proof of Tietze extension theorem

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To prove the Tietze Extension Theorem, we first need a lemma.

Lemma 1. *If X is a normal topological space and A is closed in X, then for any continuous function $f\colon A \to \mathbb{R}$ such that $|f(x)| \leq 1$, there is a continuous function $g\colon X \to \mathbb{R}$ such that $|g(x)| \leq \frac{1}{3}$ for $x \in X$, and $|f(x) - g(x)| \leq \frac{2}{3}$ for $x \in A$.*

Proof. The sets $f^{-1}\big((-\infty, -\frac{1}{3}]\big)$ and $f^{-1}\big([\frac{1}{3}, \infty)\big)$ are disjoint and closed in A. Since A is closed, they are also closed in X. Since X is normal, then by Urysohn's lemma and the fact that $[0, 1]$ is homeomorphic to $[-\frac{1}{3}, \frac{1}{3}]$, there is a continuous function $g\colon X \to [-\frac{1}{3}, \frac{1}{3}]$ such that $g\Big(f^{-1}\big((-\infty, -\frac{1}{3}c]\big)\Big) = -\frac{1}{3}$ and $g\Big(f^{-1}\big([\frac{1}{3}, \infty)\big)\Big) = \frac{1}{3}$. Thus $|g(x)| \leq \frac{1}{3}$ for $x \in X$. Now if $- \leq f(x) \leq -\frac{1}{3}$, then $g(x) = -\frac{1}{3}$ and thus $|f(x) - g(x)| \leq \frac{2}{3}$. Similarly if $\frac{1}{3} \leq f(x) \leq 1$, then $g(x) = \frac{1}{3}$ and thus $|f(x) - g(x)| \leq \frac{2}{3}$. Finally, for $|f(x)| \leq \frac{1}{3}$ we have that $|g(x)| \leq \frac{1}{3}$, and so $|f(x) - g(x)| \leq \frac{2}{3}$. Hence $|f(x) - g(x)| \leq \frac{2}{3}$ holds for all $x \in A$. $\square$

This puts us in a position to prove the main theorem.

Proof of the Tietze extension theorem. First suppose that for any continuous function on a closed subset there is a continuous extension. Let C and D be disjoint and closed in X. Define $f\colon C \cup D \to \mathbb{R}$ by $f(x) = 0$ for $x \in C$ and $f(x) = 1$ for $x \in D$. Now f is continuous and we can extend it to a continuous function $F\colon X \to \mathbb{R}$. By Urysohn's lemma, X is normal because F is a continuous function such that $F(x) = 0$ for $x \in C$ and $F(x) = 1$ for $x \in D$.

Conversely, let X be normal and A be closed in X. By the lemma, there is a continuous function $g_0\colon X \to \mathbb{R}$ such that $|g_0(x)| \leq \frac{1}{3}$ for $x \in X$ and $|f(x) - g_0(x)| \leq \frac{2}{3}$ for $x \in A$. Since $(f - g_0)\colon A \to \mathbb{R}$ is continuous, the lemma tells us there is a continuous function $g_1\colon X \to \mathbb{R}$ such that $|g_1(x)| \leq \frac{1}{3}(\frac{2}{3})$ for $x \in X$ and $|f(x) - g_0(x) - g_1(x)| \leq \frac{2}{3}(\frac{2}{3})$ for $x \in A$. By repeated application of the lemma we can construct a sequence of continuous functions $g_0, g_1, g_2, \ldots$ such that $|g_n(x)| \leq \frac{1}{3}(\frac{2}{3})^n$ for all $x \in X$, and $|f(x) - g_0(x) - g_1(x) - g_2(x) - \cdots| \leq (\frac{2}{3})^n$ for $x \in A$.

Define $F(x) = \sum_{n=0}^{\infty} g_n(x)$. Since $|g_n(x)| \leq \frac{1}{3}(\frac{2}{3})^n$ and $\sum_{n=0}^{\infty} \frac{1}{3}(\frac{2}{3})^n$ converges as a geometric series, then $\sum_{n=0}^{\infty} g_n(x)$ converges absolutely and uniformly, so F is a continuous function defined everywhere. Moreover $\sum_{n=0}^{\infty} \frac{1}{3}(\frac{2}{3})^n = 1$ implies that $|F(x)| \leq 1$.

Now for $x \in A$, we have that $\left| f(x) - \sum_{n=0}^{k} g_n(x) \right| \leq (\frac{2}{3})^{k+1}$ and as k goes to infinity, the right side goes to zero and so the sum goes to $F(x)$. Thus $|f(x) - F(x)| = 0$ Therefore F extends f. $\square$

Remarks: If f was a function satisfying $|f(x)| < 1$, then the theorem can be strengthened as follows. Find an extension F of f as above. The set $B = F^{-1}(\{-1\} \cup \{1\})$ is closed and disjoint from A because $|F(x)| = |f(x)| < 1$ for $x \in A$. By Urysohn's lemma there is a continuous function ϕ such that $\phi(A) = \{1\}$ and $\phi(B) = \{0\}$. Hence $F(x)\phi(x)$ is a continuous extension of $f(x)$, and has the property that $|F(x)\phi(x)| < 1$.

If f is unbounded, then Tietze extension theorem holds as well. To see that consider $t(x) = \tan^{-1}(x)/(\pi/2)$. The function $t \circ f$ has the property that $(t \circ f)(x) < 1$ for $x \in A$, and so it can be extended to a continuous function $h\colon X \to \mathbb{R}$ which has the property $|h(x)| < 1$. Hence $t^{-1} \circ h$ is a continuous extension of f.