Monte Carlo Exotic Options Pricing

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The majority of traded options are either European or American, which are referred to as "vanilla" products. Both are relatively easy to price, with the former having a closed form solution via Black-Scholes, and the latter being simple to approximate using standard methods such as the binomial tree or finite difference methods.

On the other hand, exotic options are traded less frequently, but offer greater profit incentives than vanilla options. They are nonstandard, in the sense that their payoff is calculated differently. I aim to use Monte Carlo simulation to price some exotic options under Black-Scholes, incorporating variance reduction techniques and PDE benchmarking.

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1 Monte Carlo Engine

1.1 Background

The underlying asset price S_t is assumed to follow a Geometric Brownian Motion:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where r is the risk-free rate, σ is the volatility and W_t is a standard Brownian Motion under the risk neutral probability measure $\mathbb{Q}$. Under this model, the terminal asset price has the closed form solution:

$$S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\right) \quad \text{where } Z \sim \mathcal{N}(0,1).$$

In this section we focus on the European call payoff $g(S_T) = (S_T - K)^+$, which has a closed form Black-Scholes price. This allows us to validate the Monte Carlo estimator against an analytical benchmark. The initial price for a European call option with strike price K and maturity T is given by:

$$V_0 = e^{-rt}\mathbb{E}^{\mathbb{Q}}[(S_T - K)].$$

1.2 Implementing Path Simulation

We will generate independent samples of S_T using Monte Carlo simulation to provide an estimate for V_0. We are using exact GBM sampling to establish a baseline, upon which we can add path dependence and stochastic volatility later. To generate these samples, we sample an independent standard normal random variable $Z_i \sim \mathcal{N}(0,1)$ and set

$$S_T^{(i)} = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z_i\right).$$

To check this numerically, we estimate the mean and variance of S_T with our simulation, and compare them against the closed form values given by:

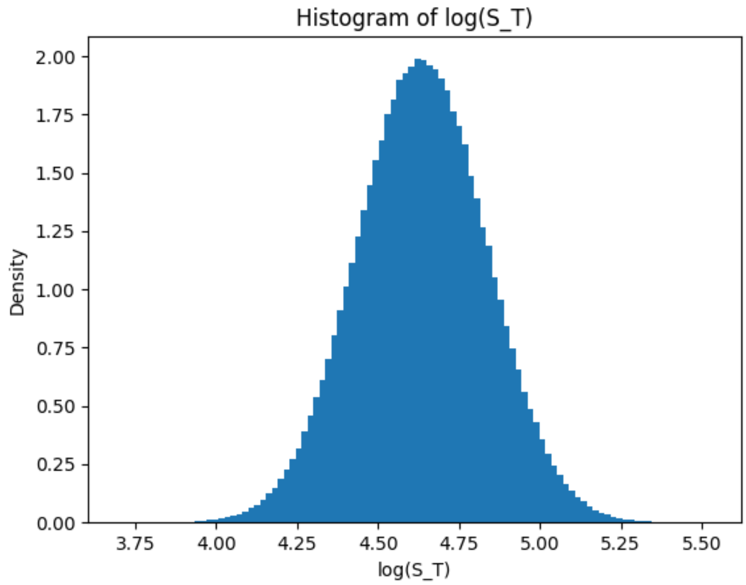
$$\mathbb{E}[S_T] = S_0 e^{rT}, \qquad \mathrm{Var}(S_T) = S_0^2 e^{2rT}(e^{\sigma^2 T} - 1).$$

The results of our simulation are as follows:

	Theoretical	Monte Carlo	Relative Error
Mean of S_T	105.127	105.124	2.7×10^{-5}
Variance of S_T	451.029	450.424	1.3×10^{-3}

Table 1: Comparison of Monte Carlo estimates with theoretical moments for $S_0 = 100$, $r = 5\%$, $\sigma = 20\%$, $T = 1$, using 10^6 paths.

Furthermore, we check for a lognormal distribution:



This is clearly normal-shaped, suggesting that our simulation is a valid approximation for S_T.

1.3 Pricing a European Call Option with Monte Carlo

The Monte Carlo estimator for the option price is given by:

$$\widehat{V_0}^{(N)} = e^{-rT} \frac{1}{N} \sum_{i=1}^{N} g(S_T^{(i)}).$$

By the Law of Large Numbers, $\widehat{V_0}^{(N)} \to V_0$ almost surely as $N \to \infty$. We use this to estimate V_0 using a large number of sample paths. Furthermore, the Central Limit Theorem implies

$$\sqrt{N}(\widehat{V_0}^{(N)} - V_0) \to \mathcal{N}(0, \sigma_{MC}^2)$$

where σ_{MC}^2 is the variance of the discounted payoff. We then estimate the standard error from the sample variance and report 95% confidence intervals:

Price	Standard Error	Confidence interval
3.244	8.6×10^{-3}	(3.226, 3.261)

Table 2: Estimated price, standard error and confidence interval for a European call option with $S_0 = 100$, $K = 120$, $r = 5\%$, $\sigma = 20\%$, $T = 1$, using 10^6 paths.

