Hausdorff space not completely Hausdorff

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On the set $\mathbb{Z}^+$ of strictly positive integers, let a and b be two different integers $b \neq 0$ and consider the set

$$S(a, b) = \{a + kb \in \mathbb{Z}^+ : k \in \mathbb{Z}\}$$

such set is the infinite arithmetic progression of positive integers with difference b and containing a. The collection of all $S(a, b)$ sets is a basis for a topology on $\mathbb{Z}^+$. We will use a coarser topology induced by the following basis:

$$\mathbb{B} = \{S(a, b) : \gcd(a, b) = 1\}$$

The collection $\mathbb{B}$ is basis for a topology on $\mathbb{Z}^+$

We first prove such collection is a basis. Suppose $x \in S(a, b) \cap S(c, d)$. By Euclid's algorithm we have $S(a, b) = S(x, b)$ and $S(c, d) = S(x, d)$ and

$$x \in S(x, bd) \subset S(x, d) \cap S(c, d)$$

besides, since $\gcd(x, b) = 1$ and $\gcd(x, d) = 1$ then $\gcd(x, bd) = 1$ so x and bd are coprimes and $S(x, bd) \in \mathbb{B}$. This concludes the proof that $\mathbb{B}$ is indeed a basis for a topology on $\mathbb{Z}^+$.

The topology on $\mathbb{Z}^+$ induced by $\mathbb{B}$ is Hausdorff

Let m, n integers two different integers. We need to show that there are open disjoint neighborhoods U_m and U_n such that $m \in U_m$ and $n \in U_n$, but it suffices to show the existence of disjoint basic open sets containing m and n.

Taking $d = |m - n|$, we can find an integer t such that $t > d$ and such that $\gcd(m, t) = \gcd(n, t) = 1$. A way to accomplish this is to take any multiple of mn greater than d and add 1.

The basic open sets $S(m, t)$ and $S(n, t)$ are disjoint, because they have common elements if and only if the diophantine equation $m + tx = n + ty$ has solutions. But it cannot have since $t(x - y) = n - m$ implies that t divides $n - m$ but $t > |n - m|$ makes it impossible.

We conclude that $S(m, t) \cap S(n, t) = \emptyset$ and this means that $\mathbb{Z}^+$ becomes a Hausdorff space with the given topology.

Some properties of $\overline{S(a,b)}$

We need to determine first some facts about $\overline{S(a,b)}$. in order to take an example, consider $S(3,5)$ first. Notice that if we had considered the former topology (where in $S(a,b)$, a and b didn't have to be coprime) the complement of $S(3,5)$ would have been $S(4,5)\cup S(5,5)\cup S(6,5)\cup S(7,5)$ which is open, and so $S(3,5)$ would have been closed. In general, in the finer topology, all basic sets were both open and closed. However, this is not true in our coarser topology (for instance $S(5,5)$ is not open).

The key fact to prove $\mathbb{Z}^+$ is not a completely Hausdorff space is: given any $S(a,b)$, then $b\mathbb{Z}^+ = \{n \in \mathbb{Z}^+ : b \text{ divides } n\}$ is a subset of $\overline{S(a,b)}$.

Indeed, any basic open set containing bk is of the form $S(bk,t)$ with t, bk coprimes. This means $\gcd(t,b) = 1$. Now $S(bk,t)$ and $S(a,b)$ have common terms if an only if $bk+tx = a+by$ for some integers x, y. But that diophantine equation can be rewritten as

$$tx - by = a - bk$$

and it always has solutions because $1 = \gcd(t,b)$ divides $a - bk$.

This also proves $S(a,b) \neq \overline{S(a,b)}$, because b is not in $S(a,b)$ but it is on the closure.

The topology on $\mathbb{Z}^+$ induced by $\mathbb{B}$ is not completely Hausdorff

We will use the closed-neighborhood sense for completely Hausdorff, which will also imply the topology is not completely Hausdorff in the functional sense.

Let m, n different positive integers. Since $\mathbb{B}$ is a basis, for any two disjoint neighborhoods U_m, U_n we can find basic sets $S(m,a)$ and $S(n,b)$ such that

$$m \in S(m,a) \subseteq U_m, \qquad n \in S(n,b) \subseteq U_n$$

and thus

$$S(m,a) \cap S(n,b) = \emptyset.$$

But then $g = ab$ is both a multiple of a and b so it must be in $\overline{S(m,a)}$ and $\overline{S(n,b)}$. This means

$$\overline{S(m,a)} \cap \overline{S(n,b)} \neq \emptyset$$

and thus $\overline{U_m} \cap \overline{U_n} \neq \emptyset$.

This proves the topology under consideration is not completely Hausdorff (under both usual meanings).