数理方程练习题答案

(第十版)

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写在前面： 此答案仅作参考之用，请读者理性食用，答案中前一部分过程较详细，后一部分笔者使用了较多的易得，但读者需注意，易得并不易得，亦需计算过程，但省略的部分在前一部分的题中已经有过了较为详细的讨论，请读者带着思考食用此答案。限于笔者水平有限，难免有纰漏之处，还请读者批评指正。

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目录

练习一

1. 解：(1) $u(0,t)=0, u(L,t)=0$

(2) $u(x,0)=f(x)$

(3) $u_t(x,0)=g(x)$

(4) $\exists 0<M<+\infty, s.t.|u(x,t)|<M, \forall x\in[0,L], t\geqslant 0$

2. 解：(1) $u(0,t)=g(t)$

(2) $u_x(L,t)=0$

3. 解：$\begin{cases} u_t=a^2u_{xx}(0<x<L, t>0), \\ u(0,t)=0, u(L,t)=0, \\ u(x,0)=\phi(x). \end{cases}$

4. 解：(1) 线性，二阶，有自由项

(2) 非线性，一阶，无自由项

(3) 线性，二阶，无自由项

(4) 非线性，三阶，无自由项

(5) 线性，二阶，有自由项

练习二

1. 解：$\dfrac{\partial u}{\partial t}=-8e^{-8t}sin2x$

$\dfrac{\partial u}{\partial x}=2e^{-8t}cos2x$

$\dfrac{\partial^2 u}{\partial x^2}=-4e^{-8t}sin2x$

则 $\dfrac{\partial u}{\partial t}=2\dfrac{\partial^2 u}{\partial x^2}$

2. 解：(1) $\dfrac{\partial u}{\partial t}=-8e^{-8t}sin2x$

$\dfrac{\partial u}{\partial x}=2e^{-8t}cos2x$

$\dfrac{\partial^2 u}{\partial x^2}=-4e^{-8t}sin2x$

则 $\dfrac{\partial u}{\partial t}=2\dfrac{\partial^2 u}{\partial x^2}$

(2) $u(0,t)=F(5t)+G(-5t)=0$

则 $F(x)+G(-x)=0$

$u(\pi t)=F(5t+2\pi)+G(-5t+2\pi)=0$

$u(x,0)=F(2x)+G(2x)=sin2x$

则 $F(x)+G(x)=sinx$

$u_t(x,0)=5F'(2x)-5G'(2x)=0$

则 $F'(x)=G'(x)$, 则 $F(x)=G(x)+C$

则 $F(x)=\dfrac{sinx+C}{2},G(x)=\dfrac{sinx-C}{2}$

则 $u(x,t)=F(2x+5t)+G(2x-5t)=$

$\dfrac{sin(2x+5t)+sin(2x-5t)}{2}=sin2xcos5t$

经验证，满足方程及定解条件。

3. 解：(1) 对 y 积分，可得：$\dfrac{\partial u}{\partial x}=\dfrac{1}{2}x^2y^2+\phi(x)$

再对 x 积分，可得：$u(x,y)=\dfrac{1}{6}x^3y^2+f(x)+g(y)$

其中，$f(x)=\displaystyle\int\phi(x)\mathrm{d}x$

(2) $u(x,0)=f(x)+g(0)=x^2$

$u(1,y)=\dfrac{1}{6}y^2+f(1)+g(y)=cosy$

则 $f(x)=x^2-g(0),g(y)=cosy-\dfrac{1}{6}y^2-f(1)$

又 $u(1,0)=f(1)+g(0)=1$

则 $u(x,y)=\dfrac{1}{6}x^3y^2+cosy-\dfrac{1}{6}y^2+x^2-1$

4. 解：$u_x = yf'(xy)$

$u_y = xf'(xy)$

则 $xu_x - yu_y = 0$

练习三

1. 解：固有值：$\lambda_n = \left(\frac{2n-1}{2l}\pi\right)^2 \quad (n = 1, 2, 3, \cdots)$

固有函数：$X_n(x) = sin\left(\frac{2n-1}{2l}\pi x\right) \quad (n = 1, 2, 3, \cdots)$

2. 解：设 $u(x,t) = X(x)T(t)$

代入可得：$X'' + \lambda X = 0, T'' + \lambda a^2 T = 0$

易得 $\lambda > 0, X(x) = Asin\sqrt{\lambda}x + Bcos\sqrt{\lambda}x$

又 $u(0,t) = 0, u_x(l,t) = 0$

则 $X(0) = 0, X'(l) = 0$

则 $B = 0, \lambda_n = \left(\frac{2n-1}{2l}\pi\right)^2 \quad (n = 1, 2, 3, \cdots)$

则 $X_n(x) = A_n sin\left(\frac{2n-1}{2l}\pi x\right) \quad (n = 1, 2, 3, \cdots)$

$T_n(t) = B_n sin\left(\frac{2n-1}{2l}a\pi t\right) + C_n cos\left(\frac{2n-1}{2l}a\pi t\right)$

则 $u(x,t) =$

$\sum_{n=1}^{+\infty}\left[a_n sin\left(\frac{2n-1}{2l}a\pi t\right) + b_n cos\left(\frac{2n-1}{2l}a\pi t\right)\right] sin\left(\frac{2n-1}{2l}\pi x\right)$

其中，$a_n = A_n B_n, b_n = A_n C_n$

则 $u(x,0) = \sum_{n=1}^{+\infty} b_n sin\left(\frac{2n-1}{2l}\pi x\right) = 3sin\frac{3\pi x}{2l} + 6sin\frac{5\pi x}{2l}$

则 $b_2 = 3, b_3 = 6, b_n = 0 \ (n \neq 2, 3)$

又 $u_t(x,0) = \sum_{n=1}^{+\infty} \frac{2n-1}{2l}a\pi \cdot a_n sin\left(\frac{2n-1}{2l}\pi x\right) = 0$

则 $a_n = 0$

则 $u_(x,t) = 3cos\frac{3a\pi t}{2l}sin\frac{3\pi x}{2l} + 6cos\frac{5a\pi t}{2l}sin\frac{5\pi x}{2l}$

3. 解： 固有值：$\lambda_n = n^2\pi^2 - 2 \quad (n = 1, 2, 3, \cdots)$

固有函数：$X_n(x) = sinn\pi x \quad (n = 1, 2, 3, \cdots)$

4. 解： 设 $u(x,t) = X(x)T(t)$

代入得：$X'' + (\lambda + 2)X = 0, T'' + \lambda\ T = 0$

易得，$\lambda_n = n^2\pi^2 - 2, X_n(x) = A_n sinn\pi x \quad (n = 1, 2, 3, \cdots)$

$T_n(t) = B_n sin\sqrt{n^2\pi^2 - 2}t + C_n cos\sqrt{n^2\pi^2 - 2}t$

则 $u(x,t) = \sum_{n=1}^{+\infty}\left(a_n sin\sqrt{n^2\pi^2 - 2}t + b_n cos\sqrt{n^2\pi^2 - 2}t\right) sinn\pi x$

其中，$a_n = A_nB_n, b_n = A_nC_n$

又 $u(x,0) = \sum_{n=1}^{+\infty} b_n sinn\pi x = 0, u_t(x,0) =$

$\sum_{n=1}^{+\infty} a_n\sqrt{n^2\pi^2 - 2}sinn\pi x = \sqrt{\pi^2 - 2}sin\pi x$

则 $a_1 = 1, a_n = 0\ (n \neq 1), b_n = 0$

则 $u(x,t) = sin\sqrt{\pi^2 - 2}tsin\pi x$

练习四

1. 解； 设 $u(x,t) = X(x)T(t)$

代入可得：$X'' + \lambda X = 0, T' + \lambda a^2 T = 0$

易得：$\lambda_n = \left(\frac{n\pi}{l}\right)^2, X_n(x) = A_n sin\frac{n\pi x}{l} \quad (n = 1, 2, 3, \cdots)$

$T_n(t) = B_n e^{-\left(\frac{n\pi a}{l}\right)^2 t}$

则 $u(x,t) = \sum_{n=1}^{+\infty} a_n e^{-\left(\frac{n\pi a}{l}\right)^2 t} sin\frac{n\pi x}{l}$

其中，$a_n = A_nB_n$

又 $u(x,0) = \sum_{n=1}^{+\infty} a_n sin\frac{n\pi x}{l} = x(l - x)$

则 $a_n = \frac{2}{l}\int_0^l x(l - x)sin\frac{n\pi x}{l}\mathrm{d}x = \frac{4l^2}{n^3\pi^3}\left[1 - (-1)^n\right]$

则 $u(x,t)=\sum_{n=1}^{+\infty}\frac{4l^2}{n^3\pi^3}\left[1-(-1)^n\right]e^{-\left(\frac{n\pi a}{l}\right)^2t}sin\frac{n\pi x}{l}$

2. 解； $\lambda<0$ 时，$X(x)=Ae^{\sqrt{\lambda}x}+Be^{-\sqrt{\lambda}x}$

又 $X'(0)=X(l)=0$，则 $A=0,B=0,X(x)=0$，平凡解，舍去

$\lambda=0$ 时，$X(x)=A+Bx$

又 $X'(0)=X(l)=0$，则 $A=0,B=0,X(x)=0$，平凡解，舍去

$\lambda=0$ 时，$X(x)=Asin\sqrt{\lambda}x+Bcos\sqrt{\lambda}x$

又 $X'(0)=X(l)=0$

则 $A=0,\lambda_n=\left(\frac{2n-1}{2l}\pi\right)^2 \quad (n=1,2,3,\cdots)$

则 $X_n(x)=B_ncos\frac{2n-1}{2l}\pi x \quad (n=1,2,3,\cdots)$

即固有值为：$\lambda_n=\left(\frac{2n-1}{2l}\pi\right)^2 \quad (n=1,2,3,\cdots)$

固有函数为：$X_n(x)=cos\frac{2n-1}{2l}\pi x \quad (n=1,2,3,\cdots)$

3. 解； 设 $u(x,t)=X(x)T(t)$

代入可得：$X''+\lambda X=0,T'+\lambda T=0$

易得，

$\lambda_n=\left(\frac{2n-1}{4}\pi\right)^2,X_n(x)=A_ncos\frac{2n-1}{4}\pi x \quad (n=1,2,3,\cdots)$

