Computer Aided Geometric Design
Fall Semester 2025

Mathematical background: Linear algebra

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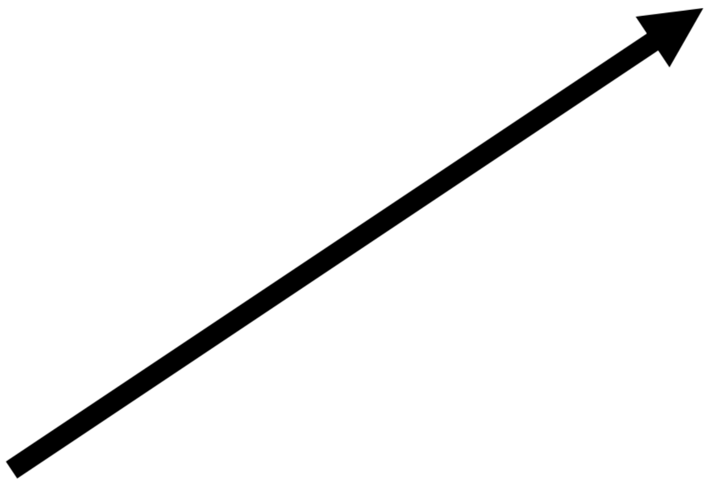
Vector Spaces

