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continuous image of a compact set is compact

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Theorem 1. *The continuous image of a compact set is also compact.*

Proof. Let X and Y be topological spaces, $f\colon X \to Y$ be continuous, A be a compact subset of X, I be an indexing set, and $\{V_\alpha\}_{\alpha\in I}$ be an open cover of $f(A)$. Thus, $f(A) \subseteq \bigcup_{\alpha\in I} V_\alpha$. Therefore, $A \subseteq f^{-1}\Big(f(A)\Big) \subseteq f^{-1}\left(\bigcup_{\alpha\in I} V_\alpha\right) = \bigcup_{\alpha\in I} f^{-1}(V_\alpha)$.

Since f is continuous, each $f^{-1}(V_\alpha)$ is an open subset of X. Since $A \subseteq \bigcup_{\alpha\in I} f^{-1}(V_\alpha)$ and A is compact, there exists $n \in \mathbb{N}$ with $A \subseteq \bigcup_{j=1}^{n} f^{-1}\left(V_{\alpha_j}\right)$ for some $\alpha_1, \ldots, \alpha_n \in I$. Hence, $f(A) \subseteq f\left(\bigcup_{j=1}^{n} f^{-1}(V_{\alpha_j})\right) = f\left(f^{-1}\left(\bigcup_{j=1}^{n} V_{\alpha_j}\right)\right) \subseteq \bigcup_{j=1}^{n} V_{\alpha_j}$. It follows that $f(A)$ is compact. $\square$