product topology

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Definition

Let $((X_\alpha, \mathcal{T}_\alpha))_{\alpha \in A}$ be a family of topological spaces, and let Y be the `http://planetmath.org/Gene` product of the sets X_α, that is

$$Y = \prod_{\alpha \in A} X_\alpha .$$

Recall that an element $y \in Y$ is a function $y\colon A \to \bigcup_{\alpha \in A} X_\alpha$ such that $y(\alpha) \in X_\alpha$ for each $\alpha \in A$, and that for each $\alpha \in A$ the projection map $\pi_\alpha\colon Y \to X_\alpha$ is defined by $\pi_\alpha(y) = y(\alpha)$ for each $y \in Y$.

The *(Tychonoff) product topology* $\mathcal{T}$ for Y is defined to be the initial topology with respect to the projection maps; that is, $\mathcal{T}$ is the smallest topology such that each π_α is `http://planetmath.org/Continuous`continuous.

Subbase

If $U \subseteq X_\alpha$ is open, then $\pi_\alpha^{-1}(U)$ is an open set in Y. Note that this is the set of all elements of Y in which the α component is restricted to U and all other components are unrestricted. The open sets of Y are the unions of finite intersections of such sets. That is,

$$\{\, \pi_\alpha^{-1}(U) \mid \alpha \in A \text{ and } U \in \mathcal{T}_\alpha \,\}$$

is a subbase for the topology on Y.

Theorems

The following theorems assume the product topology on $\prod_{\alpha \in A} X_\alpha$. Notation is as above.

Theorem 1 *Let Z be a topological space and let $f\colon Z \to \prod_{\alpha \in A} X_\alpha$ be a function. Then f is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in A$.*

Theorem 2 *The product topology on $\prod_{\alpha \in A} X_\alpha$ is the topology induced by the subbase*

$$\{\pi_\alpha^{-1}(U) \mid \alpha \in A \textit{ and } U \in \mathcal{T}_\alpha\}.$$

Theorem 3 *The product topology on $\prod_{\alpha\in A} X_\alpha$ is the topology induced by the base*

$$\left\{ \bigcap_{\alpha\in F} \pi_\alpha^{-1}(U_\alpha) \;\middle|\; F \text{ is a finite subset of } A \text{ and } U_\alpha \in \mathcal{T}_\alpha \text{ for each } \alpha \in F \right\}.$$

Theorem 4 *A net $(x_i)_{i\in I}$ in $\prod_{\alpha\in A} X_\alpha$ converges to x if and only if each coordinate $(x_i^\alpha)_{i\in I}$ converges to x^α in X_α.*

Theorem 5 *Each projection map $\pi_\alpha\colon \prod_{\alpha\in A} X_\alpha \to X_\alpha$ is continuous and `http://planetmath.org/OpenMapping`open.*

Theorem 6 *For each $\alpha \in A$, let $A_\alpha \subseteq X_\alpha$. Then*

$$\overline{\prod_{\alpha\in A} A_\alpha} = \prod_{\alpha\in A} \overline{A_\alpha}.$$

In particular, any product of closed sets is closed.

Theorem 7 *(Tychonoff's Theorem) If each X_α is compact, then $\prod_{\alpha\in A} X_\alpha$ is compact.*

Comparison with box topology

There is another well-known way to topologize Y, namely the box topology. The product topology is a subset of the box topology; if A is finite, then the two topologies are the same.

The product topology is generally more useful than the box topology. The main reason for this can be expressed in terms of category theory: the product topology is the topology of the `http://planetmath.org/CategoricalDirectProduct`direct categorical product in the category **Top** (see Theorem 1 above).

References

[1] J. L. Kelley, *General Topology*, D. van Nostrand Company, Inc., 1955.

[2] J. Munkres, *Topology* (2nd edition), Prentice Hall, 1999.