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topology via converging nets

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Given a topological space X, one can define the concept of convergence of a sequence, and more generally, the convergence of a net. Conversely, given a set X, a class of nets, and a suitable definition of "convergence" of a net, we can topologize X. The procedure is done as follows:

> Let C be the class of all pairs of the form (x,y) where x is a net in X and y is an element of X. For any subset U of X with $y \in U$, we say that a net x *converges* to y *with respect to* U if x is eventually in U. We denote this by $x \to_U y$. Let
>
> $$\mathcal{T} := \{U \subseteq X \mid (x,y) \in C \text{ and } y \in U \text{ imply } x \to_U y\}.$$
>
> Then $\mathcal{T}$ is a topology on X.
>
> *Proof.* Clearly $x \to_X y$ for any pair $(x,y) \in C$. In addition, $x \to_\varnothing y$ is vacuously true. For any $U, V \in \mathcal{T}$, we want to show that $W := U \cap V \in \mathcal{T}$. Since x is eventually in U and V, there are $i, j \in D$ (where D is the domain of x), such that $x_r \in U$ and $x_s \in V$ for all $r \geq i$ and $s \geq j$. Since D is directed, there is a $k \in D$ such that $k \geq i$ and $k \geq j$. It is clear that $x_k \in W$ and that any $t \geq k$ we have that $x_t \in W$ as well. Next, if U_α are sets in $\mathcal{T}$, we want to show their union $U := \bigcup\{U_\alpha\}$ is also in $\mathcal{T}$. If y is a point in U then y is a point in some U_α. Since $(x,y) \in C$ with x is eventually in U_α, we have that x is eventually in U as well. □

Remark. The above can be generalized. In fact, if the class of pairs (x,y) satisfies some "axioms" that are commonly found as properties of convergence, then X can be topologized. Specifically, let X be a set and C again be the class of all pairs (x,y) as described above. A subclass $\mathcal{C}$ of C is called a *convergence class* if the following conditions are satisfied

1. x is a constant net with value $y \in X$, then $(x,y) \in \mathcal{C}$
2. $(x,y) \in \mathcal{C}$ implies $(z,y) \in \mathcal{C}$ for any subnet z of x
3. if every subnet z of a net x has a subnet t with $(t,y) \in \mathcal{C}$, then $(x,y) \in \mathcal{C}$
4. suppose $(x,y) \in \mathcal{C}$ with $D = \operatorname{dom}(x)$, and for each $i \in D$, we have that $(z_i, x_i) \in \mathcal{C}$, with $D_i = \operatorname{dom}(z_i)$. Then $(z,x) \in \mathcal{C}$, where z is the net whose domain is $D \times F$ with $F := \prod\{D_i \mid i \in D\}$, given by $z(i,f) = (i, f(i))$.

If $(x, y) \in \mathcal{C}$, we write $x \to y$ or $\lim_D x = y$. The last condition can then be visualized as

$$
\begin{array}{ccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\
\cdots & z_{ia} & \vdots & z_{jf} & \vdots & z_{kp} & \cdots & & \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & & \\
\cdots & z_{ib} & \vdots & z_{jg} & \vdots & z_{kq} & \cdots & & \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & & \\
\cdots & z_{ic} & \vdots & z_{jh} & \vdots & z_{kr} & \cdots & & \\
\cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & & \\
\cdots & \downarrow & \vdots & \downarrow & \vdots & \downarrow & \cdots & & \\
\cdots & x_i & \cdots & x_j & \cdots & x_k & \cdots & \to & y
\end{array}
\quad \Rightarrow \quad
\begin{array}{cccccccc}
\ddots & & & & & & & \\
 & z_{if(i)} & & & & & & \\
 & & \ddots & & & & & \\
 & & & z_{jf(j)} & & & & \\
 & & & & \ddots & & & \\
 & & & & & z_{kf(k)} & & \\
 & & & & & & \ddots & \\
 & & & & & & & \searrow \\
 & & & & & & & & y,
\end{array}
$$

which is reminiscent of Cantor's diagonal argument.

Now, for any subset A of X, we define A^c to be the subset of X consisting of all points $y \in X$ such that there is a net x in A with $x \to y$. It can be shown that ^{c} is a closure operator, which induces a topology $\mathcal{T}_{\mathcal{C}}$ on X. Furthermore, under this induced topology, the notion of converging nets (as defined by the topology) is exactly the same as the notion of convergence described by the convergence class $\mathcal{C}$.

In addition, it may be shown that there is a one-to-one correspondence between the topologies and the convergence classes on the set X. The correspondence is order reversing in the sense that if $\mathcal{C}_1 \subseteq \mathcal{C}_2$ as convergent classes, then $\mathcal{T}_{\mathcal{C}_2} \subseteq \mathcal{T}_{\mathcal{C}_1}$ as topologies.