Seminario 3

Diffusion Processes

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Overview

1. Examples of Markov processes
2. Markov Processes and the Chapman-Kolmogorov equation
3. The Generator of a Markov Process

2.1.- Examples of Markov Processes

Random Walk in one dimension

Definition (Markov Process (Informal))

A Markov process is a stochastic process for which, given the present, the past and the future are statistically independent.

Example: Discrete-time Markov Chain

Example (Random Walk in 1 dimension)

Let $\{\xi_i\}_{i\in\mathbb{N}}$ be *iid* with $\mathbb{E}[\xi_i] = 0$ for all $i \in \mathbb{N}$, then the one-dimensional random walk is defined as:

$$X_N = \sum_{n=1}^{N} \xi_n, \quad X_0 = 0.$$

Definition (Discrete-time Markov Chain)

A stochastic process $\{X_n\}_{n\in\mathbb{N}}$ with state space $S = \mathbb{Z}$ is called a *discrete-time Markov chain* if it satisfies the *Markov property*, that is, for all integers n, m and all sequences $\{i_k\}_{k\in\mathbb{N}} \subseteq \mathbb{Z}$:

$$\mathbb{P}(X_{n+m} = i_{n+m} | X_1 = i_1, \dots, X_n = i_n) = \mathbb{P}(X_{n+m} = i_{n+m} | X_n = i_n).$$

Proposition

The one-dimensional random walk $\{X_n\}_{n\in\mathbb{N}}$ satisfies the Markov property and is therefore a discrete-time Markov chain.

Example: Continuous-time Markov Chain

Example (Poisson Process)

Let $\{N_t\}_{t\geq 0}$ be a counting process where N_t represents the number of events up to time t. A Poisson process with rate $\lambda > 0$ is defined by $N_0 = 0$ and the probability:

$$\mathbb{P}(N_{t+h} = j | N_t = i) = \begin{cases} 0, & \text{if } j < i, \\ \frac{e^{-\lambda h}(\lambda h)^{j-i}}{(j-i)!}, & \text{if } j \geq i. \end{cases}$$

Definition (Continuous-time Markov Chain)

A stochastic process $\{X_t\}_{t\geq 0}$ with state space $S = \mathbb{Z}$ is a *continuous-time Markov chain* if it satisfies the Markov property:

$$\mathbb{P}(X_{t+h} = i_{t+h} | \{X_s, s \leq t\}) = \mathbb{P}(X_{t+h} = i_{t+h} | X_t), \quad \forall h \geq 0.$$

Proposition

The Poisson process $\{N_t\}_{t\geq 0}$ satisfies the Markov property and is therefore a continuous-time Markov chain.

Definition (Continuous-time Markov Process)

A stochastic process $\{X_t\}_{t\geq 0}$ with state space $S = \mathbb{R}$ is called a *continuous-time Markov process* if for all Borel sets Γ and for all $h \geq 0$, it satisfies the Markov property:

$$\mathbb{P}(X_{t+h} \in \Gamma | \{X_s, s \leq t\}) = \mathbb{P}(X_{t+h} \in \Gamma | X_t).$$

If there exists a conditional probability density $p(y, t+h|x, t)$ such that

$$\mathbb{P}(X_{t+h} \in \Gamma | X_t = x) = \int_\Gamma p(y, t+h|x, t)\, dy \quad (\ddagger),$$

then we say that the process admits a transition density function. Additionally, we call $(\ddagger)$ the Chapman-Kolmogorov equation.

Examples of Continuous-time Markov Processes

Example 1: Brownian Motion. The Brownian motion $\{W_t\}_{t\geq 0}$ is a Markov process with transition probability density given by:

$$\mathbb{P}(W_{t+h} \in \Gamma | W_t = x) = \int_\Gamma \frac{1}{\sqrt{2\pi h}} \exp\left(-\frac{|x-y|^2}{2h}\right) dy.$$

Example 2: Ornstein-Uhlenbeck Process. The stationary Ornstein-Uhlenbeck process $V_t = e^{-t} W(e^{2t})$ is a Markov process with transition probability density:

$$p(y, t | x, s) = \frac{1}{\sqrt{2\pi(1 - e^{-2(t-s)})}} \exp\left(-\frac{|y - xe^{-(t-s)}|^2}{2(1 - e^{-2(t-s)})}\right).$$

In both cases, the Markov property follows from the fact that each motion has independent increments.

Examples of continuous-time Markov Processes

Proof.

