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The property that compact sets in a space are closed lies strictly between T1 and T2

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If a topological space is Hausdorff (T_2), then every compact subset of that space is closed. If every compact subset of a space is closed, then (since singletons are always compact) then the space is accessible (T_1). There are spaces that are T_1 and have compact sets that are not closed, and there are spaces in which compact sets are always closed but that are not T_2.

Let X be an infinite set with the finite complement topology. Singletons are finite, and therefore closed, so X is T_1. Let $S \subset X$, and let $\mathbb{F}$ be an open cover of S. Let $F \in \mathbb{F}$. Then $X \setminus F$ is finite. Choosing a member of $\mathbb{F}$ for each remaining element of S shows that $\mathbb{F}$ has a finite subcover. Thus, every subset of X is compact. An infinite subset of X will then be compact, but not closed.

Let Y be an uncountable set with the countable complement topology. No two open sets are disjoint, so Y is not Hausdorff. Let C be a compact subset of Y. I shall show that C is finite. Suppose C is infinite, and let S be an infinite sequence in C without any repetitions. For any natural number n, let U_n be all the elements of C except for all the S_k, where $k > n$. Then U_n is open for each n, and $\{U_n \mid n \in \mathbb{N}\}$ covers C, but has no finite subset that covers C, contradicting the fact that C is compact. This contradiction arose by assuming a compact subset of Y was infinite, all compact subsets of Y are finite. Y is T_1 (singleton sets are countable), so all compact subsets of Y are closed.

These examples were suggested by the person known as Polytope on EFNet's math channel.