomorphic to the Boolean algebra of clopen subsets of X.*

Proof. Let $X = B^*$, the `http://planetmath.org/DualSpaceOfABooleanAlgebra`dual space of B, which is composed of all maximal ideals of B. According to this `http://planetmath.org/DualSpaceOfABooleanAlgebra`entry, X is a Boolean space (totally disconnected compact Hausdorff) whose topology is generated by the basis

$$\mathcal{B} := \{M(a) \mid a \in B\},$$

where $M(a) = \{M \in B^* \mid a \notin M\}$.

Next, we show a general fact about the dual space B^*:

Lemma 2. *$\mathcal{B}$ is the set of* all *clopen sets in X.*

Proof. Clearly, every element of $\mathcal{B}$ is clopen, by definition. Conversely, suppose U is clopen. Then $U = \bigcup\{M(a_i) \mid i \in I\}$ for some index set I, since U is open. But U is closed, so $B^* - U = \bigcup\{M(a_j) \mid j \in J\}$ for some index set J. Hence $B^* = \bigcup\{M(a_k) \mid k \in I \cup J\}$. Since B^* is compact, there is a finite subset K of $I \cup J$ such that $B^* = \bigcup\{M(a_k) \mid k \in K\}$. Let $V = \bigcup\{M(a_i) \mid i \in K \cap I\}$. Then $V \subseteq U$. But $B^* - V \subseteq B^* - U$ also. So $U = V$. Let $y = \bigvee\{a_i \mid i \in K \cap I\}$, which exists because $K \cap I$ is finite. As a result,

$$U = V = \bigcup\{M(a_i) \mid i \in K \cap I\} = M(\bigvee\{a_i \mid i \in K \cap I\}) = M(y) \in \mathcal{B}.$$

$\square$

Finally, based on the result of `http://planetmath.org/RepresentingABooleanLatticeByFiel` entry, B is isomorphic to the field of sets

$$F := \{F(a) \mid a \in B\},$$

where $F(a) = \{P \mid P \text{ prime in } B, \text{ and } a \notin P\}$. Realizing that prime ideals and maximal ideals coincide in any Boolean algebra, the set F is precisely $\mathcal{B}$. $\square$

Remark. There is also a dual version of the Stone representation theorem, which says that every Boolean space is homeomorphic to the dual space of some Boolean algebra.