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scattered space

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A topological space X is said to be *scattered* if for every closed subset C of X, the set of isolated points of C is dense in C. Equivalently, X is a scattered space if no non-empty closed subset of X is dense in itself: for every closed subset C of X, the closure of the interior of C is not C.

A subset of a topological space is called *scattered* if it is a scattered space with the subspace topology.

Every discrete space is scattered, since every singleton is open, hence isolated.

<u>Scattered line</u>. Let $\mathbb{R}$ be the real line equipped with the usual topology T (formed by the open intervals). Let's define a new topology S on $\mathbb{R}$ as follows: a subset A is open under S ($A \in S$) if $A = B \cup C$, where B is open under T ($B \in T$) and $C \subseteq \mathbb{R} - \mathbb{Q}$, a subset of the irrational numbers. We make the following observations:

1. S is a topology on $\mathbb{R}$ which is finer than T
2. $\mathbb{R}$ is a Hausdorff space under S,
3. a singleton in $\mathbb{R}$ is clopen iff it contains an irrational number
4. any subset of irrationals is scattered under the subspace topology of $\mathbb{R}$ under S

Proof.

1. First note that every element of T is an element of S, so $\varnothing, \mathbb{R} \in S$ in particular. Suppose $A_1, A_2 \in S$ with $A_1 = B_1 \cup C_1$ and $A_2 = B_2 \cup C_2$, where B_i, C_i are defined as in the setup above. Then $A_1 \cap A_2 = B \cup C$, where $B = B_1 \cap B_2 \in T$ and $C = (C_1 \cap B_2) \cup ((B_1 \cup C_1) \cap C_2)$ is a subset of the irrationals. So $A_1 \cap A_2 \in S$. If $A_i \in S$ with $A_i = B_i \cup C_i$, then $\bigcup A_i = \bigcup B_i \cup \bigcup C_i \in S$. So S is a topology which is finer than T
2. $\mathbb{R}$ is Hausdorff under S is clear, the topological property is inherited from T.
3. First, any singleton is closed since X is Hausdorff under S. If x is irrational, then $\{x\}$ is open (under S) as well. So $\{x\}$ is clopen. If x is rational and $\{x\} \in S$, then it is the union of a T-open set B and a subset C of the irrationals. The only T-open subset of $\{x\}$ is the empty set, so $\{x\}$ is a subset of the irrationals, a contradiction.
4. Let C is a subset of the irrational numbers. and considered the subspace topology under S. Then every point r of C is isolated, since $\{r\}$ is

the open subset of C separating it from the rest. The closure of the collection of these points is clearly C itself, so C is scattered.

$\square$

The real line under the topology S is called a *scattered line*.

Remark. Every topological space is a disjoint union of a perfect set and a scattered set.