1D Linear Advection - Sine Wave

Problem Specification

Use Case. Linear Advection
Spatial domain: $0 \leq x < 1$ meters, periodic boundary conditions
Governing equations: 1D Linear Advection Equation

$$\frac{\partial \phi}{\partial t} = -\mathbf{u} \cdot \nabla \phi$$

Initial conditions:

$$\phi(x) = \sin(2\pi x)$$

$$\mathbf{u} = 1 \text{ m/s}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 1$ mm
$\Delta t = 1$ ms
$nsteps = 1000$
Forward in-time integration using Euler method: $\mathcal{O}(t)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results at three time-steps, $t = 0s$, $t = 0.5s$ and $t = 1s$ are presented in Figure 1. The L_2 norm has been computed over the ϕ variable taking the initial conditions as the analytic solution. The output simulation presents no truncation error $\mathbf{L_2 = 0}$

The Saiph's simulation animation can be checked at:
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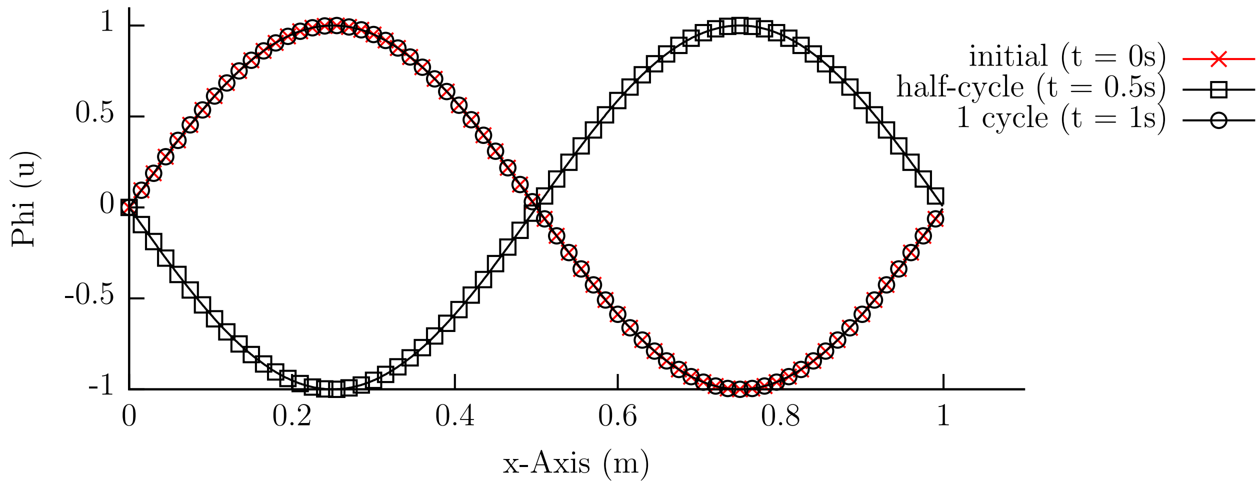

Figure 1: Phi profile at different time-steps.

1D Linear Advection - Discontinuous Waves

Problem Specification

Use Case. Linear Advection
Spatial domain: $-1 \leq x < 1$ meters, periodic boundary conditions
Governing equations: 1D Linear Advection Equation

$$\frac{\partial \phi}{\partial t} = -\mathbf{u} \cdot \nabla \phi$$

Initial conditions:

$$\phi(x) = \begin{cases} \exp\left(-\log(2)\frac{(x+7)^2}{0.0009}\right) & -0.8 \leq x \leq -0.6 \\ 1 & -0.4 \leq x \leq -0.2 \\ 1 - |10(x-0.1)| & 0 \leq x \leq 0.2 \\ \sqrt{1-100(x-0.5)^2} & 0.4 \leq x \leq 0.6 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{u} = 1 \text{ m/s}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 1$ mm
$\Delta t = 1$ ms
$nsteps = 2000$
Forward in-time integration using Euler method: $\mathcal{O}(t)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results at two time-steps, $t = 0s$ and $t = 2s$ are presented in Figure 2.
The L_2 norm has been computed over the ϕ variable taking the initial conditions as the analytic solution. The output simulation presents no truncation error $\mathbf{L_2 = 0}$

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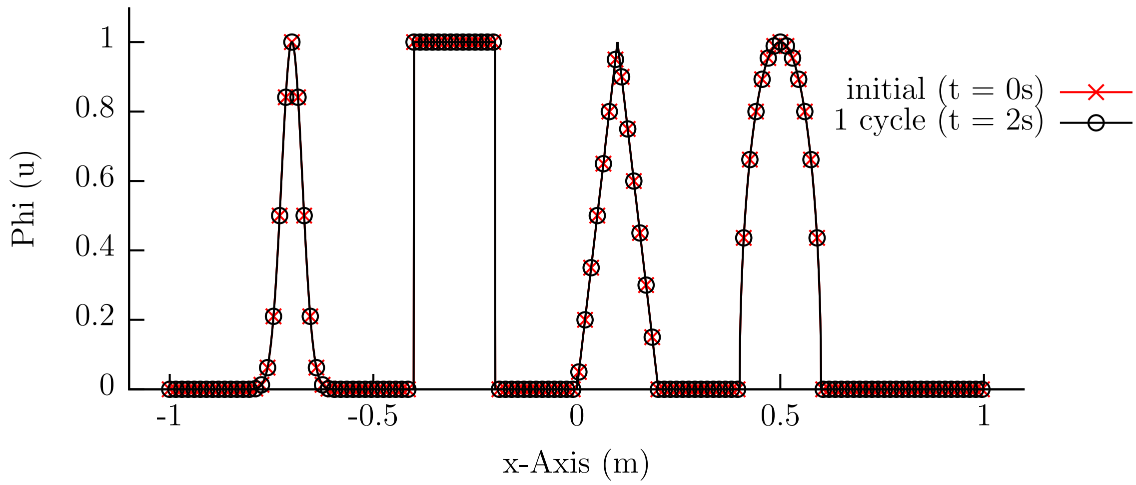

Figure 2: Phi profile at different time-steps.

