global versus local continuity

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In this entry, we establish a very basic fact about continuity:

Proposition 1. *A function $f : X \to Y$ between two topological spaces is continuous iff it is continuous at every point $x \in X$.*

Proof. Suppose first that f is continuous, and $x \in X$. Let $f(x) \in V$ be an open set in Y. We want to find an open set $x \in U$ in X such that $f(U) \subseteq V$. Well, let $U = f^{-1}(V)$. So U is open since f is continuous, and $x \in U$. Furthermore, $f(U) = f(f^{-1}(V)) = V$.

On the other hand, if f is not continuous at $x \in X$. Then there is an open set $f(x) \in V$ in Y such that no open sets $x \in U$ in X have the property

$$f(U) \subseteq V. \tag{1}$$

Let $W = f^{-1}(V)$. If W is open, then W has the property (1) above, a contradiction. Since W is not open, f is not continuous. □