the real numbers are indecomposable as a topological space

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Let $\mathbb{R}$ be the set of real numbers with standard topology. We wish to show that if $\mathbb{R}$ is homeomorphic to $X \times Y$ for some topological spaces X and Y, then either X is one point space or Y is one point space. First let us prove a lemma:

Lemma. Let X and Y be path connected topological spaces such that cardinality of both X and Y is at least 2. Then for any point $(x_0, y_0) \in X \times Y$ the space $X \times Y \setminus \{(x_0, y_0)\}$ with subspace topology is path connected.

Proof. Let $x' \in X$ and $y' \in Y$ such that $x' \neq x_0$ and $y' \neq y_0$ (we assumed that such points exist). It is sufficient to show that for any point (x_1, y_1) from $X \times Y \setminus \{(x_0, y_0)\}$ there exists a continous map $\sigma : \mathrm{I} \to X \times Y$ such that $\sigma(0) = (x_1, y_1)$, $\sigma(1) = (x', y')$ and $(x_0, y_0) \notin \sigma(\mathrm{I})$.

Let $(x_1, y_1) \in X \times Y \setminus \{(x_0, y_0)\}$. Therefore either $x_1 \neq x_0$ or $y_1 \neq y_0$. Assume that $y_1 \neq y_0$ (the other case is analogous). Choose paths $\sigma : \mathrm{I} \to X$ from x_1 to x' and $\tau : \mathrm{I} \to Y$ from y_1 to y'. Then we have induced paths:

$$\sigma' : \mathrm{I} \to X \times Y \quad \text{such that } \sigma'(t) = (\sigma(t), y_1);$$

$$\tau' : \mathrm{I} \to X \times Y \quad \text{such that } \tau'(t) = (x', \tau(t)).$$

Then the path $\sigma' * \tau' : \mathrm{I} \to X \times Y$ defined by the formula

$$(\sigma' * \tau')(t) = \begin{cases} \sigma'(2t) & \text{when } \ 0 \leq t \leq \frac{1}{2} \\ \tau'(2t-1) & \text{when } \ \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a desired path. □

Proposition. If there exist topological spaces X and Y such that $\mathbb{R}$ is homeomorphic to $X \times Y$, then either X has exactly one point or Y has exactly one point.

Proof. Assume that neither X nor Y has exactly one point. Now $X \times Y$ is path connected since it is homeomorphic to $\mathbb{R}$, so it is well known that both X and Y have to be path connected (please see `http://planetmath.org/ProductOfPathConnectedSp` entry for more details). Therefore for any point $(x, y) \in X \times Y$ the space $X \times Y \setminus \{(x, y)\}$ is also path connected (due to lemma), but there exists a real number $r \in \mathbb{R}$ such that $X \times Y \setminus \{(x, y)\}$ is homeomorphic to $\mathbb{R} \setminus \{r\}$. Contradiction, since $\mathbb{R} \setminus \{r\}$ is not path connected. □