Saiph CFD applications

Contents

i 1D Linear Advection - Sine Wave

Problem Specification

Use Case. Linear Advection
Spatial domain: $0 \leq x < 1$ meters, periodic boundary conditions
Governing equations: 1D Linear Advection Equation

$$\frac{\partial \phi}{\partial t} = -\mathbf{u} \cdot \nabla \phi$$

Initial conditions:

$$\phi(x) = \sin(2\pi x)$$

$$\mathbf{u} = 1 \text{ m/s}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 1$ mm
$\Delta t = 1$ ms
$nsteps = 1000$
Forward in-time integration using Euler method: $\mathcal{O}(\Delta t)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results at three time-steps, $t = 0s$, $t = 0.5s$ and $t = 1s$ are presented in Figure 1. The L_2 norm has been computed over the ϕ variable taking the initial conditions as the analytic solution. The output simulation presents no truncation error $\mathbf{L_2 = 0}$

The Saiph's simulation animation can be checked at:
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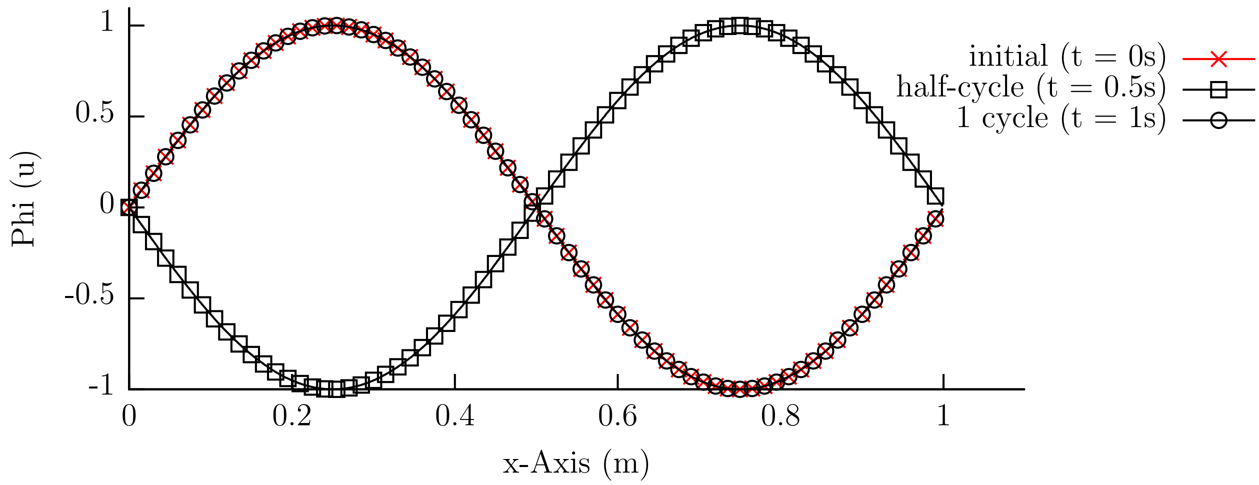

Figure 1: Phi profile at different time-steps.

ii 1D Linear Advection - Discontinuous Waves

Problem Specification

Use Case. Linear Advection
Spatial domain: $-1 \leq x < 1$ meters, periodic boundary conditions
Governing equations: 1D Linear Advection Equation

$$\frac{\partial \phi}{\partial t} = -\mathbf{u} \cdot \nabla \phi$$

Initial conditions:

$$\phi(x) = \begin{cases} \exp\left(-\log(2)\frac{(x+7)^2}{0.0009}\right) & -0.8 \leq x \leq -0.6 \\ 1 & -0.4 \leq x \leq -0.2 \\ 1 - |10(x-0.1)| & 0 \leq x \leq 0.2 \\ \sqrt{1 - 100(x-0.5)^2} & 0.4 \leq x \leq 0.6 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{u} = 1 \text{ m/s}$$

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Simulation details

$\Delta x = 1$ mm
$\Delta t = 1$ ms
$nsteps = 2000$
Forward in-time integration using Euler method: $\mathcal{O}(\Delta t)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results at two time-steps, $t = 0s$ and $t = 2s$ are presented in Figure 2.
The L_2 norm has been computed over the ϕ variable taking the initial conditions as the analytic solution. The output simulation presents no truncation error $\mathbf{L_2 = 0}$

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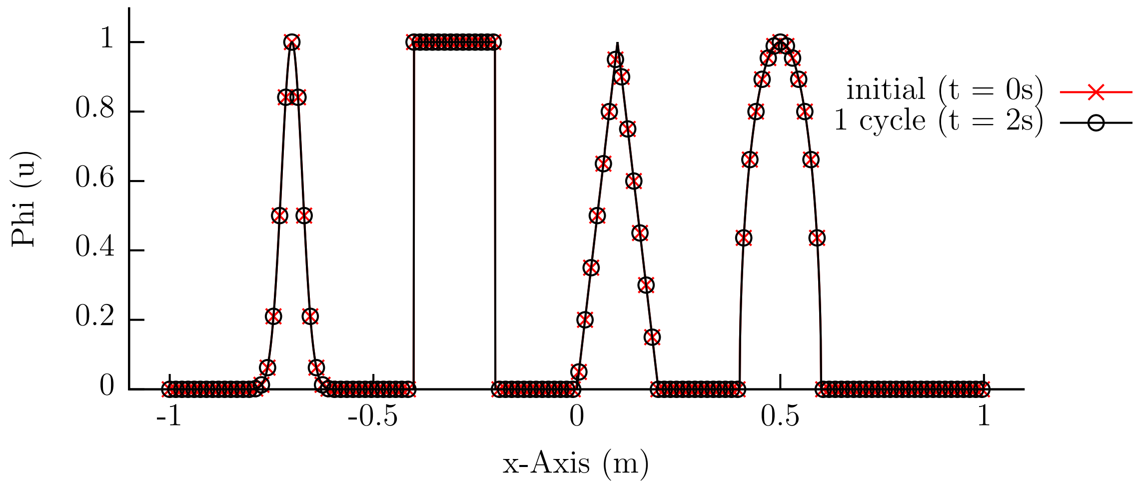

Figure 2: Phi profile at different time-steps.

iii 1D Linear Diffusion - Sine Wave

Problem Specification

Use Case. Linear Diffusion
Spatial domain: $0 \leq x < 1$ meters, periodic boundary conditions
Governing equations: 1D Linear Diffusion Equation

$$\frac{\partial \phi}{\partial t} = \boldsymbol{\nabla} \cdot (\nu \nabla \phi)$$

Initial conditions:

$$\phi(x) = \sin(2\pi x)$$

$$\nu = 0.001 \text{ m}^2/\text{s}$$

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Simulation details

$\Delta x = 12.5$ mm
$\Delta t = 50$ ms
$nsteps = 200$
Forward in-time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(\Delta t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results at three time-steps, $t = 0s$, $t = 5s$ and $t = 10s$ are presented in Figure 3. The L_2 norm has been computed over the ϕ variable taking the analytic solution as reference:

$$\phi(x, t) = e^{-\nu 4\pi^2 t} \sin(2\pi x)$$

The final output simulation ($t = 10s$), presents an error of $\mathbf{L_2 = 1.4 \cdot 10^{-7}}$

The Saiph's simulation animation can be checked at:
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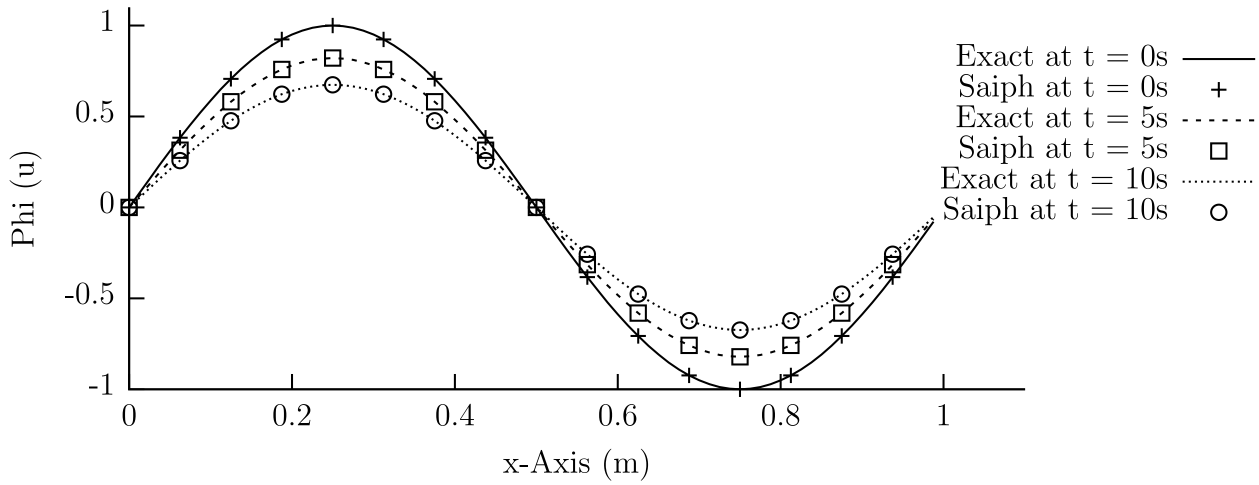

Figure 3: Phi profile at different time-steps.

iv 1D Euler Equations - Sod Shock Tube

This problem corresponds to the well known shock tube originally proposed by Sod [6]. It is a transient case consisting of a one-dimensional tube, closed at its ends and divided into two equal regions by a thin diaphragm. Each region is filled with the same gas, but with different thermodynamic parameters. The fluid is initially at rest and starts to move because of the sudden breakdown of the diaphragm, the discontinuous initial condition of the left and right states entails a shock wave that propagates to both left and right sides. Figure 4 schematizes the configuration of this test [2].