1D Linear Diffusion - Sine Wave

Problem Specification

Use Case. Linear Diffusion
Spatial domain: $0 \leq x < 1$ meters, periodic boundary conditions
Governing equations: 1D Linear Diffusion Equation

$$\frac{\partial \phi}{\partial t} = \boldsymbol{\nabla} \cdot (\nu \nabla \phi)$$

Initial conditions:

$$\phi(x) = \sin(2\pi x)$$

$$\nu = 0.001 \text{ m}^2/\text{s}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 12.5$ mm
$\Delta t = 50$ ms
$nsteps = 200$
Forward in-time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results at three time-steps, $t = 0s$, $t = 5s$ and $t = 10s$ are presented in Figure 3. The L_2 norm has been computed over the ϕ variable taking the analytic solution as reference:

$$\phi(x, t) = e^{-\nu 4\pi^2 t} \sin(2\pi x)$$

The final output simulation ($t = 10s$), presents an error of $\mathbf{L_2 = 1.4 \cdot 10^{-7}}$

The Saiph's simulation animation can be checked at:
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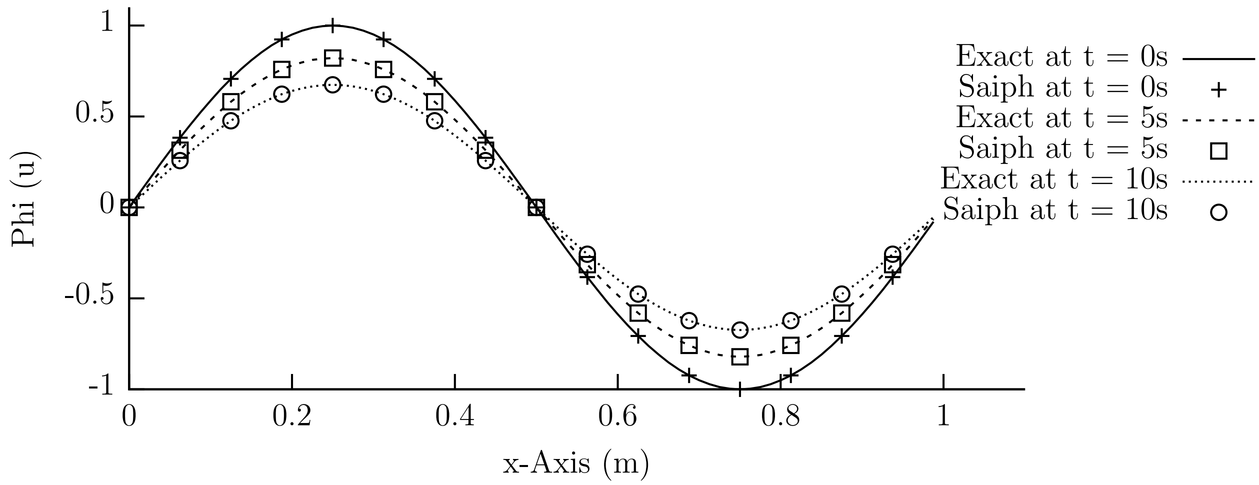

Figure 3: Phi profile at different time-steps.

1D Euler Equations - Sod Shock Tube

This problem corresponds to the well known shock tube originally proposed by Sod [3]. It is a transient case consisting of a one-dimensional tube, closed at its ends and divided into two equal regions by a thin diaphragm. Each region is filled with the same gas, but with different thermodynamic parameters. The fluid is initially at rest and starts to move because of the sudden breakdown of the diaphragm, the discontinuous initial condition of the left and right states entails a shock wave that propagates to both left and right sides. Figure 4 schematizes the configuration of this test [1].

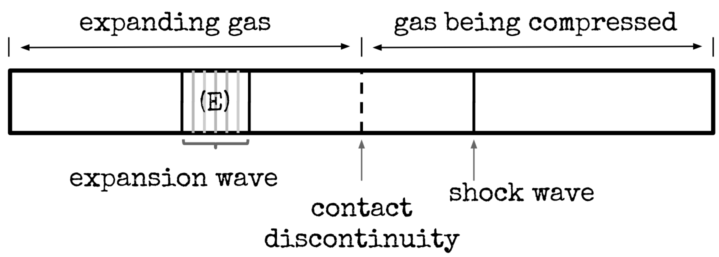

Figure 4: Schema of waves propagating in the tube after the diaphragm breakdown ($t > 0$).

Problem Specification

Use Case. Sod shock tube
Spatial domain: $0 \leq x < 1$ meters
Governing equations: 1D Euler equations, continuity, momentum, energy and state.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\left[\rho \mathbf{u} \cdot \nabla u_i + u_i \nabla \cdot (\rho \mathbf{u}) + \frac{\partial p}{\partial x_i}\right]$$

$$\frac{\partial (\rho E)}{\partial t} = -\left[\mathbf{u} \cdot \nabla \rho E + \rho E \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u}\right]$$

$$p = (\gamma - 1)\left((\rho E) - \frac{1}{2}(\rho \mathbf{u})\,\mathbf{u}\right)$$

Initial conditions:

$$\gamma = 1.4$$

$$\mathbf{u} = 0 \text{ m/s}$$

$$\rho(x) = \begin{cases} 1\, kg/m^3 & 0 \leq x < 0.5 \\ 0.125\, kg/m^3 & 0.5 \leq x \leq 1 \end{cases}$$

$$\rho E(x) = \begin{cases} 2.5\, kg/ms^2 & 0 \le x < 0.5 \\ 0.25\, kg/ms^2 & 0.5 \le x \le 1 \end{cases}$$

$$p(x) = \begin{cases} 1\, Pa & 0 \le x < 0.5 \\ 0.1\, Pa & 0.5 \le x \le 1 \end{cases}$$

