附录

1. **Theorem** 1.3.1 (2D)

   The interpolation problem has the unique solution

   $$\tilde{u}(\omega,\eta) = \frac{1}{2\pi}(e^{i\omega x}e^{i\eta y}, u)_h, \quad |\omega| \le N/2,\ |\eta| \le M/2.$$

   Proof:

   The interpolant is given by

   $$\mathrm{Int}_{N,M}u(x,y) = \frac{1}{2\pi}\sum_{\omega=-N/2}^{N/2}\sum_{\eta=-M/2}^{M/2}\tilde{u}(\omega,\eta)e^{i\omega x}e^{i\eta y}.$$

   At grid points (x_j, y_k), we have:

   $$\mathrm{Int}_{N,M}u(x_j,y_k) = \frac{1}{2\pi}\sum_{\omega,\eta}\tilde{u}(\omega,\eta)e^{i\omega x_j}e^{i\eta y_k}.$$

   Substituting $\tilde{u}(\omega,\eta) = \frac{1}{2\pi}h_x h_y \sum_{j',k'} e^{i\omega x_{j'}}e^{i\eta y_{k'}}u_{j',k'}$, we get:

   $$\mathrm{Int}_{N,M}u(x_j,y_k) = \frac{1}{4\pi^2}h_x h_y \sum_{j',k'} u_{j',k'} \sum_{\omega} e^{i\omega(x_j - x_{j'})}\sum_{\eta} e^{i\eta(y_k - y_{k'})}.$$

   Using the orthogonality relations:

   $$\sum_{\omega=-N/2}^{N/2} e^{i\omega(x_j - x_{j'})} = N\delta_{jj'}, \quad \sum_{\eta=-M/2}^{M/2} e^{i\eta(y_k - y_{k'})} = M\delta_{kk'},$$

   and noting that $h_x h_y NM = (2\pi)(2\pi) = 4\pi^2$, we obtain:

   $$\mathrm{Int}_{N,M}u(x_j,y_k) = u_{j,k}.$$

   Thus, the interpolant matches the grid function at all points. Uniqueness follows from the linear independence of the Fourier basis functions.

2. **Theorem** 1.3.2 (2D)

   Let

$$\mathrm{Int}_{N,M}u^{(j)} = \frac{1}{2\pi}\sum_{\omega=-N/2}^{N/2}\sum_{\eta=-M/2}^{M/2}\tilde{u}^{(j)}(\omega,\eta)e^{i\omega x}e^{i\eta y}, \quad j=1,2,$$

   interpolate the two grid functions. Then

$$(u^{(1)},u^{(2)})_h = \sum_{\omega=-N/2}^{N/2}\sum_{\eta=-M/2}^{M/2}\tilde{u}^{(1)}(\omega,\eta)\overline{\tilde{u}^{(2)}(\omega,\eta)} = (\mathrm{Int}_{N,M}u^{(1)},\mathrm{Int}_{N,M}u^{(2)}),$$

   where the left-hand side is the discrete inner product and the right-hand side is the continuous L^2 inner product over $[0,2\pi]\times[0,2\pi]$.

   Proof:

   By definition of the discrete inner product and Fourier coefficients:

$$(u^{(1)},u^{(2)})_h = h_x h_y \sum_{j,k} u^{(1)}_{j,k}u^{(2)}_{j,k}.$$

   From the discrete Parseval identity for 2D DFT, we have:

$$(u^{(1)},u^{(2)})_h = \sum_{\omega,\eta}\tilde{u}^{(1)}(\omega,\eta)\overline{\tilde{u}^{(2)}(\omega,\eta)}.$$

   Now, consider the continuous inner product:

$$(\mathrm{Int}_{N,M}u^{(1)},\mathrm{Int}_{N,M}u^{(2)}) = \int_0^{2\pi}\int_0^{2\pi}\mathrm{Int}_{N,M}u^{(1)}\overline{\mathrm{Int}_{N,M}u^{(2)}}\,\mathrm{d}x\mathrm{d}y.$$

   Substituting the interpolant expressions:

$$\mathrm{Int}_{N,M}u^{(1)} = \frac{1}{2\pi}\sum_{\omega,\eta}\tilde{u}^{(1)}(\omega,\eta)e^{i\omega x}e^{i\eta y}, \quad \mathrm{Int}_{N,M}u^{(2)} = \frac{1}{2\pi}\sum_{\omega',\eta'}\tilde{u}^{(2)}(\omega',\eta')e^{i\omega' x}e^{i\eta' y},$$

   we get:

$$\mathrm{Int}_{N,M}(u^{(1)},u^{(2)}) = \frac{1}{4\pi^2}\sum_{\omega,\eta}\sum_{\omega',\eta'}\tilde{u}^{(1)}(\omega,\eta)\overline{\tilde{u}^{(2)}(\omega',\eta')}\int_0^{2\pi}e^{i(\omega-\omega')x}\mathrm{d}x\int_0^{2\pi}e^{i(\eta-\eta')y}\mathrm{d}y.$$

   Using the orthogonality:

$$\int_0^{2\pi}e^{i(\omega-\omega')x}\mathrm{d}x = 2\pi\delta_{\omega\omega'}, \qquad \int_0^{2\pi}e^{i(\eta-\eta')y}\mathrm{d}y = 2\pi\delta_{\eta\eta'},$$

   we obtain:

$$(\mathrm{Int}_{N,M}u^{(1)},\mathrm{Int}_{N,M}u^{(2)}) = \sum_{\omega,\eta}\tilde{u}^{(1)}(\omega,\eta)\overline{\tilde{u}^{(2)}(\omega,\eta)}.$$

   Thus, the equality holds.