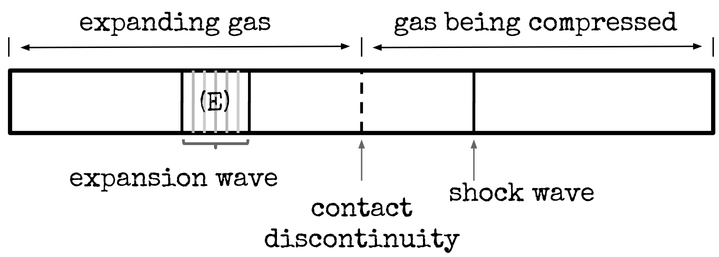

Figure 4: Schema of waves propagating in the tube after the diaphragm breakdown ($t > 0$).

Problem Specification

Use Case. Sod shock tube
Spatial domain: $0 \le x < 1$ meters
Governing equations: 1D Euler equations, continuity, momentum, energy and state.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\left[\rho \mathbf{u} \cdot \nabla u_i + u_i \nabla \cdot (\rho \mathbf{u}) + \frac{\partial p}{\partial x_i}\right]$$

$$\frac{\partial (\rho E)}{\partial t} = -\left[\mathbf{u} \cdot \nabla \rho E + \rho E \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u}\right]$$

$$p = (\gamma - 1)\left((\rho E) - \frac{1}{2}(\rho \mathbf{u})\,\mathbf{u}\right)$$

Initial conditions:

$$\gamma = 1.4$$

$$\mathbf{u} = 0 \text{ m/s}$$

$$\rho(x) = \begin{cases} 1\, kg/m^3 & 0 \le x < 0.5 \\ 0.125\, kg/m^3 & 0.5 \le x \le 1 \end{cases}$$

$$\rho E(x) = \begin{cases} 2.5\,kg/ms^2 & 0 \leq x < 0.5 \\ 0.25\,kg/ms^2 & 0.5 \leq x \leq 1 \end{cases}$$

$$p(x) = \begin{cases} 1\,Pa & 0 \leq x < 0.5 \\ 0.1\,Pa & 0.5 \leq x \leq 1 \end{cases}$$

Boundary conditions:
Symmetric boundary condition (Neumann BCs)

$$\frac{\partial \rho}{\partial x} = 0\,kg/m^4 \begin{cases} x = 0\,m, \\ x = 1\,m \end{cases}$$

$$\frac{\partial (\rho E)}{\partial x} = 0\,kg/m^2 s^2 \begin{cases} x = 0\,m, \\ x = 1\,m \end{cases}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial x} = 0\,kg/m^3 s \begin{cases} x = 0\,m, \\ x = 1\,m \end{cases}$$

$$\frac{\partial p}{\partial x} = 0\,Pa/m \begin{cases} x = 0\,m, \\ x = 1\,m \end{cases}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 2.5$ mm
$\Delta t = 0.2$ ms
$nsteps = 1000$
Forward in-time integration using 3rd order Runke-Kutta method: $\mathcal{O}(\Delta t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results at final time, $t = 0.2s$ are presented in Figure 5, against analytic solutions. The L_2 norm has been computed over the density, pressure and velocity variables taking the analytic solution as reference. The final output simulation, presents an error of $\mathbf{L_{2-\rho} = 1.2 \cdot 10^{-2}}$, $\mathbf{L_{2-p} = 1.1 \cdot 10^{-2}}$, $\mathbf{L_{2-u} = 3.7 \cdot 10^{-2}}$

The Saiph's simulation animation can be checked at:
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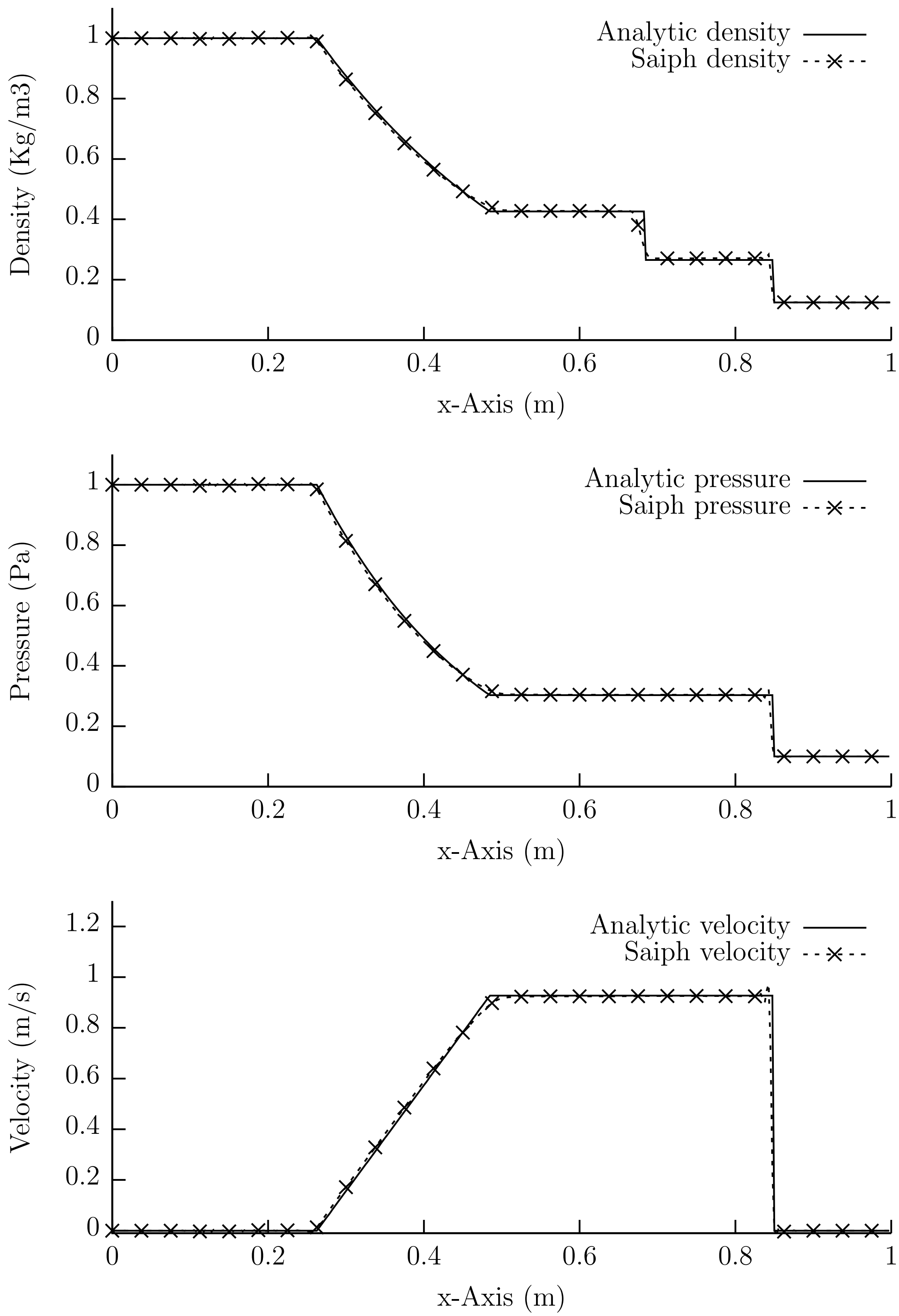

Figure 5: Variable profiles at $t = 0.2s$ for the Sod Tube problem.

v 1D Euler Equations - Lax Shock Tube

Similar to Sod shock tube use-case with different initial conditions.

Problem Specification

Use Case. Lax shock tube
Spatial domain: $0 \leq x < 1$ meters
Governing equations: 1D Euler equations, continuity, momentum, energy and state.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\left[\rho \mathbf{u} \cdot \nabla u_i + u_i \nabla \cdot (\rho \mathbf{u}) + \frac{\partial p}{\partial x_i}\right]$$

$$\frac{\partial (\rho E)}{\partial t} = -\left[\mathbf{u} \cdot \nabla \rho E + \rho E \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u}\right]$$

$$p = (\gamma - 1)\left((\rho E) - \frac{1}{2}(\rho \mathbf{u})\,\mathbf{u}\right)$$

Initial conditions:

$$\gamma = 1.4$$

$$\rho(x) = \begin{cases} 0.445\ kg/m^3 & 0 \leq x < 0.5 \\ 0.5\ kg/m^3 & 0.5 \leq x \leq 1 \end{cases}$$

$$\rho\mathbf{u}(x) = \begin{cases} 0.311\ kg/m^2 s & 0 \leq x < 0.5 \\ 0\ kg/m^2 s & 0.5 \leq x \leq 1 \end{cases}$$

$$\rho E(x) = \begin{cases} 8.928\ kg/ms^2 & 0 \leq x < 0.5 \\ 1.4275\ kg/ms^2 & 0.5 \leq x \leq 1 \end{cases}$$

Boundary conditions:
Symmetric boundary condition (Neumann BCs)

$$\frac{\partial \rho}{\partial x} = 0\ kg/m^4 \ \begin{cases} x = 0\ m, \\ x = 1\ m \end{cases}$$

$$\frac{\partial (\rho E)}{\partial x} = 0\ kg/m^3 s^2 \ \begin{cases} x = 0\ m, \\ x = 1\ m \end{cases}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial x} = 0\ kg/m^3 s \ \begin{cases} x = 0\ m, \\ x = 1\ m \end{cases}$$

$$\frac{\partial p}{\partial x} = 0\ Pa/m \ \begin{cases} x = 0\ m, \\ x = 1\ m \end{cases}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 5$ mm
$\Delta t = 1$ ms
$nsteps = 80$
Forward in time integration using 3rd order Runke-Kutta method: $\mathcal{O}(\Delta t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results at final time, $t = 0.08s$ are presented in Figure 6.