Boundary conditions:
Symmetric boundary condition (Neumann BCs)

$$\frac{\partial \rho}{\partial x} = 0\, kg/m^4 \quad \begin{cases} x = 0\, m, \\ x = 1\, m \end{cases}$$

$$\frac{\partial (\rho E)}{\partial x} = 0\, kg/m^2 s^2 \quad \begin{cases} x = 0\, m, \\ x = 1\, m \end{cases}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial x} = 0\, kg/m^3 s \quad \begin{cases} x = 0\, m, \\ x = 1\, m \end{cases}$$

$$\frac{\partial p}{\partial x} = 0\, Pa/m \quad \begin{cases} x = 0\, m, \\ x = 1\, m \end{cases}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 2.5$ mm
$\Delta t = 0.2$ ms
$nsteps = 1000$
Forward in-time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results at final time, $t = 0.2s$ are presented in Figure 5, against analytic solutions. The L_2 norm has been computed over the density, pressure and velocity variables taking the analytic solution as reference. The final output simulation, presents an error of $\mathbf{L_{2-\rho} = 1.2 \cdot 10^{-2}}$, $\mathbf{L_{2-p} = 1.1 \cdot 10^{-2}}$, $\mathbf{L_{2-u} = 3.7 \cdot 10^{-2}}$

The Saiph's simulation animation can be checked at:
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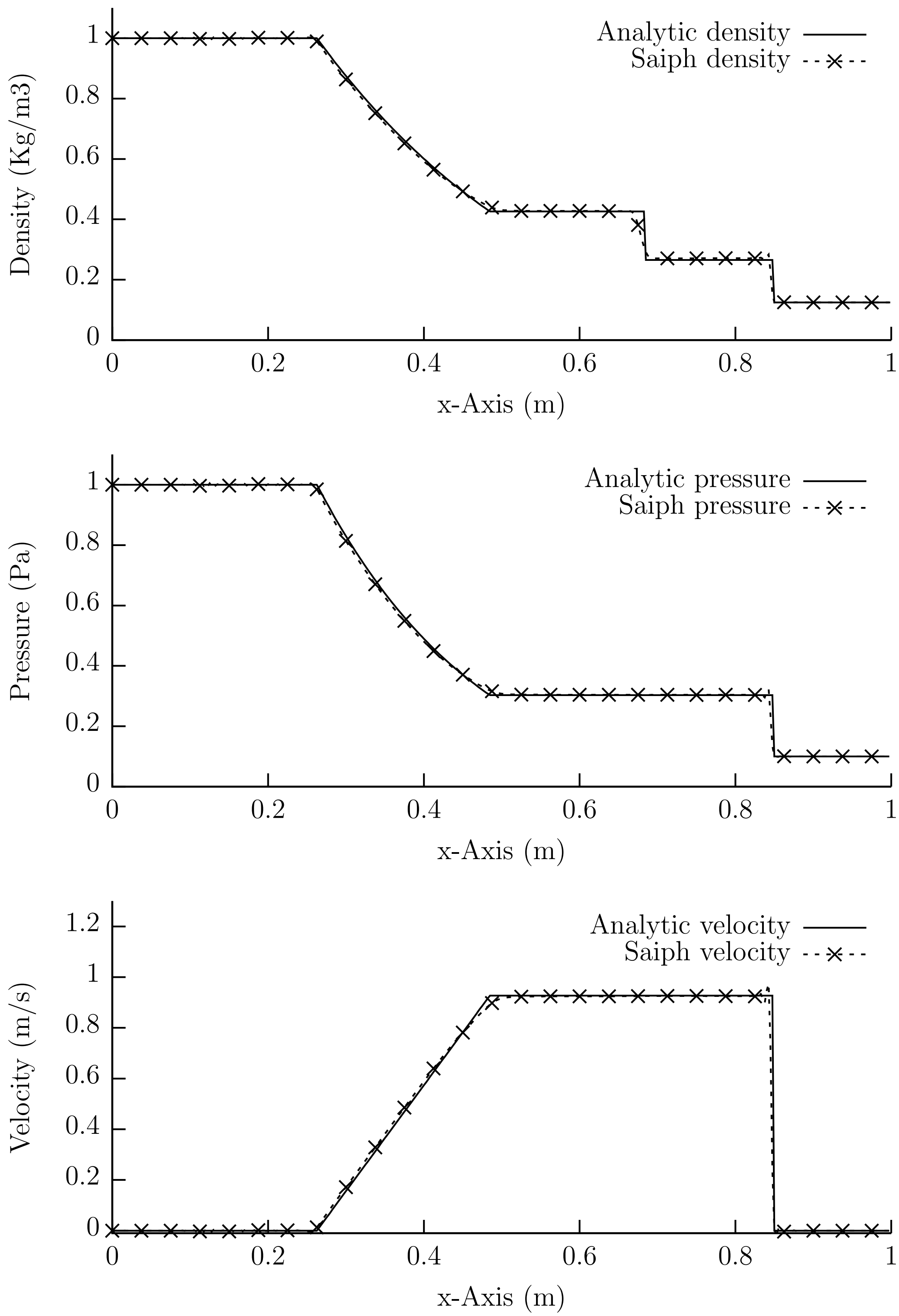

Figure 5: Variable profiles at $t = 0.2s$ for the Sod Tube problem.

1D Euler Equations - Lax Shock Tube

Similar to Sod shock tube use-case with different initial conditions.

