union of non-disjoint connected sets is connected

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Theorem 1. *Suppose A, B are connected sets in a topological space X. If A, B are not disjoint, then $A \cup B$ is connected.*

Proof. By assumption, we have two implications. First, if U, V are open in A and $U \cup V = A$, then $U \cap V \neq \emptyset$. Second, if U, V are open in B and $U \cup V = B$, then $U \cap V \neq \emptyset$. To prove that $A \cup B$ is connected, suppose U, V are open in $A \cup B$ and $U \cup V = A \cup B$. Then

$$\begin{aligned} U \cup V &= ((U \cup V) \cap A) \cup ((U \cup V) \cap B) \\ &= (U \cap A) \cup (V \cap A) \cup (U \cap B) \cup (V \cap B) \end{aligned}$$

Let us show that $U \cap A$ and $V \cap A$ are open in A. To do this, we use `http://planetmath.org/SubspaceOfASubspace`this result and notation from that entry too. For example, as $U \in \tau_{A \cup B, X}$, $U \cap A \in \tau_{A, A \cup B, X} = \tau_{A,X}$, and so $U \cap A$, $V \cap A$ are open in A. Since $(U \cap A) \cup (V \cap A) = A$, it follows that

$$\emptyset \neq (U \cap A) \cap (V \cap A) = (U \cap V) \cap A.$$

If $U \cap V = \emptyset$, then this is a contradition, so $A \cup B$ must be connected. $\square$