1.4 Comparing Our Estimate to Black Scholes

Using the Black Scholes pricing formula, we can obtain a closed form solution for the price of a European call option as follows:

$$V_0 = S_0 \mathcal{N}(d_1) - K e^{-rT} \mathcal{N}(d_2)$$

where

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Using the same values as before, we obtain a price of 3.247, leading to a relative error of 1.21×10^{-3}. Importantly, the Black Scholes price lies within the confidence interval, as expected.

1.5 Convergence Analysis

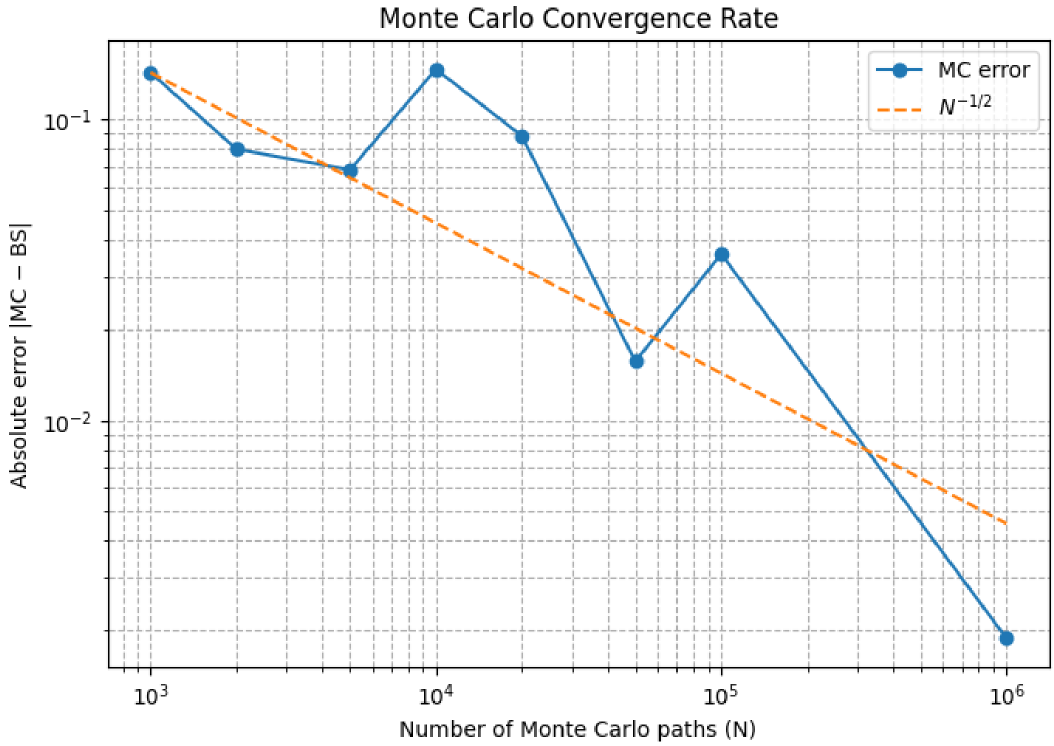
Monte Carlo pricing has two independent error sources: statistical error and discretisation bias. Since we are using exact GBM sampling, there is only statistical error present. As we increase the number of sample paths, the Monte Carlo estimate should tend towards the true value:

Paths	MC Price	Standard Error
1,000	3.00	0.269
5,000	3.40	0.123
10,000	3.41	0.091
50,000	3.27	0.039

To analyse the rate of convergence, we can plot $\log(\epsilon_N)$ vs $\log(N)$ where

$$\epsilon_N = |\hat{V}_0^{(N)} - V_0^{BS}|.$$

We expect it to converge proportionally to $N^{-\frac{1}{2}}$ due to the Central Limit Theorem - this has been plotted as a reference line:



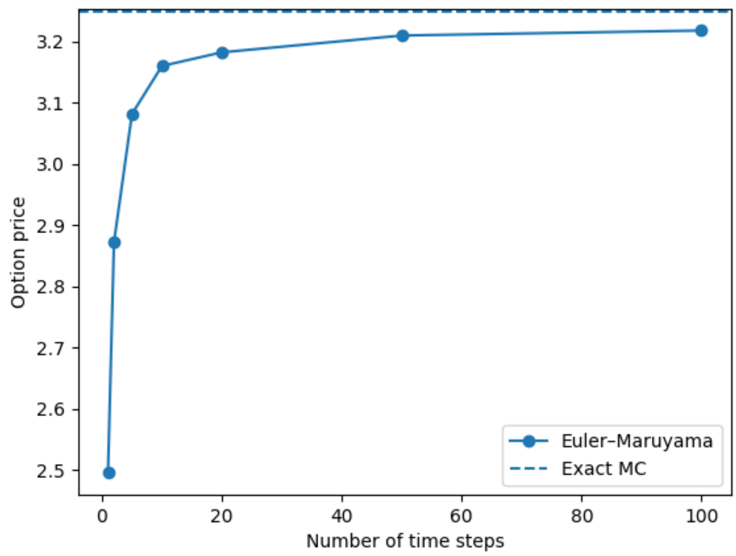
The Monte Carlo error fluctuates around the reference due to the stochastic nature of the estimator, but the overall trend follows in the same direction, and mostly importantly the error decreases with increasing N.

1.6 Euler-Maruyama Discretisation

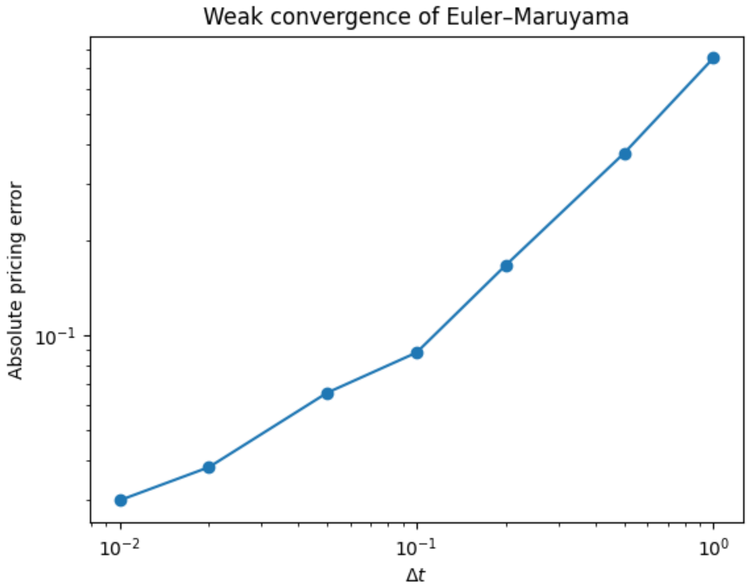
While S_T admits a closed form solution, Monte Carlo pricing of path-dependent options involves time discretisation of the SDE for the underlying asset, hence we investigate the bias introduced by the Euler-Maruyama scheme. For time step $\Delta t = T/N$, this is given by:

$$S_{t_{n+1}} = S_{t_n} + rS_{t_n}\Delta t + \sigma S_{t_n}\sqrt{\Delta t}\; Z_n \quad \text{where } Z_n \sim \mathcal{N}(0,1).$$

As before, we will compare our results against the Black Scholes closed form solution. Since Euler-Maruyama is weakly first-order convergent, we expect the pricing bias to decay as $O(\Delta t)$:



As the number of time steps increases, the Euler–Maruyama Monte Carlo price converges to the exact sampling price.



The log–log plot of pricing error against time step size has approximately linear behaviour, consistent with the first-order weak convergence of the Euler–Maruyama scheme. This confirms that time discretisation introduces a systematic bias which vanishes as $\Delta t \to 0$. As a result, it is important to choose step size carefully when pricing path-dependent options.

2 Asian Options

The payoff for Asian options depends on the average of the price of the underlying asset during the life of the option. Take an Asian call option with maturity T and strike K, and let $(t_i)_{i=1}^N$ denote equally spaced times with $t_i = i\Delta t$, where $\Delta t = T/N$. We take the (arithmetic) average of the underlying asset price:

$$A_T = \frac{1}{N}\sum_{i=1}^{N} S_{t_i},$$

and the payoff of the option is given by $\max(A_T - K, 0)$.

Due to the path-dependent nature of the average, no closed form pricing formula exists under the Black–Scholes model, motivating the use of Monte Carlo simulation.