Problem Specification

Use Case. Lax shock tube
Spatial domain: $0 \leq x < 1$ meters
Governing equations: 1D Euler equations, continuity, momentum, energy and state.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\left[\rho \mathbf{u} \cdot \nabla u_i + u_i \nabla \cdot (\rho \mathbf{u}) + \frac{\partial p}{\partial x_i}\right]$$

$$\frac{\partial (\rho E)}{\partial t} = -\left[\mathbf{u} \cdot \nabla \rho E + \rho E \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u}\right]$$

$$p = (\gamma - 1)\left((\rho E) - \frac{1}{2}(\rho \mathbf{u})\,\mathbf{u}\right)$$

Initial conditions:

$$\gamma = 1.4$$

$$\rho(x) = \begin{cases} 0.445\; kg/m^3 & 0 \leq x < 0.5 \\ 0.5\; kg/m^3 & 0.5 \leq x \leq 1 \end{cases}$$

$$\rho \mathbf{u}(x) = \begin{cases} 0.311\; kg/m^2 s & 0 \leq x < 0.5 \\ 0\; kg/m^2 s & 0.5 \leq x \leq 1 \end{cases}$$

$$\rho E(x) = \begin{cases} 8.928\; kg/ms^2 & 0 \leq x < 0.5 \\ 1.4275\; kg/ms^2 & 0.5 \leq x \leq 1 \end{cases}$$

Boundary conditions:
Symmetric boundary condition (Neumann BCs)

$$\frac{\partial \rho}{\partial x} = 0\; kg/m^4 \;\; \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial (\rho E)}{\partial x} = 0\; kg/m^3 s^2 \;\; \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial x} = 0\; kg/m^3 s \;\; \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial p}{\partial x} = 0\; Pa/m \;\; \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 5$ mm
$\Delta t = 1$ ms
$nsteps = 80$
Forward in time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results at final time, $t = 0.08s$ are presented in Figure 6.

The Saiph's simulation animation can be checked at:
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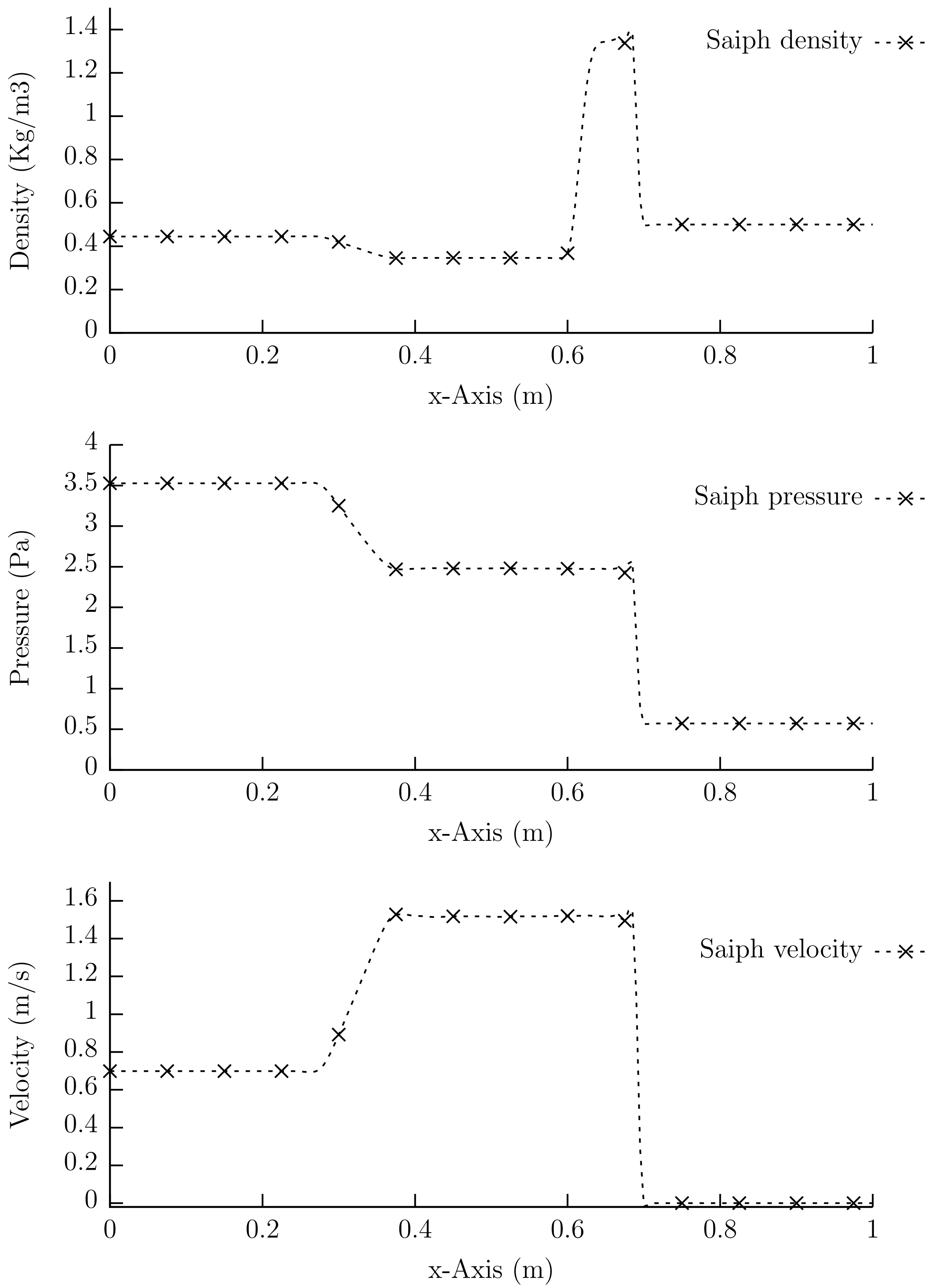

Figure 6: Variable profiles at $t = 0.08s$ for the Lax Tube problem.

1D Euler Equations - Shu Osher

A moving shock interacts with sine waves in density. Similar to Sod shock tube use-case with different initial conditions [2].

