Y is compact if and only if every open cover of Y has a finite subcover

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Theorem.
Let X be a topological space and Y a subset of X. Then the following statements are equivalent.

1. Y is compact as a subset of X.
2. Every open cover of Y (with open sets in X) has a finite subcover.

Proof. Suppose Y is compact, and $\{U_i\}_{i\in I}$ is an arbitrary open cover of Y, where U_i are open sets in X. Then $\{U_i \cap Y\}_{i\in I}$ is a collection of open sets in Y with union Y. Since Y is compact, there is a finite subset $J \subset I$ such that $Y = \cup_{i\in J}(U_i \cap Y)$. Now $Y = (\cup_{i\in J} U_i) \cap Y \subset \cup_{i\in J} U_i$, so $\{U_i\}_{i\in J}$ is finite open cover of Y.

Conversely, suppose every open cover of Y has a finite subcover, and $\{U_i\}_{i\in I}$ is an arbitrary collection of open sets (in Y) with union Y. By the definition of the subspace topology, each U_i is of the form $U_i = V_i \cap Y$ for some open set V_i in X. Now $U_i \subset V_i$, so $\{V_i\}_{i\in I}$ is a cover of Y by open sets in X. By assumption, it has a finite subcover $\{V_i\}_{i\in J}$. It follows that $\{U_i\}_{i\in J}$ covers Y, and Y is compact. $\Box$

The above proof follows the proof given in [**?**].

References

[1] B.Ikenaga, *Notes on Topology*, August 16, 2000, available online `http://www.millersv.edu/ bikenaga/topology/topnote.html`http://www.millersv.edu/ bikenaga/topology/topnote.html.