sheaf

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1 Presheaves

Let X be a topological space and let $\mathcal{A}$ be a category. A *presheaf* on X with values in $\mathcal{A}$ is a contravariant functor F from the category $\mathcal{B}$ whose objects are open sets in X and whose morphisms are inclusion mappings of open sets of X, to the category $\mathcal{A}$.

As this definition may be less than helpful to many readers, we offer the following equivalent (but longer) definition. A *presheaf* F on X consists of the following data:

1. An object $F(U)$ in $\mathcal{A}$, for each open set $U \subset X$
2. A morphism $\mathrm{res}_{V,U}\colon F(V) \to F(U)$ for each pair of open sets $U \subset V$ in X (called the *restriction morphism*), such that:
   (a) For every open set $U \subset X$, the morphism $\mathrm{res}_{U,U}$ is the identity morphism.
   (b) For any open sets $U \subset V \subset W$ in X, the diagram

$$\begin{tikzcd} F(W) \arrow[r, "\mathrm{res}_{W,V}"'] \arrow[rr, bend left, "\mathrm{res}_{W,U}"] & F(V) \arrow[r, "\mathrm{res}_{V,U}"'] & F(U) \end{tikzcd}$$

   commutes.

If the object $F(U)$ of $\mathcal{A}$ is a set, its elements are called *sections* of U.

2 Morphisms of Presheaves

Let $f\colon X \to Y$ be a continuous map of topological spaces. Suppose F_X is a presheaf on X, and G_Y is a presheaf on Y (with F_X and G_Y both having values in $\mathcal{A}$). We define a *morphism of presheaves* ϕ from G_Y to F_X, relative to f, to be a collection of morphisms $\phi_U\colon G_Y(U) \to F_X(f^{-1}(U))$ in $\mathcal{A}$, one for every open set $U \subset Y$, such that the diagram

$$\begin{tikzcd} G_Y(V) \arrow[r, "\phi_V"] \arrow[d, "\mathrm{res}_{V,U}"'] & F_X(f^{-1}(V)) \arrow[d, "\mathrm{res}_{f^{-1}(V),f^{-1}(U)}"] \\ G_Y(U) \arrow[r, "\phi_U"'] & F_X(f^{-1}(U)) \end{tikzcd}$$

commutes, for each pair of open sets $U \subset V$ in Y.

Alternatively, a morphism of presheaves can be regarded as a natural transformation from G_Y to F_Y, where F_Y is the presheaf on Y given by $F_Y(U) := F_X(f^{-1}(U))$. In the special case that f is the identity map $\mathrm{id}\colon X \to X$, we omit mention of the map f, and speak of ϕ as simply a morphism of presheaves on X.

Form the category whose objects are presheaves on X and whose morphisms are morphisms of presheaves on X. Then an *isomorphism* of presheaves ϕ on X is a morphism of presheaves on X which is an isomorphism in this category; that is, there exists a morphism ϕ^{-1} whose composition with ϕ both ways is the identity morphism.

More generally, if $f\colon X \to Y$ is any homeomorphism of topological spaces, a morphism of presheaves ϕ relative to f is an *isomorphism* if it admits a two–sided inverse morphism of presheaves ϕ^{-1} relative to f^{-1}.

3 Sheaves

We now assume that the category $\mathcal{A}$ is a concrete category. A *sheaf* is a presheaf F on X, with values in $\mathcal{A}$, such that for every open set $U \subset X$, and every open cover $\{U_i\}$ of U, the following two conditions hold:

1. Any two elements $f_1, f_2 \in F(U)$ which have identical restrictions to each U_i are equal. That is, if $\mathrm{res}_{U,U_i} f_1 = \mathrm{res}_{U,U_i} f_2$ for every i, then $f_1 = f_2$.

2. Any collection of elements $f_i \in F(U_i)$ that have common restrictions can be realized as the collective restrictions of a single element of $F(U)$. That is, if $\mathrm{res}_{U_i,U_i\cap U_j} f_i = \mathrm{res}_{U_j,U_i\cap U_j} f_j$ for every i and j, then there exists an element $f \in F(U)$ such that $\mathrm{res}_{U,U_i} f = f_i$ for all i.

4 Sheaves in abelian categories

If $\mathcal{A}$ is a concrete abelian category, then a presheaf F is a sheaf if and only if for every open subset U of X, the sequence

$$0 \longrightarrow F(U) \xrightarrow{\text{incl}} \prod_i F(U_i) \xrightarrow{\text{diff}} \prod_{i,j} F(U_i \cap U_j) \qquad (1)$$

is an exact sequence of morphisms in $\mathcal{A}$ for every open cover $\{U_i\}$ of U in X. This diagram requires some explanation, because we owe the reader a definition of the morphisms incl and diff. We start with incl (short for "inclusion"). The restriction morphisms $F(U) \to F(U_i)$ induce a morphism

$$F(U) \to \prod_i F(U_i)$$

to the categorical direct product $\prod_i F(U_i)$, which we define to be incl. The map diff (called "difference") is defined as follows. For each U_i, form the morphism

$$\alpha_i \colon F(U_i) \to \prod_j F(U_i \cap U_j).$$

By the universal properties of categorical direct product, there exists a unique morphism

$$\alpha \colon \prod_i F(U_i) \to \prod_i \prod_j F(U_i \cap U_j)$$

such that $\pi_i \alpha = \alpha_i \pi_i$ for all i, where π_i is projection onto the i^{th} factor. In a similar manner, form the morphism

$$\beta \colon \prod_j F(U_j) \to \prod_j \prod_i F(U_i \cap U_j).$$

Then α and β are both elements of the set

$$\mathrm{Hom}\left(\prod_i F(U_i), \prod_{i,j} F(U_i \cap U_j)\right),$$

which is an abelian group since $\mathcal{A}$ is an abelian category. Take the difference $\alpha - \beta$ in this group, and define this morphism to be diff.

