order topology

Canonical name	OrderTopology
Date of creation	2013-03-22 12:10:34
Last modified on	2013-03-22 12:10:34
Owner	rspuzio (6075)
Last modified by	rspuzio (6075)
Numerical id	10
Author	rspuzio (6075)
Entry type	Definition
Classification	msc 54B99
Classification	msc 06F30
Synonym	induced order topology
Related topic	OrderedSpace
Related topic	LinearContinuum
Related topic	ProofOfGeneralizedIntermediateValueTheorem
Related topic	ASpaceIsConnectedUnderTheOrderedTopologyIfAndOnlyIfItIsALinearContin

Let $(X, \leq)$ be a linearly ordered set. The *order topology* on X is defined to be the topology $\mathcal{T}$ generated by the subbasis consisting of open rays, that is sets of the form

$$(x, \infty) = \{y \in X | y > x\}$$

$$(-\infty, x) = \{y \in X | y < x\},$$

for some $x \in X$.

This is equivalent to saying that $\mathcal{T}$ is generated by the basis of open intervals; that is, the open rays as defined above, together with sets of the form

$$(x, y) = \{z \in X | x < z < y\}$$

for some $x, y \in X$.

The standard topologies on $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{N}$ are the same as the order topologies on these sets.

If Y is a subset of X, then Y is a linearly ordered set under the induced order from X. Therefore, Y has an order topology $\mathcal{S}$ defined by this ordering, the *induced order topology*. Moreover, Y has a subspace topology $\mathcal{T}'$ which it inherits as a subspace of the topological space X. The subspace topology is always finer than the induced order topology, but they are not in general the same.

For example, consider the subset $Y = \{-1\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{Q}$. Under the subspace topology, the singleton set $\{-1\}$ is open in Y, but under the order topology on Y, any open set containing -1 must contain all but finitely many members of the space.

A chain X under the order topology is Hausdorff: pick any two distinct points $x, y \in X$; without loss of generality, say $x < y$. If there is a z such that $x < z < y$, then $(-\infty, z)$ and (z, ∞) are disjoint open sets separating x and y. If no z were between x and y, then $(-\infty, y)$ and (x, ∞) are disjoint open sets separating x and y.