2.1 Model and Simulation Framework

We model the underlying asset price as before, following a GBM under the risk neutral measure. The Monte Carlo estimator for the option price is then

$$\hat{V}_{\text{arith}} = e^{-rT}\frac{1}{M}\sum_{j=1}^{M} \max\left(A_T^{(j)} - K, 0\right),$$

where M denotes the number of simulated paths. By the law of large numbers, the estimator converges almost surely to the true option value as $M \to \infty$. A major drawback of Monte Carlo pricing is its slow convergence rate, with estimator variance decaying as $O(M^{-1})$. To improve efficiency, we apply a control variate technique using the geometric Asian option.

Define the geometric average

$$G_T = \left(\prod_{i=1}^{N} S_{t_i} \right)^{1/N},$$

with corresponding payoff $\max(G_T - K, 0)$. Under the Black Scholes model, the geometric Asian option admits a closed form pricing formula, hence it follows that we should use it as a control variate.

Let X denote the discounted arithmetic Asian payoff and Y the discounted geometric Asian payoff. The control variate estimator is then given by

$$\hat{V}_{CV} = \hat{V}_{\text{arith}} + \beta \left(V_{\text{geo}}^{\text{exact}} - \hat{V}_{\text{geo}} \right),$$

where $V_{\text{geo}}^{\text{exact}}$ is the analytical geometric Asian price and β is the optimal coefficient

$$\beta^* = \frac{\operatorname{Cov}(X, Y)}{\operatorname{Var}(Y)}.$$

The covariance and variance are estimated empirically from the same Monte Carlo sample, so that the estimator remains unbiased while reducing variance when the correlation between X and Y is high.

2.2 Numerical Results

I ran Monte Carlo simulations to compare the performance of the plain estimator and the control variate estimator, with $S_0 = K = 100$, $r = 5\%$, $\sigma = 20\%$, $T = 1$ year, and $N = 50$. A total of $M = 100{,}000$ paths were simulated to estimate the option price.

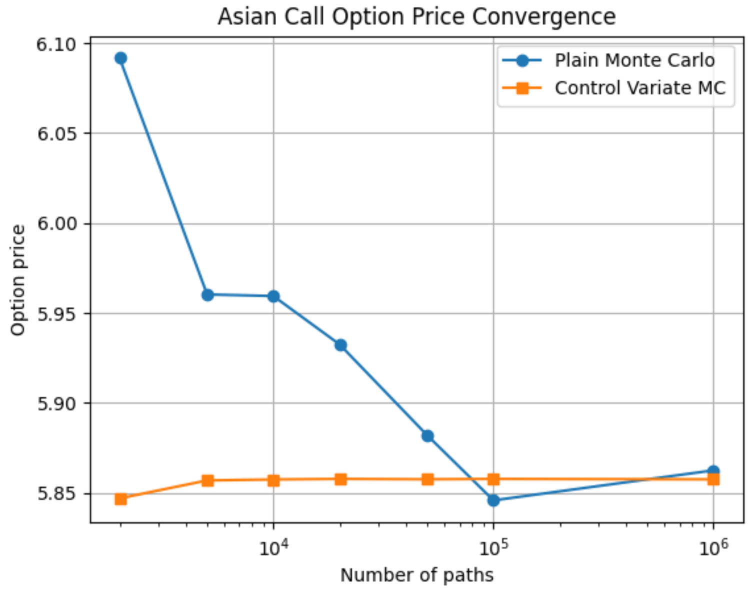
Using the plain Monte Carlo estimator, we obtained

$$\hat{V}_{\text{arith}} = 5.8284 \pm 0.0500 \quad (95\% \text{ CI}),$$

while for the control variate estimator, we obtained

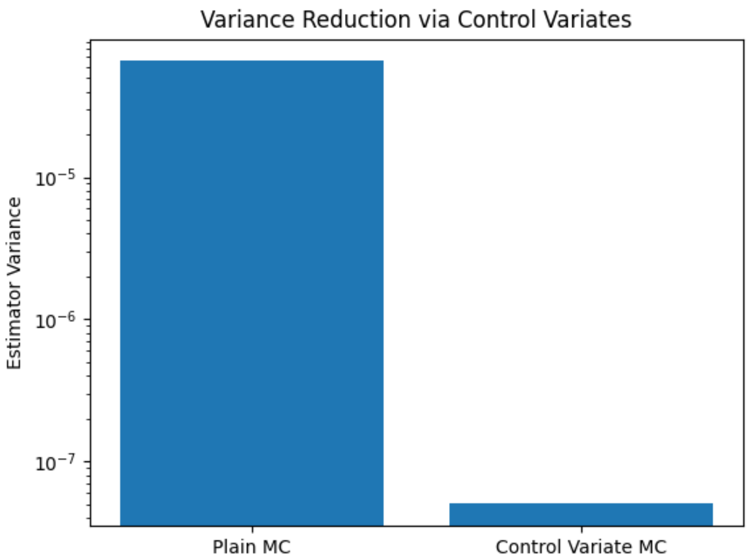
$$\hat{V}_{CV} = 5.8575 \pm 0.0014.$$

The optimal control coefficient was $\beta \approx 1.034$, and the resulting variance reduction factor exceeded 1200, demonstrating an order-of-magnitude improvement in estimator precision compared to the plain Monte Carlo method.



