connectedness is preserved under a continuous map

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Theorem Suppose $f\colon X \to Y$ is a continuous map between topological spaces X and Y. If X is a connected space, and f is surjective, then Y is a connected space.

The inclusion map for spaces $X = (0,1)$ and $Y = (0,1) \cup (2,3)$ shows that we need to assume that the map is surjective. Othewise, we can only prove that $f(X)$ is connected. See `http://planetmath.org/IfFcolonXtoYIsContinuousThenFcolonXto` page.

Proof. For a contradiction, suppose there are disjoint open sets A, B in Y such that $Y = A \cup B$. By continuity and properties of the inverse image, $f^{-1}(A)$ and $f^{-1}(B)$ are open disjoint sets in X. Since f is surjective, $Y = f(X) = A \cup B$, whence

$$X = f^{-1}f(X) = f^{-1}(A) \cup f^{-1}(B)$$

contradicting the assumption that X is connected.

References

[1] G.J. Jameson, *Topology and Normed Spaces*, Chapman and Hall, 1974.

[2] G.L. Naber, *Topological methods in Euclidean spaces*, Cambridge University Press, 1980.