$T_n(t)=B_ne^{-\left(\frac{2n-1}{4}\pi\right)^2t}$

则 $u(x,t)=\sum_{n=0}^{+\infty}a_ne^{-\left(\frac{2n-1}{4}\pi\right)^2t}cos\frac{2n-1}{4}\pi x$

又 $u(x,0)=\sum_{n=0}^{+\infty}a_ncos\frac{2n-1}{4}\pi x=4cos\frac{5\pi x}{4}$

则 $a_3=4,a_n=0\ (n\neq 3)$

则 $u(x,t)=4e^{-\frac{25}{16}\pi^2t}cos\frac{5\pi}{4}x$

4. 解； 设 $u(x,t)=X(x)T(t)$

代入可得：$X''+2X'+\lambda X=0,T'+\lambda T=0$

$\lambda < 1$ 时，$X(x) = Ae^{-1+\sqrt{1-\lambda}}$

$\lambda = 1$ 时，$X(x) = (A + Bx)e^{-1}$

$\lambda > 1$ 时，$X(x) = \left(Asin\sqrt{\lambda - 1}x + Bcos\sqrt{\lambda - 1}x\right)e^{-x}$ 又

$u(0,t) = u(1,t) = 0$，即 $X(0) = X(1) = 0$

则仅当 $\lambda > 1$ 时，$X(x)$ 有非平凡解，此时，

$\lambda_n = n^2\pi^2 + 1, X_n(x) = A_n sinn\pi x e^{-x} \quad (n = 1, 2, 3, \cdots)$

$T_n(t) = B_n e^{-\left(n^2\pi^2+1\right)t}$

则 $u(x,t) = \sum\limits_{n=1}^{+\infty} a_n e^{-\left(n^2\pi^2+1\right)t} sinn\pi x e^{-x}$

又 $u(x,0) = \sum\limits_{n=1}^{+\infty} a_n sinn\pi x e^{-x} = e^{-x} sin\pi x$

则 $a_1 = 1, a_n = 0 \ (n \neq 1)$

则 $u(x,t) = e^{-\left(\pi^2+1\right)t} sin\pi x e^{-x}$

练习五

1. 解：设 $u(x,y) = X(x)Y(y)$

代入可得：$X'' + \lambda X = 0, Y'' - \lambda Y = 0$

易得，$\lambda < 0$ 时，无非平凡解

$\lambda = 0$ 时，$X_0(x) = A_0, Y_0(y) = B_0 + C_0 y$

$\lambda > 0$ 时，$\lambda_n = n^2\pi^2, X_n(x) = A_n cosn\pi x \quad (n = 1, 2, 3, \cdots)$

$Y_n(y) = B_n e^{n\pi y} + C_n e^{-n\pi y}$

则 $u(x,y) = a_0 + b_0 y + \sum\limits_{n=1}^{+\infty} \left(a_n e^{n\pi y} + b_n e^{-n\pi y}\right) cosn\pi x$

又 $u(x,0) = a_0 + \sum\limits_{n=1}^{+\infty} (a_n + b_n) cosn\pi x = 1 + cos3\pi x$

$u(x,1) = a_0 + b_0 + \sum\limits_{n=1}^{+\infty} \left(a_n e^{n\pi} + b_n e^{-n\pi}\right) cosn\pi x = 3cos2\pi x$

则 $a_0=1,a_3+b_3=1,a_n+b_n=0\ (n\neq 3)$

$a_2e^{2\pi}+b_2e^{-2\pi}=3,a_ne^{2\pi}+b_ne^{-2\pi}=0\ (n\neq 2)$

则 $a_0=1,b_0=-1\quad a_2=\dfrac{3}{e^{2\pi}-e^{-2\pi}},b_2=-\dfrac{3}{e^{2\pi}-e^{-2\pi}}$

$a_3=\dfrac{1}{1-e^{6\pi}},b_3=\dfrac{1}{1-e^{-6\pi}}\quad a_n=b_n=0\ (n\neq 0,2,3)$

则 $u(x,y)=1-y+\dfrac{3}{e^{2\pi}-e^{-2\pi}}\left(e^{2\pi y}-e^{-2\pi y}\right)cos2\pi x+$
$\left(\dfrac{1}{1-e^{6\pi}}e^{3\pi y}+\dfrac{1}{1-e^{-6\pi}}e^{-3\pi y}\right)cos3\pi x$

2. 解：设 $u(r,\theta)=R(r)\Theta(\theta)$

代入得：$\Theta''+\lambda\Theta=0,r^2R''+rR'-\lambda R=0$

由 $\Theta(\theta)=\Theta(\theta+2\pi)$，易得：

$\lambda<0$ 时，无非平凡解

$\lambda=0$ 时，$\Theta(\theta)=A_0,R(r)=C_0+D_0lnr$

$\lambda>0$ 时，$\lambda_n=n^2,\Theta(\theta)=a_nsinn\theta+b_ncosn\theta\quad(n=1,2,3,\cdots)$

$R(r)=c_nr^n+d_nr^{-n}$

则 $u(r,\theta)=a_0+b_0lnr+\sum\limits_{n=1}^{+\infty}\left(c_nr^n+d_nr^{-n}\right)(a_nsinn\theta+b_ncosn\theta)$

又 $u(1,\theta)=a_0+\sum\limits_{n=1}^{+\infty}(c_n+d_n)(a_nsinn\theta+b_ncosn\theta)=0$

$u(2,\theta)=a_0+b_0ln2+\sum\limits_{n=1}^{+\infty}\left(c_n2^n+d_n2^{-n}\right)(a_nsinn\theta+b_ncosn\theta)=u_0$

则 $a_0=0,c_n+d_n=0,a_0+b_0ln2=u_0,c_n2^n+d_n2^{-n}=0$

则 $a_0=0,b_0=ln2,c_n=d_n=0$

则 $u(r,\theta)=\dfrac{u_0lnr}{ln2}$

3. 解：设 $u(r,\theta)=R(r)\Theta(\theta)$

代入得：$\Theta''+\lambda\Theta=0,r^2R''+rR'-\lambda R=0$

又 $\Theta(0)=0,\Theta(\dfrac{\pi}{2})=0$

则 $\lambda_n=4n^2,\Theta_n(\theta)=A_nsin2n\theta\quad(n=1,2,3,\cdots)$

$R_n(r)=B_nr^{2n}+C_nr^{-2n}$

又 $|r(0)|<+\infty$，则 $C_n=0, R_n(r)=B_nr^n$

则 $u(r,\theta)=\sum\limits_{n=1}^{+\infty}a_nr^{2n}sin2n\theta$

又 $u(1,\theta)=\sum\limits_{n=1}^{+\infty}a_nsin2n\theta=\theta\left(\frac{\pi}{2}-\theta\right)$

则 $a_n=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\theta\left(\frac{\pi}{2}-\theta\right)sin2n\theta\mathrm{d}\theta=\frac{1-(-1)^n}{n^3\pi}$

则 $u(r,\theta)=\sum\limits_{n=1}^{+\infty}\frac{1-(-1)^n}{n^3\pi}r^{2n}sin2n\theta$

4. 解： 设 $u(x,y)=X(x)Y(y)$

代入可得：$X''+\lambda X=0, Y''-\lambda Y=0$

易得：$\lambda_n=n^2, X_n(x)=A_nsinnx \quad (n=1,2,3,\cdots)$

$Y_n(y)=B_ne^{ny}+C_ne^{-ny}$，又 $\lim\limits_{y\to+\infty}Y(y)=0$，则 $B_n=0$

则 $u(x,y)=\sum\limits_{n=1}^{+\infty}a_ne^{-ny}sinnx$

又 $u(x,0)=\sum\limits_{n=1}^{+\infty}a_nsinnx=sin5x$

则 $a_5=1, a_n=0\ (n\neq 5)$

则 $u(x,y)=e^{-5y}sin5x$

练习六

1. 解： 易得，固有函数系为 $\{cosn\pi x\ (n=0,1,2,\cdots)\}$

设 $u(x,t)=\sum\limits_{n=0}^{+\infty}u_n(t)cosn\pi x$

代入得：$\sum\limits_{n=0}^{+\infty}\left(u_n'(t)+n^2\pi^2u_n(t)\right)cosn\pi x=cos\pi x$

则 $u_1'(t)+\pi^2u_1(t)=1, u_n'(t)+n^2\pi^2u_n(t)=0\ (n\neq 1)$

则 $u_0(t)=a_0, u_1(t)=a_1e^{-\pi^2t}+\frac{1}{\pi^2}, u_n(t)=a_ne^{-n^2\pi^2t}\ (n\neq 0,1)$

又 $u(x,0)=\sum_{n=0}^{+\infty}u_n(0)cosn\pi x=0$，则 $u_n(0)=0$

则 $a_1=-\frac{1}{\pi^2},a_n=0\ (n\neq 1)$

则 $u(x,t)=\frac{1}{\pi^2}\left(1-e^{-\pi^2 t}\right)cos\pi x$

2. 解： 易得，固有函数系为 $\{sin\frac{n\pi x}{l}\ (n=1,2,3,\cdots)\}$

设 $u(x,t)=\sum_{n=1}^{+\infty}u_n(t)sin\frac{n\pi x}{l}$

代入得：$u_1''(t)+\left(\frac{\pi a}{l}\right)^2u_1(t)=t,u_n''(t)+\left(\frac{n\pi a}{l}\right)^2u_n(t)=0$

则 $u_1(t)=a_1sin\frac{\pi at}{l}+b_1cos\frac{\pi at}{l}+\left(\frac{l}{\pi a}\right)^2t$

