planetmath.org

Math for the people, by the people.

metric space

Canonical name	MetricSpace
Date of creation	2013-03-22 11:53:19
Last modified on	2013-03-22 11:53:19
Owner	djao (24)
Last modified by	djao (24)
Numerical id	15
Author	djao (24)
Entry type	Definition
Classification	msc 54E35
Classification	msc 82-00
Classification	msc 83-00
Classification	msc 81-00
Related topic	Neighborhood
Related topic	VectorNorm
Related topic	T2Space
Related topic	Ultrametric
Related topic	QuasimetricSpace
Related topic	NormedVectorSpace
Related topic	PseudometricSpace
Defines	distance metric
Defines	metric
Defines	distance
Defines	metric topology
Defines	open ball
Defines	closed ball

A *metric space* is a set X together with a real valued function $d : X \times X \longrightarrow \mathbb{R}$ (called a *metric*, or sometimes a *distance* function) such that, for every $x, y, z \in X$,

- $d(x, y) \geq 0$, with equality[1] if and only if $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

For $x \in X$ and $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, the *open ball* around x of radius ε is the set $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$. An *open set* in X is a set which equals an arbitrary (possibly empty) union of open balls in X, and X together with these open sets forms a Hausdorff topological space. The topology on X formed by these open sets is called the *metric topology*, and in fact the open sets form a basis for this topology (`http://planetmath.org/PseudometricTopology`proof).

Similarly, the set $\bar{B}_\varepsilon(x) := \{y \in X \mid d(x, y) \leq \varepsilon\}$ is called a *closed ball* around x of radius ε. Every closed ball is a closed subset of X in the metric topology.

The prototype example of a metric space is $\mathbb{R}$ itself, with the metric defined by $d(x, y) := |x - y|$. More generally, any normed vector space has an underlying metric space structure; when the vector space is finite dimensional, the resulting metric space is isomorphic to Euclidean space.

References

[1] J.L. Kelley, *General Topology*, D. van Nostrand Company, Inc., 1955.

[1] This condition can be replaced with the weaker statement $d(x, y) = 0 \iff x = y$ without affecting the definition.