topology induced by uniform structure

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Let $\mathcal{U}$ be a uniform structure on a set X. We define a subset A to be open if and only if for each $x \in A$ there exists an entourage $U \in \mathcal{U}$ such that whenever $(x, y) \in U$, then $y \in A$.

Let us verify that this defines a topology on X.

Clearly, the subsets $\emptyset$ and X are open. If A and B are two open sets, then for each $x \in A \cap B$, there exist an entourage U such that, whenever $(x, y) \in U$, then $y \in A$, and an entourage V such that, whenever $(x, y) \in V$, then $y \in B$. Consider the entourage $U \cap V$: whenever $(x, y) \in U \cap V$, then $y \in A \cap B$, hence $A \cap B$ is open.

Suppose $\mathcal{F}$ is an arbitrary family of open subsets. For each $x \in \bigcup \mathcal{F}$, there exists $A \in \mathcal{F}$ such that $x \in A$. Let U be the entourage whose existence is granted by the definition of open set. We have that whenever $(x, y) \in U$, then $y \in A$; hence $y \in \bigcup \mathcal{F}$, which concludes the proof.