$u_n(t)=a_nsin\frac{n\pi at}{l}+b_ncos\frac{n\pi at}{l}$

又 $u_n(0)=0,u_n'(0)=0$

则 $a_1=-\left(\frac{l}{\pi a}\right)^3,a_n=0\ (n\neq 1),b_n=0$

则 $u(x,t)=\left(\frac{l}{\pi a}\right)^2\left(t-\frac{l}{\pi a}sin\frac{\pi at}{l}\right)sin\frac{\pi x}{l}$

3. 解： 设 $u(x,y)=v(x,y)+1$

则原问题化为：$\begin{cases} v_{xx}+v_{yy}=-2x,x^2+y^2<1 \\ v|_{x^2+y^2=1}=0 \end{cases}$

转化为极坐标，即：$\begin{cases} v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}=-2rcos\theta,r<1 \\ v(1,\theta)=0 \end{cases}$

易得，固有函数系为 $\{1,sin\theta,cos\theta,\cdots\}$

设 $v(r,\theta)=A_0(r)+\sum_{n=1}^{+\infty}(A_n(r)sinn\theta+B_n(r)cosn\theta)$

代入得：$A_0''+\frac{1}{r}A_0'=0,A_n''+\frac{1}{r}A_n'-\frac{n^2}{r^2}A_n=0\ (n\neq 0)$

$B_1''+\frac{1}{r}B_1'-\frac{1}{r^2}B_1=-2r,B_n''+\frac{1}{r}B_n'-\frac{n^2}{r^2}B_n=0\ (n\neq 1)$

则 $A_0(r)=a_0+b_0r,A_n(r)=a_nr^n+b_nr^{-n}\ (n\neq 0)$

$B_1(r)=a_1'r+b_1'r^{-1}-\frac{1}{4}r^3,B_n(r)=a_n'r^n+b_n'r^{-n}\ (n\neq 1)$

又 $v(1,\theta)=0,|v(0,\theta)|<+\infty$

则 $A_n(1)=B_n(1)=0,|A_n(0)|<+\infty,|B_n(0)|<+\infty$

则 $a_n=b_n=0,a_1'=\dfrac{1}{4},b_1'=0,a_n'=b_n'=0\ (n\neq 1)$

则 $v(r,\theta)=\left(\dfrac{r}{4}-\dfrac{r^3}{4}\right)cos\theta$，则 $v(x,y)=\dfrac{x}{4}\left(1-x^2-y^2\right)$

则 $u(x,y)=1+\dfrac{x}{4}\left(1-x^2-y^2\right)$

4. 解：易得，固有函数系为 $\{sinn\theta\ (n=1,2,3,\cdots)\}$[1]

设 $u(r,\theta)=\sum_{n=1}^{+\infty}R_n(r)sinn\theta$

代入得：$R_1''+\dfrac{1}{r}R_1'-\dfrac{1}{r^2}R_1=r,R_n''+\dfrac{1}{r}R_n'-\dfrac{n^2}{r^2}R_n=0\ (n\neq 1)$

则 $R_1(r)=a_1r+b_1r^{-1}+\dfrac{1}{8}r^3,R_n(r)=a_nr^n+b_nr^{-n}\ (n\neq 0)$

又 $R_n(1)=0,|R_n(0)|<+\infty$

则 $a_1=-\dfrac{1}{8},b_1=0,a_n=b_n=0\ (n\neq 1)$

则 $u(r,\theta)=\left(\dfrac{1}{8}r^3-\dfrac{1}{8}r\right)sin\theta$

练习七

1. 解：设 $u(x,t)=v(x,t)+w(x)$

代入得：

$$\begin{cases} v_{tt}=a^2v_{xx}+a^2w'' \quad 0<x<l,t>0, \\ v(0,t)=-w(0),\ v(l,t)=1-w(l) \quad t>0, \\ v(x,0)=sin\dfrac{3\pi x}{l}+\dfrac{x}{l},\ v_t(x,0)=x(l-x) \quad 0<x<l. \end{cases}$$

则令 $\begin{cases} w''=0 \quad 0<x<l \\ w(0)=0,\ w(l)=1 \end{cases}$ 可得，$w(x)=\dfrac{x}{l}$

[1] $\Theta(\theta)=Asin\sqrt{\lambda}\theta+Bcos\sqrt{\lambda}\theta,\Theta(0)=\Theta(\pi)$，得：$B=0,\lambda_n=n^2,\Theta_n(\theta)=A_nsinn\theta\ (n=1,2,3,\cdots)$

$$
\text{则}\begin{cases} v_{tt}=a^2v_{xx} \quad 0<x<l,t>0, \\ v(0,t)=0,\ v(l,t)=0 \quad t>0, \\ v(x,0)=sin\dfrac{3\pi x}{l},\ v_t(x,0)=x(l-x) \quad 0<x<l. \end{cases}
$$

易得，$v(x,t)=\sum_{n=1}^{+\infty}\left(a_n sin\dfrac{n\pi at}{l}+b_n cos\dfrac{n\pi at}{l}\right)sin\dfrac{n\pi x}{l}$

又 $v(x,0)=\sum_{n=1}^{+\infty}b_n sin\dfrac{n\pi x}{l}=sin\dfrac{3\pi x}{l}$

$v_t(x,0)=\sum_{n=1}^{+\infty}\dfrac{n\pi a}{l}a_n sin\dfrac{n\pi x}{l}=x(l-x)$

则 $a_n=\dfrac{4l^3\left[1-(-1)^n\right]}{n^4\pi^4 a},b_3=1,b_n=0\ (n\neq 3)$

则 $u(x,t)=$

$\dfrac{x}{l}+cos\dfrac{3\pi at}{l}+sin\dfrac{3\pi x}{l}+\sum_{n=1}^{+\infty}\dfrac{4l^3\left[1-(-1)^n\right]}{n^4\pi^4 a}sin\dfrac{n\pi at}{l}sin\dfrac{n\pi x}{l}$

2. 解： 设 $u(x,t)=v(x,t)+sint$

$$
\text{则}\begin{cases} v_t=8v_{xx}+e^x sin\dfrac{x}{2} \\ v(0,t)=0,v_x(\pi,t)=0 \\ v(x,0)=0 \end{cases}
$$

易得，$v(x,t)=\sum_{n=1}^{+\infty}v_n(t)sin\dfrac{2n-1}{2}x$

代入得：$v_1'+2v_1=e^t,v_n'+2(2n-1)^2v_n=0\ (n\neq 1)$

则 $v_1=a_1e^{-2t}+\dfrac{1}{3}e^t,v_n=a_ne^{-2(2n-1)^2t}\ (n\neq 1)$

又 $v_n(0)=0$，则 $a_1=-\dfrac{1}{3},a_n=0\ (n\neq 1)$

则 $u(x,t)=sint+\dfrac{1}{3}\left(e^t-e^{-2t}\right)sin\dfrac{x}{2}$

3. 解： 设 $u(x,t)=v(x,t)+w(x)$

$$
\text{则}\begin{cases} v_t=v_{xx}+2v_x+w''+2w'-1 \\ v(0,t)=-w(0),v(1,t)=1-w(1) \\ v(x,0)=e^{-x}sin\pi x+\dfrac{x}{2}+\dfrac{1-e^{-2x}}{2(1-e^{-2})}-w(x) \end{cases}
$$

令 $\begin{cases} w''+2w'=1 \\ w(0)=0, w(1)=1 \end{cases}$ 则 $w(x)=\dfrac{x}{2}+\dfrac{1-e^{-2x}}{2(1-e^{-2})}$

则 $\begin{cases} v_t=v_{xx}+2v_x \\ v(0,t)=0, v(1,t)=0 \\ v(x,0)=e^{-x}sin\pi x \end{cases}$

易得，$v(x,t)=\sum\limits_{n=1}^{+\infty}a_n e^{-(n^2\pi^2+1)t}sinn\pi xe^{-x}$

又 $v(x,0)=\sum\limits_{n=1}^{+\infty}a_n sinn\pi xe^{-x}=e^{-(\pi^2+1)t}sin\pi xe^{-x}$

则 $a_1=1, a_n=0\ (n\neq 2)$

则 $u(x,t)=e^{-(\pi^2+1)t}sin\pi xe^{-x}+\dfrac{x}{2}+\dfrac{1-e^{-2x}}{2(1-e^{-2})}$

练习八

1. 设 $u(x,t)=v(x,t)+w(x)$

令 $\begin{cases} w''=-e^x-2 \\ w(3)=-18-e^{-3}, w'(0)=1 \end{cases}$ 则 $w(x)=-e^{-x}-x^2-9$

则 $\begin{cases} v_t=2v_{xx} \\ v_x(0,t)=v(3,t)=0 \\ v(x,0)=-x^2+9 \end{cases}$

易得，$v(x,t)=\sum\limits_{n=1}^{+\infty}a_n e^{-\frac{(2n-1)^2\pi^2}{18}t}cos\dfrac{2n-1}{6}\pi x$

又 $v(x,0)=\sum\limits_{n=1}^{+\infty}a_n cos\dfrac{2n-1}{6}\pi x=-x^2+9$

则 $a_n=(-1)^{n+1}\dfrac{288}{(2n-1)^3\pi^3}$

则 $u(x,t)=$

$-e^{-x}-x^2-9+\sum\limits_{n=1}^{+\infty}(-1)^{n+1}\dfrac{288}{(2n-1)^3\pi^3}e^{-\frac{(2n-1)^2\pi^2}{18}t}cos\dfrac{2n-1}{6}\pi x$

2. 设 $u(x,t) = v(x,t) + w(x)$

令 $\begin{cases} w'' = 6(x-1) \\ w(0) = 0, w'(2) = 1 \end{cases}$ 则 $w(x) = x^3 - 3x^2 + x$

则 $\begin{cases} v_t = v_{xx} \\ v(0,t) = 0, v_x(2,t) = 0 \\ v(x,0) = sin\dfrac{\pi x}{4} \end{cases}$

