ordinal space

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Let α be an ordinal. The set $W(\alpha) := \{\beta \mid \beta < \alpha\}$ ordered by $\leq$ is a well-ordered set. $W(\alpha)$ becomes a topological space if we equip $W(\alpha)$ with the interval topology. An *ordinal space* X is a topological space such that $X = W(\alpha)$ (with the interval topology) for some ordinal α. In this entry, we will always assume that $W(\alpha) \neq \varnothing$, or $0 < \alpha$.

Before examining some basic topological structures of $W(\alpha)$, let us look at some of its order structures.

1. First, it is easy to see that $W(\alpha) = \uparrow y \cup W(y)$, for any $y \in W(\alpha)$. Here, $\uparrow y$ is the upper set of y.

2. Another way of saying that $W(\alpha)$ is well-ordered is that for any non-empyt subset S of $W(\alpha)$, $\bigwedge S$ exists. Clearly, $0 \in W(\alpha)$ is its least element. If in addition $1 < \alpha$, $W(\alpha)$ is also atomic, with 1 as the sole atom.

3. Next, $W(\alpha)$ is bounded complete. If $S \subseteq W(\alpha)$ is bounded from above by $a \in W(\alpha)$, then $b = \bigvee S$ is an ordinal such that $b \leq a < \alpha$, therefore $b \in W(\alpha)$ as well.

4. Finally, we note that $W(\alpha)$ is a complete lattice iff α is not a limit ordinal. If $W(\alpha)$ is complete, then $z = \bigvee W(\alpha) \in W(\alpha)$. So $z < \alpha$. This means that $z + 1 \leq \alpha$. If $z + 1 < \alpha$, then $z + 1 \in W(\alpha)$ so that $z + 1 \leq \bigvee W(\alpha) = z$, a contradiction. As a result, $z + 1 = \alpha$. On the other hand, if $\alpha = z + 1$, then $z = \bigvee W(\alpha) \in W(\alpha)$, so that $W(\alpha)$ is complete.

In any ordinal space $W(\alpha)$ where $0 < \alpha$, a typical open interval may be written (x, y), where $0 \leq x \leq y < \alpha$. If y is not a limit ordinal, we can also write $(x, y) = [x + 1, z]$ where $z + 1 = y$. This means that (x, y) is a clopen set if y is not a limit ordinal. In particular, if y is not a limit ordinal, then $\{y\} = (z, y + 1)$ is clopen, where $z + 1 = y$, so that y is an isolated point. For example, any finite ordinal is an isolated point in $W(\alpha)$.

Conversely, an isolated point can not be a limit ordinal. If y is isolated, then $\{y\}$ is open. Write $\{y\}$ as the union of open intervals (a_i, b_i). So $a_i < y < b_i$. Since $y + 1$ covers y, each b_i must be $y + 1$ or (a_i, b_i) would contain more than a point. If y is a limit ordinal, then $a_i < a_i + 1 < y$ so that, again, (a_i, b_i) would contain more than just y. Therefore, y can not be a limit ordinal and all a_i must be the same. Therefore $(a_i, b_i) = (z, y + 1)$, where z is the predecessor of y: $z + 1 = y$.

Several basic properties of an ordinal space are:

1. Isolated points in $W(\alpha)$ are exactly those points that are limit ordinals (just a summary of the last two paragraphs).

2. $W(y)$ is open in $W(\alpha)$ for any $y \in W(\alpha)$. $W(y)$ is closed iff y is not a limit ordinal.

3. For any $y \in W(\alpha)$, the collection of intervals of the form $(a, y]$ (where $a < y$) forms a neighborhood base of y.

4. $W(\alpha)$ is a normal space for any α;

5. $W(\alpha)$ is compact iff α is not a limit ordinal.

Some interesting ordinal spaces are

- $W(\omega)$, which is homeomorphic to the set of natural numbers $\mathbb{N}$.

- $W(\omega_1)$, where ω_1 is the first uncountable ordinal. $W(\omega_1)$ is often written Ω_0. Ω_0 is not a compact space.

- $W(\omega_1 + 1)$, or Ω. Ω is compact, and, in fact, a one-point compactification of Ω_0.

References

[1] S. Willard, *General Topology*, Addison-Wesley, Publishing Company, 1970.