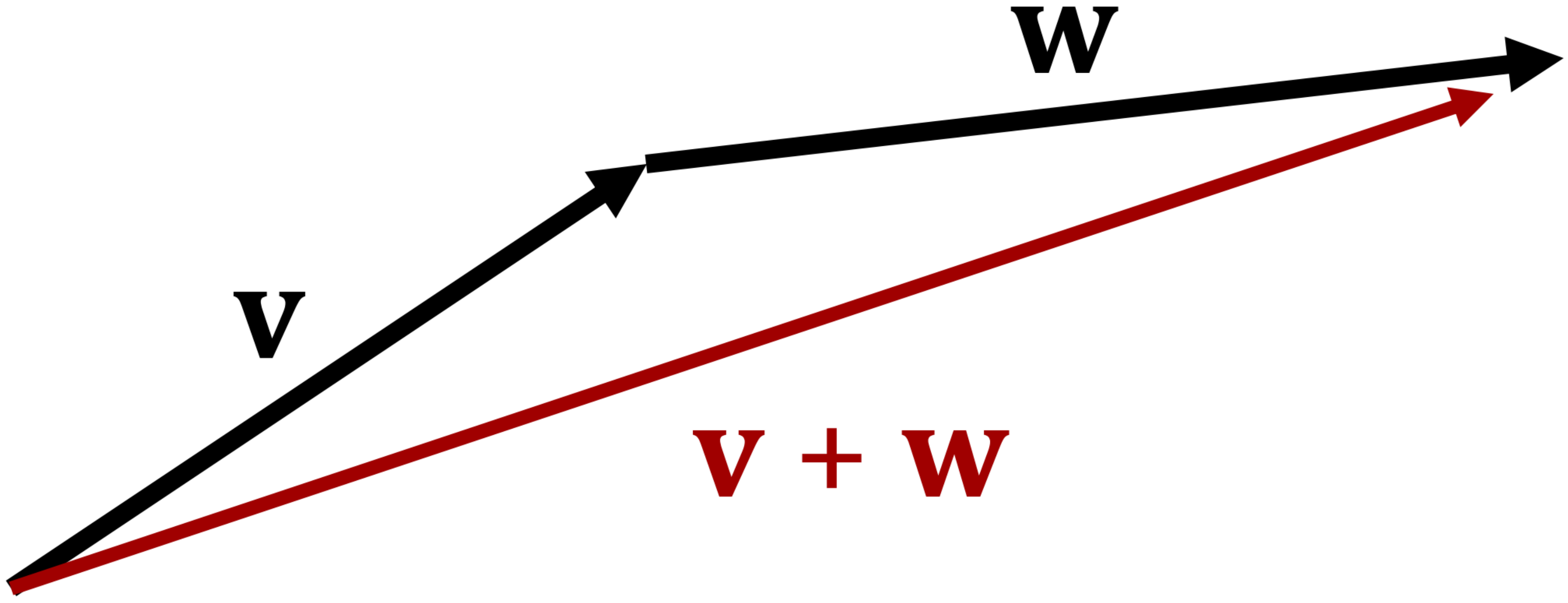
Vectors



Vectors are arrows in space

Classically: 2 or 3 dim. Euclidean space

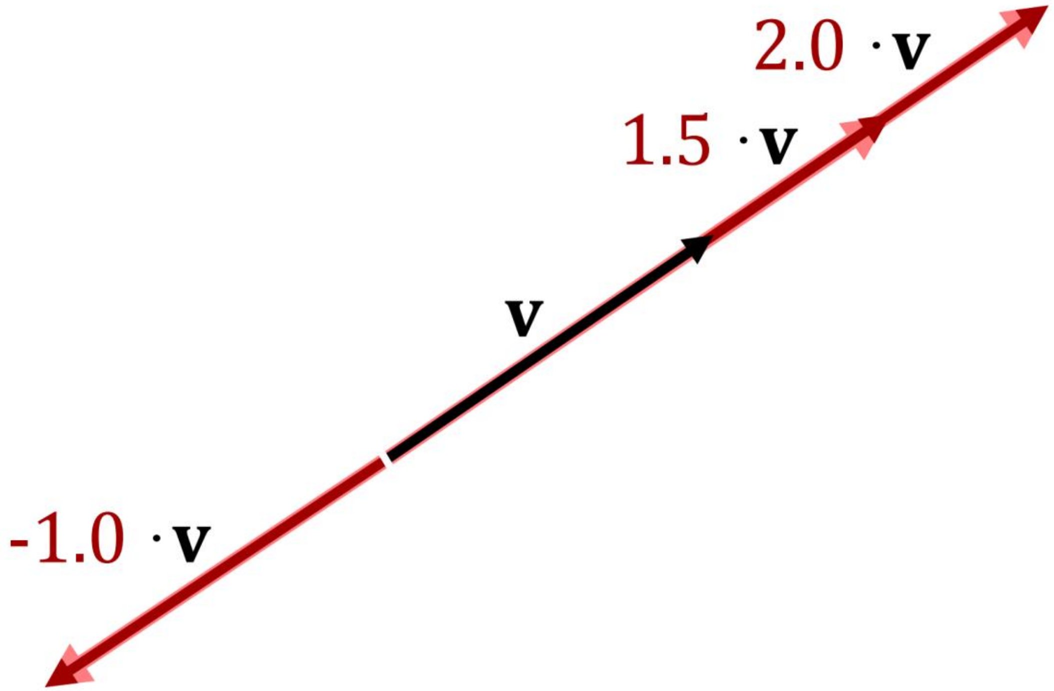
Vector Operations



“Adding” Vectors:

concatenation

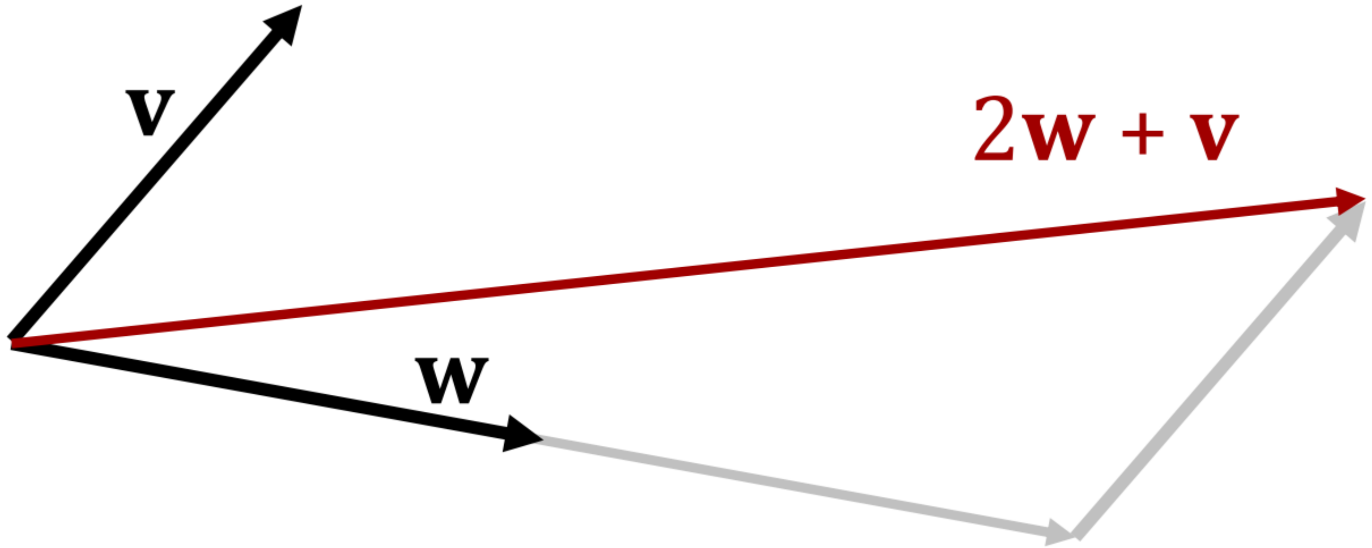
Vector Operations



Scalar Multiplication:

Scaling vectors (incl. mirroring)

You can combine it…



Linear Combinations:

This is basically all you can do.

$$\boldsymbol{r} = \sum_{i=1}^{n} \lambda_i \boldsymbol{v}_i$$

Vector Spaces

- Definition: A *vector space* over a field F (e.g. $\mathbb{R}$) is a set V together with two operations
  - Addition of vectors $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$
  - Multiplication with scalars $\boldsymbol{w} = \lambda \boldsymbol{v}$

  such that

1. $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V: (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
2. $\forall \boldsymbol{u}, \boldsymbol{v} \in V: \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
3. $\exists \boldsymbol{0}_V \in V: \forall v \in V: \boldsymbol{v} + \boldsymbol{0}_V = \boldsymbol{v}$
4. $\forall \boldsymbol{v} \in V: \exists \boldsymbol{w} \in V: \boldsymbol{v} + \boldsymbol{w} = \boldsymbol{0}_V$

$(V, +)$ is an Abelian group

5. $\forall \boldsymbol{v} \in V, \lambda, \mu \in F: \lambda(\mu \boldsymbol{v}) = (\lambda \mu) \boldsymbol{v}$
6. for $1_F \in F: \forall v \in V: 1_F \boldsymbol{v} = \boldsymbol{v}$
7. $\forall \lambda \in F: \forall \boldsymbol{v}, \boldsymbol{w} \in V: \lambda(\boldsymbol{v} + \boldsymbol{w}) = \lambda \boldsymbol{v} + \lambda \boldsymbol{w}$
8. $\forall \lambda, \mu \in F, \boldsymbol{v} \in V: (\lambda + \mu) \boldsymbol{v} = \lambda \boldsymbol{v} + \mu \boldsymbol{v}$

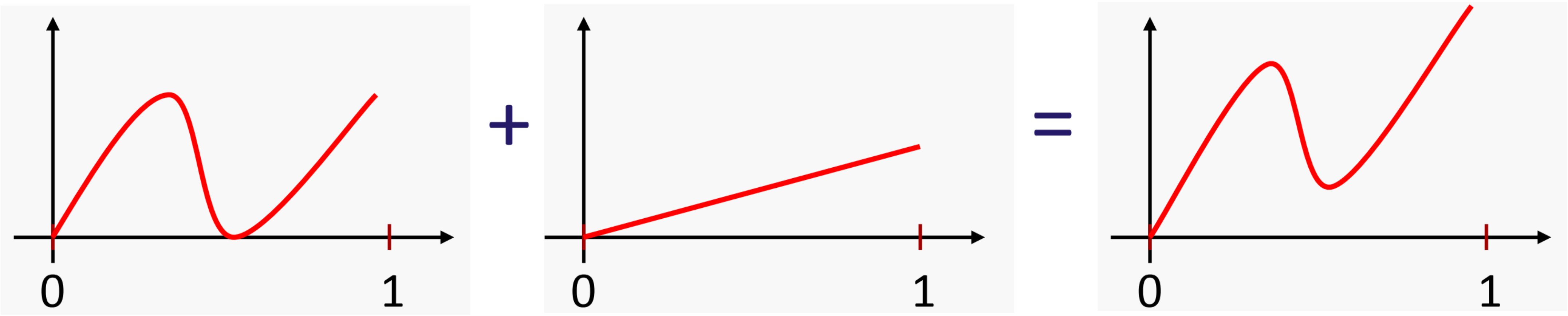
The multiplication is compatible with the addition

Vector spaces

- **Subspaces**
  - A non-empty subset $W \subset V$ is a *subspace* if W is a vector space (w.r.t the induced addition and scalar multiplication).
  - Only need to check if the addition and scalar multiplication are closed.
$$\boldsymbol{v}, \boldsymbol{w} \in W \quad \Rightarrow \boldsymbol{v} + \boldsymbol{w} \in W$$
$$\boldsymbol{v} \in W, \lambda \in F \quad \Rightarrow \lambda \boldsymbol{v} = W$$
  - What are the subspaces of $\mathbb{R}^3$?