易得，$v(x,t) = \sum\limits_{n=1}^{+\infty} a_n e^{-\frac{(2n-1)^2\pi^2}{16}t} sin\dfrac{2n-1}{4}\pi x$

其中，$a_1 = 1, a_n = 0\ (n \neq 1)$

则 $u(x,t) = x^3 - 3x^2 + x + e^{-\frac{\pi^2 t}{16}} sin\dfrac{\pi x}{4}$

3. 设 $u(x,y) = v(x,y) + w(x)$

令 $\begin{cases} w'' = sin\pi x \\ w(0) = 1, w(1)2 \end{cases}$ 则 $w(x) = -\dfrac{1}{\pi^2} sin\pi x + x + 1$

则 $\begin{cases} v_{xx} + v_{yy} = 0 \\ v(0,y) = 0, v(1,y) = 0 \\ v(x,0) = \dfrac{1}{\pi^2} sin\pi x, v(x,1) = 0 \end{cases}$

易得，$v(x,t) = \sum\limits_{n=1}^{+\infty} \left(a_n e^{n\pi y} + b_n e^{-n\pi y}\right) sinn\pi x$

其中，$a_1 = \dfrac{1}{\pi^2(1-e^{2\pi})}, b_1 = -\dfrac{e^{2\pi}}{\pi^2(1-e^{2\pi})}, a_n = b_n = 0\ (n \neq 1)$

则 $u(x,y) = -\dfrac{1}{\pi^2} sin\pi x + x + 1 + \dfrac{1}{\pi^2(1-e^2\pi)} \left(e^{\pi y} - e^{2\pi} e^{-\pi y}\right) sin\pi x$

练习九

1. 解：$\lambda < 0$ 时，$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$，$X(-\pi) = X(\pi)$，则 $A - B = 0$，$X'(-\pi) = X'(\pi)$，则 $A + B = 0$，则 $A = B = 0$，则 $X(x) \equiv 0$，平凡解，舍去

$\lambda = 0$ 时，$X(x) = Ax + B$，$X(-\pi) = X(\pi)$，$X'(-\pi) = X'(\pi)$，则 $A = 0$，则 $X(x) = B$

$\lambda > 0$ 时，$X(x) = Asin\sqrt{\lambda}x + Bcos\sqrt{\lambda}x$，，$X(-\pi) = X(\pi)$，则 $A = 0$，$X'(-\pi) = X'(\pi)$，则 $\sqrt{\lambda}\pi = n\pi$，则 $\lambda_n = n^2$ $(n = 1, 2, 3, \cdots)$，则 $X_n(x) = A_n sinnx + B_n cosnx$

综上所述，固有值为 $\lambda_n = n^2$ $(n = 0, 1, 2, 3, \cdots)$，固有函数为 $X_n(x) = A_n sinnx + B_n cosnx$ $(n = 0, 1, 2, 3, \cdots)$

2. 解：令 $x = e^t$，则方程可化为 $\dfrac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \lambda y = 0$，易得，$\lambda > 0$，$y(t) = Asin\sqrt{\lambda}t + Bcos\sqrt{\lambda}t$，则 $y(x) = Asin\left(\sqrt{\lambda}lnx\right) + Bcos\left(\sqrt{\lambda}lnx\right)$，

又 $y(1) = y(e) = 0$，则 $B = 0$，$\lambda_n = n^2\pi^2$，$y_n(x) = sin\left(n\pi lnx\right)$

又 $\displaystyle\int_1^e \frac{1}{x} sin\left(n\pi lnx\right) sin\left(m\pi lnx\right) \mathrm{d}x = \int_1^e sin\left(n\pi lnx\right) sin\left(m\pi lnx\right) \mathrm{d}\left(lnx\right) = \int_0^1 sin\left(n\pi t\right) sin\left(m\pi t\right) \mathrm{d}t = 0 \quad (n \neq m)$

即 $\displaystyle\int_1^e \frac{1}{x} y_n(x) y_m(x) = 0 \quad (n \neq m)$，则固有函数系 $\{y_n(x), n = 1, 2, \cdots\}$ 在区间 $[1, e]$ 上带权 $\dfrac{1}{x}$ 正交。[2]

3. 解：方程两边乘以 u 得：$uu_t = a^2 uu_{xx}$

对 x 积分得：$\displaystyle\int_0^l uu_t \mathrm{d}x = a^2 \int_0^l uu_{xx} \mathrm{d}x$

又 $\displaystyle\int_0^l uu_t \mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^l u^2 \mathrm{d}x\right)$，$\displaystyle\int_0^l uu_{xx} \mathrm{d}x = uu_x\big|_0^l - \int_0^l u_x^2 \mathrm{d}x = -\int_0^l u_x^2 \mathrm{d}x$

则 $\displaystyle\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^l u^2 \mathrm{d}x\right) = -a^2 \int_0^l u_x^2 \mathrm{d}x \leqslant 0$

即 $\displaystyle\int_0^l u^2 \mathrm{d}x$ 单调递减

又 $\displaystyle\int_0^l u^2(x, 0) \mathrm{d}x = 0$，则 $\displaystyle\int_0^l u^2 \mathrm{d}x \leqslant 0$

[2]对于一般的施图姆——刘维尔方程，其固有函数系正交性的证明可参见附录

又 $\int_0^l u^2\mathrm{d}x \geqslant 0$，则 $\int_0^l u^2\mathrm{d}x = 0$

则 $u(x,t) \equiv 0$

4. 解： 方程两边乘以 u_t 得，$u_t u_{tt} = a^2 u_t u_{xx}$

对 x 积分得：$\int_0^l u_t u_{tt}\mathrm{d}x = a^2\int_0^l u_t u_{xx}\mathrm{d}x$

又 $\int_0^l u_t u_{tt}\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^l u_t^2\mathrm{d}x\right), \int_0^l u_t u_{xx}\mathrm{d}x = u_t u_x\big|_0^l - \int_0^l u_{tx}u_x\mathrm{d}x =$
$-\int_0^l u_{tx}u_x\mathrm{d}x = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^l u_x^2\mathrm{d}x\right)$

则 $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^l u_t^2\mathrm{d}x\right) = -\frac{a^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^l u_x^2\mathrm{d}x\right)$

则 $\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^l \left(u_t^2 + a^2u_x^2\right)\mathrm{d}x\right] = 0$

又 $\int_0^l \left(u_t^2(x,0) + a^2u_x^2(x,0)\right)\mathrm{d}x = 0$

则 $\int_0^l \left(u_t^2 + a^2u_x^2\right)\mathrm{d}x = 0$，则 $u_t^2 + a^2u_x^2 = 0$

则 $u_t = 0$，$u_x = 0$，则 $u(x,t) \equiv 0$

练习十

1. 解： 由达朗贝尔公式可得：

$$\begin{aligned} u(x,t) &= \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)\mathrm{d}\alpha \\ &= \frac{sin(x-at)+sin(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\alpha^2\mathrm{d}\alpha \\ &= sinxcosat + x^2t + \frac{1}{3}a^2t^3 \end{aligned}$$

2. 解： 由非齐次方程达朗贝尔公式可得：

$$
\begin{aligned}
u(x,t) &= \frac{\varphi(x-at)+\varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\alpha)\mathrm{d}\alpha + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau \\
&= \frac{(x-at)+(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} sin(\alpha)\mathrm{d}\alpha + \frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} (\xi+a\tau)\mathrm{d}\xi\mathrm{d}\tau \\
&= x + \frac{1}{a}sinxsinat + \frac{1}{2}xt^2 + \frac{a}{6}t^3
\end{aligned}
$$

3. 解： $u(x,t)=f(x-at)+g(x-at)$，代入条件可得：

$u|_{x=0}=f(-at)+g(at)=\phi(t)$，$u|_{x+at=0}=f(2x)+g(0)=\psi(x)$

则 $f(-x)+g(x)=\phi\left(\frac{x}{a}\right)$，$f(x)=\psi\left(\frac{x}{2}\right)-g(0)$

则 $g(x)=\phi\left(\frac{x}{a}\right)-\psi\left(-\frac{x}{2}\right)+g(0)$

则 $u(x,t)=\psi\left(\frac{x-at}{2}\right)-\psi\left(\frac{x+at}{2}\right)+\phi\left(\frac{x+at}{2}\right)$

练习十一

1. 解： 由三维波动方程的基尔霍夫公式可得：

$$
\begin{aligned}
u(M,t) &= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\iint_{S_1^M}\varphi(M+at\omega)\mathrm{d}\omega\right) + \frac{t}{4\pi}\iint_{S_1^M}\psi(M+at\omega)\mathrm{d}\omega \\
&= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^\pi\int_0^{2\pi}(x+atsin\theta cos\varphi)(y+atsin\theta sin\varphi)sin\theta\mathrm{d}\varphi\mathrm{d}\theta\right) \\
&\quad + \frac{t}{4\pi}\int_0^\pi\int_0^{2\pi}(x+atsin\theta cos\varphi)(z+atcos\theta)sin\theta\mathrm{d}\varphi\mathrm{d}\theta \\
&= \frac{\partial}{\partial t}\left(\frac{t}{4\pi}4\pi xy\right) + \frac{t}{4\pi}4\pi xz \\
&= xy + xzt
\end{aligned}
$$

2. 解： 由二维波动方程的泊松公式可得：

$$
\begin{aligned}
u(x,y,z,t) &= \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\iint_{\Sigma_{at}^M}\frac{\varphi(\xi,\eta)\mathrm{d}\sigma}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}}\right] \\
&\quad + \frac{1}{2\pi a}\iint_{\Sigma_{at}^M}\frac{\psi(\xi,\eta)\mathrm{d}\sigma}{\sqrt{(at)^2-(\xi-x)^2-(\eta-y)^2}} \\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_0^{at}\int_0^{2\pi}\frac{(x+\rho cos\theta)^3+(x+\rho cos\theta)^2(y+\rho sin\theta)}{\sqrt{(at)^2-\rho^2}}\rho\mathrm{d}\theta\mathrm{d}\rho\right] \\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_0^{at}\frac{2\pi x^2(x+y)+\pi(3x+y)\rho^2}{\sqrt{(at)^2-\rho^2}}\rho\mathrm{d}\rho\right] \\
&= \frac{1}{2\pi a}\frac{\partial}{\partial t}\left[2\pi atx^2(x+y)+\frac{2}{3}\pi(3x+y)a^3t^3\right] \\
&= x^2(x+y)+a^2t^2(3x+y)
\end{aligned}
$$

3. 解： 三维波动方程柯西问题的齐次化原理叙述如下：

Theorem. 若 $w(x,y,z,t;\tau)$ 是初值问题

$$
\begin{cases}
w_{tt}=a^2(w_{xx}+w_{yy}+w_{zz}) \quad (t>\tau) \\
w|_{t=\tau}=0, w_t|_{t=\tau}=f(x,y,z,t,\tau)
\end{cases}
$$

的解（其中 τ 为参数），则

$$
u(x,y,z,t)=\int_0^t w(x,y,z,t;\tau)\mathrm{d}\tau
$$

就是初值问题

$$
\begin{cases}
u_{tt}=a^2(u_{xx}+u_{yy}+u_{zz}) \quad (-\infty<x,y,z<+\infty, t>0) \\
u(x,y,z,0)=0, u_t(x,y,z,0)=0
\end{cases}
$$

的解.

