totally bounded subset of a metric space is bounded

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Theorem 1. *Every totally bounded subset of a metric space is bounded.*

Proof. Let K be a totally bounded subset of a metric space. Suppose $x, y \in K$. We will show that there exists $M > 0$ such that for any x,y we have $d(x, y) < M$. From the definition of totally bounded, we can find an $\varepsilon > 0$ and a finite subset $\{x_1, x_2 ..., x_n\}$ of K such that $K \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon)$, so $x \in B(x_i, \varepsilon)$,$y \in B(x_l, \varepsilon)$, $i, l \in \{1, 2, ...n\}$. So we have that

$$\begin{aligned} d(x, y) &\leq d(x, x_i) + d(x_i, x_l) + d(x_l, y) \\ &< \varepsilon + \max_{1 \leq s,t \leq n} d(x_s, x_t) + \varepsilon = M \end{aligned}$$

$\square$