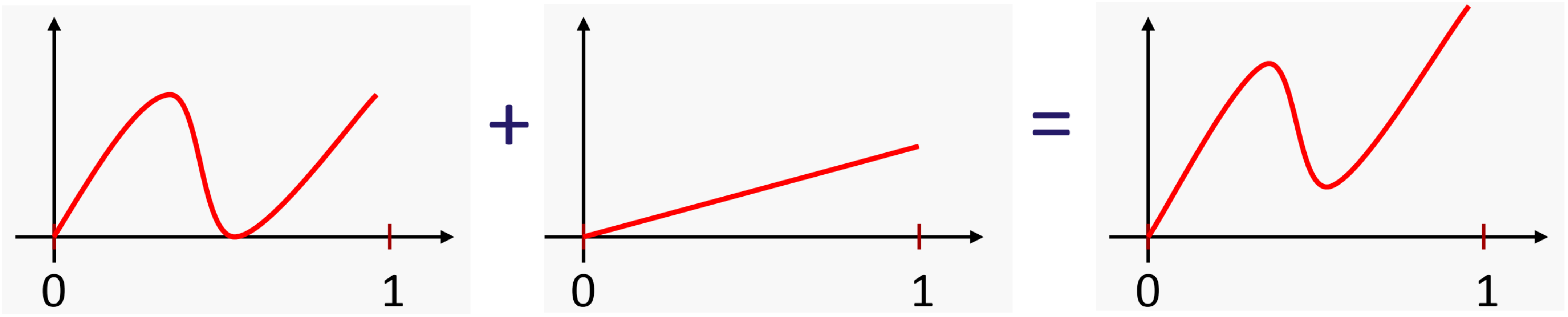
Examples Spaces

- **Function spaces:**
  - Space of all functions $f: \mathbb{R} \to \mathbb{R}$
  - Addition: $(f + g)(x) = f(x) + g(x)$
  - Scalar multiplication: $(\lambda f)(x) = \lambda f(x)$
  - Check the definition



Examples Spaces

- **Function spaces:**
  - Domains and codomain need to be $\mathbb{R}$
  - For example: space of all functions $f: [0,1]^5 \to \mathbb{R}^8$
  - Codomain must be a vector space (Why?)



Examples of Subspaces

- **Continuous / differentiable functions**
  - The continuous / differentiable functions form a subspace of the space of all functions $f: D \subset R^m \rightarrow R^n$
  - Why?

- **Polynomials**
  - The polynomials form a subspace of the space of functions $f: \mathbb{R} \rightarrow \mathbb{R}$
  - The polynomials of degree $\leq n$ again form a subspace
  - Adding polynomials

$$\sum_{i=1}^{n} a_i x^i + \sum_{i=1}^{n} b_i x^i = \sum_{i=1}^{n} (a_i + b_i) x^i$$

Constructing Spaces

Linear Span

- The *linear span* of a subset $S \subset V$ is the "smallest subspace" of V that contains S
- What does that mean?
  - For any subspace W such that $S \subset W \subset V$, we have $span(S) \subset W$
- Construction: Any $v \in span(S)$ is a finite linear combination of elements of S

$$v = \sum_{i=1}^{n} \lambda_i s^i$$

Spanning set

- A subset $S \subset V$ is a *spanning set* of V if $span(S) = V$

Vector spaces

- **Linear independence**
  - A subset $S \subset V$ is *linearly independent* if no vector of S is a finite linear combination of the other vectors of S
- **Basis**
  - A *basis* of a vector space is a linearly independent spanning set.

Dimension

- **Lemma**
  - If V has a finite basis of n elements, then all bases of V have n elements

- **Dimension**
  - If V has a finite basis, then the dimension of V is the number of elements of the basis
  - If V has no finite basis, then the dimension of V is infinite

Examples

- **Polynomials of degree $\leq \boldsymbol{n}$**
  - A basis? What is the dimension?

  Solution:
  - An example of a basis is $\{1, x, x^2, \dots, x^n\}$
  - Dimension is $n + 1$

- **Space of all polynomials**
  - A basis? What is the dimension?

  Solution:
  - An example of a basis is $\{1, x, x^2, \dots\}$
  - Dimension is infinite

Finite dimensional vector spaces

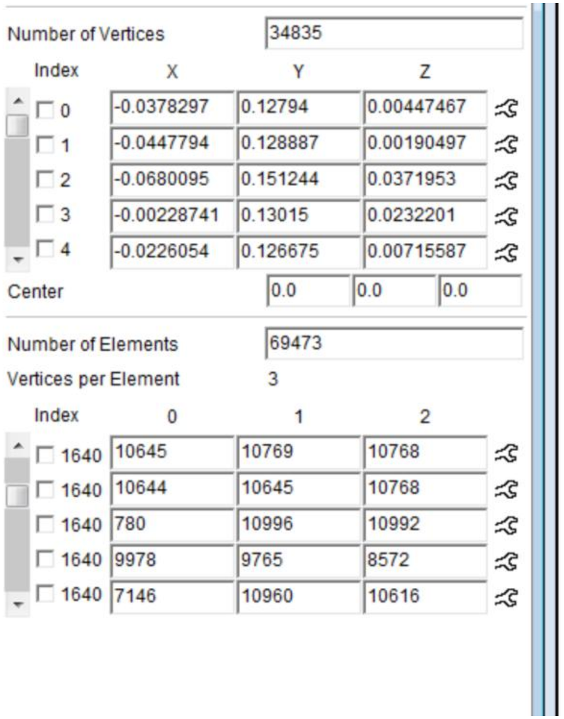
- **Vector spaces**
  - Any finite-dim., real vector space is isomorphic to $\mathbb{R}^n$
    - Array of numbers
    - Behave like arrows in a flat (Euclidean) geometry
  - Proof:
    - Construct basis
    - Represent as span of basis vectors

