PB22000093 钟佳杭

Part 0. 记号约定 与 Peskin 书中不同 我们习惯令 $[a_{\vec p}, a^\dagger_{\vec p'}]_\pm = \delta^3(\vec p-\vec p')$

这时 $\langle \vec p | \vec q\rangle = \delta^3(\vec p-\vec q)$.

为了保持正则量子化条件，场算符前面系数 $\frac{1}{(2\pi)^3} \to \frac{1}{(2\pi)^{3/2}}$

Part 1. 矢量场的拉氏密度和经典解

矢量场是指 $U(\Lambda,a)A^\mu(x)U^{-1}(\Lambda,a) = {\Lambda^{-1}}^\mu{}_\nu A^\nu(\Lambda x+a)$

可以构造的标量有（二阶导及以下）. $A_\mu A^\mu \quad (\partial_\mu A^\mu)^2 \quad (\partial_\mu A^\nu)(\partial_\nu A^\mu) \quad (\partial_\mu A_\nu)(\partial^\mu A^\nu)$

第二、第三种在分部积分下是同一种. $\partial_\mu A^\mu \partial_\nu A^\nu \leftrightarrow \partial_\mu A^\nu \partial_\nu A^\mu$

总可以 rescale 场使拉氏密度可以写作.

$$\mathcal{L} = \pm\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu + a(\partial_\mu A^\mu)^2 + b A_\mu A^\mu\right).$$

经典解. $A_\mu \propto \varepsilon_\mu e^{-ik\cdot x}$ 代入场方程. $\partial_\mu\partial^\mu A^\nu + a\,\partial_\mu\partial^\nu A^\mu - bA^\nu = 0$

得. $-k^2\varepsilon^\nu - ak^\nu(k\cdot\varepsilon) - b\varepsilon^\nu = 0$.

纵波解. $\varepsilon_\mu \propto k_\mu$. $\Rightarrow$ $k^2k^\nu + a\,k^2k^\nu + bk^\nu = 0$. $k^2 = -\frac{b}{1+a} = \mu_L^2$.

横波解. $\varepsilon_\mu k^\mu = 0$ $\Rightarrow$ $k^2\varepsilon^\nu + b\varepsilon^\nu = 0$. $k^2 = -b = \mu^2$

纵波解有一个独立解. 对应标量粒子. 横波解 ~~~~有三个独立解. 对应自旋 1 粒子. 关于这个标量粒子的量子化我放在了最后. 我们先讨论 spin 1 粒子. 为了在场中去掉标量粒子. 我们可令 $a=-1$. 这样纵波解变为 $\varepsilon^\mu = 0$.

$$\mathcal{L} = \pm\frac{1}{2}\left(\partial_\mu A_\nu\partial^\mu A^\nu - (\partial_\mu A^\mu)^2 - \mu^2 A_\mu A^\mu\right).$$

技巧：定义 $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ 则 $\mathcal{L} = \pm\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu\right)$.

我们知道量子化后. $\varepsilon^{\mu(r)}_p$ 会做为正频解系数出现. 因此解出它们是必要的.

静止系中. $k_\mu = (M,0,0,0)^T$ 令 $\varepsilon^{(r)}$ 们相互正交. $\varepsilon^{(r)*}\cdot\varepsilon^{(s)} = -\delta^{rs}$

可取. $\varepsilon^{(1)} = (0,1,0,0)$ 或者 $\varepsilon^{(1)} = \frac{1}{\sqrt2}(0,1,i,0)$ 这样 $\varepsilon^{(1)}$ $\varepsilon^{(2)}$ 还同时是

$\varepsilon^{(2)} = (0,0,1,0)$ $\varepsilon^{(2)} = \frac{1}{\sqrt2}(0,1,-i,0)$ J_z 的本征矢. ±1

$\varepsilon^{(3)} = (0,0,0,1)$ $\varepsilon^{(3)} = (0,0,0,1)$

完备性关系可以由洛伦兹变换性质简单地得到. define $P_{\mu\nu}(k)=\sum_{r=1}^{3}\varepsilon_\mu^{(r)}(k)\,\varepsilon_\nu^{(s)}(k)$.

由洛伦兹变换性质. $P_{\mu\nu}=a g_{\mu\nu}+b k_\mu k_\nu$

结合 $\begin{cases} P_{\mu\nu}k^\nu=0 \\ P_{\mu\nu}\varepsilon^{\nu(s)}=-\varepsilon_\mu^{(s)} \end{cases} \Rightarrow \begin{cases} a+bM^2=0 \\ a=-1 \end{cases} \Rightarrow P_{\mu\nu}=-g_{\mu\nu}+\frac{k_\mu k_\nu}{M^2}$

如果 $M=0$ 这个式子是矛盾的, 但我们暂时忽略它.

