uniform neighborhood

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Let X be a uniform space with uniformity $\mathcal{U}$. For each $x \in X$ and $U \in \mathcal{U}$, define the following items

- $U[x] := \{y \mid (x, y) \in U\}$, and
- $\mathfrak{N}_x := \{(x, U[x]) \mid U \in \mathcal{U}\}$
- $\mathfrak{N} = \bigcup_{x \in X} \mathfrak{N}_x$.

Proposition. $\mathfrak{N}_x$ is the abstract neighborhood system around x, hence $\mathfrak{N}$ is the abstract neighborhood system of X.

Proof. We show that all five defining conditions of a neighborhood system on a set are met:

1. For each $(x, U[x]) \in \mathfrak{N}$, $x \in U[x]$, since every entourage contains the diagonal relation.
2. Every $x \in X$ and every entourage $U \in \mathcal{U}$, $U[x] \subseteq X$ with $(x, U[x]) \in \mathfrak{N}$
3. Suppose $(x, U[x]) \in \mathfrak{N}$ and $U[x] \subseteq Y \subseteq X$. Showing that $(x, Y) \in \mathfrak{N}$ amounts to showing $Y = V[x]$ for some $V \in \mathcal{U}$. First, note that each entourage U can be decomposed into disjoint union of sets "slices" of the form $\{a\} \times U[a]$. We replace the "slice" $\{x\} \times U[x]$ by $\{x\} \times Y$. The resulting disjoint union is a set V, which is a superset of U. Since $\mathcal{U}$ is a filter, $V \in \mathcal{U}$. Furthermore, $V[x] = Y$.
4. $a \in U[x] \cap V[x]$ iff $(x, a) \in U \cap V$ iff $a \in (U \cap V)[x]$. This implies that if $(x, U[x]), (x, V[x]) \in \mathfrak{N}$, then $(x, U[x] \cap V[x]) = (x, (U \cap V)[x]) \in \mathfrak{N}$.
5. Suppose $(x, U[x]) \in \mathfrak{N}$. There is $V \in \mathcal{U}$ such that $(V \circ V)[x] \subseteq U[x]$. We show that $V[x] \subseteq X$ is what we want. Clearly, $x \in V[x]$. For any $y \in V[x]$, and any $a \in V[y]$, we have $(x, a) = (x, y) \circ (y, a) \in V \circ V$, or $a \in (V \circ V)[x] \subseteq U[x]$. So $V[y] \subseteq U[x]$ for any $y \in V[x]$. In order to show that $(y, U[x]) \in \mathfrak{N}$, we must find $W \in \mathcal{U}$ such that $U[x] = W[y]$. By the third step above, since $V[y] \subseteq U[x]$, there is $W \in \mathcal{U}$ with $W[y] = U[x]$. Thus $(y, U[x]) = (y, W[y]) \in \mathfrak{N}$.

$\square$

Definition. For each x in a uniform space X with uniformity $\mathcal{U}$, a *uniform neighborhood* of x is a set $U[x]$ for some entourage $U \in \mathcal{U}$. In general, for any $A \subseteq X$, the set

$$U[A] := \{y \in X \mid (x, y) \in U \text{ for some } x \in A\}$$

is called a *uniform neighborhood* of A.

Two immediate properties that we have already seen in the proof above are: (1). for each $U \in \mathcal{U}$, $x \in U[x]$; and (2). $U[x] \cap V[x] = (U \cap V)[x]$. More generally, $\bigcap U_i[x] = (\bigcap U_i)[x]$.

Remark. If we define $T_{\mathcal{U}} := \{A \subseteq X \mid \forall x \in A, \exists U \in \mathcal{U} \text{ such that } U[x] \subseteq A\}$, then $T_{\mathcal{U}}$ is a `http://planetmath.org/TopologyInducedByAUniformStructure`topology induced by the uniform structure $\mathcal{U}$. Under this topology, uniform neighborhoods are synonymous with neighborhoods.