练习十二

1. 解： 记 $\mathscr{F}[u(x,t)]=U(\lambda,t)$, 则 $\mathscr{F}[u_t]=\dfrac{\mathrm{d}U}{\mathrm{d}t}$, $\mathscr{F}[u_{xx}]=-\lambda^2U$, $\mathscr{F}[u(x,0)]=U(\lambda,0)=\mathscr{F}[cosx]$

则 $\begin{cases}\dfrac{\mathrm{d}U}{\mathrm{d}t}=-\lambda^2U\\ U(\lambda,0)=\mathscr{F}[cosx]\end{cases}$ 解得 $U(\lambda,t)=\mathscr{F}[cosx]e^{-\lambda^2t}$

则 $u(x,t)=cosx*\left(\dfrac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\right)=\dfrac{1}{2a\sqrt{\pi t}}\displaystyle\int_{-\infty}^{+\infty}cos\xi e^{-\frac{(x-\xi)^2}{4a^2t}}\mathrm{d}\xi$

2. 解： 记 $\mathscr{L}[u(x,t)]=U(x,s)$, 则 $\mathscr{L}[u_t]=sU$, $\mathscr{L}[u_{xx}]=\dfrac{\mathrm{d}^2U}{\mathrm{d}x^2}$, $\mathscr{L}[u(0,t)]=U(0,s)=\dfrac{1}{s}$

则 $\begin{cases}\dfrac{\mathrm{d}^2U}{\mathrm{d}x^2}=\dfrac{s}{a^2}U\\ U(0,s)=\dfrac{1}{s},|U(x,s)|\leqslant\dfrac{M}{s}\end{cases}$ 解得 $U(x,s)=\dfrac{1}{s}e^{-\frac{\sqrt{s}}{a}x}$

则 $u(x,t)=\mathscr{L}^{-1}\left[\dfrac{1}{s}e^{-\frac{\sqrt{s}}{a}x}\right]=\dfrac{2}{\sqrt{\pi}}\displaystyle\int_{\frac{x}{2a\sqrt{t}}}^{+\infty}e^{-y^2}\mathrm{d}y$

3. 解： 记 $\mathscr{L}[u(x,t)]=U(x,s)$, 则 $\mathscr{L}[u_{tt}]=s^2U-su(x,0)-u_t(x,0)=s^2U$, $\mathscr{L}[u_{xx}]=\dfrac{\mathrm{d}^2U}{\mathrm{d}x^2}$， $\mathscr{L}[u(0,t)]=U(0,s)=\mathscr{L}[Asin\omega t]$

则 $\begin{cases}\dfrac{\mathrm{d}^2U}{\mathrm{d}x^2}=\dfrac{s^2}{a^2}U\\ U(0,s)=\mathscr{L}[Asin\omega t],|U(x,s)|\leqslant\dfrac{M}{s}\end{cases}$

解得 $U(x,s)=\mathscr{L}[Asin\omega t]e^{-\frac{s}{a}x}$

则 $u(x,t)=\begin{cases}0, & t<\dfrac{x}{a}\\ Asin\omega\left(t-\dfrac{x}{a}\right), & t>\dfrac{x}{a}\end{cases}$

4. 解： 记 $\mathscr{L}[u(x,t)]=U(x,s)$， 则 $\mathscr{L}[u_{tt}]=s^2U-su(x,0)-u_t(x,0)=s^2U-scos3\pi x$, $\mathscr{L}[u_{xx}]=\dfrac{\mathrm{d}^2U}{\mathrm{d}x^2}$, $\mathscr{L}[u_x(0,t)]=\dfrac{\mathrm{d}U}{\mathrm{d}x}\Big|_{x=0}=0$, $\mathscr{L}[u_x(1,t)]=\dfrac{\mathrm{d}U}{\mathrm{d}x}\Big|_{x=1}=0$

则 $\begin{cases}\dfrac{\mathrm{d}^2U}{\mathrm{d}x^2}=\dfrac{s^2}{a^2}U-\dfrac{s}{a^2}cos3\pi x\\ \dfrac{\mathrm{d}U}{\mathrm{d}x}\Big|_{x=0}=0,\dfrac{\mathrm{d}U}{\mathrm{d}x}\Big|_{x=1}=0\end{cases}$ 解得 $U(x,s)=\dfrac{s}{s^2+9\pi^2a^2}cos3\pi x$

则 $u(x,t)=cos3\pi atcos3\pi x$

4. 另解： 根据已知边界条件,应用有限傅里叶余弦变换,记 $v(n,t)=2\int_0^1 u(x,t)cos\,(n\pi x)\,\mathrm{d}x$

则上述问题化为 $\begin{cases} \dfrac{\mathrm{d}^2v}{\mathrm{d}t^2}+n^2\pi^2a^2v=0 \\ v(n,0)=\Phi(n)=2\displaystyle\int_0^1 cos3\pi xcosn\pi x\mathrm{d}x, v_t(n,0)=0 \end{cases}$

解得 $v(n,t)=\Phi(n)cosn\pi at=\begin{cases} 0, \quad n\neq 3 \\ cos3\pi at, \quad n=3 \end{cases}$

则 $u(x,t)=cos3\pi atcos3\pi x$

练习十三

1. 解： 记 $\mathscr{F}[u(x,t)]=U(\lambda,t)$,则 $\mathscr{F}[u_{xx}]=-\lambda^2U$,$\mathscr{F}[u_t]=\dfrac{\mathrm{d}U}{\mathrm{d}t}$,$\mathscr{F}[u(x,0)]=$

$U(\lambda,0)=\mathscr{F}[\varphi(x)]$

则 $\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}=-\lambda^2t^2U \\ U(\lambda,0)=\mathscr{F}[\varphi(x)] \end{cases}$ 解得 $U(\lambda,t)=\mathscr{F}[\varphi(x)]e^{-\frac{1}{3}\lambda^2t^3}$

则 $u(x,t)=\varphi(x)*\left(\sqrt{\dfrac{3}{4\pi t^3}}e^{-\frac{3x^2}{4t^3}}\right)=\sqrt{\dfrac{3}{4\pi t^3}}\displaystyle\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{3(x-\xi)^2}{4t^3}}\mathrm{d}\xi$

2. 解： 记 $\mathscr{F}[u(x,t)]=U(\lambda,t)$,则 $\mathscr{F}[u_{xx}]=-\lambda^2U$,$\mathscr{F}[u_t]=\dfrac{\mathrm{d}U}{\mathrm{d}t}$,$\mathscr{F}[u(x,0)]=$

$U(\lambda,0)=\mathscr{F}[\varphi(x)]$

则 $\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}=-a^2\lambda^2U+kU \\ U(\lambda,0)=\mathscr{F}[\varphi(x)] \end{cases}$ 解得 $U(\lambda,t)=\mathscr{F}[\varphi(x)]e^{(k-a^2\lambda^2)t}$

则 $u(x,t)=e^{kt}\varphi(x)*\left(\dfrac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}\right)=\dfrac{e^{kt}}{2a\sqrt{\pi t}}\displaystyle\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}\mathrm{d}\xi$

3. 解： 记 $\mathscr{L}[u(x,t)]=U(x,s)$，则 $\mathscr{L}[u_x]=\dfrac{\mathrm{d}U}{\mathrm{d}x}$，$\mathscr{L}[u_t]=sU-u(x,0)=$

$sU-1$，$\mathscr{L}[u(0,t)]=U(0,s)=0$

则 $\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}x}=-sU-U+1 \\ U(0,s)=0 \end{cases}$ 解得 $U(x,s)=\dfrac{1}{s+1}-\dfrac{1}{s+1}e^{-(s+1)x}$

则 $u(x,t)=\begin{cases} e^{-t}, & 0<t<x \\ 0, & t>x \end{cases}$

4. 解： 记 $\mathscr{F}[u(x,t)]=U(\lambda,t)$, 则 $\mathscr{F}[u_{xx}]=-\lambda^2 U$, $\mathscr{F}[u_t]=\dfrac{\mathrm{d}U}{\mathrm{d}t}$, $\mathscr{F}[u(x,0)]=U(\lambda,0)=\mathscr{F}[\varphi(x)]$

则 $\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}=-\lambda^2 U-2tU \\ U(\lambda,0)=\mathscr{F}[\varphi(x)] \end{cases}$ 解得 $U(\lambda,t)=\mathscr{F}[\varphi(x)]e^{-\lambda^2 t-t^2}$

则 $u(x,t)=e^{-t^2}\varphi(x)*\left(\dfrac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right)=\dfrac{e^{-t^2}}{\sqrt{4\pi t}}\displaystyle\int_{-\infty}^{+\infty}\varphi(\xi)e^{-\frac{(x-\xi)^2}{4t}}\mathrm{d}\xi$

