Motivation: Parameter Estimation

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Machine Learning

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Machine Learning and Statistics
are **almost the same**

Overview

Probability

Statistical Modelling

Assumed: Probabilities and Densities

I assume that you know about:

- Probability spaces and random variables
- Probability densities, e.g. $P(0.4 \leqslant X \leqslant 0.5) = \int_{0.4}^{0.5} p_X(x)\mathrm{d}x$
- Joint random variables, e.g. $P(X = 3, Y = 4)$
- Joint random densities, e.g.

$$P(0.4 \leqslant X \leqslant 0.5, 0.8 \leqslant Y \leqslant 0.85) = \int_{0.4}^{0.5}\int_{0.8}^{0.85} p_{XY}(x, y)\mathrm{d}y\mathrm{d}x \quad (1)$$

See exercise sheet for recap of notation, exercises, and pointers to revision material.

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Skill: Probabilities and vectors

- For multiple joint random variables $X_1, X_2, X_3 \in \mathbb{R}$ with density

$$p_{X_1,X_2,X_3}(x_1, x_2, x_3)\,, \tag{2}$$

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- This can shorten notation when specifying densities, e.g. for $0 \leqslant x_n \leqslant 1$, we can have

$$p(x_1, x_2, x_3) = \frac{1}{C}(x_1^2 + x_2^2 + x_3^2) = \frac{1}{C}||\mathbf{x}||^2 \tag{3}$$

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- See exercise sheet for some practice.

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Probability

Statistical Modelling

Example: The shaking desk

Figure: xkcd #228

- Why is it shaking?
- How can I stop it?
- How much will it shake?

Statistical View on the World

Data Generating Process

We assume that the data we observe is the outcome of some random process. Each observation is one random variable. In this course, probabilities of the data generating process are denoted with $\mathbb{P}(\cdot)$, and which has distribution $\pi(\cdot)$.

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Example:

- We observe a dataset of 3 values $\{x_n\}_{n=1}^3$.
- This has density $\pi_{X_1,X_2,X_3}(x_1, x_2, x_3)$.

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Examples:

$$\pi_{X_1,X_2,X_3}(x_1, x_2, x_3) = \prod_{n=1}^{3} \pi(x_n)$$

$$\pi_{X_1,Y_1,X_2,Y_2,\ldots}(x_1, y_1, x_2, y_2, \ldots) = \pi_{X,Y}(\mathbf{x}, \mathbf{y}) \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^N$$

$$= \prod_{n=1}^{N} \pi(x_n, y_n)$$

Statistical Model

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A statistical model is a random process used by a statistician (i.e. “us”) to explain data coming from the data generating process. In this course, probabilities of the model are denoted with $P(\cdot)$, which has the distribution $p(\cdot)$.

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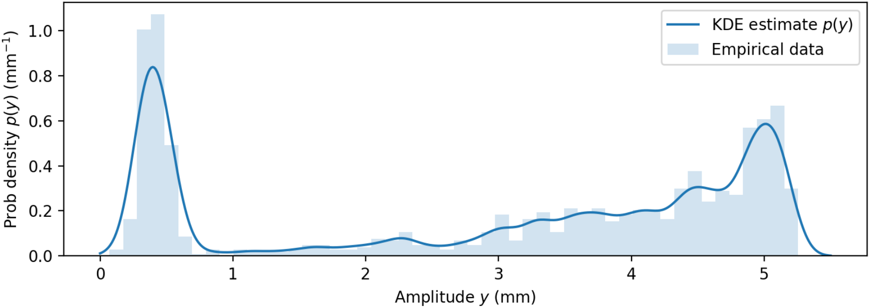
- The data generating process $\pi(\cdot)$ is unknown, and therefore usually different to our statistical model $p(\cdot)$.
- Our goal is to make them similar!
- A model often depends on some parameters $\boldsymbol{\theta}$, denoted as $p(\cdot|\boldsymbol{\theta})$, which are used to adjust the statistical model to fit the data.