Isomorphism is not unique, since we can choose different bases

Another Example of a Vector Space

Representation of a triangle mesh in $\mathbb{R}^3$

- Vertices : a finite set $\{v_1, \ldots, v_n\}$ of points in $\mathbb{R}^3$
- Faces: a list of triplets, e.g. $\{\{2, 34, 7\}, \ldots, \{14, 7, 5\}\}$



Number of Vertices 34835

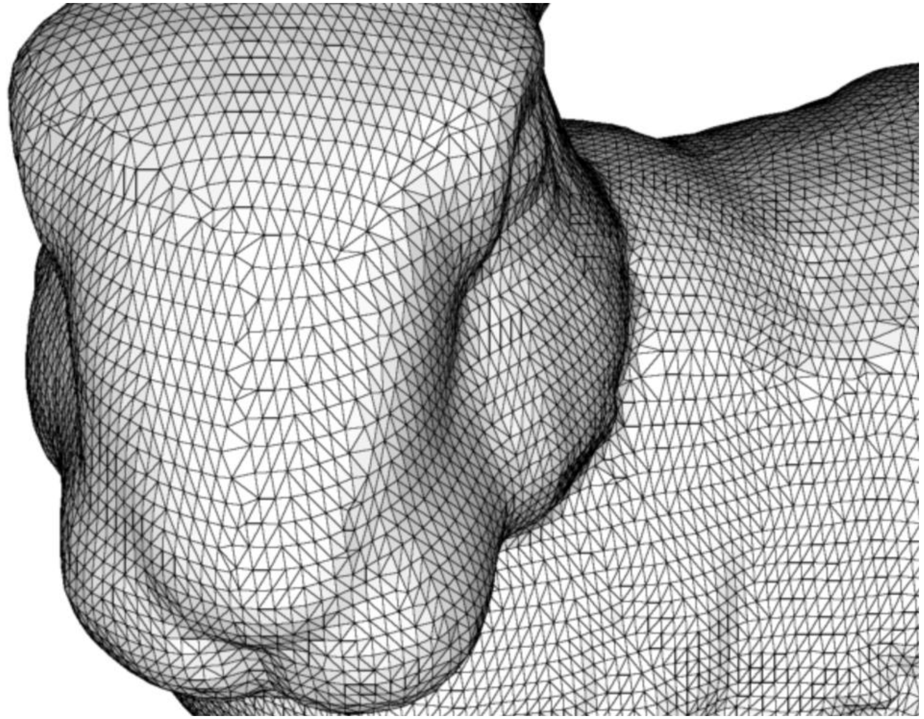
Index	X	Y	Z
0	-0.0378297	0.12794	0.00447467
1	-0.0447794	0.128887	0.00190497
2	-0.0680095	0.151244	0.0371953
3	-0.00228741	0.13015	0.0232201
4	-0.0226054	0.126675	0.00715587

Center 0.0 0.0 0.0

Number of Elements 69473

Vertices per Element 3

Index	0	1	2
1640	10645	10769	10768
1640	10644	10645	10768
1640	780	10996	10992
1640	9978	9765	8572
1640	7146	10960	10616



Another Example of a Vector Space

- **Shape space**
  - Vary the vertices, but keep the face list fixed
  - Is isomorphic to $\mathbb{R}^{3n}$

Linear Maps

Linear Maps

Definition

- A map $L: V \to W$ between vector spaces V, W is linear if
  - $\forall v_1, v_2 \in V$: $L(v_1 + v_2) = L(v_1) + L(v_2)$
  - $\forall v \in V, \lambda \in F$: $L(\lambda v) = \lambda L(v)$

This means that L is compatible with the linear structure of V and W

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  - $\forall v \in V, \lambda \in F$: $L(\lambda v) = \lambda L(v)$

Some properties

- $L(0_V) = 0_W$
- Proof: $L(0_V) = L(0\ 0_v) = 0L(0_V) = 0_W$

Linear Maps

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  - $\forall v \in V, \lambda \in F$: $\quad L(\lambda v) = \lambda L(v)$

Some properties

- The image $L(V)$ is a subspace of W
- Proof: Show addition and scalar multiplication is closed

$$L(v_1) + L(v_2) = L(v_1 + v_2) \in W$$
$$\lambda L(v) = L(\lambda v) \in W$$

Linear Maps

Definition

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  - $\forall v \in V, \lambda \in F$: $L(\lambda v) = \lambda L(v)$