The Saiph's simulation animation can be checked at:
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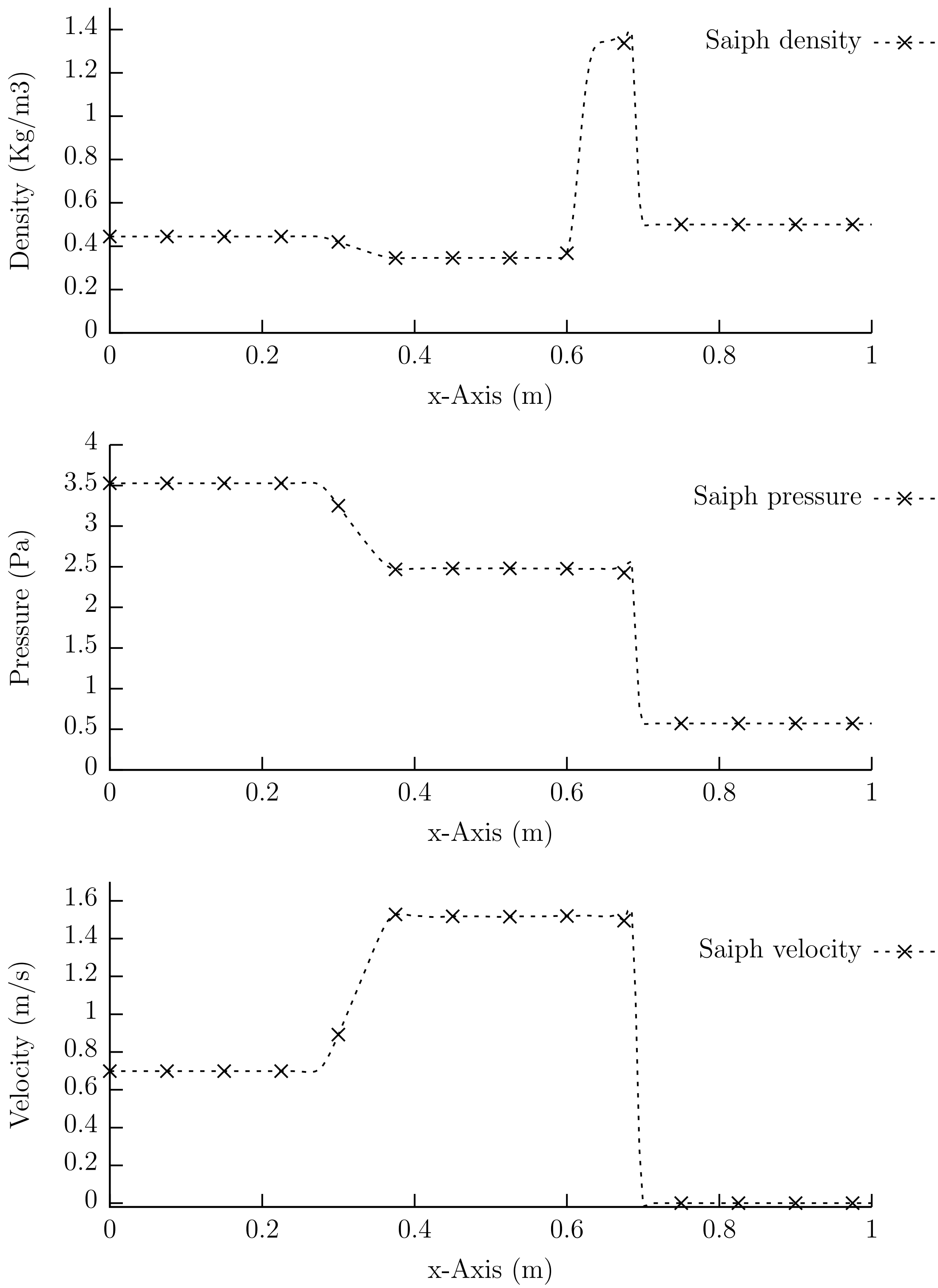

Figure 6: Variable profiles at $t = 0.08s$ for the Lax Tube problem.

vi 1D Euler Equations - Shu Osher

A moving shock interacts with sine waves in density. Similar to Sod shock tube use-case with different initial conditions [4].

Problem Specification

Use Case. Shu Osher tube
Spatial domain: $-5 \leq x < 5$ meters
Governing equations: 1D Euler equations, continuity, momentum, energy and state.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\left[\rho \mathbf{u} \cdot \nabla u_i + u_i \nabla \cdot (\rho \mathbf{u}) + \frac{\partial p}{\partial x_i}\right]$$

$$\frac{\partial (\rho E)}{\partial t} = -\left[\mathbf{u} \cdot \nabla \rho E + \rho E \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u}\right]$$

$$p = (\gamma - 1)\left((\rho E) - \frac{1}{2}(\rho \mathbf{u})\,\mathbf{u}\right)$$

Initial conditions:

$$\gamma = 1.4$$

$$\rho(x) = \begin{cases} 27/7\; kg/m^3 & -5 \leq x < -4 \\ 1 + 0.2\sin(5x)\; kg/m^3 & -4 \leq x \leq 5 \end{cases}$$

$$\rho \mathbf{u}(x) = \begin{cases} 10.14185223\; kg/m^2 s & -5 \leq x < -4 \\ 0\; kg/m^2 s & -4 \leq x \leq 5 \end{cases}$$

$$p = \begin{cases} 31/3\; Pa & -5 \leq x < -4 \\ 1\; Pa & -4 \leq x \leq 5 \end{cases}$$

Boundary conditions:
Symmetric boundary condition (Neumann BCs)

$$\frac{\partial \rho}{\partial x} = 0\; kg/m^4 \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial (\rho E)}{\partial x} = 0\; kg/m^3 s^2 \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial x} = 0\; kg/m^3 s \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial p}{\partial x} = 0\; Pa/m \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 25$ mm
$\Delta t = 1$ ms
$nsteps = 1800$
Forward in time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(\Delta t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results at final time, $t = 1.8s$ are presented in Figure 7.

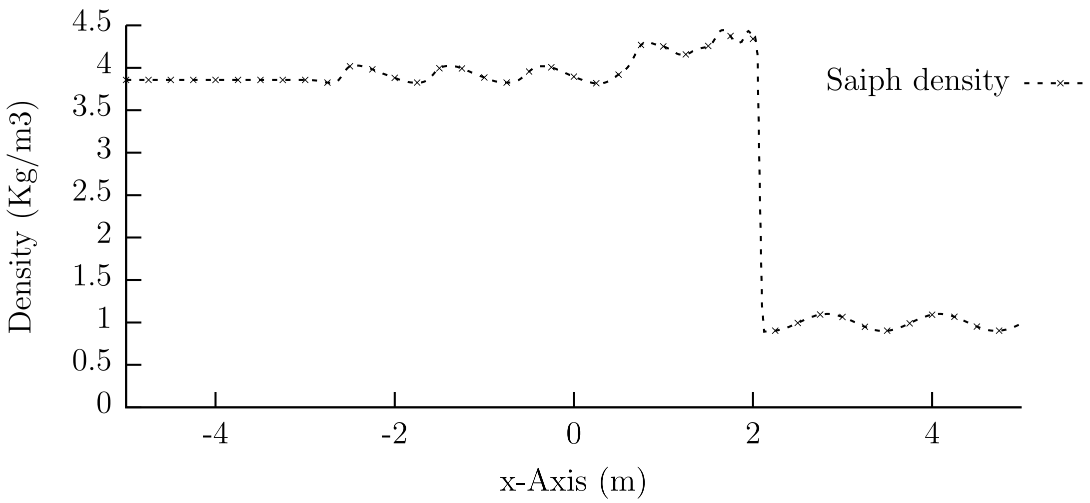

Figure 7: Density profile at $t = 1.8s$ for the Shu Osher problem.

The Saiph's simulation animation can be checked at:
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vii 1D Euler Equations - Sod Shock Tube with Gravitational Force

Similar to Sod shock tube use-case with an uniform gravitational external force $\mathbf{g}$.

Problem Specification

Use Case. Sod shock tube
Spatial domain: $0 \leq x < 1$ meters
Governing equations: 1D Euler equations, continuity, momentum, energy and state with external gravitational force.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\left[\rho \mathbf{u} \cdot \nabla u_i + u_i \nabla \cdot (\rho \mathbf{u}) + \frac{\partial p}{\partial x_i} - \rho g_i\right]$$

$$\frac{\partial (\rho E)}{\partial t} = -\left[\mathbf{u} \cdot \nabla \rho E + \rho E \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u} - \rho \mathbf{u} \cdot \mathbf{g}\right]$$

$$p = (\gamma - 1)\left((\rho E) - \frac{1}{2}(\rho \mathbf{u})\,\mathbf{u}\right)$$

Initial conditions:

$$\gamma = 1.4$$

$$\mathbf{g} = 1\; m/s^2$$

$$\mathbf{u} = 0 \text{ m/s}$$

$$\rho(x) = \begin{cases} 1\; kg/m^3 & 0 \leq x < 0.5 \\ 0.125\; kg/m^3 & 0.5 \leq x \leq 1 \end{cases}$$

$$\rho E(x) = \begin{cases} 2.5\; kg/ms^2 & 0 \leq x < 0.5 \\ 0.25\; kg/ms^2 & 0.5 \leq x \leq 1 \end{cases}$$

$$p(x) = \begin{cases} 1\; Pa & 0 \leq x < 0.5 \\ 0.1\; Pa & 0.5 \leq x \leq 1 \end{cases}$$

Boundary conditions:
Symmetric boundary condition (Neumann BCs) and zero wall velocity (slip-wall BC, Dirichlet BC)

$$\frac{\partial \rho}{\partial x} = 0\; kg/m^4 \quad \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial (\rho E)}{\partial x} = 0\; kg/m^2 s^2 \quad \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\rho \mathbf{u} = 0\; kg/m^3 s \quad \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial p}{\partial x} = 0\; Pa/m \quad \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 2.5$ mm
$\Delta t = 0.2$ ms
$nsteps = 1000$
Forward in-time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(\Delta t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results at final time, $t = 0.2s$ are presented in Figure 8.

