no continuous function switches the rational and the irrational numbers

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Let $\mathbb{J} = \mathbb{R} \setminus \mathbb{Q}$ denote the irrationals. There is no continuous function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subseteq \mathbb{J}$ and $f(\mathbb{J}) \subseteq \mathbb{Q}$.

Proof

Suppose there is such a function f.

First, $\mathbb{Q} = \bigcup_{i\in\mathbb{N}} \{q_i\}$ implies

$$f(\mathbb{Q}) = \bigcup_{i\in\mathbb{N}} \{f(q_i)\}.$$

This is because functions preserve unions (see properties of functions).

And then, $f(\mathbb{Q})$ is first category, because every singleton in $\mathbb{R}$ is nowhere dense (because $\mathbb{R}$ with the Euclidean metric has no isolated points, so the interior of a singleton is empty).

But $f(\mathbb{J}) \subseteq \mathbb{Q}$, so $f(\mathbb{J})$ is first category too. Therefore $f(\mathbb{R})$ is first category, as $f(\mathbb{R}) = f(\mathbb{Q}) \cup f(\mathbb{J})$. Consequently, we have $f(\mathbb{R}) = \bigcup_{i\in\mathbb{N}} \{z_i\}$.

But functions preserve unions in both ways, so

$$\mathbb{R} = f^{-1}(\bigcup_{i\in\mathbb{N}} \{z_i\}) = \bigcup_{i\in\mathbb{N}} f^{-1}(\{z_i\}). \tag{1}$$

Now, f is continuous, and as $\{z_i\}$ is closed for every $i \in \mathbb{N}$, so is $f^{-1}(\{z_i\})$. This means that $\overline{f^{-1}(\{z_i\})} = f^{-1}(\{z_i\})$. If $\operatorname{int}(f^{-1}(\{z_i\})) \neq \varnothing$, we have that there is an open interval $(a_i, b_i) \subseteq f^{-1}(\{z_i\})$, and this implies that there is an irrational number x_i and a rational number y_i such that both lie in $f^{-1}(\{z_i\})$, which is not possible because this would imply that $f(x_i) = f(y_i) = z_i$, and then f would map an irrational and a rational number to the same element, but by hypothesis $f(\mathbb{Q}) \subseteq \mathbb{J}$ and $f(\mathbb{J}) \subseteq \mathbb{Q}$.

Then, it must be $\operatorname{int}(f^{-1}(\{z_i\})) = \varnothing$ for every $i \in \mathbb{N}$, and this implies that $\mathbb{R}$ is first category (by (1)). This is absurd, by the Baire Category Theorem.