Introduction to Colorings of the Hypercube

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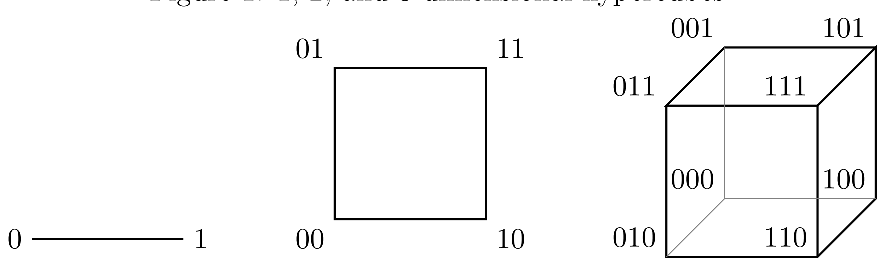
This was originally intended to be an article to cover the results of my undergraduate research to a more generic audience, but it has now become my personal project in writing an introduction to colorings on the hypercube.

Figure 1: 1, 2, and 3-dimensional hypercubes



1 Introduction to the hypercube

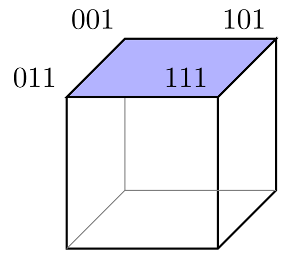
A hypercube is a generalization of a cube to n dimensions. A square and 2-dimensional hypercube are analogous, as is cube and 3-dimensional hypercube. An n-dimensional hypercube is also referred to as an n-cube or n-dimensional cube.

Definition 1.1 *For $n \in \mathbb{Z}$, $n \geq 1$, the n-dimensional hypercube, denoted by Q_n, can be expressed as the graph with vertices $V(Q_n) = \{0,1\}^n$. An edge $(u,v) \in E(Q_n) \iff u$ and v differ by 1 coordinate.*

Theorem 1.1 *For $n, m \in \mathbb{Z}$, $n \geq m \geq 1$, there are $\binom{n}{m}2^{(n-m)}$ unique m-dimensional subcubes in Q_n.*

Proof: To form an embedded Q_m in Q_n, we select m coordinates from $\{0,1\}^n$. We can do this in $\binom{n}{m}$ ways. Note that if we fix the remaining $n-m$ coordinates to some value, the m chosen coordinates can form a subcube Q_m in Q_n. See Figure 2. We can fix the remaining $n-m$ coordinates to $2^{(n-m)}$ possible values. Thus, there are a total of $\binom{n}{m}2^{(n-m)}$ subcubes Q_m in Q_n.

Figure 2: embedded Q_2 in Q_3 with last coordinate fixed



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Standard notation for denoting m-dimensional subcubes of Q_n is an n-bit binary string with m entries replaced by stars. Each vertex of the Q_m is obtained by replacing the stars with 0's and 1's. For instance $[0 * 10 * 101]$ is a Q_2 in Q_8 with vertices $[00100101], [00101101], [01100101], [01101101]$.

Definition 1.2 *We define the counting vector of subcube Q_m in Q_n to be $(v_0, \dots, v_m) \in \mathbb{Z}_{\geq 0}^{m+1}$ where v_i represents the number of 1's between the ith and $(i+1)$th stars. Let the 0th star be right before the string starts and the $(m+1)$th star be right after the string ends.*

For example, $[0 * 10 * 101]$ would have the counting vector $(0, 1, 2)$.

2 Coloring of the hypercube

Definition 2.1 *A coloring of a hypercube Q_n is a surjective function*

$$\chi : E(G) \twoheadrightarrow R$$

where $E(G) \subseteq E(Q_n)$ and set R of r colors.

We say that a Q_ℓ-coloring of Q_n is a coloring where the set of edges in $E(G)$ form a subcube Q_ℓ in Q_n.