练习十四

1. 解： 显然 u 不是常数, 由极值原理可知, u 的最小值只能在边界 Γ_R 上取到, 又 $u|_{\Gamma_R}=1+sinxy^2z^3\geqslant 0$, 则在 K_R 内有 $u>min\{u|_{\Gamma_R}\}\geqslant 0$, 即 $u>0$

2. 解： 显然 u 不是常数，由极值原理可知，u 的最大值、最小值只能在边界上取到，又 $u|_{\Gamma_r}=1, u|_{\Gamma_R}=2$，则在 $K_R\setminus\bar{K}_r$ 内，有 $min\{u|_{\Gamma_r},u|_{\Gamma_R}\}<u<max\{u|_{\Gamma_r},u|_{\Gamma_R}\}$，即 $1<u<2$

3. 解： 设 Γ_1 为上半球面，Γ_2 为下半球面，则 $0<u|_{\Gamma_1}=1-sin\theta<1$，$u|_{\Gamma_2}=0$，显然 u 不是常数，由极值原理可知，u 的最大值、最小值只能在边界取到，则在 K_1 内，有 $min\{u|_{\Gamma_1},u|_{\Gamma_2}\}<u<max\{u|_{\Gamma_1},u|_{\Gamma_2}\}$，即 $0<u<1$

又由平均值定理,有 $u|_{r=0}=\dfrac{1}{4\pi}\displaystyle\iint_{\Gamma}u\mathrm{d}S=\dfrac{1}{4\pi}\left(\iint_{\Gamma_1}u\mathrm{d}S+\iint_{\Gamma_2}u\mathrm{d}S\right)$

$=\dfrac{1}{4\pi}\displaystyle\iint_{\Gamma_1}(1-sin\theta)\mathrm{d}S=\dfrac{1}{4\pi}\int_0^{\pi}\int_0^{\frac{1}{2\pi}}(1-sin\theta)sin\theta\mathrm{d}\theta\mathrm{d}\varphi=\dfrac{1}{4}-\dfrac{\pi}{16}$

练习十五

1. 解： 平面上的格林公式为：

$$\iint_D (u\Delta v - v\Delta u)\mathrm{d}\sigma = \int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) \tag{1}$$

注意到，函数 $ln\frac{1}{r_{MM_0}} = ln\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$ 在除去 M_0 的区域内处处满足拉普拉斯方程，其中 $M_0(x_0, y_0)$ 是区域 D 内某一固定点

在式 (1) 中，令 u 为调和函数，且取 $v = ln\frac{1}{r}$，在区域 D 内挖去一个以 M_0 为中心，充分小正数 ε 为半径的圆 $K_\varepsilon^{M_0}$，在区域 $D - K_\varepsilon^{M_0}$ 上对上述的调和函数 u 和 $v = ln\frac{1}{r}$ 应用公式（1）得

$$0 = \int_{C+C_\varepsilon} \left(u\frac{\partial}{\partial n}\left(ln\frac{1}{r}\right) - ln\frac{1}{r}\frac{\partial u}{\partial n}\right)\mathrm{d}S \tag{2}$$

其中，C_ε 是圆 $K_\varepsilon^{M_0}$ 的圆周

$$\int_C \left(u\frac{\partial}{\partial n}\left(ln\frac{1}{r}\right) - ln\frac{1}{r}\frac{\partial u}{\partial n}\right)\mathrm{d}S + \int_{C_\varepsilon} \left(u\frac{\partial}{\partial n}\left(ln\frac{1}{r}\right) - ln\frac{1}{r}\frac{\partial u}{\partial n}\right)\mathrm{d}S = 0 \tag{3}$$

在圆周 C_ε 上

$$\frac{\partial}{\partial n}\left(ln\frac{1}{r}\right) = -\frac{\partial}{\partial r}\left(ln\frac{1}{r}\right) = \frac{1}{r} = \frac{1}{\varepsilon} \tag{4}$$

由此可得

$$\int_{C_\varepsilon} u\frac{\partial}{\partial n}\left(ln\frac{1}{r}\right)\mathrm{d}S = \frac{1}{\varepsilon}\int_{C_\varepsilon} u\mathrm{d}S = \frac{1}{\varepsilon}\cdot 2\pi\varepsilon\bar{u} = 2\pi\bar{u} \tag{5}$$

其中，$\bar{u}$ 是函数 u 在圆周 C_ε 上的平均值

另一方面，由于在圆周 C_ε 上

$$\begin{aligned}\int_{C_\varepsilon} ln\frac{1}{r}\frac{\partial u}{\partial n}\mathrm{d}S = ln\frac{1}{\varepsilon}\int_{C_\varepsilon}\frac{\partial u}{\partial n}\mathrm{d}S = -ln\frac{1}{\varepsilon}\int_{C_\varepsilon}\frac{\partial u}{\partial r}\mathrm{d}S \\ = -ln\frac{1}{\varepsilon}\iint_{K_\varepsilon^{M_0}} \Delta u\mathrm{d}\sigma = 0\end{aligned} \tag{6}$$

于是将式（5）与式（6）代入式（3）可得

$$\int_C \left(u\frac{\partial}{\partial n}\left(ln\frac{1}{r}\right)-ln\frac{1}{r}\frac{\partial u}{\partial n}\right)\mathrm{d}S+2\pi\bar{u}=0 \tag{7}$$

现在令 $\varepsilon\to 0$，由于 $\lim\limits_{\varepsilon\to 0}\bar{u}=u(M_0)$，由上式就可得到二维情形下，调和函数 u 的积分表达式

$$u(M_0)=-\frac{1}{2\pi}\iint_\Gamma\left[u(M)\frac{\partial}{\partial n}\left(ln\frac{1}{r_{MM_0}}\right)-ln\frac{1}{r_{MM_0}}\frac{\partial u(M)}{\partial n}\right]\mathrm{d}S \tag{8}$$

2. 解： 格林函数 $G(M,M_0)=\dfrac{1}{2\pi}ln\dfrac{1}{r_{MM_0}}-v$

由镜像法可得 $v=\dfrac{1}{2\pi}ln\dfrac{1}{r_{MM_1}}$，其中，点 M_1 为点 M_0 关于 $y=0$ 的对称点

则 $G(M,M_0)=\dfrac{1}{2\pi}\left(ln\dfrac{1}{r_{MM_0}}-ln\dfrac{1}{r_{MM_1}}\right)=\dfrac{1}{2\pi}ln\dfrac{r_{MM_1}}{r_{MM_0}}$

$=\dfrac{1}{4\pi}ln\dfrac{(x-x_0)^2+(y+y_0)^2}{(x-x_0)^2+(y-y_0)^2}$

则 $\dfrac{\partial G}{\partial n}\Big|_C=\dfrac{\partial G}{\partial y}\Big|_{y=0}=\dfrac{1}{\pi}\dfrac{y_0}{(x-x_0)^2+y_0^2}$

则 $u(M_0)=-\displaystyle\int_C f(x)\frac{\partial G}{\partial n}\mathrm{d}s=-\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{y_0 f(x)}{(x-x_0)^2+y_0^2}\mathrm{d}x$

3. 解： (1) 设 $u(r,\theta)=R(r)\Phi(\theta)$，代入方程得 $-\dfrac{r^2R''+rR'}{R}=\dfrac{\Phi''}{\Phi}=-\lambda$，

即 $r^2R''+rR'-\lambda R=0,\Phi''+\lambda\Phi=0$

易得 $u(r,\theta)=\dfrac{1}{2}a_0+\displaystyle\sum_{n=1}^{+\infty}\left(a_n cosn\theta+b_n sinn\theta\right)r^n$

则 $u(1,\theta)=\dfrac{1}{2}a_0+\displaystyle\sum_{n=1}^{+\infty}\left(a_n cosn\theta+b_n sinn\theta\right)=2cos\theta$

则 $a_1=2,a_n=0,b_n=0$

则 $u(r,\theta)=2rcos\theta$

(2) 格林函数 $G(M,M_0)=\dfrac{1}{2\pi}ln\dfrac{1}{r_{MM_0}}-v$

用镜像法可得 $v=\dfrac{1}{2\pi}ln\left(\dfrac{1}{r_0}\dfrac{1}{r_{MM_1}}\right)$，其中，$r_0=r_{OM_0}$，点 M_1 为点 M_0 关于圆周 C 的对称点

则 $G(M,M_0)=\frac{1}{2\pi}\left[ln\frac{1}{r_{MM_0}}-ln\left(\frac{1}{r_0}\frac{1}{r_{MM_1}}\right)\right]=\frac{1}{2\pi}ln\frac{r_0r_{MM_1}}{r_{MM_0}}$
$=\frac{1}{4\pi}ln\frac{r_0^2(r_1^2+r^2-2r_1rcos\gamma)}{r_0^2+r^2-2r_0rcos\gamma}=\frac{1}{4\pi}ln\frac{1+r_0^2r^2-2r_0rcos\gamma}{r_0^2+r^2-2r_0rcos\gamma}$

其中，$r_1=r_{MM_1}$，$r=r_{OM}$，$r_0r_1=1$，$coa\gamma=cos(\theta-\theta_0)$

在圆周 C 上，$\frac{\partial G}{\partial n}\Big|_C=\frac{\partial G}{\partial r}\Big|_{r=1}=\frac{1}{2\pi}\frac{r_0^2-1}{r_0^2+1-2r_0cos\gamma}$

则 $u(M_0)=-\int_C 2cos\theta\frac{\partial G}{\partial n}\mathrm{d}s=-\frac{1}{\pi}\int_0^{2\pi}\frac{(r_0^2-1)cos\theta}{r_0^2+1-2r_0cos(\theta-\theta_0)}\mathrm{d}\theta$