The shaking desk: Gathering data

Measure amplitude at various fixed points during the day.



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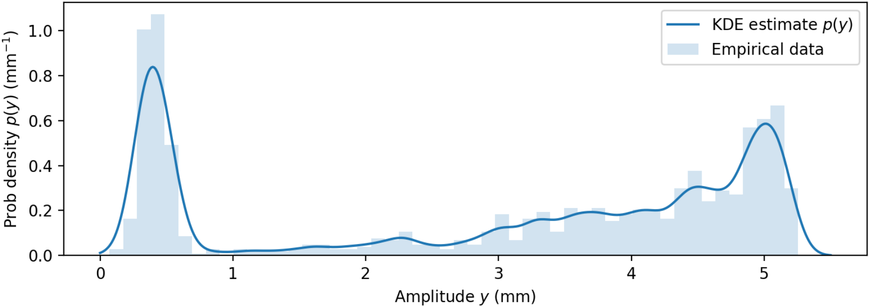

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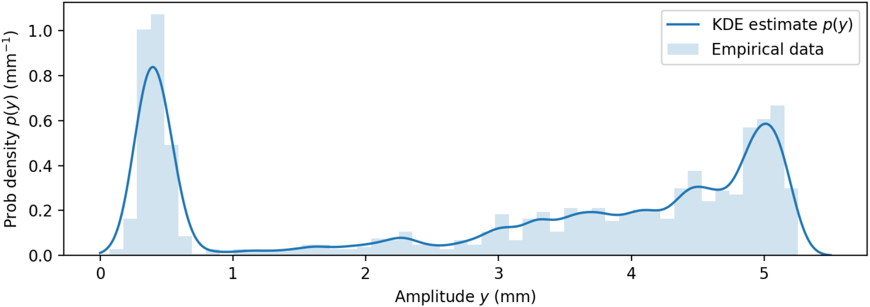

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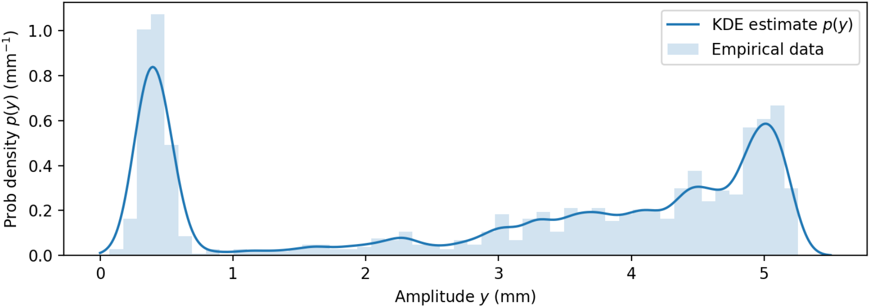

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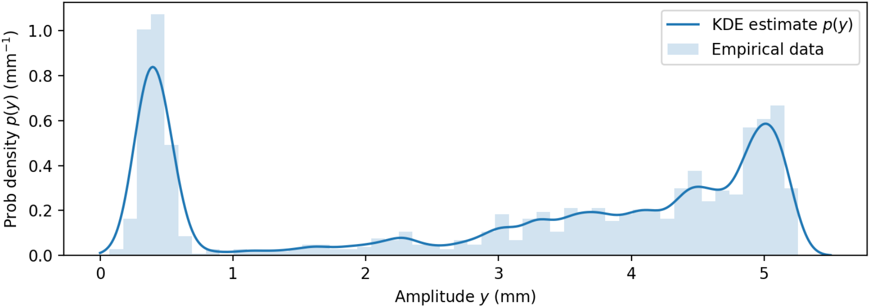