3. **Theorem** 1.3.3 (2D)

Let $\mathrm{Int}_{N,M}u$ be the interpolant of a grid function u. Then

$$\|\mathrm{Int}_{N,M}u\|^2 = \sum_{\omega=-N/2}^{N/2} \sum_{\eta=-M/2}^{M/2} |\tilde{u}(\omega,\eta)|^2 = \|u\|_h^2, \tag{1.3.4}$$

and for $l = 0, 1, \ldots,$

$$\|D_{+x}^l u\|_h^2 \le \left\| \frac{\partial^l}{\partial x^l} \mathrm{Int}_{N,M}u \right\|^2 \le \left(\frac{\pi}{2}\right)^{2l} \|D_{+x}^l u\|_h^2,$$

$$\|D_{+y}^l u\|_h^2 \le \left\| \frac{\partial^l}{\partial y^l} \mathrm{Int}_{N,M}u \right\|^2 \le \left(\frac{\pi}{2}\right)^{2l} \|D_{+y}^l u\|_h^2,$$

where D_{+x} and D_{+y} are the forward difference operators in the x- and y-directions, respectively.

Proof: The first equality (1.3.4) follows directly from the Parseval identity and the definition of the norms:

$$\|\mathrm{Int}_{N,M}u\|^2 = (\mathrm{Int}_{N,M}u, \mathrm{Int}_{N,M}u) = \sum_{\omega,\eta} |\tilde{u}(\omega,\eta)|^2,$$

and

$$\|u\|_h^2 = (u,u)_h = \sum_{\omega,\eta} |\tilde{u}(\omega,\eta)|^2.$$

For the derivative estimates, we consider the x-direction; the y-direction is similar. The partial derivative of the interpolant is:

$$\frac{\partial^l}{\partial x^l} \mathrm{Int}_{N,M}u(x,y) = \frac{1}{2\pi} \sum_{\omega,\eta} (i\omega)^l \tilde{u}(\omega,\eta) e^{i\omega x} e^{i\eta y}.$$

Thus, the L^2 norm is:

$$\left\| \frac{\partial^l}{\partial x^l} \mathrm{Int}_{N,M}u \right\|^2 = \int_{[0,2\pi]^2} \left| \frac{\partial^l}{\partial x^l} \mathrm{Int}_{N,M}u \right|^2 = \sum_{\omega,\eta} |i\omega|^{2l} |\tilde{u}(\omega,\eta)|^2 = \sum_{\omega,\eta} \omega^{2l} |\tilde{u}(\omega,\eta)|^2.$$

The forward difference operator D_{+x} is defined as:

$$D_{+x}u_{j,k} = \frac{u_{j+1,k} - u_{j,k}}{h_x},$$

and similarly for higher orders. The discrete Fourier transform of $D_{+x}^l u$ is:

$$\widetilde{D_{+x}^l u}(\omega,\eta) = \left(\frac{e^{i\omega h_x} - 1}{h_x} \right)^l \tilde{u}(\omega,\eta).$$

Therefore,

$$\|D_{+x}^l u\|_h^2 = \sum_{\omega,\eta} \left| \frac{e^{i\omega h_x} - 1}{h_x} \right|^{2l} |\tilde{u}(\omega,\eta)|^2.$$

Now, note that for $|\omega| \le N/2$, we have $|\omega h_x| \le \pi$. Using the inequalities:

$$\left| \frac{e^{i\omega h_x} - 1}{h_x} \right| = \frac{2|\sin(\omega h_x/2)|}{h_x} \le |\omega|,$$

and

$$|\omega| \le \frac{\pi}{2} \left| \frac{e^{i\omega h_x} - 1}{h_x} \right|,$$

which follows from $|\sin(\theta)| \ge \frac{2|\theta|}{\pi}$ for $|\theta| \le \pi/2$ (with $\theta = \omega h_x/2$).

Thus,

$$\|D_{+x}^l u\|_h^2 = \sum_{\omega,\eta} \left| \frac{e^{i\omega h_x} - 1}{h_x} \right|^{2l} |\tilde{u}|^2 \le \sum_{\omega,\eta} |\omega|^{2l} |\tilde{u}|^2 = \left\| \frac{\partial^l}{\partial x^l} \mathrm{Int}_{N,M} u \right\|^2,$$

and

$$\left\| \frac{\partial^l}{\partial x^l} \mathrm{Int}_{N,M} u \right\|^2 = \sum_{\omega,\eta} |\omega|^{2l} |\tilde{u}|^2 \le \left(\frac{\pi}{2}\right)^{2l} \sum_{\omega,\eta} \left| \frac{e^{i\omega h_x} - 1}{h_x} \right|^{2l} |\tilde{u}|^2 = \left(\frac{\pi}{2}\right)^{2l} \|D_{+x}^l u\|_h^2.$$

The same reasoning applies to the y-direction.