proof of Urysohn's lemma

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First we construct a family U_p of open sets of X indexed by the rationals such that if $p < q$, then $\bar{U_p} \subseteq U_q$. These are the sets we will use to define our continuous function.

Let $P = \mathbb{Q} \cap [0, 1]$. Since P is countable, we can use induction (or recursive definition if you prefer) to define the sets U_p. List the elements of P is an infinite sequence in some way; let us assume that 1 and 0 are the first two elements of this sequence. Now, define $U_1 = X \backslash D$ (the complement of D in X). Since C is a closed set of X contained in U_1, by normality of X we can choose an open set U_0 such that $C \subseteq U_0$ and $\bar{U_0} \subseteq U_1$.

In general, let P_n denote the set consisting of the first n rationals in our sequence. Suppose that U_p is defined for all $p \in P_n$ and

$$\text{if } p < q, \text{then } \bar{U_p} \subseteq U_q. \tag{1}$$

Let r be the next rational number in the sequence. Consider $P_{n+1} = P_n \cup \{r\}$. It is a finite subset of $[0, 1]$ so it inherits the usual ordering $<$ of $\mathbb{R}$. In such a set, every element (other than the smallest or largest) has an immediate predecessor and successor. We know that 0 is the smallest element and 1 the largest of P_{n+1} so r cannot be either of these. Thus r has an immediate predecessor p and an immediate successor q in P_{n+1}. The sets U_p and U_q are already defined by the inductive hypothesis so using the normality of X, there exists an open set U_r of X such that

$$\bar{U_p} \subseteq U_r \text{ and } \bar{U_r} \subseteq U_q.$$

We now show that (1) holds for every pair of elements in P_{n+1}. If both elements are in P_n, then (1) is true by the inductive hypothesis. If one is r and the other $s \in P_n$, then if $s \leq p$ we have

$$\bar{U_s} \subseteq \bar{U_p} \subseteq U_r$$

and if $s \geq q$ we have

$$\bar{U_r} \subseteq U_q \subseteq U_s.$$

Thus (1) holds for ever pair of elements in P_{n+1} and therefore by induction, U_p is defined for all $p \in P$.

We have defined U_p for all rationals in $[0, 1]$. Extend this definition to every rational $p \in \mathbb{R}$ by defining

$$\begin{array}{ll} U_p = \emptyset & \text{if } p < 0 \\ U_p = X & \text{if } p > 1. \end{array}$$

Then it is easy to check that (1) still holds.

Now, given $x \in X$, define $\mathbb{Q}(x) = \{p : x \in U_p\}$. This set contains no number less than 0 and contains every number greater than 1 by the definition of U_p for $p < 0$ and $p > 1$. Thus $\mathbb{Q}(x)$ is bounded below and its infimum is an element in $[0,1]$. Define

$$f(x) = \inf \mathbb{Q}(x).$$

Finally we show that this function f we have defined satisfies the conditions of lemma. If $x \in C$, then $x \in U_p$ for all $p \geq 0$ so $\mathbb{Q}(x)$ equals the set of all nonnegative rationals and $f(x) = 0$. If $x \in D$, then $x \notin U_p$ for $p \leq 1$ so $\mathbb{Q}(x)$ equals all the rationals greater than 1 and $f(x) = 1$.

To show that f is continuous, we first prove two smaller results:

(a) $x \in \bar{U}_r \Rightarrow f(x) \leq r$

Proof. If $x \in \bar{U}_r$, then $x \in U_s$ for all $s > r$ so $\mathbb{Q}(x)$ contains all rationals greater than r. Thus $f(x) \leq r$ by definition of f.

(b) $x \notin U_r \Rightarrow f(x) \geq r$.

Proof. If $x \notin U_r$, then $x \notin U_s$ for all $s < r$ so $\mathbb{Q}(x)$ contains no rational less than r. Thus $f(x) \geq r$.

Let $x_0 \in X$ and let (c, d) be an open interval of $\mathbb{R}$ containing $f(x)$. We will find a neighborhood U of x_0 such that $f(U) \subseteq (c, d)$. Choose $p, q \in \mathbb{Q}$ such that

$$c < p < f(x_0) < q < d.$$

Let $U = U_q \backslash \bar{U}_p$. Then since $f(x_0) < q$, (b) implies that $x \in U_q$ and since $f(x_0) > p$, (a) implies that $x_0 \notin \bar{U}_p$. Hence $x_0 \in U$.

Finally, let $x \in U$. Then $x \in U_q \subseteq \bar{U}_q$, so $f(x) \leq q$ by (a). Also, $x \notin \bar{U}_p$ so $x \notin U_p$ and $f(x) \geq p$ by (b). Thus

$$f(x) \in [p, q] \subseteq (c, d)$$

as desired. Therefore f is continuous and we are done.