The Saiph's simulation animation can be checked at:
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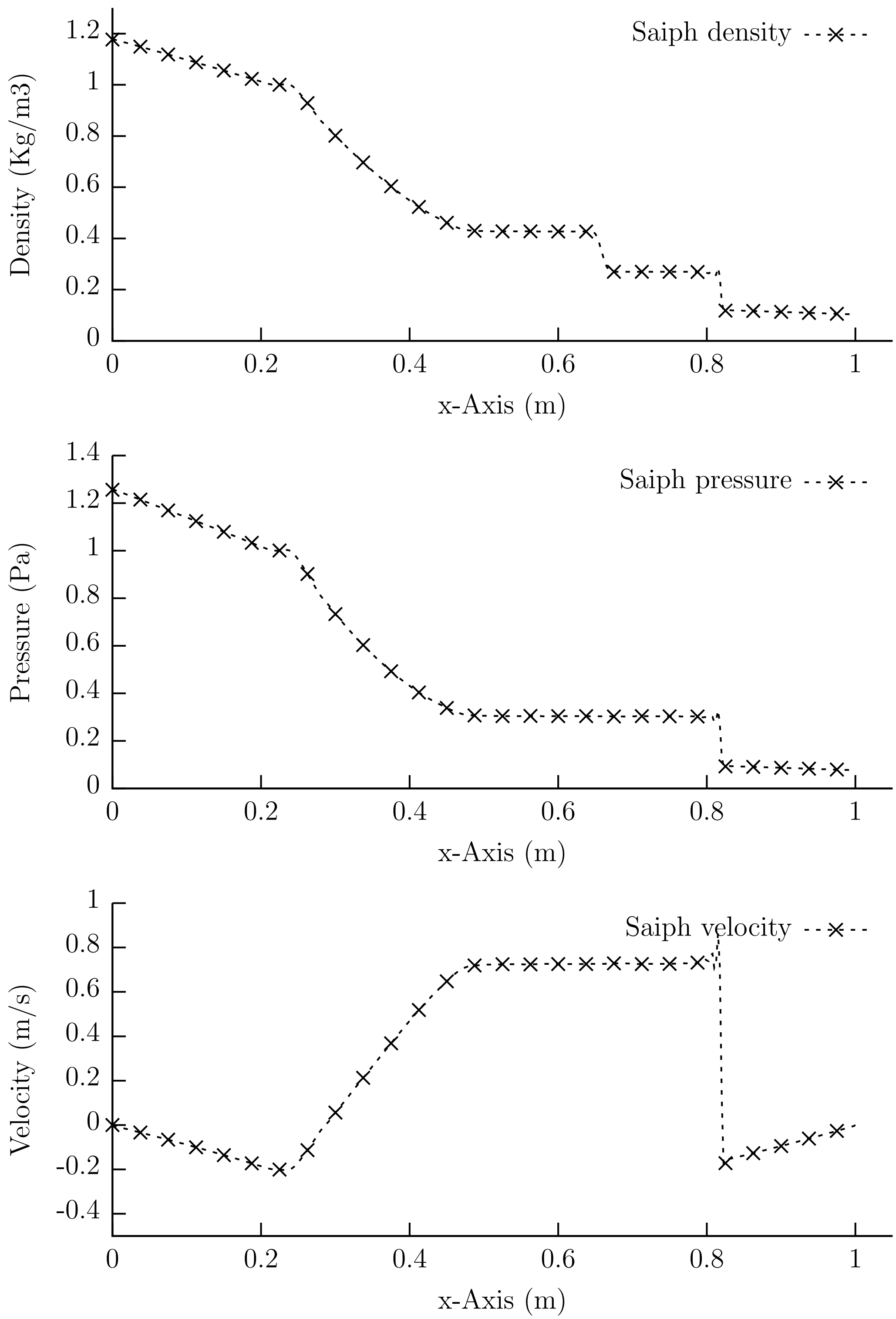

Figure 8: Variable profiles at $t = 0.2s$ for the Sod Tube problem with gravitational force.

viii 2D Linear Advection - Gaussian Pulse

Problem Specification

Use Case. Linear Advection
Spatial domain: $-6 \leq x < 6$ meters, $-3 \leq y < 3$ meters periodic boundary conditions in both directions
Governing equations: 2D Linear Advection Equation

$$\frac{\partial \phi}{\partial t} = -\mathbf{u} \cdot \nabla \phi$$

Initial conditions:

$$\phi(x) = e^{-\left(\frac{x^2}{2} + \frac{y^2}{2}\right)}$$

$$\mathbf{u} = (1, 0) \text{ m/s}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 40$ mm
$\Delta t = 40$ ms
$nsteps = 300$
Forward in-time integration using Euler method: $\mathcal{O}(\Delta t)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results at three time-steps, $t = 0s$, $t = 6s$ and $t = 12s$ are presented in Figure 9. The L_2 norm has been computed over the ϕ variable taking the initial conditions as the analytic solution. The output simulation presents no truncation error $\mathbf{L_2 = 0}$

The Saiph's simulation animation can be checked at:
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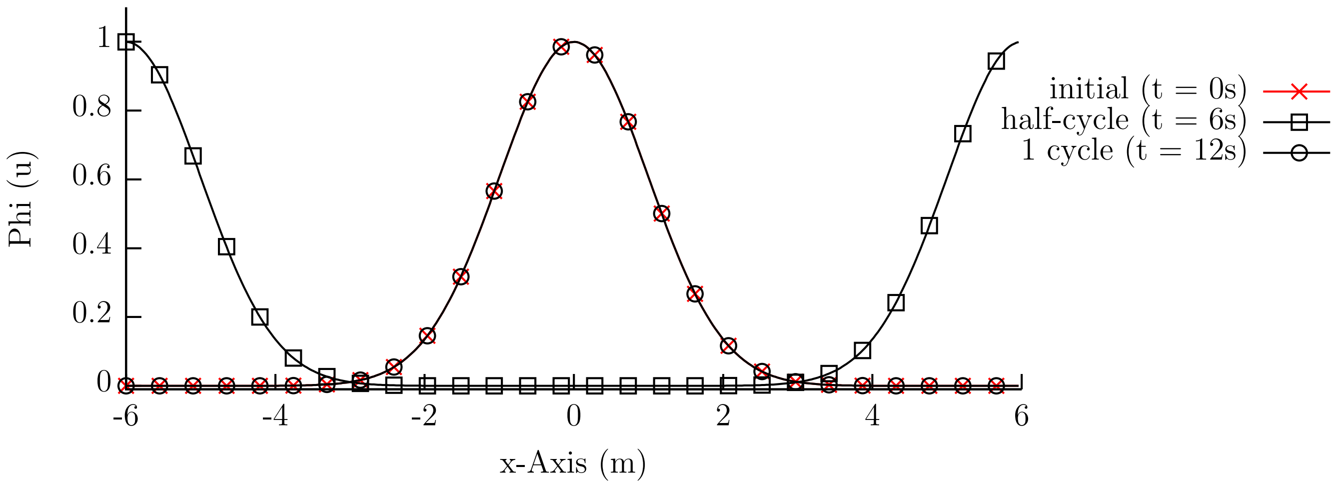

Figure 9: $\phi(x, y)$ profile at different time-steps.

ix 2D Linear Diffusion - Sine Wave

Problem Specification

Use Case. Linear Diffusion
Spatial domain: $0 \leq x, y < 1$ meters, periodic boundary conditions in both directions
Governing equations: 2D Linear Diffusion Equation

$$\frac{\partial \phi}{\partial t} = \boldsymbol{\nabla} \cdot (\nu \nabla \phi)$$

Initial conditions:

$$\phi(x) = \sin(2\pi x)\sin(2\pi y)$$

$$\nu = 0.001 \text{ m}^2/\text{s}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 12.5$ mm
$\Delta t = 20$ ms
$nsteps = 500$
Forward in-time integration using 4^{th} order Runke-Kutta method: $\mathcal{O}(\Delta t^4)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results of the ϕ diagonals profiles at three time-steps, $t = 0s$, $t = 5s$ and $t = 10s$ are presented in Figure 10.
The L_2 norm has been computed over the ϕ variable taking the analytic solution as reference:

$$\phi(x,t) = e^{-\nu 8\pi^2 t}\sin(2\pi x)\sin(2\pi y)$$

The final output simulation ($t = 10s$), presents an error of $\mathbf{L_2 = 1.04 \cdot 10^{-7}}$