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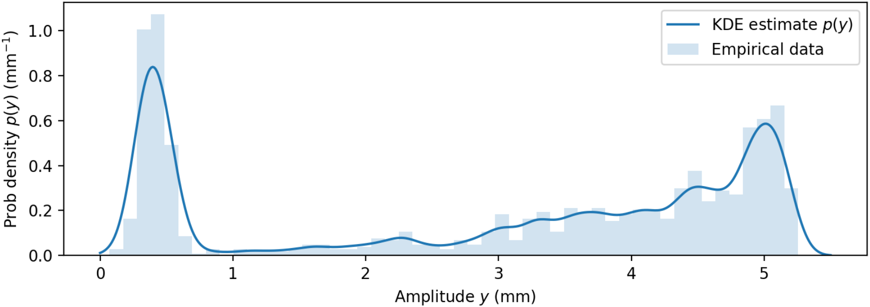

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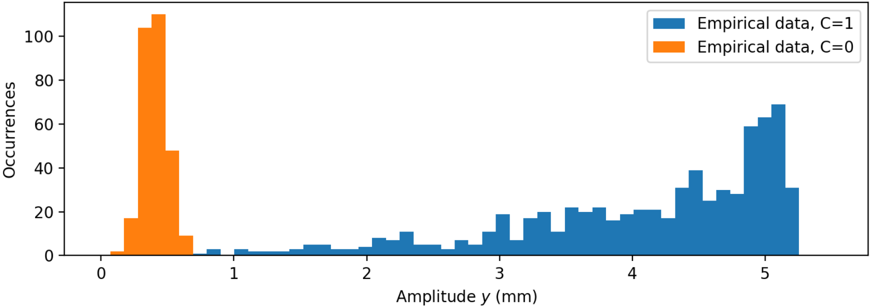
- Can we predict whether the shaking will happen? ⏩ No.
- We can at least predict: Unlikely to be small amplitude.

The shaking desk: Gathering data

More data: Also measure whether your colleague is present (C).



- Now we observe pairs c_n, y_n from a generating process $\pi(c, y)$
- We now find two models $p(y|C = 0, \boldsymbol{\theta})$ and $p(y|C = 1, \boldsymbol{\theta})$
- It seems that $C = 1$ indicates larger shaking
- Given C, we can now predict with more certainty!

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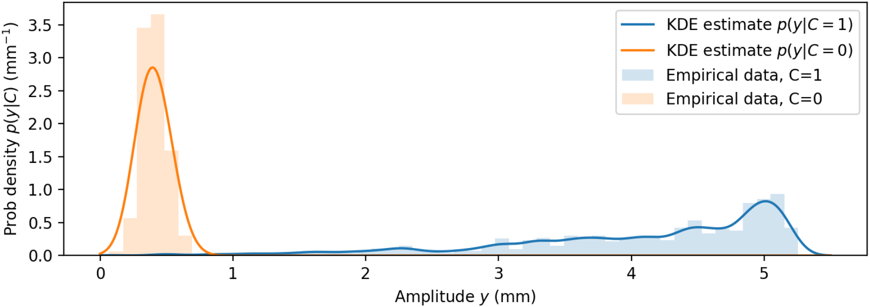

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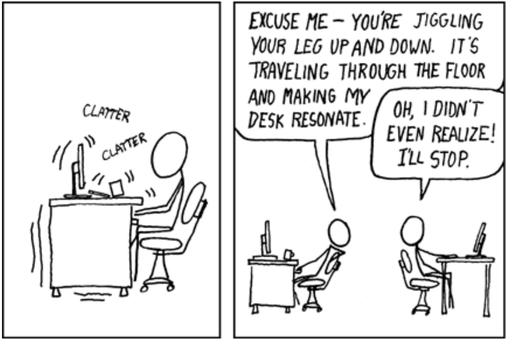