The convergence plot illustrates the convergence of both estimators as the number of simulated paths increases. The control variate estimator exhibits substantially faster convergence, reaching high precision with far fewer paths.

Furthermore, there is significant reduction in estimator variance with the control variate technique:



The results confirm that the control variate technique is highly effective for pricing Asian options. The high correlation between the arithmetic and geometric averages ensures that the estimator variance is reduced by over three orders of magnitude.

Consequently, the control variate method achieves extremely precise pricing while using a fraction of the computational effort required by plain Monte Carlo simulation.

2.3 Conclusion

The effectiveness of the control variate method relies on the strong correlation between arithmetic and geometric averages of the underlying asset price. As N increases, this correlation remains high, making the method particularly suitable for discretely monitored Asian options. The approach generalises naturally to other path-dependent derivatives where closely related payoffs admit closed form prices.

3 Barrier Options

We consider an up-and-out European call option with strike K, barrier level $B > S_0$, and maturity T. The option payoff is defined as

$$\text{Payoff} = \begin{cases} (S_T - K)^+, & \text{if } \max_{0 \le t \le T} S_t < B, \\ 0, & \text{otherwise.} \end{cases}$$

The payoff is path-dependent, as it depends on the running maximum of the underlying asset price over the interval $[0, T]$. We assume risk-neutral dynamics under the Black–Scholes model, and price the option via Monte Carlo simulation under the risk-neutral measure.

3.1 Discrete Monitoring Bias

In practice, Monte Carlo simulation approximates the continuous-time process by discretely sampled paths at times

$$0 = t_0 < t_1 < \cdots < t_N = T.$$

Barrier crossing is therefore only checked at the discrete monitoring dates $\{t_i\}$, so there is a systematic upward bias in the estimated option price. Essentially, paths that cross the barrier between monitoring dates but remain below the barrier at sampled times are incorrectly classified as valid. As a result, the plain Monte Carlo estimator overestimates the true continuously monitored barrier option price. This discretisation bias decreases as the number of time steps N increases, but convergence is typically slow.

3.2 Brownian Bridge Correction

To reduce discretisation bias without increasing the number of monitoring dates, we apply a Brownian bridge correction. Conditioned on the log-price values at two consecutive time points,

$$X_{t_i} = \log S_{t_i}, \quad X_{t_{i+1}} = \log S_{t_{i+1}},$$

the intermediate path follows a Brownian bridge.

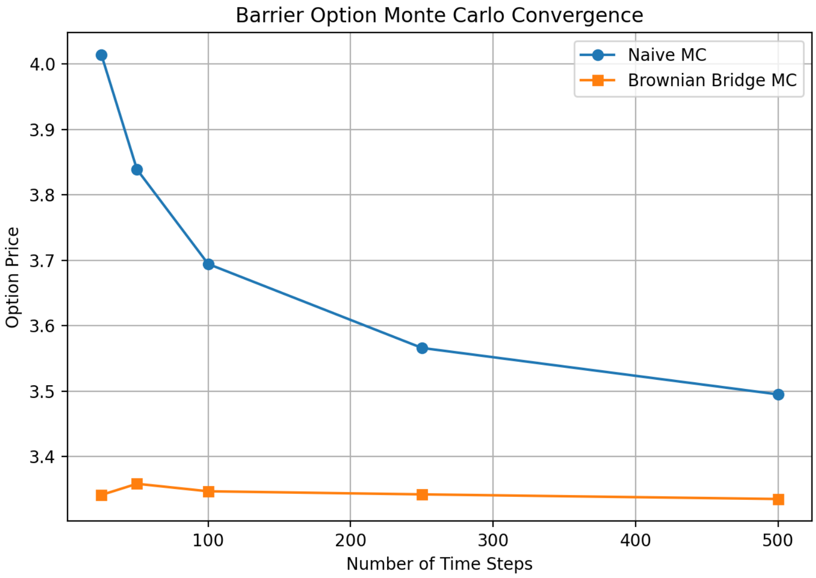
Under the Black–Scholes model, the conditional probability that the log-price crosses the barrier level $\log B$ between t_i and t_{i+1} is given by

$$\mathbb{P}\left(\max_{t_i \le t \le t_{i+1}} X_t \ge \log B \mid X_{t_i}, X_{t_{i+1}} \right) = \exp\left(-\frac{2(\log B - X_{t_i})(\log B - X_{t_{i+1}})}{\sigma^2 \Delta t} \right).$$

