closed subsets of a compact set are compact

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Theorem 1. *Suppose X is a topological space. If K is a compact subset of X, C is a closed set in X, and $C \subseteq K$, then C is a compact set in X.*

The below proof follows `http://planetmath.org/Eg`e.g. [**?**]. A proof based on the finite intersection property is given in [**?**].

Proof. Let I be an indexing set and $F = \{V_\alpha \mid \alpha \in I\}$ be an arbitrary open cover for C. Since $X \setminus C$ is open, it follows that F together with $X \setminus C$ is an open cover for K. Thus, K can be covered by a finite number of sets, say, $V_1, \ldots, V_N$ from F together with possibly $X \setminus C$. Since $C \subset K$, $V_1, \ldots, V_N$ cover C, and it follows that C is compact. □

The following proof uses the `http://planetmath.org/ASpaceIsCompactIfAndOnlyIfTheSpace` intersection property.

Proof. Let I be an indexing set and $\{A_\alpha\}_{\alpha \in I}$ be a collection of X-closed sets contained in C such that, for any finite $J \subseteq I$, $\bigcap_{\alpha \in J} A_\alpha$ is not empty. Recall that, for every $\alpha \in I$, $A_\alpha \subseteq C \subseteq K$. Thus, for every $\alpha \in I$, $A_\alpha = K \cap A_\alpha$. Therefore, $\{A_\alpha\}_{\alpha \in I}$ are K-closed subsets of K (see `http://planetmath.org/ClosedSetInASubspa` page) such that, for any finite $J \subseteq I$, $\bigcap_{\alpha \in J} A_\alpha$ is not empty. As K is compact, $\bigcap_{\alpha \in I} A_\alpha$ is not empty (again, by `http://planetmath.org/ASpaceIsCompactIfAndOnlyIfTheSpaceH` result). This proves the claim. □

References

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