continuity and convergent nets

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Theorem. *Let X and Y be topological spaces. A function $f : X \to Y$ is continuous at a point $x \in X$ if and only if for each net $(x_\alpha)_{\alpha \in A}$ in X converging to x, the net $(f(x_\alpha))_{\alpha \in A}$ converges to $f(x)$.*

Proof. If f is continuous, $(x_\alpha)_{\alpha \in A}$ converges to x, and V is an open neighborhood of $f(x)$ in Y, then $f^{-1}(V)$ is an open neighborhood of x in X, so there exists $\alpha_0 \in A$ such that $x_\alpha \in f^{-1}(V)$ for $\alpha \geq \alpha_0$. It follows that $f(x_\alpha) \in V$ for $\alpha \geq \alpha_0$, hence that $f(x_\alpha) \to f(x)$. Conversely, suppose there exists a net $(x_\alpha)_{\alpha \in A}$ in X converging to x such that $(f(x_\alpha))_{\alpha \in A}$ does not converge to $f(x)$, so that, for some open subset V of Y containing $f(x)$ and every $\alpha_0 \in A$, there exists $\alpha \geq \alpha_0 \in A$ such that $f(x_\alpha) \notin V$, hence such that $x_\alpha \notin f^{-1}(V)$; as $x_\alpha \to x$ by hypothesis, this implies that $f^{-1}(V)$ cannot be a neighborhood of x, and thus that f fails to be continuous at x. $\square$