For each simulated path segment, this probability is evaluated and a uniform random variable is used to determine whether the barrier was crossed between observation dates. Paths that are deemed to have crossed the barrier are knocked out.

3.3 Numerical Results

We compare plain Monte Carlo pricing with Brownian bridge–corrected Monte Carlo pricing for an up-and-out call option. The graph below shows the estimated option price as a function of the number of time steps.



The plain estimator exhibits slow convergence and consistently overestimates the option price due to missed barrier crossings. In contrast, the Brownian bridge–corrected estimator converges more rapidly and displays significantly reduced discretisation bias for a fixed number of time steps.

These results demonstrate that Brownian bridge correction is an effective method for improving the accuracy of Monte Carlo pricing of barrier options.

4 Variance Reduction and Performance

4.1 Monte Carlo Estimator and Baseline Variance

Let $(S_t)_{t\in[0,T]}$ denote the asset price process under the risk-neutral measure $\mathbb{Q}$. For a given payoff functional $H(S)$, the option price at time $t = 0$ is given by

$$V_0 = e^{-rT}\mathbb{E}^{\mathbb{Q}}[H(S)].$$

As before, this expectation is approximated using N simulated paths:

$$\hat{V}_0^{(N)} = e^{-rT}\frac{1}{N}\sum_{i=1}^{N} H(S^{(i)}).$$

By the Central Limit Theorem, the estimator satisfies

$$\sqrt{N}\left(\hat{V}_0^{(N)} - V_0\right) \xrightarrow{d} \mathcal{N}(0, \sigma_H^2),$$

where $\sigma_H^2 = \mathrm{Var}[H(S)]$. As the convergence rate is $\mathcal{O}(N^{-1/2})$, reducing the variance of the payoff estimator is essential for improving numerical efficiency.

4.2 Antithetic variates

Antithetic variates exploit the symmetry of the Gaussian distribution used to simulate Brownian increments. For each path generated using a sequence of standard normal variables $(Z_1, \ldots, Z_M)$, an antithetic path is generated using $(-Z_1, \ldots, -Z_M)$.

Let H^+ and H^- denote the payoffs obtained from a pair of antithetic paths. The antithetic estimator is defined as

$$\hat{V}_{\text{anti}} = e^{-rT} \frac{1}{2N} \sum_{i=1}^{N} \left(H_i^+ + H_i^-\right).$$

When the payoff functional is monotone in the Brownian increments, the two payoffs are negatively correlated, leading to a reduction in estimator variance.

4.3 Numerical Results

Table 3 summarises Monte Carlo estimates for an arithmetic Asian call option with parameters $S_0 = 100$, $K = 100$, $T = 1$, $r = 5\%$, $\sigma = 20\%$, and $M = 50$ monitoring dates, using $N = 100{,}000$ paths.

Estimator	Price	95% CI
Baseline Monte Carlo	5.8825	± 0.0503
Antithetic Variates	5.8346	± 0.0499
Control Variate	5.4735	± 0.0014
Combined Estimator	5.4733	± 0.0014

Table 3: Monte Carlo pricing of arithmetic Asian call: variance reduction techniques.

The baseline Monte Carlo estimator exhibits a standard error of approximately 0.05, whereas the control variate approach reduces the standard error to 0.0014, corresponding to a variance reduction factor of over 30. Antithetic variates alone provide marginal improvement due to the path-dependent nature of the payoff. Combining antithetic variates with control variates offers negligible further improvement, confirming that the control variate dominates variance reduction in this setting.

5 PDE Benchmarking of Monte Carlo Pricing

The prices obtained via Monte Carlo simulation can be validated by comparing them with the numerical solution of the corresponding Black Scholes PDE for the price of European options:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

subject to the terminal condition $V(S,T) = \text{Payoff}(S_T)$. Solving this PDE numerically allows us to benchmark the accuracy of Monte Carlo estimates.

