ε-net

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Definition Suppose X is a metric space with a metric d, and suppose S is a subset of X. Let ε be a positive real number. A subset $N \subset S$ is an *ε-net for S* if, for all $x \in S$, there is an $y \in N$, such that $d(x, y) < \varepsilon$.

For any $\varepsilon > 0$ and $S \subset X$, the set S is trivially an ε-net for itself.

Theorem Suppose X is a metric space with a metric d, and suppose S is a subset of X. Let ε be a positive real number. Then N is an ε-net for S, if and only if

$$\{B_\varepsilon(y) \mid y \in N\}$$

is a cover for S. (Here $B_\varepsilon(x)$ is the open ball with center x and radius ε.)

Proof. Suppose N is an ε-net for S. If $x \in S$, there is an $y \in N$ such that $x \in B_\varepsilon(y)$. Thus, x is covered by some set in $\{B_\varepsilon(x) \mid x \in N\}$. Conversely, suppose $\{B_\varepsilon(y) \mid y \in N\}$ is a cover for S, and suppose $x \in S$. By assumption, there is an $y \in N$, such that $x \in B_\varepsilon(y)$. Hence $d(x, y) < \varepsilon$ with $y \in N$. $\square$

Example In $X = \mathbb{R}^2$ with the usual Cartesian metric, the set

$$N = \{(a, b) \mid a, b \in \mathbb{Z}\}$$

is an ε-net for X assuming that $\varepsilon > \sqrt{2}/2$. $\square$

The above definition and example can be found in [**?**], page 64-65.

References

[1] G. Bachman, L. Narici, *Functional analysis*, Academic Press, 1966.