# proof of Tychonoff's theorem in finite case



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| Owner | stevecheng (10074) |
| Last modified by | stevecheng (10074) |
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| Author | stevecheng (10074) |
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(The finite case of Tychonoff's Theorem is of course a subset of the infinite case, but the proof is substantially easier, so that is why it is presented here.)

To prove that $X_1 \times \cdots \times X_n$ is compact if the $X_i$ are compact, it suffices (by induction) to prove that $X \times Y$ is compact when $X$ and $Y$ are. It also suffices to prove that a finite subcover can be extracted from every open cover of $X \times Y$ by only the *basis sets* of the form $U \times V$, where $U$ is open in $X$ and $V$ is open in $Y$.

*Proof.* The proof is by the straightforward strategy of composing a finite subcover from a lower-dimensional subcover. Let the open cover $\mathcal{C}$ of $X \times Y$ by basis sets be given.

The set $X \times \{y\}$ is compact, because it is the image of a continuous embedding of the compact set $X$. Hence $X \times \{y\}$ has a finite subcover in $\mathcal{C}$: label the subcover by $\mathcal{S}^y = \{U_1^y \times V_1^y, \ldots, U_{k_y}^y \times V_{k_y}^y\}$. Do this for each $y \in Y$.

To get the desired subcover of $X \times Y$, we need to pick a finite number of $y \in Y$. Consider $V^y = \bigcap_{i=1}^{k_y} V_i^y$. This is a finite intersection of open sets, so $V^y$ is open in $Y$. The collection $\{V^y : y \in Y\}$ is an open covering of $Y$, so pick a finite subcover $V^{y_1}, \ldots, V^{y_l}$. Then $\bigcup_{j=1}^{l} \mathcal{S}^{y_j}$ is a finite subcover of $X \times Y$. $\square$