Hausdorff metric

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Defines	Hausdorff hemimetric

Let (X, d) be a metric space, and let $\mathcal{F}_X$ be the family of all closed and bounded subsets of X. Given $A \in \mathcal{F}_X$, we will denote by $N_r(A)$ the neighborhood of A of radius r, i.e. the set $\cup_{x \in A} B(x, r)$.

The *upper Hausdorff hemimetric* is defined by

$$\delta^*(A, B) = \inf\{r > 0 : B \subset N_r(A)\}.$$

Analogously, the *lower Hausdorff hemimetric* is

$$\delta_*(A, B) = \inf\{r > 0 : A \subset N_r(B)\}.$$

Finally, the *Hausdorff metric* is given by

$$\delta(A, B) = \max\{\delta^*(A, B), \delta_*(A, B)\}.$$

for $A, B \in \mathcal{F}_X$.

The following properties follow straight from the definitions:

1. $\delta^*(A, B) = \delta_*(B, A)$;

2. $\delta^*(A, B) = 0$ if and only if $B \subset A$;

3. $\delta_*(A, B) = 0$ if and only if $A \subset B$;

4. $\delta^*(A, C) \leq \delta^*(A, B) + \delta^*(B, C)$, and similarly for δ_*.

From this it is clear that δ is a metric: the triangle inequality follows from that of δ_* and δ^*; symmetry follows from $\delta^*(A, B) = \delta_*(A, B)$; and $\delta(A, B) = 0$ iff both $\delta_*(A, B)$ and $\delta^*(A, B)$ are zero iff $A \subset B$ and $B \subset A$ iff $A = B$.

Hausdorff metric inherits completeness; i.e. if (X, d) is complete, then so is $(\mathcal{F}_X, \delta)$. Also, if (X, d) is totally bounded, then so is $(\mathcal{F}_X, \delta)$.

Intuitively, the Hausdorff hemimetric δ^* (resp. δ_*) measure how much bigger (resp. smaller) is a set compared to another. This allows us to define hemicontinuity of correspondences.