another proof of Dini's theorem

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This is the version of the **Dini's theorem** I will prove: Let K be a compact metric space and $(f_n)_{n\in N} \subset C(K)$ which converges pointwise to $f \in C(K)$.

Besides, $f_n(x) \geq f_{n+1}(x) \quad \forall x \in K, \forall n$.

Then $(f_n)_{n\in N}$ converges uniformly in K.

Proof

Suppose that the sequence does not converge uniformly. Then, by definition,

$$\exists \varepsilon > 0 \text{ such that } \forall m \in N\ \exists\, n_m > m,\ x_m \in K \text{ such that } \quad |f_{n_m}(x_m) - f(x_m)| \geq \varepsilon.$$

So,

$$\begin{array}{l} \text{For } \ m = 1\ \exists\, n_1 > 1\ ,\ x_1 \in K \text{ such that } \ |f_{n_1}(x_1) - f(x_1)| \geq \varepsilon \\ \exists\, n_2 > n_1\ ,\ x_2 \in K \text{ such that } \ |f_{n_2}(x_2) - f(x_2)| \geq \varepsilon \\ \vdots \\ \exists\, n_m > n_{m-1}\ ,\ x_m \in K \text{ such that } \ |f_{n_m}(x_m) - f(x_m)| \geq \varepsilon \end{array}$$

Then we have a sequence $(x_m)_m \subset K$ and $(f_{n_m})_m \subset (f_n)_n$ is a subsequence of the original sequence of functions. K is compact, so there is a subsequence of $(x_m)_m$ which converges in K, that is, $\left(x_{m_j}\right)_j$ such that

$$x_{m_j} \longrightarrow x \in K$$

I will prove that f is not continuous in x (A contradiction with one of the hypothesis).

To do this, I will show that $f\left(x_{m_j}\right)_j$ does not converge to $f(x)$, using above's ε.

Let j_0 such that $j \geq j_0 \Rightarrow \left|f_{n_{m_j}}(x) - f(x)\right| < {}^{\varepsilon}\!/_4$, which exists due to the punctual convergence of the sequence. Then, particularly, $\left|f_{n_{m_{j_o}}}(x) - f(x)\right| < {}^{\varepsilon}\!/_4$.

Note that

$$\left|f_{n_{m_j}}(x_{m_j}) - f(x_{m_j})\right| = f_{n_{m_j}}(x_{m_j}) - f(x_{m_j})$$

because (using the hypothesis $f_n(y) \geq f_{n+1}(y) \quad \forall y \in K, \forall n$) it's easy to see that

$$f_n(y) \geq f(y) \quad \forall y \in K, \forall n$$

Then, $f_{n_{m_j}}(x_{m_j}) - f(x_{m_j}) \geq \varepsilon \; \forall j$. And also the hypothesis implies

$$f_{n_{m_j}}(y) \geq f_{n_{m_{j+1}}}(y) \;\; \forall y \in K, \forall j$$

So, $j \geq j_0 \Rightarrow f_{n_{m_{j_0}}}(x_{m_j}) \geq f_{n_{m_j}}(x_{m_j})$, which implies

$$\left| f_{n_{m_{j_0}}}(x_{m_j}) - f(x_{m_j}) \right| \geq \varepsilon$$

Now,

$$\left| f_{n_{m_{j_0}}}(x_{m_j}) - f(x) \right| + \left| f(x_{m_j}) - f(x) \right| \geq \left| f_{n_{m_{j_0}}}(x_{m_j}) - f(x_{m_j}) \right| \geq \varepsilon \;\; \forall j \geq j_0$$

and so

$$\left| f(x_{m_j}) - f(x) \right| \geq \varepsilon - \left| f_{n_{m_{j_0}}}(x_{m_j}) - f(x) \right| \;\; \forall j \geq j_0.$$

On the other hand,

$$\left| f_{n_{m_{j_0}}}(x_{m_j}) - f(x) \right| \leq \left| f_{n_{m_{j_0}}}(x_{m_j}) - f_{n_{m_{j_0}}}(x) \right| + \left| f_{n_{m_{j_0}}}(x) - f(x) \right|$$

And as $f_{n_{m_{j_0}}}$ is continuous, there is a j_1 such that

$$j \geq j_1 \Rightarrow \left| f_{n_{m_{j_0}}}(x_{m_j}) - f_{n_{m_{j_0}}}(x) \right| < \varepsilon/4$$

Then,

$$j \geq j_1 \Rightarrow \left| f_{n_{m_{j_0}}}(x_{m_j}) - f(x) \right| \leq \left| f_{n_{m_{j_0}}}(x_{m_j}) - f_{n_{m_{j_0}}}(x) \right| + \left| f_{n_{m_{j_0}}}(x) - f(x) \right| < \varepsilon/2,$$

which implies

$$\left| f(x_{m_j}) - f(x) \right| \geq \varepsilon - \left| f_{n_{m_{j_0}}}(x_{m_j}) - f(x) \right| \geq \varepsilon/2 \;\; \forall j \geq \max(j_0, j_1).$$

Then, particularly, $f(x_{m_j})_j$ does not converge to $f(x)$. QED.