equivalent condition for being a fundamental system of entourages

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Lemma. *Let X be a set and let $\mathcal{B}$ be a nonempty family of subsets of $X \times X$. Then $\mathcal{B}$ is a fundamental system of entourages of a uniformity on X if and only if it satisfies the following axioms.*

- *(B1) If S, $T \in \mathcal{B}$, then $S \cap T$ contains an element of $\mathcal{B}$.*
- *(B2) Each element of $\mathcal{B}$ contains the diagonal $\Delta(X)$.*
- *(B3) For any $S \in \mathcal{B}$, the inverse relation of S contains an element of $\mathcal{B}$.*
- *(B4) For any $S \in \mathcal{B}$, there is an element $T \in \mathcal{B}$ such that the relational composition $T \circ T$ is contained in S.*

Proof. Suppose $\mathcal{B}$ is a fundamental system of entourages for a uniformity $\mathcal{U}$. Verification of axiom (B2) is immediate, since $\mathcal{B} \subseteq \mathcal{U}$ and each entourage is already required to contain the diagonal of X. We will prove that $\mathcal{B}$ satisfies (B1); the proofs that (B3) and (B4) hold are analogous.

Let S, T be entourages in $\mathcal{B} \subseteq \mathcal{U}$. Since $\mathcal{U}$ is closed under binary intersections, $S \cap T \in \mathcal{U}$. By the definition of fundamental system of entourages, since $S \cap T \in \mathcal{U}$, there exists an entourage $B \in \mathcal{B}$ such that $B \subseteq S \cap T$. Thus $\mathcal{B}$ satisfies axioms (B1) through (B4).

To prove the converse, define a family of subsets of $X \times X$ by

$$\mathcal{U} = \{S \subseteq X \times X : B \subseteq S \text{ for some } B \in \mathcal{B}\}.$$

By construction, each element of $\mathcal{U}$ contains an element of $\mathcal{B}$, so all that remains is to show that $\mathcal{U}$ is a uniformity. Suppose T is a subset of $X \times X$ that contains an element $S \in \mathcal{U}$. By the definition of $\mathcal{U}$, there exists some $B \in \mathcal{B}$ such that $B \subseteq S$. Since $S \subseteq T$, it follows that $B \subseteq T$, so T satisfies the requirement for membership in $\mathcal{U}$. Thus $\mathcal{U}$ is closed under taking supersets. The remaining axioms for a uniformity follow directly from the appropriate axioms for the fundamental system of entourages by applying the axiom we have just checked. Hence $\mathcal{B}$ is a fundamental system of entourages for a uniformity on X. $\square$