proof of ham sandwich theorem

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This proof uses the Borsuk-Ulam theorem, which states that any continuous function from S^n to $\mathbb{R}^n$ maps some pair of antipodal points to the same point.

Let A be a measurable bounded subset of $\mathbb{R}^n$. Given any unit vector $\hat{n} \in S^{n-1}$ and $s \in \mathbb{R}$, there is a unique $n-1$ dimensional hyperplane normal to $\hat{n}$ and containing $s\hat{n}$.

Define $f : S^{n-1} \times \mathbb{R} \to [0, \infty)$ by sending $(\hat{n}, s)$ to the measure of the subset of A lying on the side of the plane corresponding to $(\hat{n}, s)$ in the direction in which $\hat{n}$ points. Note that $(\hat{n}, s)$ and $(-\hat{n}, -s)$ correspond to the same plane, but to different sides of the plane, so that $f(\hat{n}, s) + f(-\hat{n}, -s) = m(A)$.

Since A is bounded, there is an $r > 0$ such that A is contained in $\overline{B_r}$, the closed ball of radius r centered at the origin. For sufficiently small changes in $(\hat{n}, s)$, the measure of the portion of $\overline{B_r}$ between the different corresponding planes can be made arbitrarily small, and this bounds the change in $f(\hat{n}, s)$, so that f is a continuous function.

Finally, it's easy to see that, for fixed $\hat{n}$, $f(\hat{n}, s)$ is monotonically decreasing in s, with $f(\hat{n}, -s) = m(A)$ and $f(\hat{n}, s) = 0$ for s sufficiently large.

Given these properties of f, we see by the intermediate value theorem that, for fixed $\hat{n}$, there is an interval $[a, b]$ such that the set of s with $f(\hat{n}, s) = m(A)/2$ is $[a, b]$. If we define $g(\hat{n})$ to be the midpoint of this interval, then, since f is continuous, we see g is a continuous function from S^{n-1} to $\mathbb{R}$. Also, since $f(\hat{n}, s) + f(-\hat{n}, -s) = m(A)$, if $[a, b]$ is the interval corresponding to $\hat{n}$, then $[-b, -a]$ is the interval corresponding to $-\hat{n}$, and so $g(\hat{n}) = -g(-\hat{n})$.

Now let $A_1, A_2, ..., A_n$ be measurable bounded subsets of $\mathbb{R}^n$, and let f_i, g_i be the maps constructed above for A_i. Then we can define $h : S^{n-1} \to R^{n-1}$ by:

$$h(\hat{n}) = (f_1(\hat{n}, g_n(\hat{n})), f_2(\hat{n}, g_n(\hat{n})), ...f_{n-1}(\hat{n}, g_n(\hat{n})))$$

This is continuous, since each coordinate function is the composition of continuous functions. Thus we can apply the Borsuk-Ulam theorem to see there is some $\hat{n} \in S^{n-1}$ with $h(\hat{n}) = h(-\hat{n})$, ie, with:

$$f_i(\hat{n}, g_n(\hat{n})) = f_i(-\hat{n}, g_n(-\hat{n})) = f_i(-\hat{n}, -g_n(\hat{n}))$$

where we've used the property of g mentioned above. But this just means that for each A_i with $1 \leq i \leq n-1$, the measure of the subset of A_i lying on one side of the plane corresponding to $(\hat{n}, g_n(\hat{n}))$, which is $f_i(\hat{n}, g_n(\hat{n}))$, is the same as the measure of the subset of A_i lying on the other side of the plane, which is $f_i(-\hat{n}, -g_n(\hat{n}))$. In other words, the plane corresponding to $(\hat{n}, g_n(\hat{n}))$ bisects each A_i with $1 \leq i \leq n-1$. Finally, by the definition of g_n, this plane also bisects A_n, and so it bisects each of the A_i as claimed.