Some properties

- The set of linear maps from V to W forms a **subspace** of the space of all functions
- Proof: If $L, \tilde{L}$ are linear, then $L + \tilde{L}$ is linear

  If L is linear, then λL is linear

Linear Map Representation

Construction

- A linear map $L: V \to W$ is uniquely determined if we specify the image of each basis vector of a basis of V
- Proof: We have $v = \sum_j \alpha_\mathrm{j} v_j$, hence

$$L(v) = L\left(\sum_j \alpha_j v_j\right) = \sum_j \alpha_j L(v_j)$$

Matrix Representation

- Let V and W be vector spaces with respective bases $v = (v_1, v_2, \dots, v_n)$ and $w = (w_1, w_2, \dots, w_m)$
- Suppose $L: V \to W$ is a linear mapping, such that

$$L(v_1) = a_{11}w_1 + a_{21}w_2 + \cdots + a_{m1}w_m$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$L(v_n) = a_{1n}w_1 + a_{2n}w_2 + \cdots + a_{mn}w_m$$

- The matrix representation of L w.r.t. the basis v and w is

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

The j^{th}-column of A is formed by the coefficients of $L(v_j)$

Example

- $L: \mathbb{R}^2 \to \mathbb{R}^3, s.t. (x, y) \to (x + 3y, 2x + 5y, 7x + 9y)$

- Find the matrix representation of L w.r.t the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$

- Answer: $L(1,0) = (1,2,7), L(0,1) = (3,5,9)$, hence the matrix of L, w.r.t the standard bases is the 3×2 matrix
$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

Matrix Representation

Explicitly

- The coefficients α_j and β_i are related by $\beta_i = \sum_j a_{ij}\alpha_j$

$$L(v) = L\left(\sum_j \alpha_j v_j\right) = \sum_j \alpha_j L(v_j) = \sum_j \alpha_j \sum_i a_{ij} w_i$$

$$= \sum_i \left(\boxed{\sum_j a_{ij}\alpha_j}\right) w_i = \sum_i \boxed{\beta_i} w_i = w$$

This can be written as a matrix-vector product

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix}$$

Example Matrices

Shearing

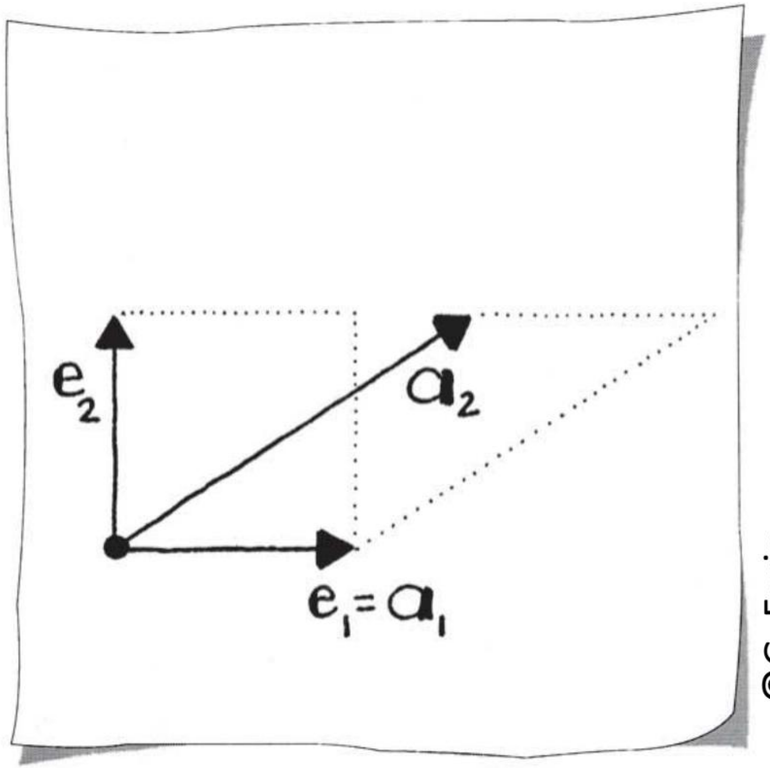
- Consider the standard basis of $\mathbb{R}^2$
  - Matrix?
  - First row

