a compact metric space is second countable

Canonical name	ACompactMetricSpaceIsSecondCountable
Date of creation	2013-03-22 17:00:49
Last modified on	2013-03-22 17:00:49
Owner	azdbacks4234 (14155)
Last modified by	azdbacks4234 (14155)
Numerical id	17
Author	azdbacks4234 (14155)
Entry type	Theorem
Classification	msc 54D70
Related topic	MetricSpace
Related topic	Compact
Related topic	Lindelof
Related topic	Ball
Related topic	basisTopologicalSpace
Related topic	Cover
Related topic	BasisTopologicalSpace

Proposition. *Every compact metric space is second countable.*

Proof. Let (X,d) be a compact metric space, and for each $n \in \mathbb{Z}^+$ define $\mathcal{A}_n = \{B(x,1/n) : x \in X\}$, where $B(x,1/n)$ denotes the open ball centered about x of `http://planetmath.org/node/1296`radius $1/n$. Each such collection is an open cover of the compact space X, so for each $n \in \mathbb{Z}^+$ there exists a finite collection $\mathcal{B}_n \subseteq \mathcal{A}_n$ that X. Put $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$. Being a countable union of finite sets, it follows that $\mathcal{B}$ is countable; we assert that it forms a basis for the metric topology on X. The first property of a basis is satisfied trivially, as each set $\mathcal{B}_n$ is an open cover of X. For the second property, let $x, x_1, x_2 \in X$, $n_1, n_2 \in \mathbb{Z}^+$, and suppose $x \in B(x_1, 1/n_1) \cap B(x_2, 1/n_2)$. Because the sets $B(x_1, 1/n_1)$ and $B(x_2, 1/n_2)$ are open in the metric topology on X, their intersection is also open, so there exists $\epsilon > 0$ such that $B(x,\epsilon) \subseteq B(x_1, 1/n_1) \cap B(x_2, 1/n_2)$. Select $N \in \mathbb{Z}^+$ such that $1/N < \epsilon$. There must exist $x_3 \in X$ such that $x \in B(x_3, 1/2N)$ (since $\mathcal{B}_{2N}$ is an open cover of X). To see that $B(x_3, 1/2N) \subseteq B(x_1, 1/n_1) \cap B(x_2, 1/n_2)$, let $y \in B(x_3, 1/2N)$. Then we have

$$d(x,y) \leq d(x,x_3) + d(x_3,y) < \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N} < \epsilon, \tag{1}$$

so that $y \in B(x,\epsilon)$, from which it follows that $y \in B(x_1, 1/n_1) \cap B(x_2, 1/n_2)$, hence that $B(x_3, 1/2N) \subseteq B(x_1, 1/n_1) \cap B(x_2, 1/n_2)$. Thus the countable collection $\mathcal{B}$ forms a basis for a topology on X; the verification that the topology by $\mathcal{B}$ is in fact the metric topology follows by an to that used to verify the second property of a basis, and completes the proof that X is second countable. □

It is worth nothing that, because a countable union of countable sets is countable, it would have been sufficient to assume that (X,d) was a Lindelöf space.