Baire category theorem

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In a non-empty complete metric space, any countable intersection of dense, open subsets is non-empty.

In fact, such countable intersections of dense, open subsets are dense. So the theorem holds also for any non-empty open subset of a complete metric space.

Alternative formulations: Call a set *first category*, or a *meagre* set, if it is a countable union of nowhere dense sets, otherwise *second category*. The Baire category theorem is often stated as "no non-empty complete metric space is of first category", or, trivially, as "a non-empty, complete metric space is of second category". In short, this theorem says that every nonempty complete metric space is a Baire space.

In functional analysis, this important property of complete metric spaces forms the basis for the proofs of the important principles of Banach spaces: the open mapping theorem and the closed graph theorem.

It may also be taken as giving a concept of "small sets", similar to sets of measure zero: a countable union of these sets remains "small". However, the real line $\mathbb{R}$ may be partitioned into a set of measure zero and a set of first category; the two concepts are distinct.

Note that, apart from the requirement that the set be a complete metric space, all conditions and conclusions of the theorem are phrased topologically. This "metric requirement" is thus something of a disappointment. As it turns out, there are two ways to reduce this requirement.

First, if a topological space $\mathcal{T}$ is homeomorphic to a non-empty open subset of a complete metric space, then we can transfer the Baire property through the homeomorphism, so in $\mathcal{T}$ too any countable intersection of open dense sets is non-empty (and, in fact, dense). The other formulations also hold in this case.

Second, the Baire category theorem holds for a locally compact, Hausdorff[1] topological space $\mathcal{T}$.

[1]Some authors only define a locally compact space to be a Hausdorff space; that is the sense required for this theorem.