closure space

Canonical name	ClosureSpace
Date of creation	2013-03-22 16:48:08
Last modified on	2013-03-22 16:48:08
Owner	CWoo (3771)
Last modified by	CWoo (3771)
Numerical id	12
Author	CWoo (3771)
Entry type	Derivation
Classification	msc 54A05
Defines	closure topology

Call a set X with a closure operator defined on it a *closure space*.

Every topological space is a closure space, if we define the closure operator of the space as a function that takes any subset to its closure. The converse is also true:

Proposition 1. *Let X be a closure space with c the associated closure operator. Define a "closed set" of X as a subset A of X such that $A^c = A$, and an "open set" of X as the complement of some closed set of X. Then the collection $\mathcal{T}$ of all open sets of X is a topology on X.*

Proof. Since $\varnothing^c = \varnothing$, $\varnothing$ is closed. Also, $X \subseteq X^c$ and $X^c \subseteq X$ imply that $X^c = X$, or X is closed. If $A, B \subseteq X$ are closed, then $(A \cup B)^c = A^c \cup B^c = A \cup B$ is closed as well. Finally, suppose A_i are closed. Let $B = \bigcap A_i$. For each i, $A_i = B \cup A_i$, so $A_i = A_i^c = (B \cup A_i)^c = B^c \cup A_i^c = B^c \cup A_i$. This means $B^c \subseteq A_i$, or $B^c \subseteq \bigcap A_i = B$. But $B \subseteq B^c$ by definition, so $B = B^c$, or that $\bigcap A_i$ is closed. $\square$

$\mathcal{T}$ so defined is called the *closure topology* of X with respect to the closure operator c.

Remarks.

1. A closure space can be more generally defined as a set X together with an operator $\mathrm{cl} : P(X) \to P(X)$ such that cl satisfies all of the Kuratowski's closure axioms where the equal sign "=" is replaced with set inclusion "$\subseteq$", and the preservation of $\varnothing$ is no longer assumed.

2. Even more generally, a closure space can be defined as a set X and an operator cl on $P(X)$ such that

   - $A \subseteq \mathrm{cl}(A)$,
   - $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$, and
   - cl is order-preserving, i.e., if $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$.

   It can be easily deduced that $\mathrm{cl}(A) \cup \mathrm{cl}(B) \subseteq \mathrm{cl}(A \cup B)$. In general however, the equality fails. The three axioms above can be shown to be equivalent to a single axiom:

   $$A \subseteq \mathrm{cl}(B) \quad \text{iff} \quad \mathrm{cl}(A) \subseteq \mathrm{cl}(B).$$

3. In a closure space X, a subset A of X is said to be closed if $\mathrm{cl}(A) = A$. Let $C(X)$ be the set of all closed sets of X. It is not hard to see that if $C(X)$ is closed under $\cup$, then cl "distributes over" $\cup$, that is, we have the equality $\mathrm{cl}(A) \cup \mathrm{cl}(B) = \mathrm{cl}(A \cup B)$.

4. Also, $\mathrm{cl}(\varnothing)$ is the smallest closed set in X; it is the bottom element in $C(X)$. This means that if there are two disjoint closed sets in X, then $\mathrm{cl}(\varnothing) = \varnothing$. This is equivalent to saying that $\varnothing$ is closed whenever there exist $A, B \subseteq X$ such that $\mathrm{cl}(A) \cap \mathrm{cl}(B) = \varnothing$.

5. Since the distributivity of cl over $\cup$ does not hold in general, and there is no guarantee that $\mathrm{cl}(\varnothing) = \varnothing$, a closure space under these generalized versions is a more general system than a topological space.

References

[1] N. M. Martin, S. Pollard: *Closure Spaces and Logic*, Springer, (1996).