Definition 2.2 *A Q_ℓ-coloring of Q_n is d-polychromatic if every subcube $Q_d \in Q_n$ contains a Q_ℓ of every color.*

Definition 2.3 *For $1 \leq \ell \leq d$, $p^\ell(d)$ is the maximum number of colors such that for all $n \geq d$ there is a d-polychromatic Q_ℓ-coloring of Q_n.*

As an example, let us observe $p^1(2)$. $p^1(2) = r \implies r$ is the largest number of colors we can use such that there is a way to paint each of the edges (Q_1) of Q_n for $n \geq 2$ such that for every face (Q_2) in Q_n, there are r unique edge colors.

Trivially, $r \leq 4$ because there are 4 unique Q_1's in a Q_2 by Theorem 1.1 (a square has 4 edges). Similarly, $r \geq 1$ since we can always assign every edge the same color. It turns out that $p^1(2) = 2$.

Motivated by Turán type problems on the hypercube, Alon, Krech, and Szabó [1] proved bounds on $p^1(d)$.

Theorem 2.1 (Alon, Krech, and Szabó [1]) *For all $d \geq 1$,*

$$\binom{d+1}{2} \geq p^1(d) \geq \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$$

The lower bound is given by what Chen [2] called *basic colorings*, which is defined later in this section. Offner demonstrated in [5] that the basic coloring is optimal for edge colorings of the hypercube.

Theorem 2.2 (Offner [5]) *For all $d \geq 1$,*

$$p^1(d) = \left\lfloor \frac{(d+1)^2}{4} \right\rfloor$$

Later, Özkahya and Stanton generalized the bounds of Theorem 2.1 to $p^\ell(d)$ for $\ell > 1$.

Theorem 2.3 (Özkahya and Stanton [6]) *For all $d \geq \ell \geq 1$, let $0 < r \leq \ell + 1$ be such that $r \equiv d+1 \pmod{\ell+1}$. Then*

$$\binom{d+1}{\ell+1} \geq p^\ell(d) \geq \left\lceil \frac{d+1}{\ell+1} \right\rceil^r \left\lfloor \frac{d+1}{\ell+1} \right\rfloor^{\ell+1-r}$$

As with Theorem 2.1, the lower bound in Theorem 2.3 is achieved by basic colorings by [5].

Definition 2.4 *A Q_ℓ coloring χ is called simple if all Q_ℓ's with the same counting vector are assigned the same color, i.e. if χ is a simple coloring and $x, y \in \mathbb{Z}_{\geq 0}^{\ell+1}$ with $x = y$, then $\chi(x) = \chi(y)$.*

An application of Ramsey's theorem, see Lemma 3 from [4], implies that when studying polychromatic colorings on the hypercube, we need only consider simple colorings. Thus, for the rest of this article, Q_ℓ-colorings χ will have the form

$$\chi : \mathbb{Z}_{\geq 0}^{\ell+1} \twoheadrightarrow R$$

Definition 2.5 *A Q_ℓ coloring χ is called linear if the set of colors is a finite abelian group Z and the coloring is induced by an additive map*

$$\chi : \mathbb{Z}_{\geq 0}^{\ell+1} \twoheadrightarrow Z$$

Thus, we let $p^\ell_{lin}(d)$ have the same definition for $p^\ell(d)$ but for linear colorings only.

Definition 2.6 *A linear Q_ℓ-coloring χ is called (ℓ, d)-basic (or just basic when ℓ and d are clear from context) when the finite abelian group Z is a direct sum of quotient groups of Z.*

$$\chi : \mathbb{Z}_{\geq 0}^{\ell+1} \twoheadrightarrow \bigoplus_{i=0}^{\ell} Z/m_i$$

where $\sum_{i=0}^{\ell} m_i = d + 1$ and all $m_i \in \mathbb{Z}^+$.

It was shown in [6] that a (ℓ, d)-basic coloring is always d-polychromatic. Note that all basic colorings are linear, so

$$p^\ell_{bas}(d) \leq p^\ell_{lin}(d)$$

3 Acknowledgements

I worked with and was advised by David Offner on the research [3] done towards hypercubes. I would like to thank David Offner for providing me the opportunity to work with him and for all the valuable guidance given to me.

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