Problem Specification

Use Case. Shu Osher tube
Spatial domain: $-5 \leq x < 5$ meters
Governing equations: 1D Euler equations, continuity, momentum, energy and state.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\left[\rho \mathbf{u} \cdot \nabla u_i + u_i \nabla \cdot (\rho \mathbf{u}) + \frac{\partial p}{\partial x_i}\right]$$

$$\frac{\partial (\rho E)}{\partial t} = -\left[\mathbf{u} \cdot \nabla \rho E + \rho E \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u}\right]$$

$$p = (\gamma - 1)\left((\rho E) - \frac{1}{2}(\rho \mathbf{u})\,\mathbf{u}\right)$$

Initial conditions:

$$\gamma = 1.4$$

$$\rho(x) = \begin{cases} 27/7 \; kg/m^3 & -5 \leq x < -4 \\ 1 + 0.2 \sin(5x) \; kg/m^3 & -4 \leq x \leq 5 \end{cases}$$

$$\rho \mathbf{u}(x) = \begin{cases} 10.14185223 \; kg/m^2 s & -5 \leq x < -4 \\ 0 \; kg/m^2 s & -4 \leq x \leq 5 \end{cases}$$

$$p = \begin{cases} 31/3 \; Pa & -5 \leq x < -4 \\ 1 \; Pa & -4 \leq x \leq 5 \end{cases}$$

Boundary conditions:
Symmetric boundary condition (Neumann BCs)

$$\frac{\partial \rho}{\partial x} = 0 \; kg/m^4 \; \begin{cases} x = 0 \; m, \\ x = 1 \; m \end{cases}$$

$$\frac{\partial (\rho E)}{\partial x} = 0 \; kg/m^3 s^2 \; \begin{cases} x = 0 \; m, \\ x = 1 \; m \end{cases}$$

$$\frac{\partial (\rho \mathbf{u})}{\partial x} = 0 \; kg/m^3 s \; \begin{cases} x = 0 \; m, \\ x = 1 \; m \end{cases}$$

$$\frac{\partial p}{\partial x} = 0 \; Pa/m \; \begin{cases} x = 0 \; m, \\ x = 1 \; m \end{cases}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 25$ mm
$\Delta t = 1$ ms
$nsteps = 1800$
Forward in time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results at final time, $t = 1.8s$ are presented in Figure 7.

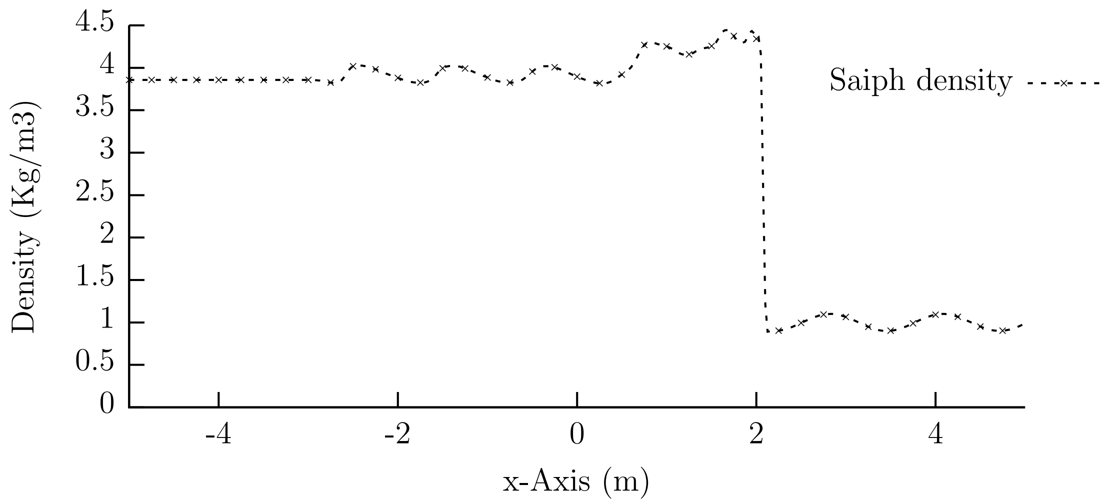

Figure 7: Density profile at $t = 1.8s$ for the Shu Osher problem.

The Saiph's simulation animation can be checked at:
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1D Euler Equations - Sod Shock Tube with Gravitational Force

Similar to Sod shock tube use-case with an uniform gravitational external force $\mathbf{g}$.

Problem Specification

Use Case. Sod shock tube
Spatial domain: $0 \leq x < 1$ meters
Governing equations: 1D Euler equations, continuity, momentum, energy and state with external gravitational force.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{u})$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\left[\rho \mathbf{u} \cdot \nabla u_i + u_i \nabla \cdot (\rho \mathbf{u}) + \frac{\partial p}{\partial x_i} - \rho g_i\right]$$

$$\frac{\partial (\rho E)}{\partial t} = -\left[\mathbf{u} \cdot \nabla \rho E + \rho E \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p + p \nabla \cdot \mathbf{u} - \rho \mathbf{u} \cdot \mathbf{g}\right]$$

$$p = (\gamma - 1)\left((\rho E) - \frac{1}{2}(\rho \mathbf{u})\,\mathbf{u}\right)$$

Initial conditions:

$$\gamma = 1.4$$

$$\mathbf{g} = 1\; m/s^2$$

$$\mathbf{u} = 0 \text{ m/s}$$

$$\rho(x) = \begin{cases} 1\; kg/m^3 & 0 \leq x < 0.5 \\ 0.125\; kg/m^3 & 0.5 \leq x \leq 1 \end{cases}$$

$$\rho E(x) = \begin{cases} 2.5\; kg/ms^2 & 0 \leq x < 0.5 \\ 0.25\; kg/ms^2 & 0.5 \leq x \leq 1 \end{cases}$$

$$p(x) = \begin{cases} 1\; Pa & 0 \leq x < 0.5 \\ 0.1\; Pa & 0.5 \leq x \leq 1 \end{cases}$$

Boundary conditions:
Symmetric boundary condition (Neumann BCs) and zero wall velocity (slip-wall BC, Dirichlet BC)

$$\frac{\partial \rho}{\partial x} = 0\; kg/m^4 \; \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial (\rho E)}{\partial x} = 0\; kg/m^2 s^2 \; \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\rho \mathbf{u} = 0\; kg/m^3 s \; \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

$$\frac{\partial p}{\partial x} = 0\; Pa/m \; \begin{cases} x = 0\; m, \\ x = 1\; m \end{cases}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 2.5$ mm
$\Delta t = 0.2$ ms
$nsteps = 1000$
Forward in-time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results at final time, $t = 0.2s$ are presented in Figure 8.

