Appendix

Proof of Theorem 1: Let vector $\overrightarrow{y}^*$ be

$$\overrightarrow{y}^*(j) \triangleq \begin{cases} 1, & 1 \le j \le B_0 \\ 0, & B_0 < j \le m \end{cases}$$

thus,

$$\sum_{j=1}^{m}(\overrightarrow{\beta}(j) - \overrightarrow{y}^*(j)) = 0$$

If $1 \le j \le B_0$, $\overrightarrow{\beta}(j) - \overrightarrow{y}^*(j) \le 0$. Similarly, if $B_0 < j \le m$ holds, $\overrightarrow{\beta}(j) - \overrightarrow{y}^*(j) \ge 0$. thus,

$$\begin{aligned}
&\overrightarrow{D}^T\overrightarrow{\beta} - \overrightarrow{D}^T\overrightarrow{y}^* \\
&= \sum_{j=1}^{B_0}\overrightarrow{D}(j)(\overrightarrow{\beta}(j) - \overrightarrow{y}^*(j)) \\
&+ \sum_{j=B_0+1}^{m}\overrightarrow{D}(j)(\overrightarrow{\beta}(j) - \overrightarrow{y}^*(j)) \\
&\ge \overrightarrow{D}(B_0)\sum_{j=1}^{B_0}(\overrightarrow{\beta}(j) - \overrightarrow{y}^*(j)) \\
&+ \overrightarrow{D}(B_0+1)\sum_{j=B_0+1}^{m}(\overrightarrow{\beta}(j) - \overrightarrow{y}^*(j)) \\
&\ge \overrightarrow{D}(B_0+1) - \overrightarrow{D}(B_0)]\sum_{j=B_0+1}^{m}(\overrightarrow{\beta}(j) - \overrightarrow{y}^*(j)) \ge 0
\end{aligned}$$

Finally,we can get

$$\overrightarrow{D}^T\overrightarrow{\beta} - \overrightarrow{D}^T\overrightarrow{\beta}_0 \ge \overrightarrow{D}^T\overrightarrow{y}^* - \overrightarrow{D}^T\overrightarrow{\beta}_0 = B^*$$

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Proof of Lemma 1: Similar to Theorem 1, let

$$\overrightarrow{z}^*(j) \triangleq \begin{cases} 1, & 1 \le j \le m - B_0 \\ 0, & m - B_0 < j \le m \end{cases}$$

and

$$\forall\overrightarrow{\beta} : \sum_{j=1}^{m}\beta(j) = B_0, \quad \overrightarrow{D}^T\overrightarrow{\beta} \ge \sum_{j=m-B_0+1}^{m}\overrightarrow{D}(j)$$

we can get

$$\overrightarrow{D}^T\overrightarrow{\beta_1} - \overrightarrow{D}^T\overrightarrow{\beta_2} \le \sum_{j=m-B_0+1}^{m}\overrightarrow{D}(j) - \sum_{j=1}^{B_0}\overrightarrow{D}(j) = B^\times$$

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Deduction of equation (1):
We can write expression under event $\mathcal{F}_t$ in **BLAG** as

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\mathcal{F}_t}[\overrightarrow{D}^T\overrightarrow{\Delta\beta}^t_{et} - \alpha\overrightarrow{D}^T\overrightarrow{\Delta\beta}^*] \\
&\le \mathop{\mathbb{E}}_{\mathcal{F}_t}\left[\overrightarrow{D}^T\overrightarrow{\Delta\beta}^t_{et} - \sum_{i\in\mathcal{S}_t}\mu_{i,t}\right] + \mathop{\mathbb{E}}_{\mathcal{F}_t}\left[\sum_{i\in\mathcal{S}_t}\mu_{i,t}\right] - \alpha\overrightarrow{D}^T\overrightarrow{\Delta\beta}^* \\
&\le \mathop{\mathbb{E}}_{\mathcal{F}_t}\left[\sum_{i\in\mathcal{S}_t}\left|\overrightarrow{D}^T\overrightarrow{\beta}_i - \mu_{i,t}\right|\right] + \alpha\left(\mathop{\mathbb{E}}_{\mathcal{F}_t}\left[\sum_{i\in\mathcal{S}^*_t}\mu_{i,t}\right] - \overrightarrow{D}^T\overrightarrow{\Delta\beta}^*\right) \\
&\le c\sigma\sum_{i\in\mathcal{S}_t}\frac{1}{\sqrt{T_{i,t}}}
\end{aligned}$$