4. 解： 设 $u(r,\theta)=A'rcos\theta+B'rsin\theta+C$，显然满足方程

代入边界条件可得，$A'=\frac{A}{R},B'=\frac{B}{R},C=0$

则 $u(r,\theta)=\frac{A}{R}cos\theta+\frac{B}{R}sin\theta$

练习十六

1. 解： 假设该问题有两个解 u_1 和 u_2，设 $u=u_1-u_2$

则 u 满足 $\begin{cases}\Delta u=0, & (x,y,z)\in\Omega \\ \frac{\partial u}{\partial n}+ku\Big|_\Gamma=0, & (x,y,z)\in\Gamma\end{cases}$

由格林第一公式 $\iiint_\Omega u\Delta v\mathrm{d}\Omega=\iint_\Gamma u\frac{\partial v}{\partial n}\mathrm{d}S-\iiint_\Omega\nabla u\cdot\nabla v\mathrm{d}\Omega$

取 $v=u$，则 $\iiint_\Omega u\Delta u\mathrm{d}\Omega=\iint_\Gamma u\frac{\partial u}{\partial n}\mathrm{d}S-\iiint_\Omega(\nabla u)^2\mathrm{d}\Omega$

又 $(\nabla u)^2=\Delta u=0$，$\frac{\partial u}{\partial n}+ku\Big|_\Gamma=0$

则 $\iint_\Gamma ku^2\mathrm{d}S=0$，又 $k>0$，则 $u\equiv0$，即 $u_1=u_2$

所以该问题的解是唯一的

2. 解： 由格林第二公式 $\iiint_{\Omega_R}(u\Delta v-v\Delta u)\mathrm{d}\Omega=\iint_\Gamma\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)\mathrm{d}S$

其中 Ω_R 为区域 Ω 内的任意球体，Γ 为其球面，取 $v=1$

则 $\iiint_{\Omega_R}\Delta u\mathrm{d}\Omega=\iint_\Gamma\frac{\partial u}{\partial n}\mathrm{d}S=0$

由 Ω_R 的任意性, 且其可取遍整个 Ω 区域, 则 $\Delta u\equiv0,\forall(x,y,z)\in\Omega$

即 u 是 Ω 上的调和函数

3. 解： 先证必要性

由格林第二公式 $\iiint_{\Omega}(u\Delta v - v\Delta u)\mathrm{d}\Omega = \iint_{\Gamma}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S$

其中 Ω 为任意球体，Γ 为其球面，取 $v=1$

则 $\iiint_{\Omega}\Delta u\mathrm{d}\Omega = \iint_{\Gamma}\frac{\partial u}{\partial n}\mathrm{d}S$

又 u 在 $\mathbb{R}^3$ 中下调和, 则 $\Delta u \geqslant 0$, 则 $\iint_{\Gamma}\frac{\partial u}{\partial n}\mathrm{d}S = \iiint_{\Omega}\Delta u\mathrm{d}\Omega \geqslant 0$

再证充分性

由格林第二公式 $\iiint_{\Omega}(u\Delta v - v\Delta u)\mathrm{d}\Omega = \iint_{\Gamma}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\mathrm{d}S$

其中 Ω 为任意球体，Γ 为其球面，取 $v=1$

则 $\iiint_{\Omega}\Delta u\mathrm{d}\Omega = \iint_{\Gamma}\frac{\partial u}{\partial n}\mathrm{d}S \geqslant 0$

由 Ω 的任意性，则 $\Delta u \geqslant 0, \forall(x,y,z)\in\mathbb{R}^3$，即 u 在 $\mathbb{R}^3$ 中下调和

则 u 在 $\mathbb{R}^3$ 中下调和的充分必要条件是对任何球面 Γ 都有

$$\iint_{\Gamma}\frac{\partial u}{\partial n}\mathrm{d}S \geqslant 0$$

练习十七

1. 解： (1) $\frac{\mathrm{d}}{\mathrm{d}x}[xJ_0(x)J_1(x)] = J_0(x)[xJ_1(x)]' + xJ_1(x)J_0'(x)$

$= xJ_0^2(x) - xJ_1^2(x) = x\left[J_0^2(x) - J_1^2(x)\right]$

(2) $\int x^2J_1(x)\mathrm{d}x = -\int x^2\mathrm{d}(J_0(x)) = -x^2J_0(x) + 2\int xJ_0(x)\mathrm{d}x$

$= 2xJ_1(x) - x^2J_0(x) + C$

(3) $J_2(x) - J_0(x) = -2J_1'(x) = 2J_0''(x)$

(4) $\int x^nJ_0(x)\mathrm{d}x = \int x^{n-1}\mathrm{d}(xJ_1(x)) = x^nJ_1(x) - (n-1)\int x^{n-1}J_1(x)\mathrm{d}x$

$= x^nJ_1(x) + (n-1)\int x^{n-1}\mathrm{d}(J_0(x))$

$= x^nJ_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2\int x^{n-2}J_0(x)\mathrm{d}x$

2. 解：（1） $\displaystyle\int_0^3 (3-x)J_0\left(\frac{\mu_2^{(0)}}{3}x\right)\mathrm{d}x = 3\int_0^3 J_0\left(\frac{\mu_2^{(0)}}{3}x\right)\mathrm{d}x - \int_0^3 xJ_0\left(\frac{\mu_2^{(0)}}{3}x\right)\mathrm{d}x$

$$= \frac{9}{\mu_2^{(0)}}\int_0^{\mu_2^{(0)}} J_0(x)\mathrm{d}x - \frac{9}{(\mu_2^{(0)})^2}\int_0^{\mu_2^{(0)}} xJ_0(x)\mathrm{d}x$$

$$= \frac{9}{\mu_2^{(0)}}\int_0^{\mu_2^{(0)}} J_0(x)\mathrm{d}x - \frac{9}{\mu_2^{(0)}}J_1(\mu_2^{(0)})$$

（2） $\displaystyle\int_0^R r(R^2-r^2)J_0\left(\frac{\mu_m^{(0)}}{R}r\right)\mathrm{d}r = R^2\int_0^R rJ_0\left(\frac{\mu_m^{(0)}}{R}r\right)\mathrm{d}r - \int_0^R r^3J_0\left(\frac{\mu_m^{(0)}}{R}r\right)\mathrm{d}r$

$$= \frac{R^4}{(\mu_m^{(0)})^2}\int_0^{\mu_m^{(0)}} xJ_0(x)\mathrm{d}x - \frac{R^4}{(\mu_m^{(0)})^4}\int_0^{\mu_m^{(0)}} x^3J_0(x)\mathrm{d}x$$

$$= \frac{R^4}{\mu_m^{(0)}}J_1(\mu_m^{(0)}) - \frac{R^4}{(\mu_m^{(0)})^4}\left[(\mu_m^{(0)})^3J_1(\mu_m^{(0)}) - 2(\mu_m^{(0)})^2J_2(\mu_m^{(0)})\right]$$

$$= \frac{2R^4}{(\mu_m^{(0)})^2}J_2(\mu_m^{(0)}) = \frac{4R^4}{(\mu_m^{(0)})^3}J_1(\mu_m^{(0)})$$

3. 解： 将 x 展成 $J_1(\mu_m^{(1)})$ 的傅里叶-贝塞尔级数：$x = \displaystyle\sum_{m=1}^{+\infty} C_m J_1(\mu_m^{(1)}x)$

其中，$C_m = \dfrac{\int_0^1 x^2J_1(\mu_m^{(1)}x)\mathrm{d}x}{\frac{1}{2}J_0^2(\mu_m^{(1)})}$

又 $\displaystyle\int_0^1 x^2J_1(\mu_m^{(1)}x)\mathrm{d}x = \frac{1}{(\mu_m^{(1)})^3}\int_0^{\mu_m^{(1)}} x^2J_1(x)\mathrm{d}x = -\frac{J_0(\mu_m^{(1)})}{\mu_m^{(1)}}$

则 $C_m = -\dfrac{2}{\mu_m^{(1)}J_0(\mu_m^{(1)})}$

则 $x = -\displaystyle\sum_{m=1}^{+\infty}\frac{2}{\mu_m^{(1)}J_0(\mu_m^{(1)})}J_1(\mu_m^{(1)}x)$

练习十八

1. 解： 设 $u(r,t) = R(r)T(t)$，代入方程得 $\dfrac{T'}{T} = \dfrac{R'' + \frac{1}{r}R'}{R} = -\lambda$，则

$T' + \lambda T = 0, r^2R'' + rR' + \lambda r^2R = 0$

$R(r)$ 满足 0 阶贝塞尔方程,其通解为 $R(r) = AJ_0(\sqrt{\lambda}r) + BY_0(\sqrt{\lambda}r)$,

又 $|u(r,t)| < +\infty$，则 $|R(0)| < +\infty$，则 $B = 0$

又 $u(1,t) = 0$，则 $R(1) = 0$，则 $J_0(\sqrt{\lambda}) = 0$，则 $\lambda_m = (\mu_m^{(0)})^2$

则 $R_m(r) = J_0(\mu_m^{(0)}r)$，$T_m(t) = C_me^{-(\mu_m^{(0)})^2t}$

则 $u(r,t)=\sum\limits_{m=1}^{+\infty}C_m e^{-(\mu_m^{(0)})^2 t}J_0(\mu_m^{(0)}r)$

则 $u(r,0)=\sum\limits_{m=1}^{+\infty}C_m J_0(\mu_m^{(0)}r)=1-r^2$

则 $C_m=\dfrac{\int_0^1 r(1-r^2)J_0(\mu_m^{(0)}r)\mathrm{d}r}{\frac{1}{2}J_1^2(\mu_m^{(0)})}$

又 $\displaystyle\int_0^1 rJ_0(\mu_m^{(0)}r)\mathrm{d}r=\frac{1}{\mu_m^{(0)}}J_1(\mu_m^{(0)}),\int_0^1 r^3J_0(\mu_m^{(0)}r)\mathrm{d}r=\frac{1}{\mu_m^{(0)}}J_1(\mu_m^{(0)})-\frac{2}{(\mu_m^{(0)})^2}J_2(\mu_m^{(0)})$

