section filter

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Let X be a set and $(x_i)_{i\in D}$ a non-empty net in X. For each $j \in D$, define $S(j) := \{x_i \mid i \le j\}$. Then the set

$$S := \{S(j) \mid j \in D\}$$

is a filter basis: S is non-empty because $(x_i) \neq \varnothing$, and for any $j, k \in D$, there is a ℓ such that $j \le \ell$ and $k \le \ell$, so that $S(\ell) \subseteq S(j) \cap S(k)$.

Let $\mathcal{A}$ be the family of all filters containing S. $\mathcal{A}$ is non-empty since the filter generated by S is in $\mathcal{A}$. Order $\mathcal{A}$ by inclusion so that $\mathcal{A}$ is a poset. Any chain $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ has an upper bound, namely,

$$\mathcal{F} := \bigcup_{i=1}^{\infty} \mathcal{F}_i.$$

By Zorn's lemma, $\mathcal{A}$ has a maximal element $\mathcal{X}$.

Definition. $\mathcal{X}$ defined above is called the *section filter* of the net (x_i) in X.

Remark. A section filter is obviously a filter. The name "section" comes from the elements $S(j)$ of S, which are sometimes known as "sections" of the net (x_i).