Thus, we can get the expectation in overall situation

$$\mathbb{E}[\overrightarrow{D}^T\overrightarrow{\Delta\beta}^t_{et} - \alpha\overrightarrow{D}^T\overrightarrow{\Delta\beta}^*] \le c\sigma\sum_{i\in\mathcal{S}_t}\frac{1}{\sqrt{T_{i,t}}} + B^\times\mathcal{P}(\overline{\mathcal{F}_t})$$

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Deduction of equation (2):

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=1}\cdot\overrightarrow{D}\cdot\overrightarrow{\Delta\beta}^t_{et} - \alpha\overrightarrow{D}\cdot\overrightarrow{\Delta\beta}^*\right] \le c\sigma\sum_{t=1}^{T}\sum_{i\in\mathcal{S}_t}\frac{1}{\sqrt{T_{i,t}}} + B^\times\sum_{t=1}\cdot\mathcal{P}(\overline{\mathcal{F}_t}) \\
&\le c\sigma\sum_{i\in\mathcal{V}}\sum_{l=1}^{T_{i,T}-1}\frac{1}{\sqrt{l}} + B^\times\sum_{t=1}\cdot\mathcal{P}(\overline{\mathcal{F}_t}) \le 2c\sigma M\sqrt{T} + B^\times\sum_{t=1}\cdot\mathcal{P}(\overline{\mathcal{F}_t})
\end{aligned}$$

The third equality exchanges the summation order, and notice when $i \in \mathcal{S}_t$, $T_{i,t+1} = T_{i,t} + 1$. The fourth equality holds because $T_{i,T} - 1 \le T$. M is the initial size of ASG where $M \propto \binom{m}{2}$ ■.

Deduction of equation (3):
We can write expression under event $\mathcal{F}_t$ in **CUCB** as

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\mathcal{F}_t}[\overrightarrow{D}^T\overrightarrow{\Delta\beta}^t - \alpha\overrightarrow{D}^T\overrightarrow{\Delta\beta}^*] \le \mathop{\mathbb{E}}_{\mathcal{F}_t}\left[\overrightarrow{D}^T\overrightarrow{\Delta\beta}^t - \sum_{i\in\mathcal{S}_t}\left(\mu_{i,t} - \frac{c\sigma}{\sqrt{T_{i,t}}}\right)\right] \\
&+ \mathop{\mathbb{E}}_{\mathcal{F}_t}\left[\sum_{i\in\mathcal{S}_t}\left(\mu_{i,t} - \frac{c\sigma}{\sqrt{T_{i,t}}}\right)\right] - \alpha\overrightarrow{D}^T\overrightarrow{\Delta\beta}^* \\
&\le \mathop{\mathbb{E}}_{\mathcal{F}_t}\left[\sum_{i\in\mathcal{S}_t}\left|\overrightarrow{D}^T\overrightarrow{\beta}_i - \left(\mu_{i,t} - \frac{c\sigma}{\sqrt{T_{i,t}}}\right)\right|\right] \\
&+ \alpha\left(\mathop{\mathbb{E}}_{\mathcal{F}_t}\left[\sum_{i\in\mathcal{S}^*_t}\left(\mu_{i,t} - \frac{c\sigma}{\sqrt{T_{i,t}}}\right)\right] - \overrightarrow{D}^T\overrightarrow{\Delta\beta}^*\right) \le 2c\sigma\sum_{i\in\mathcal{S}_t}\frac{1}{\sqrt{T_{i,t}}}
\end{aligned}$$

Thus, we can get the expectation in overall situation

$$\mathbb{E}[\overrightarrow{D}^T\overrightarrow{\Delta\beta}^t - \alpha\overrightarrow{D}^T\overrightarrow{\Delta\beta}^*] \le 2c\sigma\sum_{i\in\mathcal{S}_t}\frac{1}{\sqrt{T_{i,t}}} + B^\times\mathcal{P}(\overline{\mathcal{F}_t})$$

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