For $t > s$:

$$
\begin{aligned}
\mathbb{P}(V_t \leq y | V_s = x) &= \mathbb{P}(e^{-t}W(e^{2t}) \leq y | e^{-s}W(e^{2s}) = x) \\
&= \mathbb{P}(W(e^{2t}) \leq e^t y | W(e^{2s}) = e^s x) \\
&= \int_{-\infty}^{e^t y} \frac{1}{\sqrt{2\pi(e^{2t} - e^{2s})}} \exp\left(-\frac{|z - xe^s|^2}{2(e^{2t} - e^{2s})} \right) dz \\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi e^{2t}(1 - e^{-2(t-s)})}} \exp\left(-\frac{|\rho e^t - xe^s|^2}{2(e^{2t}(1 - e^{-2(t-s)}))} \right) d\rho \\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi(1 - e^{-2(t-s)})}} \exp\left(-\frac{|\rho - xe^{-(t-s)}|^2}{2(1 - e^{-2(t-s)})} \right) d\rho.
\end{aligned}
$$

□

Examples of continuous-time Markov Processes and Chapman-Kolmogorov Equation

Proof.

Then, the *transition conditional probability density* is given by:

$$
\begin{aligned}
p(y,t|x,s) &= \frac{\partial}{\partial y}\mathbb{P}\left(V_t \leq y | V_s = x\right) \\
&= \frac{1}{\sqrt{2\pi(1-e^{-2(t-s)})}} \exp\left(-\frac{|y - xe^{-(t-s)}|^2}{2(1-e^{-2(t-s)})}\right).
\end{aligned}
$$

$\square$

The Markov property enables us to obtain an evolution of the Markov chain in terms of the transition probability density for a discrete-time or continuous-time Markov chain. The evolution is given by the *Chapman-Kolmogorov equation*.

Chapman-Kolmogorov Equation

For a time-homogeneous Markov process, the transition matrix P is:

$$p_{ij} = \mathbb{P}(X_{n+1} = j | X_n = i).$$

The matrix P is stochastic, meaning:

- $p_{ij} \geq 0$ for all $i, j \in S$.
- $\sum_{j \in S} p_{ij} = 1$ for all $i \in S$.

Moreover, we can define the n-step transition matrix $P(n)$ as:

$$p_{ij}(n) = \mathbb{P}(X_{m+n} = j | X_m = i).$$

The Chapman-Kolmogorov equation for time-homogeneous processes is:

$$p_{ij}(m+n) = \sum_{k \in S} p_{ik}(m) p_{kj}(n).$$

Chapman-Kolmogorov Equation

The above is equivalent to:

$$P(m+n) = P(m)P(n).$$

Then, starting with an initial probability vector $\mu^{(0)}$, we can obtain the probability distribution at time n as:

$$\mu^{(n)} = \mu^{(0)} P^n.$$

The Generator of a Markov Process

Consider a continous-time Markov Chain with transition probability:

$$p_{ij}(s,t) = \mathbb{P}(X_t = j | X_s = i).$$

If the chain is homogeneous, then:

$$p_{ij}(s,t) = p_{ij}(0,t), \quad \forall s,t \geq 0.$$

So we can define $p_{ij}(s,t) := p_{ij}(0,t)$ as the transition probability density. The Chapman-Kolmogorov equation can be written as:

$$\frac{\partial}{\partial t} p_{ij}(t) = \sum_{k \in S} p_{ik}(t) g_{kj} \quad (\ddagger)$$

Where G is called the generator of the Markov chain and it is defined as:

$$G = \lim_{h \to 0} \frac{1}{h}(P_h - I)$$

The Generator of a Markov Process

Thus, we have that (‡) is equivalent to:

$$\frac{\partial P_t}{\partial t} = P_t G.$$

Let define $\mu_t^i := \mathbb{P}(X_t = i)$ This vector is the distribution of the Markov chain at time t.
Then, we have that:

$$\mu_t = \mu_0 P_t.$$

2.2.- Markov Processes and the Chapman-Kolmogorov equation

Filtration

In order to give the definition of a Markov process with $T = \mathbb{R}^+$ and $S = \mathbb{R}^d$, we need to use the conditional expectation of the stochastic process conditioned on all past values. We can encode all past information about a stochastic process into an appropriate collection of σ-algebras.

Definition (σ-algebra generated by a stochastic process)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\{X_t\}_{t \in T}$ be a stochastic process on it with space state $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. The σ-algebra generated by the process is the smallest σ-algebra $\sigma(X_t, t \in T)$ on Ω such that X_t is measurable.

Filtration

Definition (Filtration)

Definimos una filtración en $(\Omega, \mathcal{F})$ como una familia no decreciente $\{\mathcal{F}_t, t \in \mathbb{R}_+\}$ de sub-σ-álgebras de $\mathcal{F}$:

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad \text{para } s \leq t.$$

Definimos $\mathcal{F}_\infty = \sigma(\cup_{t \in T} \mathcal{F}_t)$. La filtración generada por nuestro proceso estocástico X_t es:

$$\mathcal{F}_t^X := \sigma(X_s, s \leq t).$$

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2.3.- The Generator of a Markov Process

Theorem

Theorem (Mass–energy equivalence)

$E = mc^2$ *aa*

References

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