Part 2 有质量 spin 1 粒子. $M\neq 0$. $\mathcal{L}=\pm\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}M^2A_\mu A^\mu\right)$.

场方程. $\partial_\mu F^{\mu\nu}-M^2A^\nu=0$. ∂_ν: $\partial_\nu\partial_\mu F^{\mu\nu}-M^2\partial_\nu A^\nu=0$. $\Rightarrow$ $\partial_\nu A^\nu=0$.

代回得. $(\partial_\mu\partial^\mu-M^2)A^\nu=0$ 满足 K-G 方程.

这是加号

在做哈密顿手续之前. $\partial_\nu A^\nu=0$ 提示我们真独立的 A_μ 只有3个.

*命题. 协时刻下 $\{A_i,\frac{\partial\mathcal{L}}{\partial(\partial_0A_i)}\}$ 在所有空间上的取值. 确定了场的演化.

证明: $\pi^i=\frac{\partial\mathcal{L}}{\partial(\partial_0A_i)}=sF^{0i}$ s代表正负号.

由场方程. $A^0=\frac{1}{M^2}\partial_iF^{i0}=\cancel{\frac{1}{M^2}\partial_iF^{0i}}=-s\frac{1}{M^2}\partial_i\pi^i$ 确定

所以 $\partial^0A^i=F^{0i}+\partial^iA^0=s\pi-\frac{1}{M^2}s\,\nabla(\nabla\cdot\vec{\pi})$ 确定

$\partial_0A^0=\partial_\nu A^\nu-\partial_iA^i$ 确定.

于是. $\{A_i,F^{0i}\}$ 构成了场方程的初值.

将 $\mathcal{L}$ 重新写成 $\mathcal{L}=\pm\left(\frac{1}{2}F_{0i}F^{0i}+\frac{1}{4}F_{ij}F^{ij}-\frac{1}{2}M^2A_0A^0-\frac{1}{2}M^2A_iA^i\right)$

$\mathcal{H}=F^{0i}\partial_0A_i-\mathcal{L}=\pm\left[\vec{\pi}\cdot\left(-\vec{\pi}+\frac{1}{M^2}\nabla(\nabla\cdot\vec{\pi})\right)+\frac{1}{2}\vec{\pi}^2-\frac{1}{4}(F_{ij})^2+\frac{1}{2}\frac{1}{M^2}(\nabla\cdot\vec{\pi})^2-\frac{1}{2}M^2(A_i)^2\right]$

$=\pm\left[-\frac{1}{2}\vec{\pi}^2-\frac{1}{2M^2}(\nabla\cdot\vec{\pi})^2-\frac{1}{4}(F_{ij})^2-\frac{1}{2}M^2(A_i)^2\right]$.

要求 $\mathcal{H}$ 有下界. 则应取负号. 所以 $\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}M^2A_\mu A^\mu$

正则量子化条件为.

$[A_i(\vec{x},t),A_j(\vec{y},t)]_\pm=0$ 这里 $[\ ]$ $[\ ]_\pm$ 分别代表对易子

$[A_i(\vec{x},t),-F^{0j}(\vec{y},t)]_\pm=i\delta^3(\vec{x}-\vec{y})\delta_i^j$ 和反对易子. 将在下面确定.

$[-F^{0i}(\vec{x},t),-F^{0j}(\vec{y},t)]_\pm=0$

Fourier 变换下 解出场算符

$A_\mu=\int\frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}}\sum_{r=1}^{3}\left(\varepsilon_\mu^{(r)}(p)\,a_p^r\,e^{-ip\cdot x}+\varepsilon_\mu^{r*}(p)\,a_p^{r\dagger}\,e^{ip\cdot x}\right)$

可以解出产生湮灭算子代数. $[a_p^r, a_{p'}^s]_t=0$ $[a_p^r, a_{p'}^{s\dagger}]_t = \delta^3(\vec p-\vec p')\delta^{rs}$ $[a_p^{r\dagger}, a_{p'}^{s\dagger}]_t=0$.

为了确定是对易子还是反对易子. 我们利用一般性的结论.

$$\mathcal{H} = \int d^3p \sum_r \omega_p a_p^{r\dagger} a_p^r$$ (温伯格 场论I 4.2节 (4.2.11)式)

得到应该取对易子.