The Saiph's simulation animation can be checked at:
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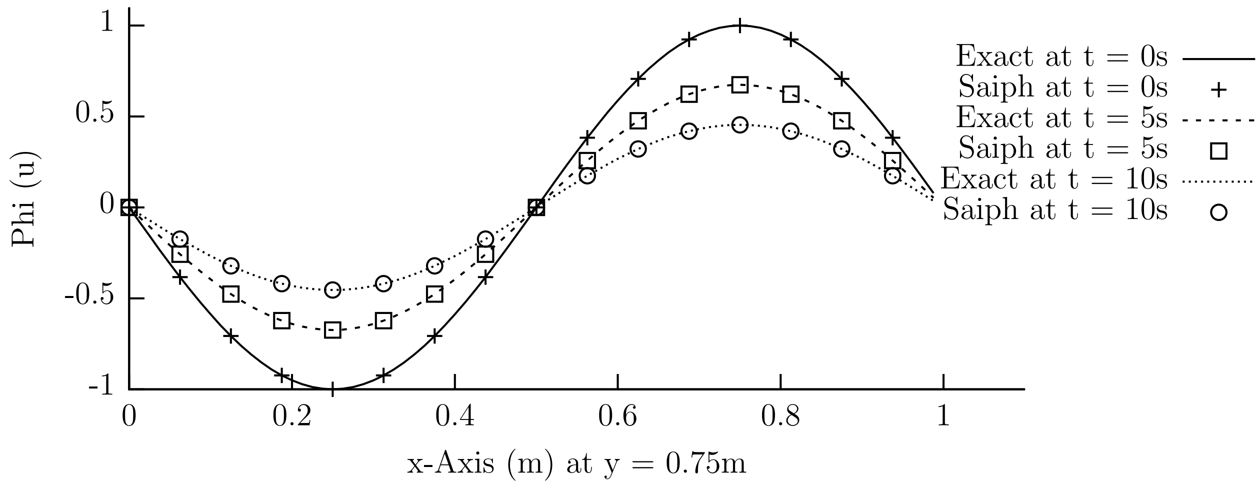

Figure 10: Phi x-profile at different time-steps.

x 2D Euler Equations - Inviscid Vortex Convection

Problem Specification

Use Case. Inviscid Vortex Convection
Spatial domain: $0 \leq x, y < 10$ meters, periodic boundary conditions in both directions
Governing equations: 2D Euler Equation

$$\frac{\partial \rho}{\partial t} = -\mathbf{u} \cdot \nabla \rho - \rho \boldsymbol{\nabla} \cdot \mathbf{u}$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\mathbf{u} \cdot \nabla (\rho u_i) - (\rho u_i) \boldsymbol{\nabla} \cdot \mathbf{u} - (\nabla p)_i$$

$$\frac{\partial (\rho E)}{\partial t} = -\mathbf{u} \cdot \nabla (\rho E) - (\rho E) \boldsymbol{\nabla} \cdot \mathbf{u} - \mathbf{u} \cdot \nabla p - p \boldsymbol{\nabla} \cdot \mathbf{u}$$

$$p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho u^2\right)$$

Initial conditions:
Adiabatic index $\gamma = 1.4$
Vortex strength $b = 0.5$
Vortex initial center $(x_c, y_c) = (5, 5)$
Distance from the vortex center $r = \left[((x - x_c)^2 + (y - y_c)^2)\right]^{1/2}$

$$\rho = \left[1 - \frac{(\gamma - 1) b^2}{8 \gamma \pi^2} e^{1 - r^2}\right]^{\frac{1}{\gamma - 1}}$$

$$u_x = \frac{b}{2\pi} e^{\frac{1}{2}(1 - r^2)} (y - y_c)$$

$$u_y = 0.1 - \frac{b}{2\pi} e^{\frac{1}{2}(1 - r^2)} (x - x_c)$$

$$p = 1$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 19.53125$ mm
$\Delta t = 3.125$ ms
$nsteps = 3200$
Forward in-time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(\Delta t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results at initial time $t = 0s$ and final $t = 10s$ are presented in Figure 11. After one convection cycle, the L_2 norm of the pressure quantity, taking the initial conditions as reference results is $\boldsymbol{L_2 = 4 \cdot 10^{-6}}$Pa.

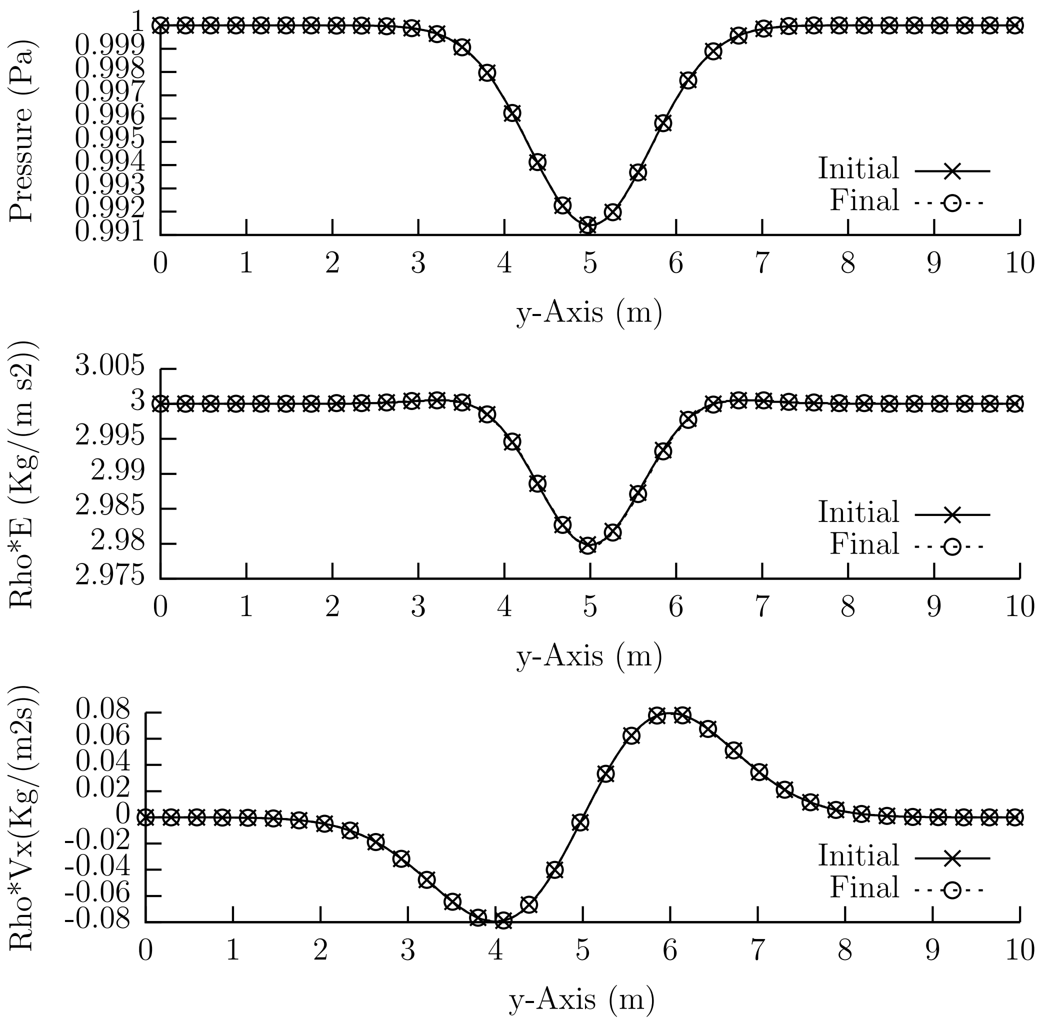

Figure 11: Variable profiles after one convection cycle for the Inviscid Vortex Convection application.

The Saiph's simulation animation can be checked at:
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xi 2D Non-linear Advection - Smith Hutton

This application corresponds to a specific test problem presented by Smith and Hutton from 1982 [5]. It corresponds to a two-dimensional convection test problem where a scalar field such as ϕ reaches a steady-state in a prescribed velocity field $\boldsymbol{u}(x, y)$ with no diffusion. Figure 12 schematizes this test problem.

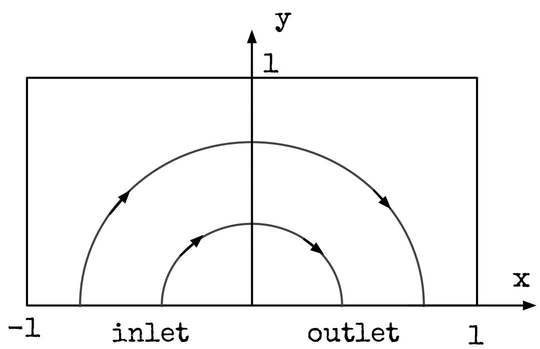

Figure 12: Schema of Smith Hutton problem.