The Saiph's simulation animation can be checked at:
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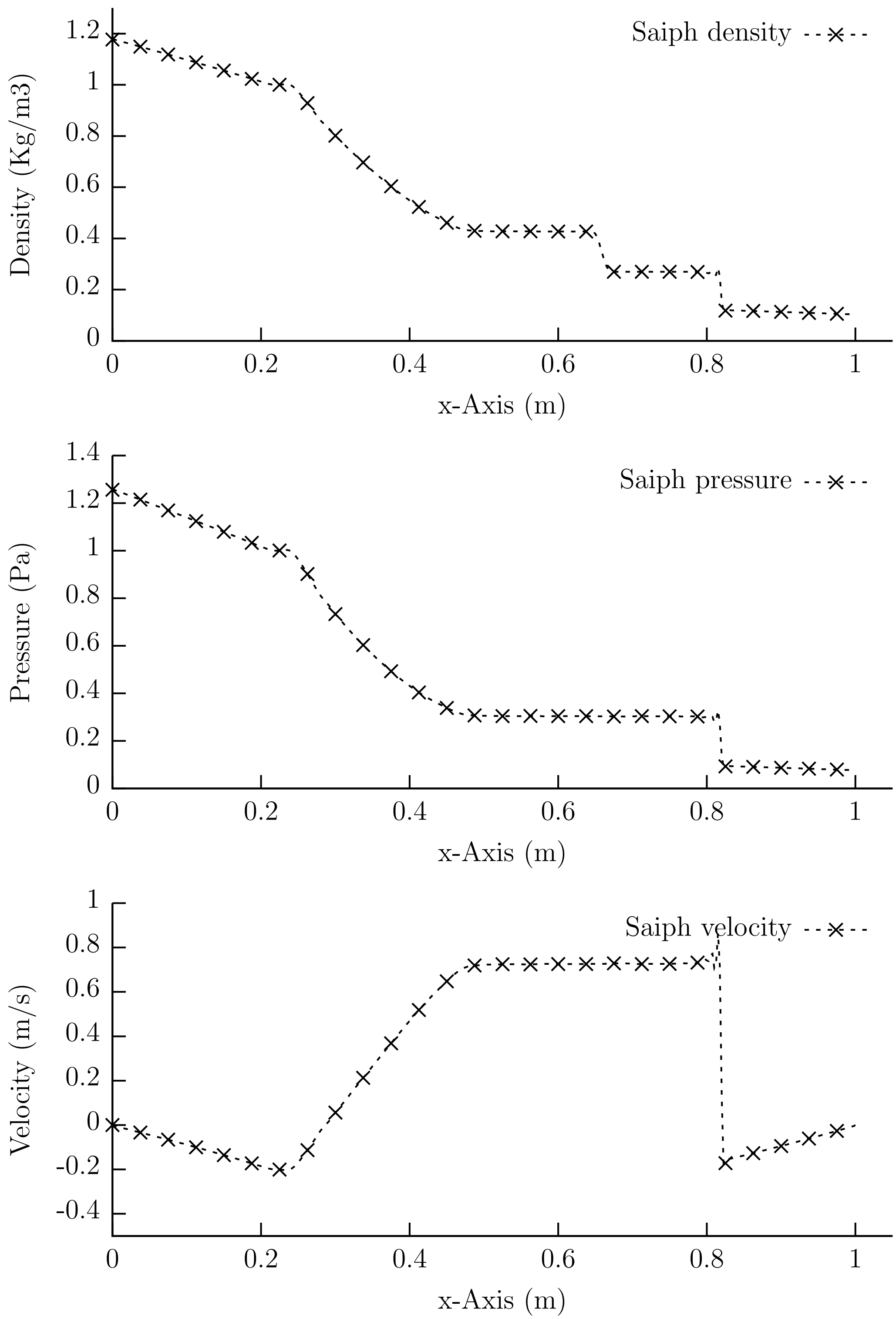

Figure 8: Variable profiles at $t = 0.2s$ for the Sod Tube problem with gravitational force.

2D Linear Advection - Gaussian Pulse

Problem Specification

Use Case. Linear Advection
Spatial domain: $-6 \leq x < 6$ meters, $-3 \leq y < 3$ meters periodic boundary conditions in both directions
Governing equations: 2D Linear Advection Equation

$$\frac{\partial \phi}{\partial t} = -\mathbf{u} \cdot \nabla \phi$$

Initial conditions:

$$\phi(x) = e^{-\left(\frac{x^2}{2} + \frac{y^2}{2}\right)}$$

$$\mathbf{u} = (1, 0) \text{ m/s}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 40$ mm
$\Delta t = 40$ ms
$nsteps = 300$
Forward in-time integration using Euler method: $\mathcal{O}(t)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results at three time-steps, $t = 0s$, $t = 6s$ and $t = 12s$ are presented in Figure 9. The L_2 norm has been computed over the ϕ variable taking the initial conditions as the analytic solution. The output simulation presents no truncation error $\mathbf{L_2 = 0}$

