separated uniform space

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Let X be a uniform space with uniformity $\mathcal{U}$. X is said to be *separated* or *Hausdorff* if it satisfies the following *separation axiom*:

$$\bigcap \mathcal{U} = \Delta,$$

where Δ is the diagonal relation on X and $\bigcap \mathcal{U}$ is the intersection of all elements (entourages) in $\mathcal{U}$. Since $\Delta \subseteq \bigcap \mathcal{U}$, the separation axiom says that the only elements that belong to every entourage of $\mathcal{U}$ are precisely the diagonal elements (x, x). Equivalently, if $x \neq y$, then there is an entourage U such that $(x, y) \notin U$.

The reason for calling X separated has to do with the following assertion:

> X is separated iff X is a Hausdorff space under the topology $T_{\mathcal{U}}$ `http://planetmath.org/TopologyInducedByAUniformStructure`induced by $\mathcal{U}$.

Recall that $T_{\mathcal{U}} = \{A \subseteq X \mid \text{for each } x \in A, \text{ there is } U \in \mathcal{U}, \text{ such that } U[x] \subseteq A\}$, where $U[x]$ is some uniform neighborhood of x where, under $T_{\mathcal{U}}$, $U[x]$ is also a neighborhood of x. To say that X is Hausdorff under $T_{\mathcal{U}}$ is the same as saying every pair of distinct points in X have disjoint uniform neighborhoods.

Proof. ($\Rightarrow$). Suppose X is separated and $x, y \in X$ are distinct. Then $(x, y) \notin U$ for some $U \in \mathcal{U}$. Pick $V \in \mathcal{U}$ with $V \circ V \subseteq U$. Set $W = V \cap V^{-1}$, then W is symmetric and $W \subseteq V$. Furthermore, $W \circ W \subseteq V \circ V \subseteq U$. If $z \in W[x] \cap W[y]$, then $(x, z), (y, z) \in W$. Since W is symmetric, $(z, y) \in W$, so $(x, y) = (x, z) \circ (z, y) \in W \circ W \subseteq U$, which is a contradiction.

($\Leftarrow$). Suppose X is Hausdorff under $T_{\mathcal{U}}$ and $(x, y) \in U$ for every $U \in \mathcal{U}$ for some $x, y \in X$. If $x \neq y$, then there are $V[x] \cap W[y] = \varnothing$ for some $V, W \in \mathcal{U}$. Since $(x, y) \in V$ by assumption, $y \in V[x]$. But $y \in W[y]$, contradicting the disjointness of $V[x]$ and $W[y]$. Therefore $x = y$. $\square$