boundary of an open set is nowhere dense

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This entry provides another example of a nowhere dense set.

Proposition 1. *If A is an open set in a topological space X, then ∂A, the boundary of A is nowhere dense.*

Proof. Let $B = \partial A$. Since $B = \overline{A} \cap \overline{A^{\complement}}$, it is closed, so all we need to show is that B has empty interior $\operatorname{int}(B) = \varnothing$. First notice that $B = \overline{A} \cap A^{\complement}$, since A is open. Now, we invoke one of the interior axioms, namely $\operatorname{int}(U \cap V) = \operatorname{int}(U) \cap \operatorname{int}(V)$. So, by direct computation, we have

$$\operatorname{int}(B) = \operatorname{int}(\overline{A}) \cap \operatorname{int}(A^{\complement}) = \operatorname{int}(\overline{A}) \cap \overline{A}^{\complement} \subseteq \overline{A} \cap \overline{A}^{\complement} = \varnothing.$$

The second equality and the inclusion follow from the general properties of the interior operation, the proofs of which can be found `http://planetmath.org/DerivationOfPro`

□

Remark. The fact that A is open is essential. Otherwise, the proposition fails in general. For example, the rationals $\mathbb{Q}$, as a subset of the reals $\mathbb{R}$ under the usual order topology, is not open, and its boundary is not nowhere dense, as $\overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^{\complement}} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$, whose interior is $\mathbb{R}$ itself, and thus not empty.