The Saiph's simulation animation can be checked at:
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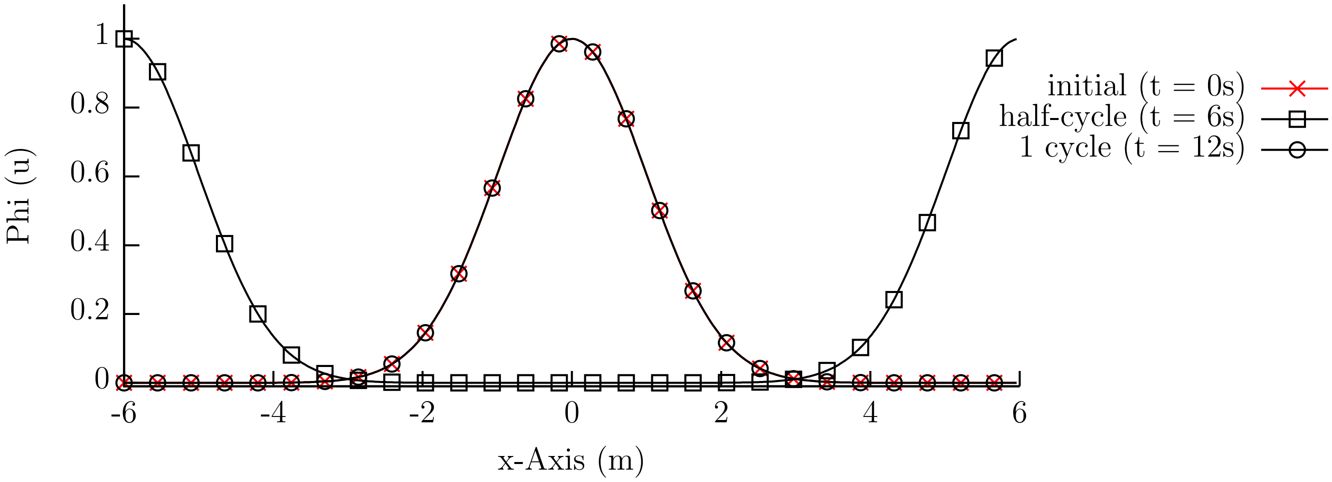

Figure 9: $\phi(x, y)$ profile at different time-steps.

2D Linear Diffusion - Sine Wave

Problem Specification

Use Case. Linear Diffusion
Spatial domain: $0 \leq x, y < 1$ meters, periodic boundary conditions in both directions
Governing equations: 2D Linear Diffusion Equation

$$\frac{\partial \phi}{\partial t} = \boldsymbol{\nabla} \cdot (\nu \nabla \phi)$$

Initial conditions:

$$\phi(x) = \sin(2\pi x)\sin(2\pi y)$$

$$\nu = 0.001 \text{ m}^2/\text{s}$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 12.5$ mm
$\Delta t = 20$ ms
$nsteps = 500$
Forward in-time integration using 4^{th} order Runke-Kutta method: $\mathcal{O}(t^4)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results of the ϕ diagonals profiles at three time-steps, $t = 0s$, $t = 5s$ and $t = 10s$ are presented in Figure 10.
The L_2 norm has been computed over the ϕ variable taking the analytic solution as reference:

$$\phi(x, t) = e^{-\nu 8\pi^2 t}\sin(2\pi x)\sin(2\pi y)$$

The final output simulation ($t = 10s$), presents an error of $\mathbf{L_2 = 1.04 \cdot 10^{-7}}$

The Saiph's simulation animation can be checked at:
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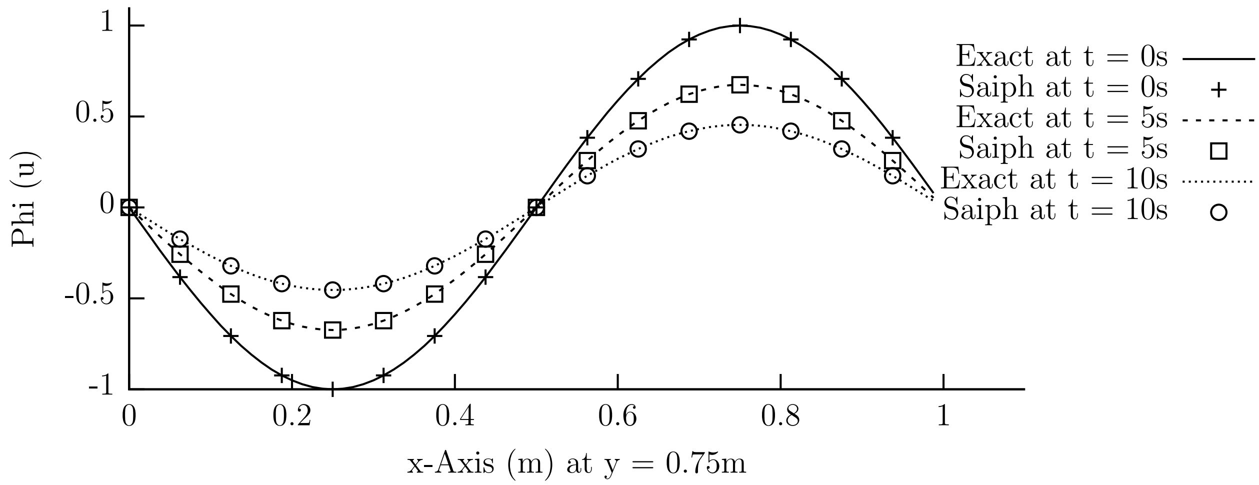

Figure 10: Phi x-profile at different time-steps.

2D Euler Equations - Inviscid Vortex Convection

Problem Specification

Use Case. Inviscid Vortex Convection
Spatial domain: $0 \leq x, y < 10$ meters, periodic boundary conditions in both directions
Governing equations: 2D Euler Equation

$$\frac{\partial \rho}{\partial t} = -\mathbf{u} \cdot \nabla \rho - \rho \boldsymbol{\nabla} \cdot \mathbf{u} \tag{1}$$

$$\frac{\partial (\rho u_i)}{\partial t} = -\mathbf{u} \cdot \nabla (\rho u_i) - (\rho u_i) \boldsymbol{\nabla} \cdot \mathbf{u} - (\nabla p)_i \tag{2}$$

$$\frac{\partial (\rho E)}{\partial t} = -\mathbf{u} \cdot \nabla (\rho E) - (\rho E) \boldsymbol{\nabla} \cdot \mathbf{u} - \mathbf{u} \cdot \nabla p - p \boldsymbol{\nabla} \cdot \mathbf{u} \tag{3}$$

$$p = (\gamma - 1) \left(\rho E - \frac{1}{2} \rho u^2 \right) \tag{4}$$

Initial conditions:
Adiabatic index $\gamma = 1.4$
Vortex strength $b = 0.5$
Vortex initial center $(x_c, y_c) = (5, 5)$
Distance from the vortex center $r = \left[((x - x_c)^2 + (y - y_c)^2) \right]^{1/2}$

$$\rho = \left[1 - \frac{(\gamma - 1) b^2}{8 \gamma \pi^2} e^{1 - r^2} \right]^{\frac{1}{\gamma - 1}}$$

$$u_x = \frac{b}{2\pi} e^{\frac{1}{2}(1 - r^2)} (y - y_c)$$

$$u_y = 0.1 - \frac{b}{2\pi} e^{\frac{1}{2}(1 - r^2)} (x - x_c)$$

$$p = 1$$

The Saiph's code specification can be checked at:
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Simulation details

