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every net has a universal subnet

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Theorem - (Kelley's theorem) - Let X be a non-empty set. Every net $(x_\alpha)_{\alpha\in\mathcal{A}}$ in X has a `http://planetmath.org/Ultranet`universal subnet. That is, there is a subnet such that for every $E \subseteq X$ either the subnet is eventually in E or eventually in $X - E$.

Proof : Let $\mathcal{F}$ be a section filter for the net $(x_\alpha)_{\alpha\in\mathcal{A}}$.

Let $\mathcal{D} = \{(\alpha, U) : \alpha \in \mathcal{A}\ ,\ U \in \mathcal{F},\ x_\alpha \in U\}$. $\mathcal{D}$ is a directed set under the order relation given by

$$(\alpha, U) \leq (\beta, V) \Longleftrightarrow \begin{cases} \alpha \leq \beta \\ V \subseteq U \end{cases}$$

The map $f : \mathcal{D} \longrightarrow \mathcal{A}$ defined by $f(\alpha, U) := \alpha$ is order preserving and cofinal. Therefore there is a subnet $(y_{(\alpha,U)})_{(\alpha,U)\in\mathcal{D}}$ of $(x_\alpha)_{\alpha\in\mathcal{A}}$ associated with the map f (that is, $y_{(\alpha,U)} = x_\alpha$).

We now prove that $(y_{(\alpha,U)})_{(\alpha,U)\in\mathcal{D}}$ is a net.

Let $E \subseteq X$. We have that $(y_{(\alpha,U)})_{(\alpha,U)\in\mathcal{D}}$ is frequently in E or frequently in $X - E$.

Suppose $(y_{(\alpha,U)})_{(\alpha,U)\in\mathcal{D}}$ is frequently in E.

Let $A \in \mathcal{F}$ and $S(\alpha) := \{x_\beta : \alpha \leq \beta\}$. We have that $S(\alpha) \in \mathcal{F}$ by definition of section filter.

As $\mathcal{F}$ is a filter, $A \cap S(\alpha) \neq \emptyset$ and so there exists β with $\alpha \leq \beta$ such that $x_\beta \in A$. Hence, $(\beta, A) \in \mathcal{D}$.

As $(y_{(\alpha,U)})_{(\alpha,U)\in\mathcal{D}}$ is frequently in E, there exists $(\gamma, B) \in \mathcal{D}$ with $(\beta, A) \leq (\gamma, B)$ such that $y_{(\gamma,B)} \in E$.

Also, $y_{(\gamma,B)}$ is in B, and therefore, in A. So $A \cap E \neq \emptyset$.

We conclude that $E \cap A \neq \emptyset$ for every $A \in \mathcal{F}$. Therefore, $\mathcal{F} \cup \{E\}$ a filter in X. As $\mathcal{F}$ is a maximal filter we conclude that $E \in \mathcal{F}$, and consequently, $(\gamma, E) \in \mathcal{D}$.

We can now see that for every (δ, C) with $(\gamma, E) \leq (\delta, C)$, $y_{(\delta,C)}$ is in C and so is in E. Therefore, $(y_{(\alpha,U)})_{(\alpha,U)\in\mathcal{D}}$ is eventually in E.

Remark: If $(y_{(\alpha,U)})_{(\alpha,U)\in\mathcal{D}}$ is frequently in $X - E$, by an analogous we can conclude that it is eventually in $X - E$.

This proves that $(y_{(\alpha,U)})_{(\alpha,U)\in\mathcal{D}}$ is a subnet of $(x_\alpha)_{\alpha\in\mathcal{A}}$. $\square$