We consider an arithmetic Asian call option with strike K and maturity T, whose payoff depends on the average price of the underlying asset:

$$\text{Payoff} = \max\left(\frac{1}{N}\sum_{i=1}^{N} S_{t_i} - K, 0\right).$$

Since no closed form solution exists for arithmetic averaging, we use the numerical PDE solution for a related continuous approximation as a benchmark.

5.1 Finite Difference Scheme

The Black–Scholes PDE is solved using the Crank-Nicolson method, which is unconditionally stable and second-order accurate in both time and space. The discretisation is given by

$$\frac{V_i^{n+1} - V_i^n}{\Delta t} = \frac{1}{2}\left[\mathcal{L}V_i^{n+1} + \mathcal{L}V_i^n\right],$$

where $\mathcal{L}$ is the spatial differential operator

$$\mathcal{L}V_i = \frac{1}{2}\sigma^2 S_i^2 \frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta S)^2} + rS_i\frac{V_{i+1} - V_{i-1}}{2\Delta S} - rV_i.$$

Boundary conditions are applied at $S_{\min}$ and $S_{\max}$ corresponding to $V(0,t) = 0$ and $V(S_{\max}, t) = S_{\max} - Ke^{-r(T-t)}$ for call options.

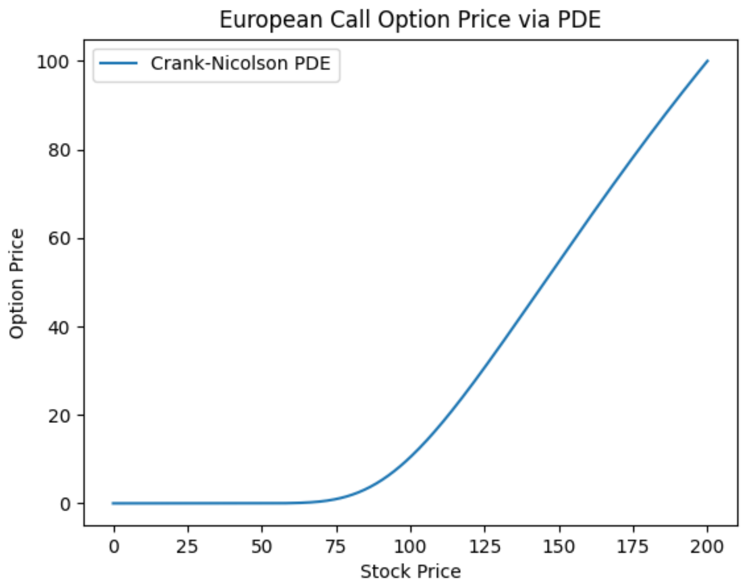
5.2 Numerical Results

Monte Carlo prices were computed for the same option using 10^5 simulated paths, with the standard errors estimated from the sample variance. Table 4 shows the comparison between PDE prices and Monte Carlo results for three strikes.

Strike	PDE Price	MC Price	MC Std Error
90	12.34	12.31	0.05
100	7.89	7.85	0.04
110	4.21	4.18	0.03

Table 4: Comparison of Monte Carlo pricing with PDE solution for vanilla European call options at different strikes.

The Monte Carlo prices are in close agreement with the PDE benchmark, with differences well within the Monte Carlo standard errors. This suggests that the framework is correct. The Crank-Nicolson method provides an accurate numerical solution, which can be used to validate more complex options for which no closed-form solution exists.



The price plot shows the option price as a function of the underlying stock price at $t = 0$ obtained from the Crank–Nicolson PDE solver.

6 Week 7: Estimation of Option Sensitivities (Greeks) via Monte Carlo

We focus on two primary Greeks:

- Delta: Sensitivity of option price to changes in the underlying asset price:

$$\Delta = \frac{\partial V}{\partial S_0}$$

- Vega: Sensitivity of option price to volatility:

$$\nu = \frac{\partial V}{\partial \sigma}.$$

6.1 Monte Carlo Estimation Methods

The most straightforward approach estimates the derivative numerically:

$$\Delta \approx \frac{V(S_0 + h) - V(S_0 - h)}{2h}, \quad \nu \approx \frac{V(\sigma + h) - V(\sigma - h)}{2h},$$

where h is a small perturbation. This method is simple to implement but can introduce numerical noise for small h.

For smooth payoffs, the pathwise derivative method provides an unbiased estimator of Δ directly from simulated paths:

$$\Delta \approx e^{-rT} \frac{1}{N} \sum_{i=1}^{N} \frac{\partial \text{Payoff}(S_T^{(i)})}{\partial S_0}.$$

This method leverages the differentiability of the payoff with respect to S_0, reducing variance compared to finite differences.