$\Delta x = 19.53125$ mm
$\Delta t = 3.125$ ms
$nsteps = 3200$
Forward in-time integration using 3$^{\text{rd}}$ order Runke-Kutta method: $\mathcal{O}(t^3)$
Spatial differentiation accuracy (default): $\mathcal{O}(x^4)$

Results

Output results at initial time $t = 0s$ and final $t = 10s$ are presented in Figure 11. After one convection cycle, the L_2 norm of the pressure quantity, taking the initial conditions as reference results is $\boldsymbol{L_2 = 4 \cdot 10^{-6}}$Pa.

The Saiph's simulation animation can be checked at:
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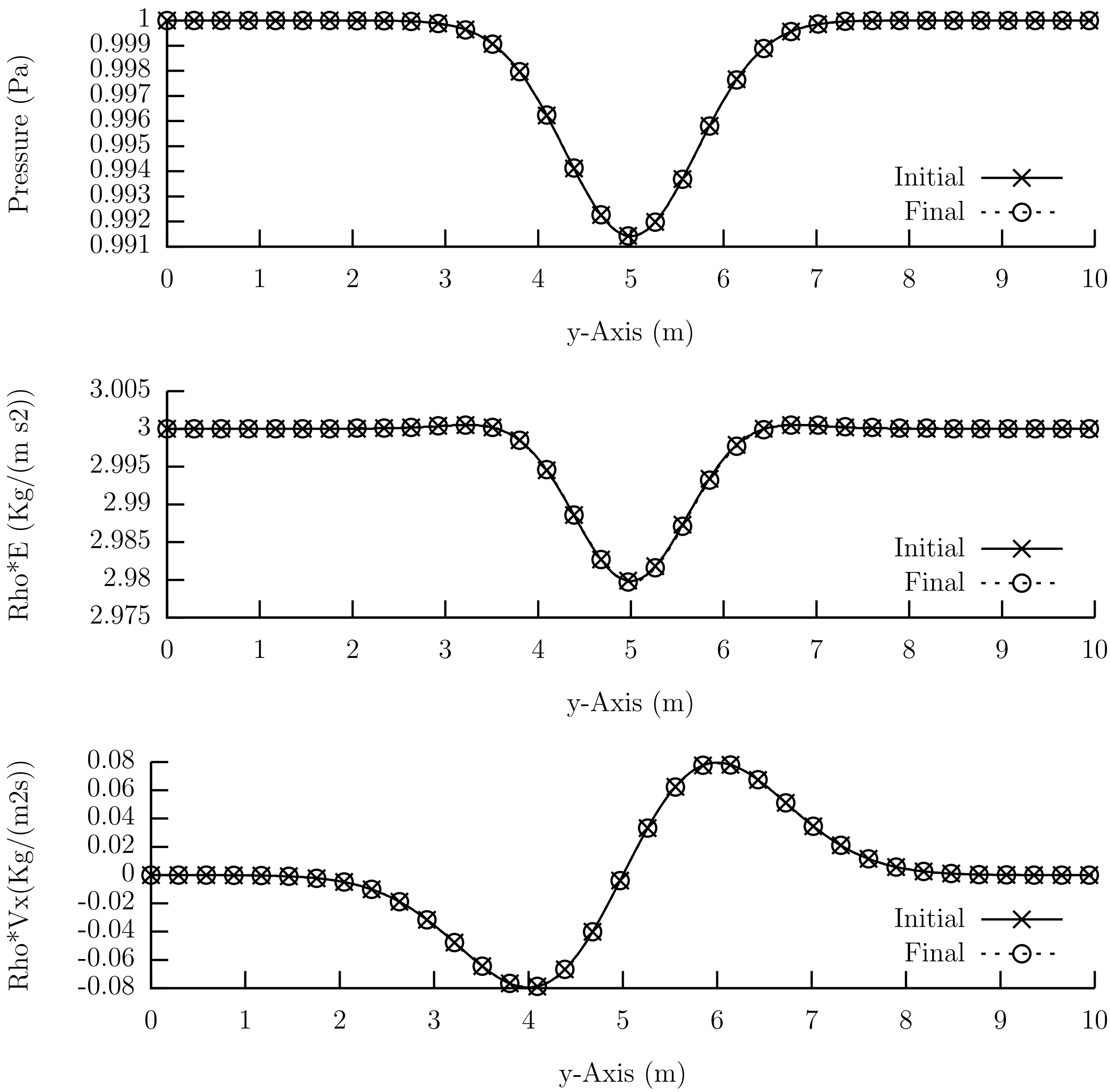

Figure 11: Variable profiles after one convection cycle for the Inviscid Vortex Convection application.

References

[1] Macià, S., Mateo, S., Martínez-Ferrer, P. J., Beltran, V., Mira, D., and Ayguadé, E. Saiph: Towards a dsl for high-performance computational fluid dynamics. In *Proceedings of the Real World Domain Specific Languages Workshop 2018* (2018), ACM, p. 6.

[2] Shu, C.-W., and Osher, S. Efficient implementation of essentially non-oscillatory shock-capturing schemes. *Journal of computational physics 77*, 2 (1988), 439–471.

[3] Sod, G. A survey of several finite difference methods for systems of nonlinear hyperbolic conservation laws. *Journal of Computational Physics 27*, 1 (1978), 1–31.