Problem Specification

Use Case. Smith Hutton problem
Spatial domain: $-1 \leq x < 1$ meters, $0 \leq y < 1$ meters.
Governing equations: 2D Advection Equation

$$\frac{\partial \phi}{\partial t} = -\mathbf{u} \cdot \boldsymbol{\nabla} \phi - \phi \boldsymbol{\nabla} \mathbf{u}$$

Initial conditions:

$$\phi = 1 - \tanh(\alpha)$$

$$\boldsymbol{u} = (2y(1 - x^2), -2x(1 - y^2))m/s$$

Boundary conditions:
Wall inlet and outlet boundary conditions.

$$\begin{array}{ll} \phi = 1 + \tanh(\alpha(2x + 1)) & \left\{ \begin{array}{l} y = 0 \\ x < 0 \end{array} \right. (inlet) \\ \frac{\partial \phi}{\partial y} = 0 & \left\{ \begin{array}{l} y = 0 \\ x \geq 0 \end{array} \right. (outlet) \\ \phi = 1 - \tanh(\alpha) & \quad \text{elsewhere} \end{array}$$

The Saiph's code specification can be checked at:
[From local repository] **Click to Saiph code**
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Simulation details

$\Delta x = 10$ mm
$\Delta t = 1$ ms
$nsteps = 3000$
Forward in-time integration using 3rd order Runke-Kutta method: $\mathcal{O}(\Delta t^4)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^4)$

Results

Output results of the ϕ y-profile at $x = 0m$ and converged stationary state, are presented in Figure 13.
The L_2 norm has been computed over the ϕ variable taking the analytic solution as reference:

$$\phi(x_0, y) = 1 + \tanh(\alpha(1 - 2y))$$

The final output simulation, presents an error of $\mathbf{L_2 = 9.6 \cdot 10^{-3}}$

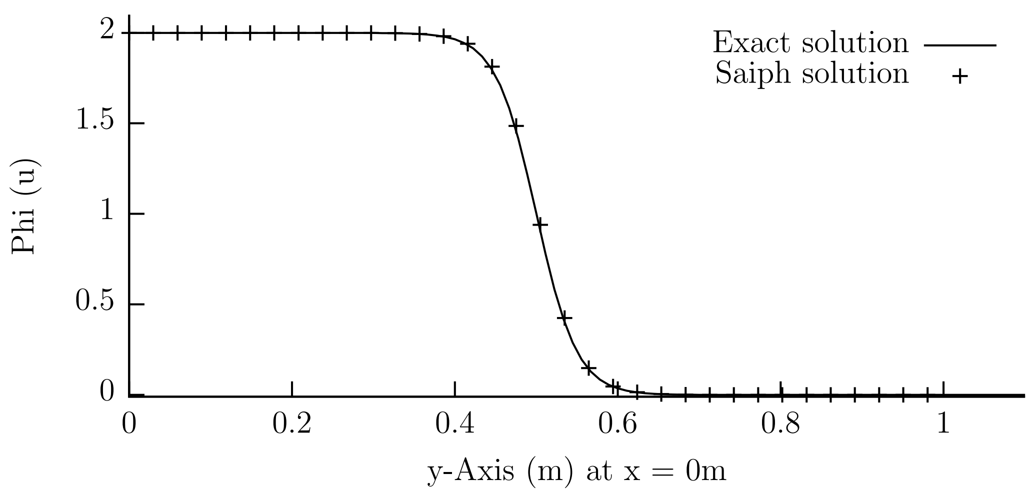

Figure 13: Phi y-profile at $x = 0m$, converged state.

The Saiph's simulation animation can be checked at:
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xii 2D Euler Equations with gravitational force - Rising Thermal Bubble

The rising thermal bubble application simulates the evolution of a warm bubble in a constant potential temperature environment [1]. The initial bubble is warmer than the ambient air. The air mass is initially at rest and in hydrostatic balance. A potential temperature perturbation is then added, so at $t0$ the bubble starts to move up, deforming as a consequence of the shearing motion caused by the velocity field gradients.

Problem Specification

Use Case. Rising Thermal Bubble
Spatial domain: $0 \leq x < 1000$ meters, $0 \leq y < 1000$ meters.
Governing equations: 2D Euler Equations in conservative form using density, momentum, and potential temperature.

$$\frac{\partial \rho}{\partial t} = -\mathbf{u} \cdot \nabla \rho - \rho \boldsymbol{\nabla} \cdot \mathbf{u}$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\mathbf{u} \cdot \nabla (\rho u_i) - (\rho u_i) \boldsymbol{\nabla} \cdot \mathbf{u} - (\nabla p)_i - \rho g \hat{k}_i$$

$$\frac{\partial (\rho \theta)}{\partial t} = -\mathbf{u} \cdot \nabla (\rho \theta) - (\rho \theta) \boldsymbol{\nabla} \cdot \mathbf{u}$$

$$p = p_0 \left(\frac{\rho R \theta}{p_0} \right)^{c_p / c_v}$$

Initial conditions:
Gas constant $R = c_p - c_v = 287.058\, m^2/Ks^2$
Specific heats for constant pressure $c_p = 717.645\, m^2/Ks^2$
Specific heats for constant volume $c_v = 1004.703\, m^2/Ks^2$
Adiabatic index $\gamma = 1.4$
Gravitational constant $g = 9.8\, m/s^2$
Pressure at the surface $p_0 = 100000\, Pa$
Directional vector along the vertical $\hat{k} = (0, 1)$
Mean potential temperature $\bar{\theta} = 300\, K$
Exner pressure (hydrostatic balance) $p_{exner}(y) = 1 - \frac{gy}{c_p \bar{\theta}}\, u.a.$
Perturbation temperature amplitude $\theta_c = 0.5\, K$
Perturbation center $(x_c, y_c) = (500, 350)\, m$
Perturbation radius $r_c = 250\, m$
With $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$,

$$\theta(x, y) = \begin{cases} \bar{\theta} & r > r_c \\ \bar{\theta} + \frac{1}{2}\theta_c (1 + \cos \frac{\pi r}{r_c}) & r \leq r_c \end{cases}$$

$$\rho = \left(\frac{p_0}{R \theta} \right) p_{exner}^{c_v / R}$$

$$\boldsymbol{u} = (0, 0) m/s$$

$$p = p_0 \left(\frac{\rho R \theta}{p_0} \right)^{c_p/c_v}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 20$ m
$\Delta t = 5$ ms
$nsteps = 14000$
Forward in-time integration using 3rd order Runke-Kutta method: $\cdot\mathcal{O}(t^4)$
Spatial differentiation accuracy (default): $\mathcal{O}(\Delta x^3)$

Results

The following plots shows the potential temperature perturbation $d\theta = \frac{\rho\theta}{\rho} - \bar{\theta}$ at the initial and final time $t = 0 - 700s$.

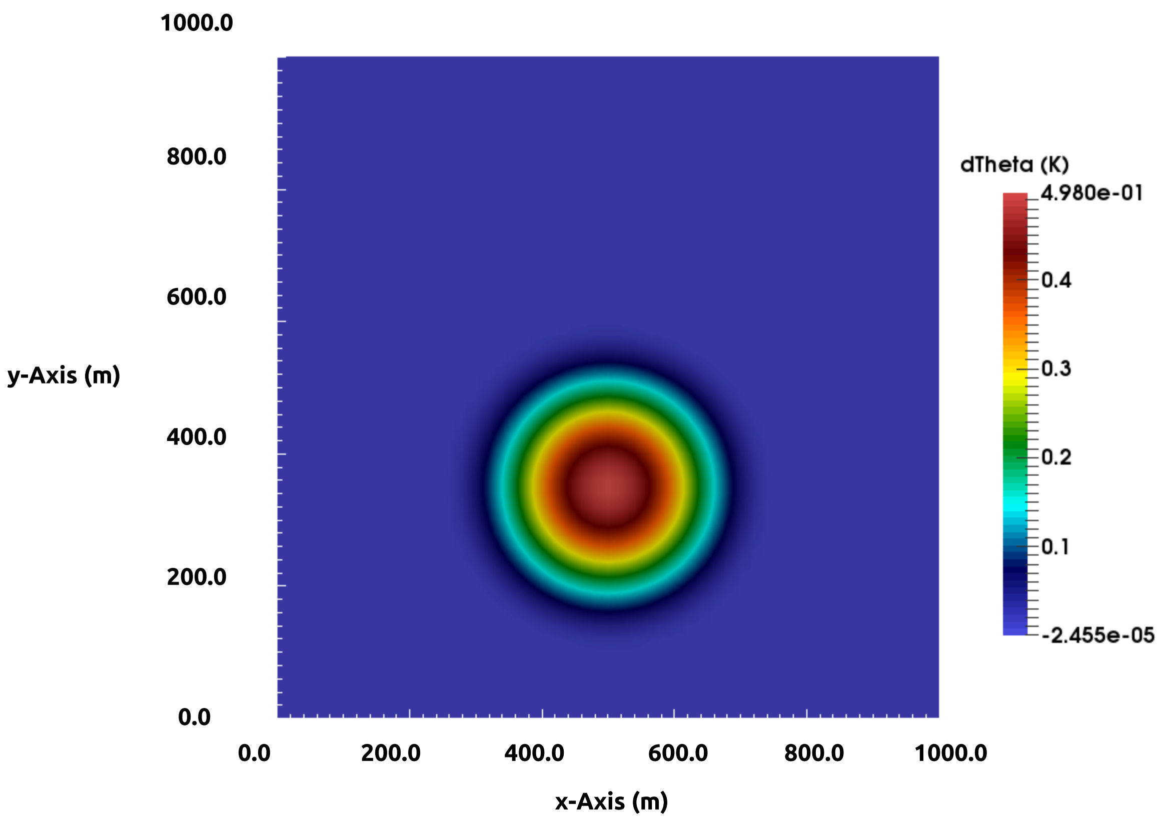

Figure 14: Potential temperature perturbation at $t = 0s$.