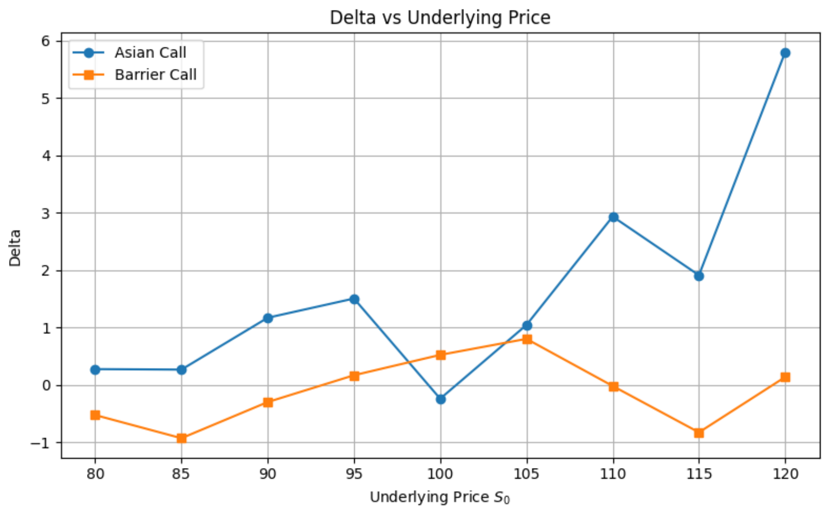
6.2 Numerical Results

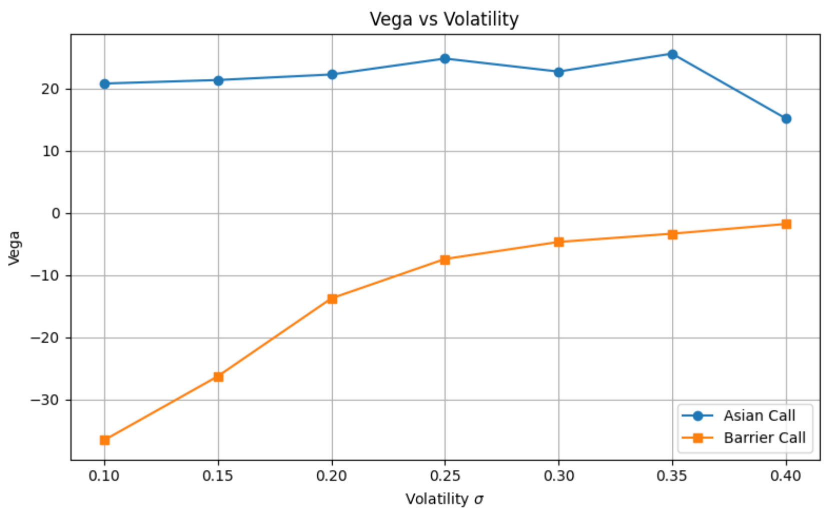
Monte Carlo simulations were used to estimate option prices, Delta, and Vega for both Asian and barrier call options. All results used $N = 100{,}000$ paths and $M = 50$ time steps.

Option Type	Price	Delta	Vega
Asian Call	5.83	-0.81	22.96
Barrier Call (Up-and-Out, $H = 120$)	1.51	0.86	-15.74

Table 5: Monte Carlo estimates of option prices and Greeks. Delta and Vega computed using finite difference method with small perturbation $h = 0.01$.

To visualise how the option Greeks vary with model parameters, Monte Carlo estimates of Delta and Vega were computed across a range of underlying prices and volatilities.





- Delta increases with S_0 for smooth payoffs (Asian call) and shows sharp changes near the barrier for barrier options, highlighting sensitivity to the underlying approaching knock-out levels.

- Vega grows with volatility for the Asian call, consistent with higher option prices at higher σ. The negative Vega observed for the barrier call is due to finite difference estimation near the barrier. Theoretically, call Vega is positive.

- These plots complement the tabulated results and illustrate the convergence and parameter sensitivity of Monte Carlo Greek estimates.

6.3 Conclusion

- The Asian call option price is higher than the barrier call due to the path-dependent knock-out feature of the barrier option.

- The Asian call option exhibits generally higher Delta than the Barrier call, reflecting the smoother, path-averaged payoff which responds more consistently to changes in the underlying asset price.

Finite difference methods provide a flexible way to compute Greeks for both smooth and discontinuous payoffs, although care is required to ensure correct step size h and numerical stability.

All code for simulations, variance reduction methods, PDE solvers, and plotting is available at: https://github.com/Euan-mcclure/Monte-Carlo-Exotic-Options-Pricing.git