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* 计算如下： $\mathcal{L} = -[\frac{1}{2} F_{0i}F^{0i} - \mu^2 A_0A^0 - \frac{1}{4} F_{ij}F^{ij} + \frac{1}{2}\mu^2 A_\mu A^\mu]$

$$H = \int d^3\vec{x}\, [-\tfrac{1}{2} F_{0i}F^{0i} + \tfrac{1}{4} F_{ij}F^{ij} + \tfrac{1}{2}\mu^2 A_0A^0 - \tfrac{1}{2}\mu^2 A_iA^i]$$

$$A_\mu = \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} \sum_r (\varepsilon_\mu^r(p)\, a_p^r e^{-ip\cdot x} + \varepsilon_\mu^{r*}(p)\, a_p^{r\dagger} e^{ip\cdot x})$$

$$F_{\mu\nu} = i \sum_r \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{2E_p}} [(p_\nu \varepsilon_\mu^r - p_\mu \varepsilon_\nu^r) a_p^r e^{-ip\cdot x} - (p_\nu \varepsilon_\mu^{r*} - p_\mu \varepsilon_\nu^{r*}) a_p^{r\dagger} e^{ip\cdot x}]$$

由于计算繁复. 我们只考虑含有 $a_p^r a_{p'}^{s\dagger}$ 或 $a_p^{r\dagger} a_{p'}^s$ 的项. 且这对证明对易子是足够的, 其它的项也会 cancell掉

$$\int d^3\vec{x}\, F^{0i}F_{0i} = -\int \frac{d^3\vec p}{2E_p} \sum_{r,s} [-(p^i \varepsilon^{0r} - p^0 \varepsilon^{ir}) a_p^r (p_i \varepsilon_0^{s*} - p_0 \varepsilon_i^{s*}) a_p^{s\dagger} - (p^i \varepsilon^{0r*} - p^0 \varepsilon^{ir*}) a_p^{r\dagger} (p_i \varepsilon_0^s - p_0 \varepsilon_i^s) a_p^s]$$

$$p^\mu \varepsilon_\mu^r = 0 \Rightarrow p^i \varepsilon_i^r = -p^0 \varepsilon_0^r$$

于是 $$\int d^3\vec{x}\, F^{0i}F_{0i} = -\sum_{r,s}\int \frac{d^3\vec p}{2E_p} \{[(|\vec p|^2 - 2p_0^2)\varepsilon^{0r}\varepsilon_0^{s*} - p_0^2 \varepsilon^{ir}\varepsilon_i^{s*}] a_p^r a_p^{s\dagger} + [(|\vec p|^2 - 2p_0^2)\varepsilon^{0r*}\varepsilon_0^s - p_0^2 \varepsilon^{ir*}\varepsilon_i^s] a_p^{r\dagger} a_p^s$$

同理计算出

$$\int d^3\vec{x}\, F^{ij}F_{ij} = \sum_{r,s}\int \frac{d^3\vec p}{2E_p} \{[2p_0^2 \varepsilon^{0r}\varepsilon_0^{s*} + 2|\vec p|^2 \varepsilon^{ir}\varepsilon_i^{s*}] a_p^r a_p^{s\dagger} + [2p_0^2 \varepsilon^{0r*}\varepsilon_0^s + 2|\vec p|^2 \varepsilon^{ir*}\varepsilon_i^s] a_p^{r\dagger} a_p^s\}$$

$$\int d^3\vec{x}\, A^0A_0 = \sum_{r,s}\int \frac{d^3\vec p}{2E_p} (\varepsilon^{0r}\varepsilon_0^{s*} a_p^r a_p^{s\dagger} + \varepsilon^{0r*}\varepsilon_0^s a_p^{r\dagger} a_p^s)$$

$$\int d^3\vec{x}\, A^iA_i = \sum_{r,s}\int \frac{d^3\vec p}{2E_p} (\varepsilon^{ir}\varepsilon_i^{s*} a_p^r a_p^{s\dagger} + \varepsilon^{ir*}\varepsilon_i^s a_p^{r\dagger} a_p^s)$$

最终

$$H = \sum_{r,s}\int \frac{d^3\vec p}{2E_p} k_0^2 \sum_r (a_p^r a_p^{r\dagger} + a_p^{r\dagger} a_p^r)$$ 应当取对易子.

Part 3 零质量矢量场. $\mathcal{L}_{free} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}$

规范不变性作为第一原理. $A_\mu \to A_\mu + \partial_\mu \varepsilon$ $\delta S=0$.

可见 $\delta S_{free}=0$ 满足. 考虑与场 A_μ 的耦合.

$$\delta S_{I} = \int d^4x \frac{\partial \mathcal{L}_I}{\partial A_\mu}\partial_\mu \varepsilon = 0 \Rightarrow \partial_\mu \frac{\partial \mathcal{L}_I}{\partial A_\mu} = 0.$$ $J^\mu \triangleq \frac{\partial \mathcal{L}_I}{\partial A_\mu}$ 为守恒流.

