proof that every filter is contained in an ultrafilter

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Let Y be the set of all non-empty subsets of X which are not contained in $\mathcal{F}$. By Zermelo's well-orderding theorem, there exists a relation '$\succ$' which well-orders Y. Define $Y' = \{0\} \cup Y$ and extend the relation '$\succ$' to Y' by decreeing that $0 \prec y$ for all $y \in Y$.

We shall construct a family of filters S_i indexed by Y' using transfinite induction. First, set $S_0 = \mathcal{F}$. Next, suppose that, for some $j \in Y$, S_i has already been defined when $i \prec j$. Consider the set $\bigcup_{i \prec j} S_i$; if A and B are elements of this set, there must exist an $i \prec j$ such that $A \in S_i$ and $B \in S_i$; hence, $A \cap B$ cannot be empty. If, for some $i \prec j$ there exists an element $f \in S_i$ such that $f \cap j$ is empty, let S_j be the filter generated by the filter subbasis $\bigcup_{i \prec j} S_i$. Otherwise $\{j\} \cup \bigcup_{i \prec j} S_i$ is a filter subbasis; let S_j be the filter it generates.

Note that, by this definition, whenever $i \prec j$, it follows that $S_i \subseteq S_j$; in particular, for all $i \in Y'$ we have $\mathcal{F} \subseteq S_i$. Let $\mathcal{U} = \bigcup_{i \prec j} S_i$. It is clear that $\emptyset \notin \mathcal{F}$ and that $\mathcal{F} \subseteq \mathcal{U}$.

It is easy to see that $\mathcal{U}$ is a filter. Suppose that $A \cap B \in \mathcal{U}$. Then there must exist an $i \in Y'$ such that $A \cap B \in S_i$. Since S_i is a filter, $A \in S_i$ and $B \in S_i$, hence $A \in \mathcal{U}$ and $B \in \mathcal{U}$. Conversely, if $A \in \mathcal{U}$ and $B \in \mathcal{U}$, then there exists an $i \in Y'$ such that $A \in S_i$ and $B \in S_i$. Since S_i is a filter, $A \cap B \in S_i$, hence $A \cap B \in \mathcal{U}$. By the alternative characterization of a filter, $\mathcal{U}$ is a filter.

Moreover, $\mathcal{U}$ is an ultrafilter. Suppose that $A \in \mathcal{U}$ and $B \in \mathcal{U}$ are disjoint and $A \cup B = X$. If either $A \in \mathcal{F}$ or $B \in \mathcal{F}$, then either $A \in \mathcal{U}$ or $B \in \mathcal{U}$ because $\mathcal{F} \subset \mathcal{U}$. If $A \in Y$ and $A \in S_A$, then $A \in \mathcal{U}$ because $S_A \subset \mathcal{U}$. If $A \in Y$ and $A \notin S_A$, there must exist $x \in \mathcal{U}$ such that $A \cap x$ is empty. Because B is the complement of A, this means that $x \subset B$ and, hence $B \in \mathcal{U}$.

This completes the proof that $\mathcal{U}$ is an ultrafilter — we have shown that $\mathcal{U}$ meets the criteria given in the alternative characterization of ultrafilters.