Figure: xkcd #228

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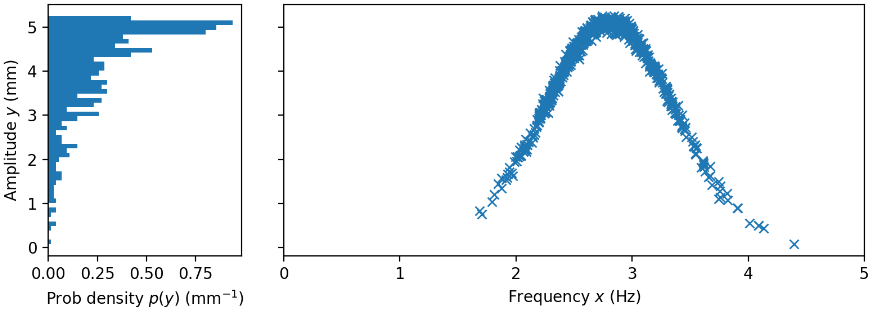
- How can I stop it?
- How much will it shake?

The shaking desk: Gathering more data

Data: For colleague present ($C = 1$), measure jiggling frequency x.



- Could we estimate on model per x? I.e. $p(y|x, \boldsymbol{\theta})$.
- We should be able to predict the amplitude very accurately!
- Uncertainty reduces, predictions improve

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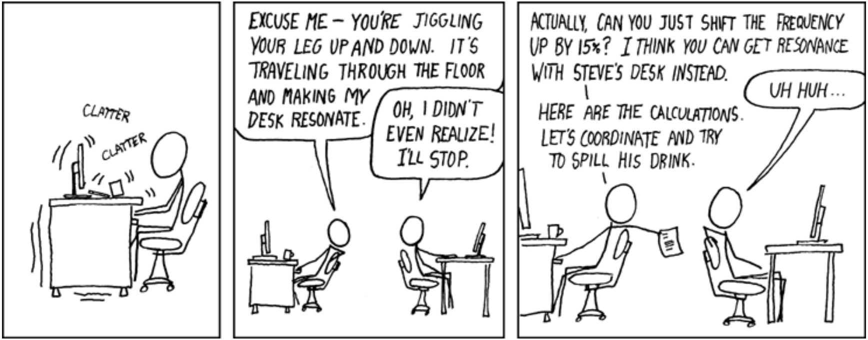

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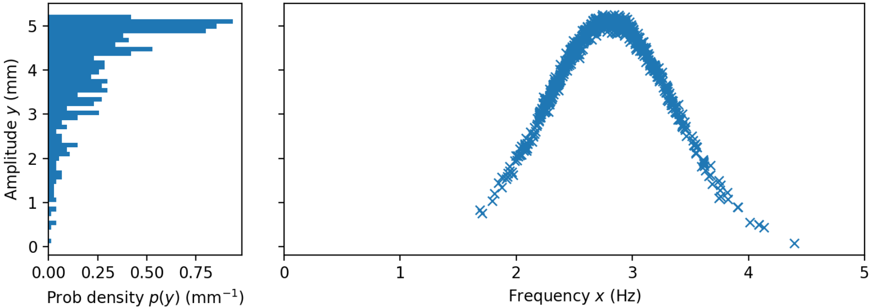

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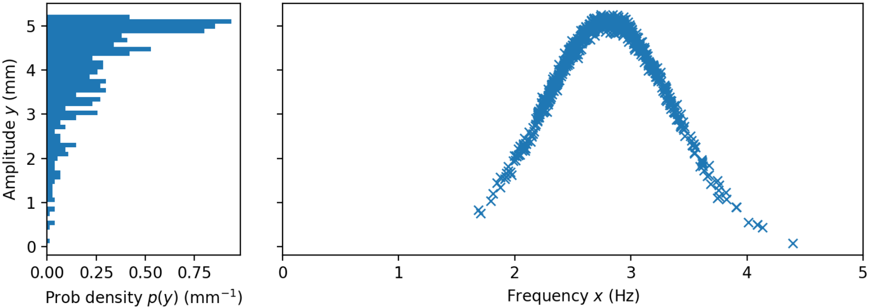

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Regression