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C-embedding

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Let X be a topological space, and $C(X)$ the ring of continuous functions on X. A subspace $A \subseteq X$ is said to be C-*embedded* (in X) if every function in $C(A)$ can be extended to a function in $C(X)$. More precisely, for every real-valued continuous function $f : A \to \mathbb{R}$, there is a real-valued continuous function $g : X \to \mathbb{R}$ such that $g(x) = f(x)$ for all $x \in A$.

If $A \subseteq X$ is C-embedded, $f \mapsto g$ (defined above) is an embedding of $C(A)$ into $C(X)$ by axiom of choice, and hence the nomenclature.

Similarly, one may define C^*-*embedding* on subspaces of a topological space. Recall that for a topological space X, $C^*(X)$ is the ring of bounded continuous functions on X. A subspace $A \subseteq X$ is said to be C^*-*embedded* (in X) if every $f \in C^*(A)$ can be extended to some $g \in C^*(X)$.

Remarks. Let A be a subspace of X.

1. If A is C-embedded in X, and $A \subseteq Y \subseteq X$, then A is C-embedded in Y. This is also true for C^*-embeddedness.

2. If A is C-embedded, then A is C^*-embedded: for if f is a bounded continuous function on A, say $-n \leq f \leq n$, and g is its continuous extension on X, then $-n \vee (g \wedge n)$ is a bounded continuous extension of f on X.

3. The converse, however, is not true in general. A necessary and sufficient condition that a C^*-embedded set A is C-embedded is:

   if a zero set is disjoint from A, the it is completely separated from A.

   Since any pair of disjoint zero sets are completely separated, we have that if A is a C^*-embedded zero set, then A is C-embedded.

References

[1] L. Gillman, M. Jerison: *Rings of Continuous Functions*, Van Nostrand, (1960).