nested interval theorem

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Owner	pahio (2872)
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Author	pahio (2872)
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Proposition 1. If

$$[a_1,\, b_1] \;\supseteq\; [a_2,\, b_2] \;\supseteq\; [a_3,\, b_3] \;\supseteq \ldots$$

is a sequence of nested closed intervals, then

$$\bigcap_{n=1}^{\infty} [a_n,\, b_n] \;\neq\; \varnothing.$$

If also $\lim_{n\to\infty}(b_n - a_n) = 0$, then the infinite intersection consists of a unique real number.

Proof. There are two consequences to nesting of intervals: $[a_m,\, b_m] \subseteq [a_n,\, b_n]$ for $n \leq m$:

1. first of all, we have the inequality $a_n \leq a_m$ for $n \leq m$, which means that the sequence $a_1, a_2, \ldots, a_n, \ldots$ is nondecreasing;

2. in addition, we also have two inequalities: $a_m \leq b_n$ and $a_n \leq b_m$. In either case, we have that $a_i \leq b_j$ for all i, j. This means that the sequence $a_1, a_2, \ldots, a_n, \ldots$ is bounded from above by all b_i, where $i = 1, 2, \ldots$.

Therefore, the limit of the sequence (a_i) exists, and is just the supremum, say a (see proof `http://planetmath.org/NondecreasingSequenceWithUpperBound`here). Similarly the sequence (b_i) is nonincreasing and bounded from below by all a_i, where $i = 1, 2, \ldots$, and hence has an infimum b.

Now, as the supremum of (a_i), $a \leq b_i$ for all i. But because b is the infimum of (b_i), $a \leq b$. Therefore, the interval $[a, b]$ is non-empty (containing at least one of a, b). Since $a_i \leq a \leq b \leq b_i$, every interval $[a_i, b_i]$ contains the interval $[a, b]$, so their intersection also contains $[a, b]$, hence is non-empty.

If c is a point outside of $[a, b]$, say $c < a$, then there is some a_i, such that $c < a_i$ (by the definition of the supremum a), and hence $c \notin [a_i, b_i]$. This shows that the intersection actually coincides with $[a, b]$.

Now, since $\lim_{n\to\infty}(b_n - a_n) = 0$, we have that $b - a = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty}(b_n - a_n) = 0$. So $a = b$. This means that the intersection of the nested intervals contains a single point a. $\square$

Remark. This result is called the *nested interval theorem*. It is a restatement of the *finite intersection property* for the compact set $[a_1, b_1]$. The result may also be proven by elementary methods: namely, any number lying in between the supremum of all the a_n and the infimum of all the b_n will be in all the nested intervals.