Note that exactness of the sequence (**??**) is an element free condition, and therefore makes sense for any abelian category $\mathcal{A}$, even if $\mathcal{A}$ is not concrete. Accordingly, for any abelian category $\mathcal{A}$, we define a sheaf to be a presheaf F for which the sequence (**??**) is always exact.

5 Examples

It's high time that we give some examples of sheaves and presheaves. We begin with some of the standard ones.

Example 1. If F is a presheaf on X, and $U \subset X$ is an open subset, then one can define a presheaf $F|_U$ on U by restricting the functor F to the subcategory of open sets of X in U and inclusion morphisms. In other words, for open subsets of U, define $F|_U$ to be exactly what F was, and ignore open subsets of X that are not open subsets of U. The resulting presheaf is called, for obvious reasons, the *restriction presheaf* of F to U, or the *restriction sheaf* if F was a sheaf to begin with.

Example 2. For any topological space X, let $\mathcal{C}_X$ be the presheaf on X, with values in the category of rings, given by

- $\mathcal{C}_X(U) :=$ the ring of continuous real–valued functions $U \to \mathbb{R}$,
- $\mathrm{res}_{V,U} f :=$ the restriction of f to U, for every element $f \colon V \to \mathbb{R}$ of $\mathcal{C}_X(V)$ and every subset U of V.

Then $\mathcal{C}_X$ is actually a sheaf of rings, because continuous functions are uniquely specified by their values on an open cover. The sheaf $\mathcal{C}_X$ is called the *sheaf of continuous real–valued functions* on X.

Example 3. Let X be a smooth differentiable manifold. Let $\mathcal{D}_X$ be the presheaf on X, with values in the category of real vector spaces, defined by setting $\mathcal{D}_X(U)$ to be the space of smooth real–valued functions on U, for each open set U, and with the restriction morphism given by restriction of functions as before. Then $\mathcal{D}_X$ is a sheaf as well, called the *sheaf of smooth real–valued functions* on X.

Much more surprising is that the construct $\mathcal{D}_X$ can actually be used to **define** the concept of smooth manifold! That is, one can define a smooth manifold to be a locally Euclidean n–dimensional second countable topological space X, together with a sheaf F, such that there exists an open cover $\{U_i\}$ of X where:

> For every i, there exists a homeomorphism $f_i \colon U_i \to \mathbb{R}^n$ and an isomorphism of sheaves $\phi_i \colon \mathcal{D}_{\mathbb{R}^n} \to F|_{U_i}$ relative to f_i.

The idea here is that not only does every smooth manifold X have a sheaf $\mathcal{D}_X$ of smooth functions, but specifying this sheaf of smooth functions is sufficient to fully describe the smooth manifold structure on X. While this phenomenon may seem little more than a toy curiousity for differential geometry, it arises in full force in the field of algebraic geometry where the coordinate functions are often unwieldy and algebraic structures in many cases can only be satisfactorily described by way of sheaves and schemes.

Example 4. Similarly, for a complex analytic manifold X, one can form the sheaf $\mathcal{H}_X$ of holomorphic functions by setting $\mathcal{H}_X(U)$ equal to the complex vector space of $\mathbb{C}$–valued holomorphic functions on U, with the restriction morphism being restriction of functions as before.

Example 5. The algebraic geometry analogue of the sheaf $\mathcal{D}_X$ of differential geometry is the prime spectrum $\operatorname{Spec}(R)$ of a commutative ring R. However, the construction of the sheaf $\operatorname{Spec}(R)$ is beyond the scope of this discussion and merits a separate article.

Example 6. For an example of a presheaf that is not a sheaf, consider the presheaf F on X, with values in the category of real vector spaces, whose sections on U are locally constant real–valued functions on U modulo constant functions on U. Then every section $f \in F(U)$ is locally zero in some fine enough open cover $\{U_i\}$ (it is enough to take a cover where each U_i is connected), whereas f may be nonzero if U is not connected.

We conclude with some interesting examples of morphisms of sheaves, chosen to illustrate the unifying power of the language of schemes across various diverse branches of mathematics.

1. For any continuous function $f\colon X \to Y$, the map $\phi_U\colon \mathcal{C}_Y(U) \to \mathcal{C}_X(f^{-1}(U))$ given by $\phi_U(g) := gf$ defines a morphisms of sheaves from $\mathcal{C}_Y$ to $\mathcal{C}_X$ with respect to f.

2. For any continuous function $f\colon X \to Y$ of smooth differentiable manifolds, the map given by $\phi_U(g) := gf$ has the property

$$g \in \mathcal{D}_Y(U) \implies \phi_U(g) \in \mathcal{D}_X(f^{-1}(U))$$

   if and only if f is a smooth function.

3. For any continuous function $f\colon X \to Y$ of complex analytic manifolds, the map given by $\phi_U(g) := gf$ has the property

$$g \in \mathcal{H}_Y(U) \implies \phi_U(g) \in \mathcal{H}_X(f^{-1}(U))$$

   if and only if f is a holomorphic function.

4. For any Zariski continuous function $f\colon X \to Y$ of algebraic varieties over a field k, the map given by $\phi_U(g) := gf$ has the property

$$g \in \mathcal{O}_Y(U) \implies \phi_U(g) \in \mathcal{O}_X(f^{-1}(U))$$

if and only if f is a regular function. Here $\mathcal{O}_X$ denotes the sheaf of k–valued regular functions on the algebraic variety X.

References

[1] David Mumford, *The Red Book of Varieties and Schemes, Second Expanded Edition*, Springer–Verlag, 1999 (LNM **1358**).

[2] Charles Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, 1994.