HW 2 CME 241

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Problem 3

Part A

First we will setup our MDP. Our state space for each time t is $\mathcal{S}_t = \{(t, I_t)\}$ for $t \in \{0, \dots, T\}$ and where I_t is an integer that is the store inventory at time t. Our action space is $\mathcal{A} = \{p_1, \dots, p_N\}$. Our non-terminating states are just the $\mathcal{S}_t$. The Bellman Optimality Equation tells us that the optimal value function is one that has

$$V_t^*(s) = \max_{a \in A} \sum_{s', r} \mathcal{P}(r, s' | s_t, a_t)(r + \gamma \cdot V_{t+1}^*(s')). \tag{3.a.1}$$

We know that $a_t = p_k$ for some k. For a given $s = (t, I_t), s_{t+1} = (t+1, I_{t+1})$, r is just the revenue, meaning $r = I_t - I_{t+1}$ (note that $I_t - I_{t+1} \geq 0$ because we never add any masks in). All that is left is to compute $\mathcal{P}(s_t, a_t, r, s') = \mathbb{P}(s_{t+1} = s' | s_t, a_t)$.

Suppose that $s_t = (t, I_t)$, $s' = (t+1, I_{t+1})$, and $a_t = p_\ell$. Then the customer demand was $dI_t = I_t - I_{t+1}$.

If $I_{t+1} > 0$, then we have

$$\begin{aligned}
\mathbb{P}(s_{t+1} = s' | s_t, a_t) &= \mathbb{P}(dt = I_t - I_{t+1} | s_t = (t, I_t), a_t = p_\ell) \\
&= \frac{e^{-\lambda_\ell} \lambda_\ell^{dI_t}}{dI_t!} \\
&= \frac{\exp(-\lambda_\ell) \lambda_\ell^{I_t - I_{t+1}}}{(I_t - I_{t+1})!}.
\end{aligned}$$

If $I_{t+1} = 0$, then it is because $dI_t \geq I_t$, which means

$$\begin{aligned}
\mathbb{P}(s_{t+1} = s' | s_t, a_t) &= \mathbb{P}(dt \geq I_t | s_t = (t, I_t), a_t = p_\ell) \\
&= \sum_{k=I_t}^{\infty} \frac{\exp(-\lambda) \lambda_\ell^k}{k!} \\
&= 1 - \sum_{k=0}^{I_t - 1} \frac{\exp(-\lambda) \lambda_\ell^k}{k!}.
\end{aligned}$$

We can now substitute these into (3.a.1) (we don't substitute for $\mathcal{P}$ to avoid clutter). Note that the inner sum is only over s' because the revenue for a given day (our reward) is determined by what s_{t+1} is. We further parametrize s' by I_{t+1} because that is what determines what our reward and subsequent state is.

$$
\begin{aligned}
V_t^*(s) = \max_{p_k \in \{p_1, \dots, p_N\}} \sum_{I_{t+1}=0}^{I_t} & \big[\mathbb{P}(s_{t+1} = (t+1, I_{t+1}) | s_t = (t, I_t), a_t = p_k) \\
& \cdot \big((I_t - I_{t+1})p_k + \gamma V_t^*((t+1, I_{t+1}))\big)\big]
\end{aligned}
$$

Part B

On day T, the revenue we can expect going forward is just what we expect to earn on day T given an optimal price. Let $s_T = (t, I_T)$ and suppose that D is the amount we sell (note that $D = \min(I_T, dI_T)$).

$$
V_T^*(s) = \max_{p_k \in \{p_1, \dots, p_N\}} \sum_{\ell=0}^{I_T} \mathbb{P}(D = \ell | s_T, a_T = p_k) p_k D
$$

where

$$
\mathbb{P}(D = \ell) = \begin{cases} f_{\lambda_k}(\ell) : & \text{if } \ell > 0 \\ 1 - F_{\lambda_k}(\ell - 1) : & \text{if } \ell = 0. \end{cases}
$$

using the same math as in Part A.