$$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

  - Second row

$$A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1.3\\1\end{pmatrix}$$

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$



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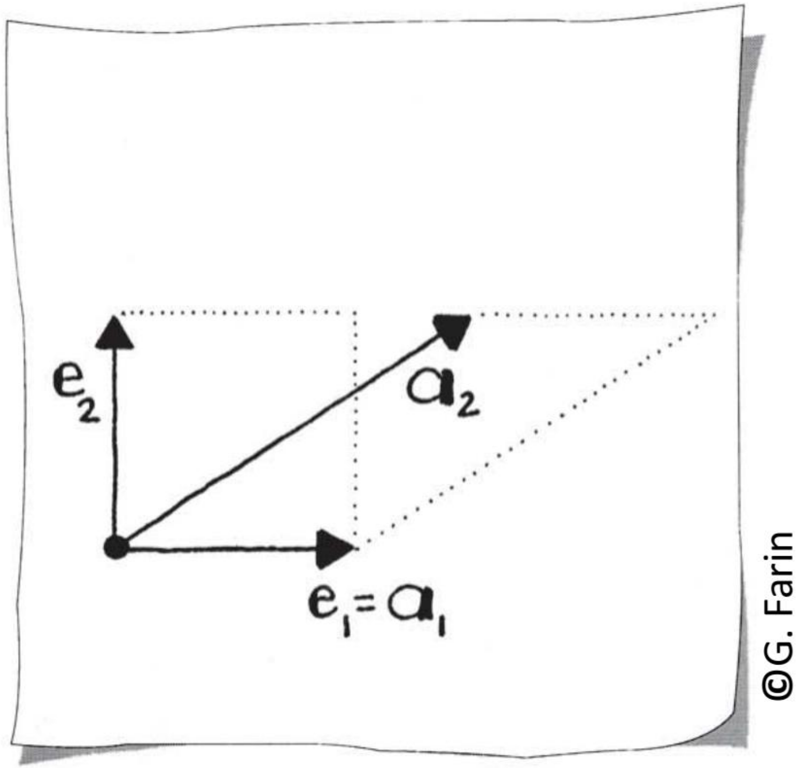
$$A=\begin{pmatrix}1 & 1.3\\0 & 1\end{pmatrix}$$



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Reminder: Properties of Matrices

Symmetric

- $A^T = A$

Orthogonal

$A^T = A^{-1}$

Product is not commutative!

- Find an example with $AB \neq BA$

Product of symmetric matrices may not be symmetric

- Find an example

Product of orthogonal matrices *is* orthogonal

$$(AB)^T = B^T A^T = B^{-1} A^{-1} = (AB)^{-1}$$

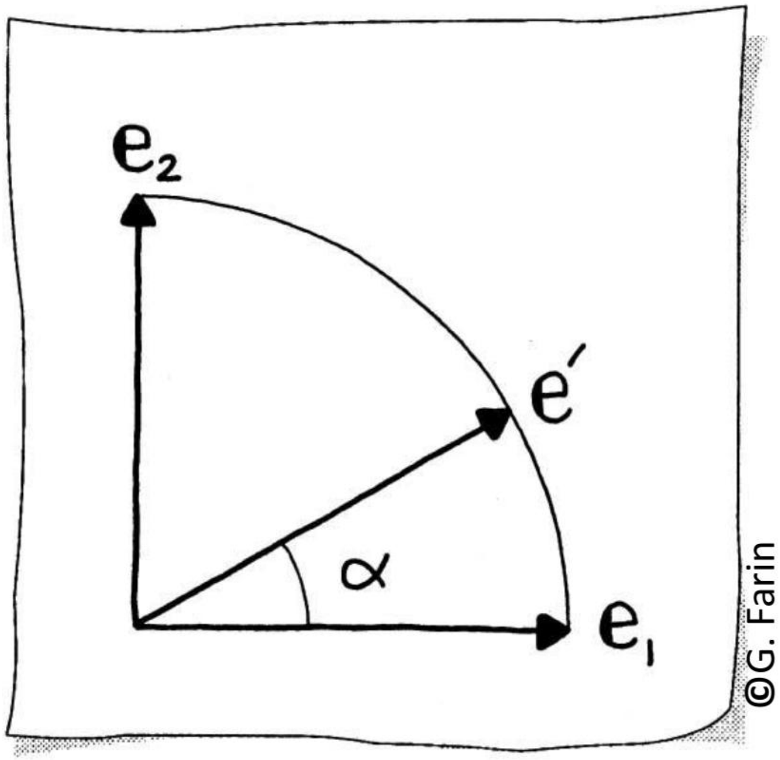
Example of Matrices

Rotation of the plane

- Linear?
- Consider standard basis of $\mathbb{R}^2$

  Matrix?

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$



- Transposition reverses orientation of the rotation

$$\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

Hence matrix is orthogonal $A^T = A^{-1}$

Examples of Linear Maps

Linear operators on a function space

Derivatives

- Differentiation maps functions to functions

$$\frac{\partial}{\partial x} : C^{i}(\mathbb{R}) \mapsto C^{i-1}(\mathbb{R})$$

$$f \mapsto \frac{\partial}{\partial x} f$$

Why is it linear?