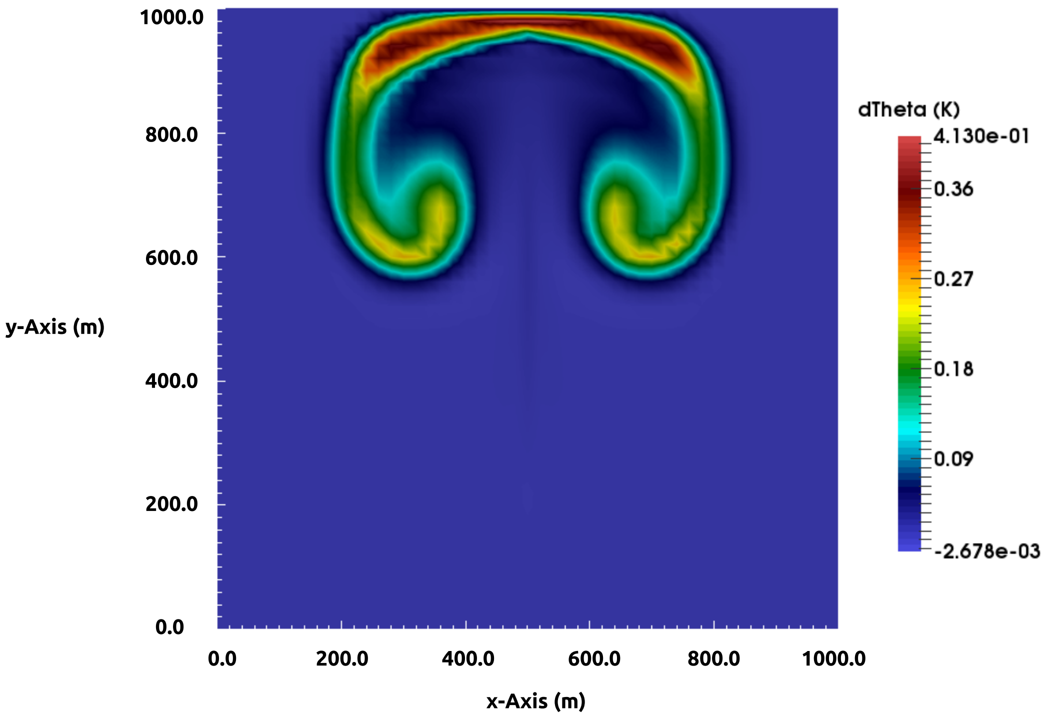

Figure 15: Potential temperature perturbation at $t = 700s$.

The Saiph's simulation animation can be checked at:
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xiii 2D Taylor-Green Vortex

This application is a simplification of the test problem presented by Taylor and Green in 1937 [7]. It corresponds to a two-dimensional unsteady flow of a decaying vortex.

Problem Specification

Use Case. Taylor-Green vortex
Spatial domain: $-\pi L \leq x, y \leq \pi L$, with $\pi L = 0.016$ meters, periodic boundary conditions in both directions.
Governing equations: 2D Navier-Stokes

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{u}\rho) \tag{1}$$

$$\frac{\partial (\rho u)_i}{\partial t} = -\nabla \cdot ((\rho u)_i \cdot \mathbf{u} + p) + \mu \nabla^2 u_i \tag{2}$$

$$\frac{\partial (\rho E)}{\partial t} = -\nabla \cdot ((\rho E) \cdot \mathbf{u} + p \cdot \mathbf{u}) + \mu(u_x \nabla^2 u_x + u_y \nabla^2 u_y) + \frac{\mu c_p}{Pr} \nabla^2 T \tag{3}$$

$$p = \rho R T \ \text{, with } T = \frac{1}{c_v}\left(\frac{\rho E}{\rho} - \frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right) \tag{4}$$

For simulation output purposes we compute the fluid vorticity quantity:

$$\omega = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \tag{5}$$

Initial conditions:

Initial flow fields
Density $\rho = \frac{p}{RT}$
Velocity $\mathbf{u} = \begin{cases} u_x = U_0 \sin \frac{x}{L} \cos \frac{y}{L} \\ u_y = -U_0 \cos \frac{x}{L} \sin \frac{y}{L} \end{cases}$
Pressure $p = p_0 + \frac{3\rho_0 U_0^2}{16}\left(\cos \frac{2x}{L} + \cos \frac{2y}{L}\right)$
Energy $E = c_v T_0 + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}$

Initial flow parameters
Initial velocity value $U_0 = 33.7 m/s$
Initial temperature $T_0 = 273K$
Initial density $\rho_0 = 194.60 \cdot 10^{-3} kg/m^3$
Initial pressure $p_0 = \rho_0 R T_0 = 15778.49 Pa$
Gas constant for nitrogen $R = 297 J/(kgK)$
Mach number for the ICs $Ma = \frac{U_0}{c_{so}} = 0.1$
Speed of sound for nitrogen (at the initial temperature) $c_{so} = \sqrt{\gamma R T_0}$
Specific heat ratio $\gamma = \frac{c_p}{c_v} = 1.4$
Reynolds number $Re = \frac{\rho_0 U_0 L}{\mu_0} = 2000$

Dynamic shear viscosity for nitrogen (at the initial temperature) $\mu_0 = 1.67 \cdot 10^{-5} kg/(ms)$
Prandtl number $Pr = 0.71$
Convective time scale $t_c = \frac{L}{U_0} = 1.5 \cdot 10^{-4} s$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 8 \cdot 10^{-5}$ m
$\Delta t = 7,6 \cdot 10^{-7}$ s
$nsteps = 800000$
Forward in-time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(\Delta t^3)$
Spatial differentiation accuracy: $\mathcal{O}(\Delta x^8)$

Results

The problem has a time-dependent analytical solution derived from the incompressible two-dimensional stream function [3].
We compare Saiph numerical output results with the time-dependent analytical solution given in terms of the vorticity as

$$\omega(x, y, t) = \omega_{init} e^{-\frac{2}{Re}\frac{t}{t_c}}$$

with $\omega_{init} = \frac{2U_0}{L} \sin\frac{x}{L} \sin\frac{y}{L}$
Figure 16 shows the vorticity contours of the Saiph simulation.
Finally, the L_2 norm has been computed over the ω variable taking the analytic solution as reference. Figure 17 reports the errors obtained.

The Saiph's simulation animation can be checked at:
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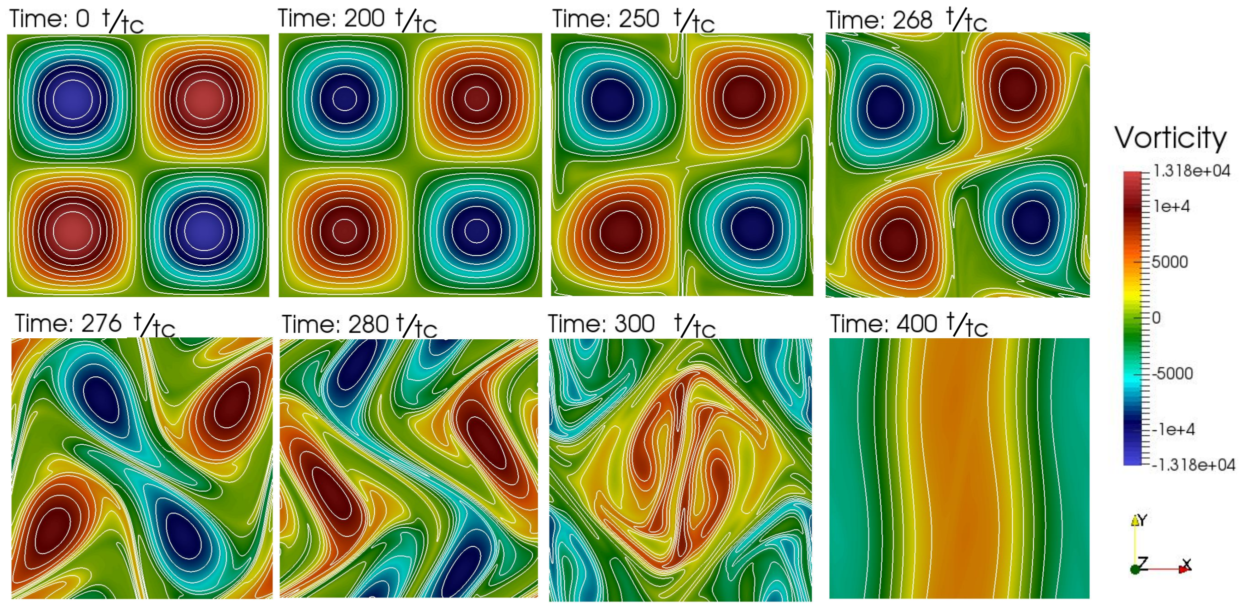

Figure 16: Saiph 2D-TGV vorticity contours at different time-steps using a RK3 time-integration method in a 400×400 mesh at $Re = 2000$

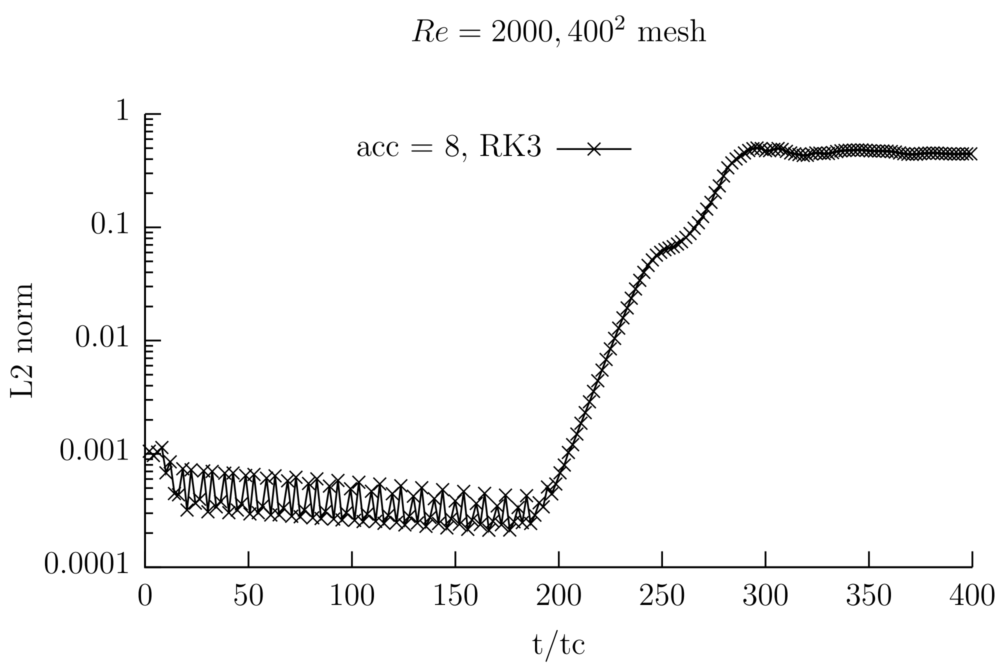

Figure 17: Saiph 2D-TGV vorticity L2 norm

xiv 3D Linear Diffusion - Heat Equation

Problem Specification

Use Case. Linear Diffusion
Spatial domain: $0 \leq x, y, z < 3$ meters
Governing equations: 3D Linear Diffusion Equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T$$

