partial ordering in a topological space

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Let X be a topological space. For any $x, y \in X$, we define a binary relation $\leq$ on X as follows:

$$x \leq y \qquad \text{iff} \qquad x \in \overline{\{y\}}.$$

Proposition 1. *$\leq$ is a preorder.*

Proof. Clearly $x \leq x$. Next, suppose $x \leq y$ and $y \leq z$. Let C be a closed set containing z. Since y is in the closure of $\{z\}$, $y \in C$. Since x is in the closure of $\{y\}$, $x \in C$ also. So $x \leq z$. $\square$

We call $\leq$ the *specialization preorder* on X. If $x \leq y$, then x is called a *specialization point* of y, and y a *generization point* of x. For any set $A \subseteq X$,

- the set of all specialization points of points of A is called the *specialization* of A, and is denoted by $\mathrm{Sp}(A)$;
- the set of all generization points of points of A is called the *generization* of A, and is denoted by $\mathrm{Gen}(A)$.

Proposition 2. *. If X is `http://planetmath.org/T0`T_0, then $\leq$ is a partial order.*

Proof. Suppose next that $x \leq y$ and $y \leq x$. If $x \neq y$, then there is an open set A such that $x \in A$ and $y \notin A$. So $y \in A^c$, the complement of A, which is a closed set. But then $x \in A^c$ since it is in the closure of $\{y\}$. So $x \in A \cap A^c = \varnothing$, a contradition. Thus $x = y$. $\square$

This turns a T_0 topological space into a poset, where $\leq$ here is called the *specialization order* of the space.

Given a T_0 space, we have the following:

Proposition 3. *$x \leq y$ iff $x \in U$ implies $y \in U$ for any open set U in X.*

Proof. $(\Rightarrow)$: if $x \in U$ and $y \notin U$, then $y \in U^c$. Since $x \leq y$, we have $x \in U^c$, a contradiction. $(\Leftarrow)$: if $x \notin \overline{\{y\}}$, then for some closed set C, we have $y \in C$ and $x \notin C$. But then $x \in C^c$, so that $y \in C^c$, a contradiction. $\square$

Remarks.

- $\overline{\{x\}} = \downarrow x$, the lower set of x. ($z \in \downarrow x$ iff $z \leq x$ iff $z \in \overline{\{x\}}$).
- But if X is `http://planetmath.org/T1`T_1, then the partial ordering just defined is trivial (the diagonal set), since every point is a closed point (for verification, just modify the antisymmetry portion of the above proof).