- Basic rules of differentiation

$$\frac{\partial}{\partial x}(f+g) = \frac{\partial}{\partial x} f + \frac{\partial}{\partial x} g \quad \text{and} \quad \frac{\partial}{\partial x}(\lambda f) = \lambda \frac{\partial}{\partial x} f$$

Matrix Representation

Derivative on a space of polynomials

- Consider polynomials of degree ≤ 3 and the monomial basis
- What is the matrix representation of the derivative?
- Solution: Evaluate $\frac{\partial}{\partial x}$ on the basis
- $\frac{\partial}{\partial x} 1 = 0$, $\frac{\partial}{\partial x} x = 1$, $\frac{\partial}{\partial x} x^2 = 2x$, $\frac{\partial}{\partial x} x^3 = 3x^2$

Results are the columns of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Examples of Linear Maps

Integrals on $\boldsymbol{C^0([a, b])}$

- Integration maps a continuous function to a number

$$I: C^0([a, b]) \mapsto \mathbb{R}$$

$$I(f) = \int_a^b f dx$$

- The map is linear:

$$\int_a^b (f + g) dx = \int_a^b f dx + \int_a^b g dx$$

$$\int_a^b \lambda f dx = \lambda \int_a^b f dx$$

Matrix Representation

Integrals on a space of polynomials

- Consider polynomials of degree≤ 3 over the interval $[0,1]$ and the monomial basis.
- What is the matrix representation of the integral?
- Solution: Evaluate $\int_0^1 dx$ on the basis

$$\int_0^1 1dx = 1, \qquad \int_0^1 xdx = \frac{1}{2}, \qquad \int_0^1 x^2dx = \frac{1}{3}, \qquad \int_0^1 x^3dx = \frac{1}{4}$$

Results are the columns of the matrix

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \end{pmatrix}$$

Basis Transformations

Matrix representation of L

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "M"] \arrow[dr, "\Phi_A"'] & & & \mathbb{R}^m \arrow[dl, "\Phi_B"] \\
& \mathrm{V} \arrow[r, "L"] & \mathrm{W} &
\end{tikzcd}
$$

- $A = \{v_1, v_2, \dots, v_n\}$ $\qquad$ $B = \{w_1, w_2, \dots, w_n\}$
- $\Phi_A(e_i) = v_i$ $\qquad$ $\Phi_B(e_i) = w_i$
- M maps e_i to $\Phi_B^{-1} \circ L \circ \Phi_A(e_i)$

Basis Transformations

- Basis transformation

$$\begin{tikzcd}
\mathbb{R}^n \arrow[dr, "\Phi_A"] \arrow[dd, "T"'] & \\
& V \\
\mathbb{R}^n \arrow[ur, "\Phi_{\tilde{A}}"'] &
\end{tikzcd}$$

- $A = \{v_1, v_2, \dots, v_n\}$ $\qquad$ $\tilde{A} = \{\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_n\}$
- $\Phi_A(e_i) = v_i$ $\qquad$ $\Phi_{\tilde{A}}(e_i) = \tilde{v}_i$
- T maps e_i to $\Phi_{\tilde{A}}^{-1} \circ \Phi_A(e_i)$

Basis Transformations

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "M"] \arrow[dr, "\Phi_A"] \arrow[dd, "T"'] & & & \mathbb{R}^m \arrow[dl, "\Phi_B"] \arrow[dd, "S"] \\
& V \arrow[r, "L"] & W & \\
\mathbb{R}^n \arrow[ur, "\Phi_{\tilde{A}}"'] & & & \mathbb{R}^m \arrow[ul, "\Phi_{\tilde{B}}"']
\end{tikzcd}
$$

Basis Transformations

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "M"] \arrow[dr, "\Phi_A"] \arrow[dd, "T"'] & & & \mathbb{R}^m \arrow[dl, "\Phi_B"'] \arrow[dd, "S"] \\
& V \arrow[r, "L"] & W & \\
\mathbb{R}^n \arrow[rrr, "\tilde{M}"'] \arrow[ur, "\Phi_{\tilde{A}}"'] & & & \mathbb{R}^m \arrow[ul, "\Phi_{\tilde{B}}"]
\end{tikzcd}
$$

$$\tilde{M} = SMT^{-1}$$

Basis Transformations

In the special case that V equals W:

$$
\begin{tikzcd}
\mathbb{R}^n \arrow[rrr, "M"] \arrow[dr, "\Phi_A"] \arrow[dd, "T"'] & & & \mathbb{R}^n \arrow[dl, "\Phi_A"'] \arrow[dd, "T"] \\
& \mathrm{V} \arrow[r, "L"] & \mathrm{V} & \\
\mathbb{R}^n \arrow[rrr, "\widetilde{M}"'] \arrow[ur, "\Phi_{\tilde{A}}"'] & & & \mathbb{R}^n \arrow[ul, "\Phi_{\tilde{A}}"]
\end{tikzcd}
$$

$$\widetilde{M} = TMT^{-1}$$