the union of a locally finite collection of closed sets is closed

Canonical name	TheUnionOfALocallyFiniteCollectionOfClosedSetsIsClosed
Date of creation	2013-03-22 16:14:45
Last modified on	2013-03-22 16:14:45
Owner	yark (2760)
Last modified by	yark (2760)
Numerical id	10
Author	yark (2760)
Entry type	Theorem
Classification	msc 54A99

The union of a collection of closed subsets of a topological space need not, of course, be closed. However, we do have the following result:

Theorem. *The union of a locally finite collection of closed subsets of a topological space is itself closed.*

Proof. Let $\mathcal{S}$ be a locally finite collection of closed subsets of a topological space X, and put $Y = \cup\mathcal{S}$. Let $x \in X \setminus Y$. By local finiteness there is an open neighbourhood U of x that meets only finitely many members of $\mathcal{S}$, say $A_1, \ldots, A_n$. So $U \setminus Y = U \setminus \bigcup_{i=1}^{n} A_i$, which is open. Thus $U \setminus Y$ is an open neighbourhood of x that does not meet Y. It follows that Y is closed. $\square$

One use for this result can be found in the entry on gluing together continuous functions.