equivalent formulation of the tube lemma

Canonical name	EquivalentFormulationOfTheTubeLemma
Date of creation	2013-03-22 19:15:18
Last modified on	2013-03-22 19:15:18
Owner	joking (16130)
Last modified by	joking (16130)
Numerical id	4
Author	joking (16130)
Entry type	Theorem
Classification	msc 54D30

Let us recall the thesis of the tube lemma. Assume, that X and Y are topological spaces.

(TL) If $U \subseteq X \times Y$ is open (in product topology) and if $x \in X$ is such that $x \times Y \subseteq U$, then there exists an open neighbourhood $V \subseteq X$ of x such that $V \times Y \subseteq U$.

We wish to give a relation between (TL) and the the following thesis, concering closed projections:

(CP) The projection $\pi : X \times Y \to X$ given by $\pi(x,y) = x$ is a closed map.

The following theorem relates these two statements:

Theorem. (TL) is equivalent to (CP).

Proof. ,,$\Rightarrow$" Let $F \subseteq X \times Y$ be a closed set and let $U = (X \times Y) \backslash F$ be its open complement. We will show, that $\pi(F)$ is closed, by showing that $V = X \backslash \pi(F)$ is open. So assume, that $x \in V$. Obviously

$$\big(\pi^{-1}(x) = x \times Y\big) \cap F = \emptyset.$$

Therefore $x \times Y \subseteq U$ and by (TL) there exists open neighbourhood $V' \subseteq X$ of x such that $V' \times Y \subseteq U$. It easily follows, that $V' \subseteq V$ and it is open, so (since x was chosen arbitrary) V is open.

,,$\Leftarrow$" Let $U \subseteq X \times Y$ be an open subset such that $x \times Y \subseteq U$ for some $x \in X$. Let $F = (X \times Y) \backslash U$. Then F is closed and by (CP) we have that $\pi(F) \subseteq X$ is closed. Also $x \notin \pi(F)$ and thus $V = X \backslash \pi(F)$ is an open neighbourhood of x. It can be easily checked, that $V \times Y \subseteq U$, which completes the proof. $\square$

Remark. The theorem doesn't state that any of statements is true. It is well known (see tha parent object), that if both X and Y are Hausdorff with Y compact, then both are true. On the other hand, for example for $X = Y = \mathbb{R}$, where $\mathbb{R}$ denotes reals with standard topology, they are both false. For example consider

$$F = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}.$$

Of course F is closed, but $\pi(F) = \mathbb{R} \backslash \{0\}$ is not closed, so the (CP) is false.