Question 4

Part A

Expanding out our wealth, we have that

$$
\begin{aligned}
W &= (1 + r)(1 - \pi) + (1 + R)\pi \\
&= 1 + r - \pi r + \pi R.
\end{aligned}
$$

Since $R \sim \mathcal{N}(\mu, \sigma^2)$, we know that $\pi R \sim \mathcal{N}(\pi\mu, \pi^2\sigma^2)$, which means (because the left hand sum of W is just a scalar) that

$$
\begin{aligned}
W &\sim \mathcal{N}(1 + r - \pi r + \pi\mu, \pi^2\sigma^2) \\
&\sim \mathcal{N}(1 + r + \pi(\mu - r), \pi^2\sigma^2). \qquad (4.a.1)
\end{aligned}
$$

We will use the fact that W is distributed normally in a second. For now, notice that

$$
\begin{aligned}
\mathbb{E}(U(x)) &= \mathbb{E}(\frac{1 - e^{-aW}}{a}) \\
&= \frac{1}{a} - \frac{1}{a}(\mathbb{E}(e^{-aW})). \qquad (4.a.2)
\end{aligned}
$$

Notice that W is normally distributed and that $\mathbb{E}(e^{-aW})$ is precisely the definition of the MGF of a random variable evaluated at $-a$. If $X \sim \mathcal{N}(\mu', \sigma'^2)$, recall that the MGF is

$$
f(a) = e^{\mu' a + \frac{\sigma'^2 a^2}{2}}.
$$

Substituting in the appropriate mean and variance from (4.a.1), we have

$$\mathbb{E}(e^{aW}) = \exp[a(1+r+\pi(\mu-r)) + \frac{\pi^2\sigma^2a^2}{2}].$$

Using this in (4.a.2), we get

$$\mathbb{E}(U(W)) = \frac{1}{a} - \frac{\exp[-a(1+r+\pi(\mu-r)) + \frac{\pi^2\sigma^2a^2}{2}]}{a}. \tag{4.a.3}$$

We're now ready to solve for W_{CE}. Recall the definition of W_{CE} as

$$U(W_{CE}) = \mathbb{E}(U(W)).$$

Using (4.a.3), we have

$$\begin{aligned}
\implies \quad & \frac{1}{a} - \frac{\exp(-aW_{CE})}{a} = \frac{1}{a} - \frac{\exp[-a(1+r+\pi(\mu-r)) + \frac{\pi^2\sigma^2a^2}{2}]}{a} \\
\implies \quad & \exp(-aW_{CE})) = \exp[-a(1+r+\pi(\mu-r)) + \frac{\pi^2\sigma^2a^2}{2}] \\
\implies \quad & W_{CE} = (1+r+\pi(\mu-r)) - \frac{\pi^2\sigma^2a}{2}. \qquad (4.a.4)
\end{aligned}$$

Part B

Recall from part A (4.a.3) that

$$\mathbb{E}(U(W)) = \frac{1}{a} - \frac{\exp(-a(1+r+\pi(\mu-r)) + \frac{\pi^2\sigma^2a^2}{2})}{a}.$$

Note that by 4.a.1, $\mathbb{E}(W) = 1+r+\pi(\mu-r)$ and $\mathrm{var}(W) = \pi^2\sigma^2$. Taking the derivative with respect to π and setting that equal to 0, we have

$$\begin{aligned}
& 0 = \frac{d}{d\pi}\left[\frac{1}{a} - \frac{\exp(-a(1+r+\pi(\mu-r)) + \frac{\pi^2\sigma^2a^2}{2})}{a}\right] \\
\implies \quad & 0 = -\frac{1}{a}\cdot\frac{d}{d\pi}\exp(-a(1+r+\pi(\mu-r)) + \frac{\pi^2\sigma^2a^2}{2}) \\
\implies \quad & 0 = -\frac{1}{a}\exp\left(-a\mathbb{E}(W) + \frac{\mathrm{var}(W)a^2}{2}\right)\frac{d}{d\pi}\left(-a(1+r+\pi(\mu-r)) + \frac{\pi^2\sigma^2a^2}{2}\right) \\
\implies \quad & 0 = \exp\left(-a\mathbb{E}(W) + \frac{\mathrm{var}(W)a^2}{2}\right)\left[(\mu-r) - \pi\sigma^2a\right].
\end{aligned}$$

Noting in the last line that exp is nonnegative, we have

$$\begin{aligned}
& 0 = \left[(\mu-r) - \pi\sigma^2a\right] \\
\implies \quad & \pi = \frac{\mu-r}{\sigma^2a}.
\end{aligned}$$

