proof of spaces homeomorphic to Baire space

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We show that a topological space X is homeomorphic to Baire space, $\mathcal{N}$, if and only if the following are satisfied.

1. It is a nonempty Polish space.
2. It is zero dimensional.
3. No nonempty and open subsets are compact.

As Baire space is easily shown to satisfy these properties, we just need to show that if they are satisfied then there exists a homeomorphism $f\colon \mathcal{N} \to X$. By property **??** there is a complete metric d on X.

We choose subsets $C(n_1, \ldots, n_k)$ of X for integers $k \geq 0$ and $n_1, \ldots, n_k$ satisfying the following.

(i) $C(n_1, \ldots, n_k)$ is a nonempty clopen set with diameter no more than 2^{-k}.

(ii) $C() = X$.

(iii) For any $n_1, \ldots, n_k$ then $C(n_1, \ldots, n_k, m)$ are pairwise disjoint as m ranges over the natural numbers and,

$$\bigcup_{m=1}^{\infty} C(n_1, \ldots, n_k, m) = C(n_1, \ldots, n_k). \tag{1}$$

This can be done inductively. Suppose that $S = C(n_1, \ldots, n_k)$ has already been chosen. As it is open, condition **??** says that it is not compact. Therefore, there is a $\delta > 0$ such that S has no finite open cover consisting of sets of diameter no more than δ (see `http://planetmath.org/ProofThatAMetricSpaceIsCompactIfAndO`
However, as Polish spaces are separable, there is a countable sequence $S_1, S_2, \ldots$ of open sets with diameter less than δ and covering S. As the space is zero dimensional, these can be taken to be clopen. By replacing S_j by $S_j \cap S$ we can assume that $S_j \subseteq S$. Then, replacing by $S_j \setminus \bigcup_{i<j} S_i$, the sets S_j can be taken to be pairwise disjoint.

By eliminating empty sets we suppose that $S_j \neq \emptyset$ for each j, and since S has no finite open cover consisting of sets of diameter less than δ, the sequence S_j will still be infinite. Defining

$$C(n_1, \ldots, n_k, n_{k+1}) = S_{n_{k+1}}$$

satisfies the required properties.

We now define a function $f\colon \mathcal{N} \to X$ such that $f(n) \in C(n_1, \ldots, n_k)$ for each $n \in \mathcal{N}$ and $k \geq 0$. Choose any $n \in \mathcal{N}$ there is a sequence $x_k \in C(n_1, \ldots, n_k)$. This set has diameter bounded by 2^{-k} and, so, $d(x_k, x_j) \leq 2^{-k}$ for $j \geq k$. This sequence is `http://planetmath.org/CauchySequence`Cauchy and, by completeness of the metric, must converge to a limit x. As $C(n_1, \ldots, n_k)$ is closed, it contains x for each k and therefore

$$\bigcap_k C(n_1, \ldots, n_k) \neq \emptyset.$$

In fact, as it has zero diameter, this set must contain a single element, which we define to be $f(n)$.

So, we have defined a function $f\colon \mathcal{N} \to X$. If $m, n \in \mathcal{N}$ satisfy $m_j = n_j$ for $j \leq k$ then $f(m), f(n)$ are both contained in $C(m_1, \ldots, m_k)$ and $d(f(m), f(n)) \leq 2^{-k}$. Therefore, f is continuous.

It only remains to show that f has continuous inverse. Given any $x \in X$ then $x \in C()$ and equation (**??**) allows us to choose a sequence $n_k \in \mathbb{N}$ such that $x \in C(n_1, \ldots, n_k)$ for each k. Then, $f(n) = x$ showing that f is onto.

If $m \neq n \in \mathcal{N}$ then, letting k be the first integer for which $m_k \neq n_k$, the sets $C(m_1, \ldots, m_k)$ and $C(n_1, \ldots, n_k)$ are disjoint and, therefore, $f(m) \neq f(n)$ and f is one to one.

Finally, we show that f is an open map, so that its inverse is continuous. Sets of the form

$$\mathcal{N}(n_1, \ldots, n_k) = \{m \in \mathcal{N} \colon m_j = n_j \text{ for } j \leq k\}$$

form a basis for the topology on $\mathcal{N}$. Then, $f\left(\mathcal{N}(n_1, \ldots, n_k)\right) = C(n_1, \ldots, n_k)$ is open and, therefore, f is an open map.