则 $C_m=\dfrac{4J_2(\mu_m^{(0)})}{(\mu_m^{(0)})^2J_1^2(\mu_m^{(0)})}=\dfrac{8}{(\mu_m^{(0)})^3J_1(\mu_m^{(0)})}$

则 $u(r,t)=\sum\limits_{m=1}^{+\infty}\frac{8}{(\mu_m^{(0)})^3J_1(\mu_m^{(0)})}e^{-(\mu_m^{(0)})^2 t}J_0(\mu_m^{(0)}r)$

2. 解：设 $u(r,z)=R(r)Z(z)$，代入方程得 $-\dfrac{Z''}{Z}=\dfrac{R''+\frac{1}{r}R'}{R}=-\lambda$，即 $Z''-\lambda Z=0,r^2R''+rR'+\lambda r^2R=0$

则 $R(r)$ 满足 0 阶贝塞尔方程，其通解为 $R(r)=AJ_0(\sqrt{\lambda}r)+BJ_0(\sqrt{\lambda}r)$，又 $u(1,z)=0,|u(0,z)|<+\infty$，即 $R(1)=0,|R(0)|<+\infty$，则 $B=0,\lambda_m=(\mu_m^{(0)})^2$

则 $R_m(r)=J_0(\mu_m^{(0)}r),Z_m(z)=C_me^{\mu_m^{(0)}z}+D_me^{-\mu_m^{(0)}z}$

则 $u(r,z)=\sum\limits_{m=1}^{+\infty}\left(C_me^{\mu_m^{(0)}z}+D_me^{-\mu_m^{(0)}z}\right)J_0(\mu_m^{(0)}r)$

又 $u_z(r,0)=\sum\limits_{m=1}^{+\infty}\left(C_m\mu_m^{(0)}-D_m\mu_m^{(0)}\right)J_0(\mu_m^{(0)}r)=0$，

$u(r,1)=\sum\limits_{m=1}^{+\infty}\left(C_me^{\mu_m^{(0)}}+D_me^{-\mu_m^{(0)}}\right)J_0(\mu_m^{(0)}r)=1$

则 $C_m\mu_m^{(0)}-D_m\mu_m^{(0)}=0$，$C_me^{\mu_m^{(0)}}+D_me^{-\mu_m^{(0)}}=\dfrac{\int_0^1 rJ_0(\mu_m^{(0)}r)\mathrm{d}r}{\frac{1}{2}J_1^2(\mu_m^{(0)})}=\dfrac{2}{\mu_m^{(0)}J_1(\mu_m^{(0)})}$

则 $C_m=D_m=\dfrac{\int_0^1 rJ_0(\mu_m^{(0)}r)\mathrm{d}r}{\frac{1}{2}J_1^2(\mu_m^{(0)})}=\dfrac{2}{\left(e^{\mu_m^{(0)}}+e^{-\mu_m^{(0)}}\right)\mu_m^{(0)}J_1(\mu_m^{(0)})}$

则 $u(r,z)=\sum\limits_{m=1}^{+\infty}\dfrac{2}{\left(e^{\mu_m^{(0)}}+e^{-\mu_m^{(0)}}\right)\mu_m^{(0)}J_1(\mu_m^{(0)})}\left(e^{\mu_m^{(0)}z}+e^{-\mu_m^{(0)}z}\right)J_0(\mu_m^{(0)}r)$

3. 解：易得，固有函数系为 $\{J_0(\mu_m^{(0)}r),m=1,2,\cdots\}$

设 $u(r,t)=\sum\limits_{m=1}^{+\infty}u_m(t)J_0(\mu_m^{(0)}r)$，又 $A=\sum\limits_{m=1}^{+\infty}\dfrac{2A}{\mu_m^{(0)}J_1(\mu_m^{(0)})}J_0(\mu_m^{(0)}r)$，

代入方程，

得 $\sum\limits_{m=1}^{+\infty}u_m''(t)J_0(\mu_m^{(0)}r)-\sum\limits_{m=1}^{+\infty}a^2u_m(t)\left\{\left[J_0(\mu_m^{(0)}r)\right]''+\dfrac{1}{r}\left[J_0(\mu_m^{(0)}\right]'\right\}=$
$\sum\limits_{m=1}^{+\infty}\dfrac{2A}{\mu_m^{(0)}J_1(\mu_m^{(0)})}J_0(\mu_m^{(0)}r)$

由零阶贝塞尔方程可知，

$\left[J_0(\mu_m^{(0)}r)\right]''+\dfrac{1}{r}\left[J_0(\mu_m^{(0)}\right]'+(\mu_m^{(0)})^2J_0(\mu_m^{(0)}r)=0$

则 $\sum\limits_{m=1}^{+\infty}\left[u_m''(t)+(\mu_m^{(0)}a)^2u_m(t)-\dfrac{2A}{\mu_m^{(0)}J_1(\mu_m^{(0)})}\right]J_0(\mu_m^{(0)}r)=0$

则 $u_m''(t)+(\mu_m^{(0)}a)^2u_m(t)-\dfrac{2A}{\mu_m^{(0)}J_1(\mu_m^{(0)})}=0$

又 $u(r,0)=0,u_t(r,0)=0$，即 $u_m(0)=0,u_m'(0)=0$

解得 $u_m(t)=\dfrac{2A}{(\mu_m^{(0)})^3a^2J_1(\mu_m^{(0)})}\left[1-cos(\mu_m^{(0)}at)\right]$

则 $u(r,t)=\sum\limits_{m=1}^{+\infty}\dfrac{2A}{(\mu_m^{(0)})^3a^2J_1(\mu_m^{(0)})}\left[1-cos(\mu_m^{(0)}at)\right]J_0(\mu_m^{(0)}r)$

4. 解：易得，固有函数系为 $\{J_0(\mu_m^{(0)}r),m=1,2,\cdots\}$

设 $u(r,t)=\sum\limits_{m=1}^{+\infty}u_m(t)J_0(\mu_m^{(0)}r)$，代入方程，

得 $\sum\limits_{m=1}^{+\infty}u_m'(t)J_0(\mu_m^{(0)}r)-\sum\limits_{m=1}^{+\infty}a^2u_m(t)\left\{\left[J_0(\mu_m^{(0)}r)\right]''+\dfrac{1}{r}\left[J_0(\mu_m^{(0)}\right]'\right\}=$
$\sum\limits_{m=1}^{+\infty}u_m(t)J_0(\mu_m^{(0)}r)$

由零阶贝塞尔方程可知，

$\left[J_0(\mu_m^{(0)}r)\right]''+\dfrac{1}{r}\left[J_0(\mu_m^{(0)}\right]'+(\mu_m^{(0)})^2J_0(\mu_m^{(0)}r)=0$

则 $\sum\limits_{m=1}^{+\infty}\left\{u_m'(t)+\left[(\mu_m^{(0)}a)^2-1\right]u_m(t)\right\}J_0(\mu_m^{(0)}r)=0$

则 $u_m'(t)+\left[(\mu_m^{(0)}a)^2-1\right]u_m(t)=0$

又 $u(r,0)=\sum\limits_{m=1}^{+\infty}u_m(0)J_0(\mu_m^{(0)}r)=1-r$,则 $u_m(0)=\dfrac{2}{(\mu_m^{(0)})^3J_1^2(\mu_m^{(0)})}\displaystyle\int_0^{\mu_m^{(0)}}J_0(x)\mathrm{d}x$

解得 $u_m(t)=\dfrac{2\int_0^{\mu_m^{(0)}}J_0(x)\mathrm{d}x}{(\mu_m^{(0)})^3J_1^2(\mu_m^{(0)})}e^{-\left[(\mu_m^{(0)}a)^2-1\right]t}$

则 $u(r,t)=\sum\limits_{m=1}^{+\infty}\dfrac{2\int_0^{\mu_m^{(0)}}J_0(x)\mathrm{d}x}{(\mu_m^{(0)})^3J_1^2(\mu_m^{(0)})}e^{-\left[(\mu_m^{(0)}a)^2-1\right]t}J_0(\mu_m^{(0)}r)$

附录

Theorem. 对于方程 $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)-q(x)y+\lambda\rho(x)y=0$，其固有函数系 $\{y_n(x)\}$ 带权 $\rho(x)$ 正交。[3]

证明. 设函数 $y_n(x)$ 与 $y_m(x)$ 是是对应固有值 λ_n 与 λ_m 的两个解，其中 $n\neq m$，则有

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right)-q(x)y_n+\lambda_n\rho(x)y_n=0 \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\right)-q(x)y_m+\lambda_m\rho(x)y_m=0 \tag{10}$$

方程 (1) 乘以 $y_m(x)$ 减去方程 (2) 乘以 $y_n(x)$ 可得

$$y_m\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right)-y_n\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\right)+\rho(x)y_ny_m(\lambda_n-\lambda_m)=0$$

积分得

$$\begin{aligned}(\lambda_m-\lambda_n)\int_a^b\rho(x)y_ny_m\mathrm{d}x&=\int_a^b\left[y_m\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right)-y_n\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\right)\right]\mathrm{d}x\\&=\int_a^b\left[y_m\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right)\right]\mathrm{d}x-\int_a^b\left[y_n\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\right)\right]\mathrm{d}x\\&=\left[y_mp(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\right]\Bigg|_a^b-\int_a^b p(x)\frac{\mathrm{d}y_n}{\mathrm{d}x}\frac{\mathrm{d}y_m}{\mathrm{d}x}\mathrm{d}x-\left[y_np(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\right]\Bigg|_a^b\\&\quad+\int_a^b p(x)\frac{\mathrm{d}y_m}{\mathrm{d}x}\frac{\mathrm{d}y_n}{\mathrm{d}x}\mathrm{d}x\\&=0\end{aligned}$$

又 $\lambda_n\neq\lambda_m$，则有

$$\int_a^b\rho(x)y_ny_m\mathrm{d}x=0$$

即固有函数系 $\{y_n(x)\}$ 带权 $\rho(x)$ 正交 □

[3]在讨论其固有函数问题时，有条件 $y(a)=y(b)=0$