(Sol) HW10 P vs NP

1. [10 points] **Clique Problem:** Given an undirected graph $G = (V, E)$ and an integer k, the problem is to determine whether there exists a *clique* (a set of vertices where every pair of distinct vertices is adjacent) of size or more in G. Choose one of the three problems we discussed in class on Nov. 15th (Independent Set, Vertex Cover, and Set Cover), and show how the chosen problem poly-time reduces to the Clique problem.

    a. [2 points] Show the construction steps.
    Answer: Given an instance of the Vertex Cover problem with graph $G = (V, E)$ and integer k, construct a new graph $G' = (V, E')$ as follows:

        i. For each pair of distinct vertices u and v, if (u, v) is not in E, add then add (u, v) to E'. Note that G is undirected so (u, v) is equivalent to (v, u).

        ii. $k' = |V| - k$

    b. [2 points] Analyze the time complexity of the construction steps.
    Answer: $\Theta(|V|^2)$

    c. [3 points] Show the conversion steps.
    Answer: Given a clique of size k' or more in G', let's call the clique C. $V \setminus C$ is a vertex cover for G. Create an empty set VC. For every vertex v in , if $v \notin C$ then add to VC .

    d. [2 points] Explain how after the conversion you'd have the solution to the chosen problem. (Example: why is the independent set S you get from the vertex cover $V \setminus S$ of size $|V| - k$ is indeed an independent set of size k?)
    Answer: For VC to be a vertex cover, every edge in G should have at least one incident vertex in VC. For every edge (u, v) in G, if u is in C, there is no edge between u and v in G', so v cannot be in the clique C. Thus v has to be in VC. if u is not in C, then u is in VC, so at least one of the incident vertices is in VC. Thus VC is a vertex cover. Since the clique C's size is k' $= |V| - k$, the VC's size is k, so given a clique C of size k' in G', $VC = V \setminus C$ is a vertex cover of size k in G.

e. [2 points] Analyze the time complexity of the conversion steps.
   Answer: $\Theta(|V|)$.

2. [10 points] NP-complete Problems
   The following are well-known NP-complete problems that every computer scientist should know. You can refer to the book chapter where the problem is formally defined, or you can refer to KT Chapter 8.10 (which provides the taxonomy of NP-C problems). You don't need to know about NP-completeness to answer this question.

   Feel free to include a short explanation/justification, but that is not necessary.
   For each of the following, decide whether the input instance is a yes-instance or a no-instance.
   Answer: they are all yes-instances. Please see the examples inline.

   a. Consider the following input instance of the 3-SAT problem (see KT Chapter 8.2).
      $n = 5$ (five variables) and $m = 4$ (four clauses): $(x1 \vee x2 \vee x3) \wedge (x2 \vee \bar{x4} \vee \bar{x1}) \wedge (\bar{x2} \vee x4 \vee \bar{x5}) \wedge (\bar{x2} \vee x5 \vee \bar{x3})$.
      Example: $x_1 = 1$ $x_2 = 1$, $x_4 = 1$, $x_5 = 1$, and any value for x_3.

   b. Consider the following input instance of the TSP (see KT Chapter 8.5). $G = (V, E)$ with $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2, 2), (v_2, v_3, 4), (v_3, v_4, 3), (v_4, v_1, 2), (v_1, v_3, 2), (v_3, v_2, 2), (v_2, v_4, 2), (v_1, v_4, 5), (v_3, v_1, 5)\}$, and $D = 8$. In E, (v_x, v_y, w) describes an edge from v_x to v_y with weight w.
      Example: $v_2 \to v_4 \to v_1 \to v_3 \to v_2$.

   c. Consider the following input instance of the Hamiltonian Cycle problem (see KT Chapter 8.5).
      G = (V, E) with V = {v1, v2, v3, v4, v5} and E = {(v1, v2), (v2, v3), (v3, v1), (v4, v1), (v3, v5), (v5, v4)} (where (x, y) means an edge is oriented from x to y).
      Example: $v_1 \to v_2 \to v_3 \to v_5 \to v_4 \to v_1$.

   d. Consider the following input instance of the 3-D Matching problem (see KT Chapter 8.6).
      Let n = 3 and T = {(x1, y1, z2), (x1, y3, z2), (x2, y3, z2), (x2, y1, z3), (x3, y2, z1), (x3, y2, z2)}.
      Example: $\{(x_1, y_3, z_2), (x_2, y_1, z_3), (x_3, y_2, z_1)\}$

e. Consider the following input instance of the k-coloring problem (see KT Chapter 8.7).
G = (V,E) with V = {v1,v2,v3,v4,v5} and E = {(v1,v2),(v2,v3),(v3,v1),(v2,v4),(v3,v5)} and k = 3.
Example: color1=$\{v_1, v_4\}$, color2=$\{v_2, v_5\}$, color3=$\{v3\}$.

f. Consider the following input instance of the Subset Sum problem (see KT Chapter 8.8).
n = 8 with w = [1,4,16,32,64,128,256,512] and W = 673.
Example: $\{1, 32, 128, 512\}$

g. Consider the following input instance of the Independent Set problem (see KT Chapter 8.1).
G = (V,E) with V = {v1,v2,v3,v4,v5} and E = {(v1,v2),(v2,v3),(v3,v1),(v2,v4),(v3,v5)} and k = 3.
Example: $\{v_1, v_4, v_5\}$

h. Consider the following input instance of the Set Packing problem (see KT Chapter 8.1).
U={1,2,3,4,5}(n=5),S1 ={1,3},S2 ={1,2,4},S3 ={2,3,5},S4 ={4},S5 ={5}(so m=5),and
k = 3.
Example: $\{s_1, s_4, s_5\}$

i. Consider the following input instance of the Vertex Cover problem (see KT Chapter 8.1).
G = (V,E) with V = {v1,v2,v3,v4,v5} and E = {(v1,v2),(v2,v3),(v3,v1),(v2,v4),(v3,v5)} and k = 2.
Example: $\{v_2, v_3\}$

j. Consider the following input instance of the Set Cover problem (see KT Chapter 8.1).
U={1,2,3,4,5}(n=5),S1 ={1,3},S2 ={1,2,4},S3 ={2,3,5},S4 ={4},S5 ={5}(so m=5),and
k = 2.
Example: $\{v_2, v_3\}$