So our optimal investment fraction is

$$\pi^* = \frac{\mu-r}{\sigma^2a}. \tag{4.b.1}$$

Part C

Recall the definition of absolute risk premium, we have

$$W_{CE} = \mathbb{E}(W) - \pi_A.$$

Matching up terms with (4.a.4), and recalling that $\mathbb{E}(W) = 1 + r + \pi(\mu - r)$ from the distribution of W in part A, we see that

$$\pi_A = \frac{\pi^2 \sigma^2 a}{2}.$$

Part D

We have that $r = 0.02$, $\mu = 0.08$, $\sigma^2 = 0.04$, and $a = 3$. Using (4.b.1),

$$\begin{aligned} \pi^* &= \frac{0.08 - 0.02}{0.04 \cdot 3} \\ &= 0.5. \end{aligned}$$

Using (4.a.4), and assuming we are using an optimal allocation strategy, we have

$$\begin{aligned} W_{CE} &= (1 + 0.02 + 0.5(0.08 - 0.02)) - \frac{0.5^2 \cdot 0.04 \cdot 3}{2} \\ &= \underbrace{1.05}_{\mathbb{E}(W)} - \underbrace{0.0075}_{\pi_A} \\ &= 1.0425 \end{aligned}$$

Using Part C and realizing that the risk premium is just the second sum of our calculation for W_{CE}, we have

$$\pi_A = 0.0075.$$

Interpretation: The proportion of my portfolio that I should allocate to the risky asset to maximize my expected utility of wealth $\mathbb{E}(U(W))$ is precisely half of my portfolio.

Since $U(W_{CE}) = \mathbb{E}(U(W))$, W_{CE} is precisely how much money you'd have to give me so that the amount I value W_{CE} is the same as the expected utility from the investments. In other words, paying me 1.0425 dollars right now gives me the same value as investing my money.

In practice the expected value of my investment $\mathbb{E}(W)$ is different from our certainty equivalent wealth W_{CE}. The latter being the amount of money you'd have to give me so that I have no preference between receiving W_{CE} and pursuing the risky investment. Instead, W_{CE} is slightly lower because I'm ok with receiving a little less money if my return is certain.

The difference between $\mathbb{E}(W)$ and W_{CE} is π_A, is the risk premium. In this case, \$0.0075 is how much I am willing to pay on top of $\mathbb{E}(W)$ such that pursuing the risky investment with π^* is as enticing as W_{CE} riskless.

Finally, notice that the risk premium is proportional to changes in our risk tolerance a and the variance σ^2 while the allocation percentage is inversely proportional to a and σ^2. This makes sense since we expect our risk premium to go up when the the investment is riskier (more variance) or when we are less tolerant of risk; and we expect that the amount we invest in the risky asset goes down when our risk tolerance goes down (a goes up) or the riskiness of the investment goes up (variance).

Part E

Using the hint, we know that

$$\mathbb{E}(U(W)) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} U(W) dW. \tag{4.e.1}$$

First we'll just solve for the inner integral.

$$\begin{aligned} \int_{x_1}^{x_2} U(x) dx &= \int_{x_1}^{x_2} \left(\frac{1 - e^{-ax}}{a} \right) dx \\ &= \frac{x}{a} + \frac{e^{-ax}}{a^2} \bigg|_{x_1}^{x_2}. \end{aligned} \tag{4.e.2}$$

Using (4.e.2) in (4.e.1), and substituting for x_1 and x_2,

$$\begin{aligned} \mathbb{E}(U(W)) &= \frac{1}{\beta - \alpha} \left(\frac{x}{a} + \frac{e^{-ax}}{a^2} \bigg|_{\alpha}^{\beta} \right) \\ &= \frac{1}{\beta - w_\alpha} \left(\frac{\beta}{a} - \frac{\alpha}{a} + \frac{\exp(-a\beta)}{a^2} - \frac{\exp(-a\alpha)}{a^2} \right) \\ &= \frac{1}{\beta - w_\alpha} \left(\frac{\beta - \alpha}{a} + \frac{\exp(-a\beta) - \exp(-a\alpha)}{a^2} \right) \\ &= \frac{1}{a} + \frac{\exp(-a\beta) - \exp(-a\alpha)}{a^2(\beta - \alpha)} \end{aligned}$$