在接下来的讨论中, 为了一般性, 我们不要求物质场和电磁场是最小耦合方案. 只是要求、耦合项只含 A_μ 而不含有 ∂A_μ, 换句话讲, 和 $F_{\mu\nu}$ 有关的项, 只有 $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

正则量子化. $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu + \mathcal{L}_m$

$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\mu)} = -F^{0\mu}$ 得到初级约束 $\Pi_0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0$. 令 $\chi_1 = \Pi_0$

$\dot{\pi}_0 = [H, \pi^0]_P = 0$. 得到次级约束. $\partial_i \pi^i = -J^0$ 令 $\chi_2 = \partial_i\pi^i + J^0$.
没有更高级约束.

可以验证两个约束对易. $[\chi_1, \chi_2]_P = 0$ 属于第一类约束. 需施加规范条件才能正则量子化

库仑规范 $\nabla\cdot\vec{A} = 0$. (简单起见, 这次只讨论库仑规范下的量子化)

此时 $J^0 = -\partial_i\pi^i = -\partial_i F^{i0} = -\partial_i(\partial^i A^0 - \partial^0 A^i) = -\partial_i\partial^i A^0 = \nabla^2 A^0$

可解出 $A^0 = -\int d^3y \frac{J^0(\vec{y},t)}{4\pi|\vec{x}-\vec{y}|}$ 代回到 $\mathcal{L}$ 中.

并且此时约束变为 $\chi_1(x) = \partial_i A^i(x) = 0$ $\chi_2(x) = \partial_i\pi^i(x) + J^0(x) = 0$

它们不对易. 是第二类约束.

现在计算狄拉克括号 由 $[\chi_1(\vec{x}), \chi_2(\vec{y})]_P = -\nabla^2\delta^3(\vec{x}-\vec{y})$ 知

$$\begin{pmatrix} C_{1\vec{x},1\vec{y}} & C_{1\vec{x},2\vec{y}} \\ C_{2\vec{x},1\vec{y}} & C_{2\vec{x},2\vec{y}} \end{pmatrix} = \begin{pmatrix} 0 & -\nabla^2\delta^3(\vec{x}-\vec{y}) \\ \nabla^2\delta^3(\vec{x}-\vec{y}) & 0 \end{pmatrix} \Rightarrow (C^{-1})_{ab} = \begin{pmatrix} 0 & -\frac{1}{4\pi|\vec{x}-\vec{y}|} \\ \frac{1}{4\pi|\vec{x}-\vec{y}|} & 0 \end{pmatrix}$$

Dirac 量子化 $[A^i(\vec{x}), \pi^j(\vec{y})] = i[A^i(\vec{x}), \pi^j(\vec{y})]_P - i[A^i, \chi_N]_P [C^{-1}]^{NM}[\chi_M, \pi_j]_P$

$$= i\delta^{ij}\delta^3(\vec{x}-\vec{y}) + i\frac{\partial^2}{\partial x^i\partial x^j}\left(\frac{1}{4\pi|\vec{x}-\vec{y}|}\right)$$

$$[A^i(\vec{x}), A^j(\vec{y})] = [\pi^i(\vec{x}), \pi^j(\vec{y})] = 0$$

然而这样的选择会使物质场和 $\pi^i(x)$ 不对易. 令 F 为物质场函数

计算 $[F, \vec{\pi}(\vec{z})]_D = -\int d^3x\, d^3y\, [F, \chi_2(x)]_P \frac{1}{4\pi|\vec{x}-\vec{y}|}[\chi_1(\vec{y}), \vec{\pi}(\vec{z})]_P$

$$= -\int d^3x\, d^3y\, [F, J^0(\vec{x})]_P \frac{1}{4\pi|\vec{x}-\vec{y}|}\nabla\delta^3(\vec{y}-\vec{z})$$

$$= \int d^3y\, [F, A^0(\vec{y})]_P \nabla\delta^3(\vec{y}-\vec{z})$$

$$= -[F, \nabla A^0(\vec{z})]_P = -[F, \nabla A^0(\vec{z})]_D$$

这时令 $\vec{\pi}_\perp = \vec{\pi} + \nabla A^0 = -\partial_0\vec{A}$ 可以验证 $\vec{\pi}_\perp$ 满足 $\vec{\pi}$ 所有的量子化条件.
且 $[\vec{\pi}_\perp, F]_D = 0$. 即与物质场对易. 同时它有一个简单约束. $\nabla\cdot\vec{\pi}_\perp = 0$

使用 $\{\vec{\pi}_\perp, \vec{A}\}$ 作为正则变量, 体系的哈密顿量可写成 $\mathcal{H} = \vec{\pi}_\perp\cdot\dot{\vec{A}} - \mathcal{L}$ *

$$H = \int d^3x \left[-\pi_\perp^2 - \frac{1}{2}(\pi_\perp - \nabla A^0)^2 + \frac{1}{2}(\nabla\times\vec{A})^2 - J^0A^0 + \vec{J}\cdot\vec{A}\right] + H_M$$

* Steven Weinberg, The Quantum thoery of Fields. Vol 1. Chapter 7. Appendix

完成了量子化