uniform continuity

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In this entry, we extend the usual definition of a uniformly continuous function between metric spaces to arbitrary uniform spaces.

Let $(X,\mathcal{U}),(Y,\mathcal{V})$ be uniform spaces (the second component is the uniformity on the first component). A function $f:X\to Y$ is said to be *uniformly continuous* if for any $V\in\mathcal{V}$ there is a $U\in\mathcal{U}$ such that for all $x\in X$, $U[x]\subseteq f^{-1}(V[f(x)])$.

Sometimes it is useful to use an alternative but equivalent version of uniform continuity of a function:

Proposition 1. *Suppose $f:X\to Y$ is a function and $g:X\times X\to Y\times Y$ is defined by $g(x_1,x_2)=(f(x_1),f(x_2))$. Then f is uniformly continuous iff for any $V\in\mathcal{V}$, there is a $U\in\mathcal{U}$ such that $U\subseteq g^{-1}(V)$.*

Proof. Suppose f is uniformly continuous. Pick any $V\in\mathcal{V}$. Then $U\in\mathcal{U}$ exists with $U[x]\subseteq f^{-1}(V[f(x)])$ for all $x\in X$. If $(a,b)\in U$, then $b\in U[a]\subseteq f^{-1}(V[f(a)])$, or $f(b)\subseteq V[f(a)]$, or $g(a,b)=(f(a),f(b))\in V$. The converse is straightforward. $\square$

Remark. Note that we could have picked U so the inclusion becomes an equality.

Proposition 2. . *If $f:X\to Y$ is uniformly continuous, then it is continuous under the uniform topologies of X and Y.*

Proof. Let A be open in Y and set $B=f^{-1}(A)$. Pick any $x\in B$. Then $y=f(x)$ has a uniform neighborhood $V[y]\subseteq A$. By the uniform continuity of f, there is an entourage $U\in\mathcal{U}$ with $x\in U[x]\subseteq f^{-1}(V[y])\subseteq f^{-1}(A)=B$. $\square$

Remark. The converse is not true, even in metric spaces.