Initial conditions:

$$T_0(x, y, z) = 400 \cdot e^{-((x-1.5)^2+(y-1.5)^2+(z-1.5)^2)} \text{ K}$$

$$\kappa = 1.0 \text{ m}^2/\text{s}$$

Boundary conditions:
Fixed condition along all the boundaries of the domain (Dirichlet BCs), using the initial value at the boundaries:

$$T_0 = 400 \cdot e^{(-3*1.5^2)} \begin{cases} x = 0\,m, \\ x = 3\,m \\ y = 0\,m, \\ y = 3\,m \\ z = 0\,m, \\ z = 3\,m \end{cases}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 10.0$ mm
$\Delta t = 0.01$ ms
$nsteps = 25000$
Forward in-time integration using Euler's method: $\mathcal{O}(\Delta t)$
Spatial differentiation accuracy: $\mathcal{O}(\Delta x^8)$

Results

Output results at three four-steps, $t = 0ms$, $t = 50ms$, $t = 100ms$ and $t = 200ms$ are presented in Figure 18.

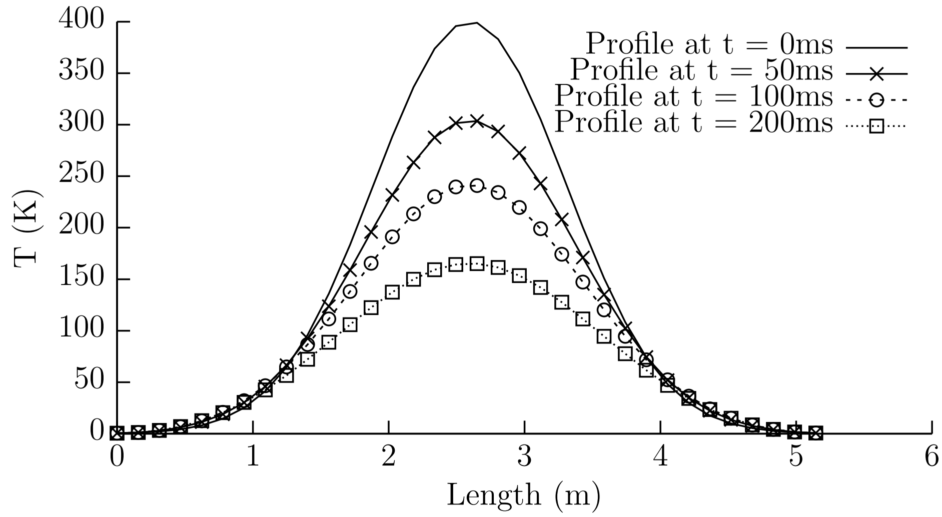

Figure 18: Temperature profile at the mesh diagonal for different time-steps.

The Saiph's simulation animation can be checked at:
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xv 3D Taylor-Green Vortex

The Taylor-Green vortex test case simulates the evolution of a vortex flow from a laminar to a turbulent state. The application was originally presented by Taylor and Green in 1937 [7].

Problem Specification

Use Case. Taylor-Green vortex
Spatial domain: $-\pi L \leq x, y, z \leq \pi L$, with $\pi L = 0.016$ meters, periodic boundary conditions in all directions.
Governing equations: 3D Navier-Stokes

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{u}\rho) \tag{6}$$

$$\frac{\partial (\rho u)_i}{\partial t} = -\nabla \cdot ((\rho u)_i \cdot \mathbf{u} + p) + \mu \nabla^2 u_i \tag{7}$$

$$\begin{aligned} \frac{\partial (\rho e)}{\partial t} =& -\nabla \cdot ((\rho e) \cdot \mathbf{u} + p \cdot \mathbf{u}) \\ &+ \mu (u_x \nabla^2 u_x + u_y \nabla^2 u_y + u_z \nabla^2 u_z) + \frac{\mu c_p}{Pr} \nabla^2 T \end{aligned} \tag{8}$$

$$p = \rho R T \ , \ T = \frac{1}{c_v} \left(\frac{\rho e}{\rho} - \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) \tag{9}$$

For simulation output purposes we compute the fluid kinetic energy E_k and enstrophy ξ quantities:

$$E_k = \frac{1}{\rho_0 \Omega U_0^2} \int_\Omega \rho \frac{\mathbf{u} \cdot \mathbf{u}}{2} d\Omega \quad , \quad \xi = \frac{t_c^2}{\rho_0 \Omega} \int_\Omega \rho \frac{\boldsymbol{\omega} \cdot \boldsymbol{\omega}}{2} d\Omega \tag{10}$$

with $t_c = \frac{L}{U_0}$ the convective time scale and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ the fluid vorticity:

$$\omega_x = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} \ , \ \omega_y = -\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \ , \ \omega_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \tag{11}$$

Initial conditions:

Initial flow fields
Density $\rho = \frac{p}{RT}$
Velocity $\mathbf{u} = \begin{cases} u_x = U_0 \sin \frac{x}{L} \cos \frac{y}{L} \cos \frac{z}{L} \\ u_y = -U_0 \cos \frac{x}{L} \sin \frac{y}{L} \cos \frac{z}{L} \\ u_z = 0 \end{cases}$
Pressure $p = p_0 + \frac{\rho_0 U_0^2}{16} \left(\cos \frac{2x}{L} + \cos \frac{2y}{L} \right) \left(\cos \frac{2z}{L} + 2 \right)$
Energy $E = c_v T_0 + \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u}$

Initial flow parameters
Initial velocity value $U_0 = 33.7m/s$
Initial temperature $T_0 = 273K$
Initial density $\rho_0 = 155.68 \cdot 10^{-3} kg/m^3$
Initial pressure $p_0 = \rho_0 R T_0 = 12622.80Pa$
Gas constant for nitrogen $R = 297J/(kgK)$
Mach number for the ICs $Ma = \frac{U_0}{c_{so}} = 0.1$
Speed of sound for nitrogen (at the initial temperature) $c_{so} = \sqrt{\gamma R T_0}$
Specific heat ratio $\gamma = \frac{c_p}{c_v} = 1.4$
Reynolds number $Re = \frac{\rho_0 U_0 L}{\mu_0} = 1600$
Dynamic shear viscosity for nitrogen (at the initial temperature) $\mu_0 = 1.67 \cdot 10^{-5} kg/(ms)$
Prandtl number $Pr = 0.71$
Convective time scale $t_c = \frac{L}{U_0} = 1.5 \cdot 10^{-4} s$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 0,625 \cdot 10^{-4}$ m
$\Delta t = 0,375 \cdot 10^{-7}$ s
$nsteps = 80000$
Forward in-time integration using 3rd order Runke-Kutta method: $\mathcal{O}(\Delta t^3)$
Spatial differentiation accuracy: $\mathcal{O}(\Delta x^8)$

Results

We compare Saiph numerical output results against a provided reference flow solution [8] given in terms of the enstrophy. Figure 19 shows the evolution of the enstrophy for the Saiph simulation and the reference result.
Finally, we compute the L_2 norm for the ξ variable taking the solution from [8] as reference. The output simulation presents an error of $\mathbf{L_2 = 7,0 \cdot 10^{-3}}$

The Saiph's simulation animation can be checked at:
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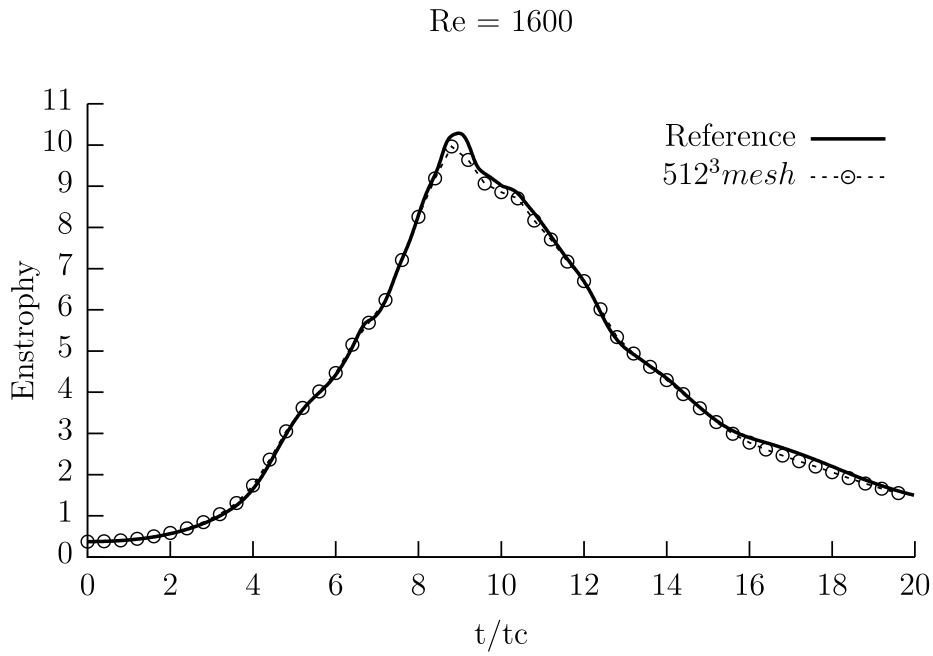

Figure 19: Saiph 3D-TGV enstrophy numerical results

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