2D Taylor-Green Vortex Application

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1 PROBLEM STATEMENT

The two-dimensional Taylor-Green Vortex (2D-TGV) application simplifies the 3D problem [1] using a square domain where vortices analytically decay without generating turbulence. The simplified problem has an analytical time-dependent equilibrium solution. However, this equilibrium solution is not numerically stable [2]. The 2D specification serve as benchmark for testing and validating Navier-Stokes codes.

2 GOVERNING EQUATIONS

We use nitrogen as the working gas, described by the density $\rho(x, y)$, velocity $\mathbf{u}(x, y)$, total energy $E(x, y)$ and pressure $p(x, y)$ fields. The following 2D Navier-Stokes formulation as given by the continuity Eq.(1), momentum Eq.(2) and energy Eq.(3) equations defines the system of PDEs describing the fluid dynamics.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{u}\rho) \tag{1}$$

$$\frac{\partial (\rho u)_i}{\partial t} = -\nabla \cdot ((\rho u)_i \cdot \mathbf{u} + p) + \mu \nabla^2 u_i \tag{2}$$

$$\begin{aligned}\frac{\partial (\rho E)}{\partial t} = &- \nabla \cdot ((\rho E) \cdot \mathbf{u} + p \cdot \mathbf{u}) \\ &+ \mu(u_x \nabla^2 u_x + u_y \nabla^2 u_y) + \frac{\mu c_p}{Pr} \nabla^2 T\end{aligned} \tag{3}$$

To close the system, we use additional thermodynamic relations between state variables, including the equation of state and the velocity and temperature relations (4).

$$p = \rho RT \quad , \quad u_i = \frac{(\rho u)_i}{\rho} \quad , \quad T = \frac{1}{c_v}\left(\frac{\rho E}{\rho} - \frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right) \tag{4}$$

Finally, for simulation output purposes we add to the equation system the relation for the computation of the fluid vorticity $\boldsymbol{\omega}$ quantity (5).

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{5}$$

3 INITIAL CONDITIONS

Flow fields' initial conditions are determined by thermodynamics relations, and geometric and flow parameters:

$$\rho = \frac{p}{RT_0} \tag{6}$$

$$\mathbf{u} = \begin{cases} u_x = U_0 \sin \frac{x}{L} \cos \frac{y}{L} \\ u_y = -U_0 \cos \frac{x}{L} \sin \frac{y}{L} \end{cases} \tag{7}$$

$$p = p_0 + \frac{3\rho_0 U_0^2}{16}\left(\cos \frac{2x}{L} + \cos \frac{2y}{L}\right) \tag{8}$$

$$E = c_v T_0 + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \tag{9}$$

With initial temperature $T_0 = 273K$, initial pressure $p_0 = \rho_0 RT_0$, gas constant $R = 297J/(kgK)$, Mach number $Ma = \frac{U_0}{C_{so}} = 0.1$, speed of sound for nitrogen $c_{so} = \sqrt{\gamma RT_0}$, specific heat ratio $\gamma = \frac{c_p}{c_v} = 1.4$, Reynolds number $Re = \frac{\rho_0 U_0 L}{\mu_0}$, dynamic shear viscosity for nitrogen $\mu_0 = 1.67 \cdot 10^{-5} kg/(ms)$ and Prandtl number $Pr = 0.71$.

4 SIMULATION SETUP

We simulate the Taylor-Green vortex decay in nitrogen at a temperature of standard conditions for turbulent (Re = 1600, 2000) decay regimes [3]. We set the following configuration parameters for the 2D-TGV simulation:

Table 1: 2D-TGV initial flow parameters

Re	U_0 (m/s)	ρ_0 (kg/m^3)	p_0 ($Pa \equiv J/m^3$)
1600	33, 7	$155, 68 \cdot 10^-3$	12622, 80
2000	33, 7	$194, 60 \cdot 10^-3$	15778, 49

We use enough time steps to ensure the apparition of numerical instabilities. Table 2 shows the chosen simulation parameters based on geometry and stability conditions.

Table 2: 2D-TGV simulation parameters

Mesh points	Δx_i (m)	β	Δt (s)	nsteps
64^2	$5, 0 \cdot 10^-4$	0, 4	$6, 0 \cdot 10^-7$	250000
128^2	$2, 5 \cdot 10^-4$	0, 4	$3, 0 \cdot 10^-7$	500000
400^2	$8, 0 \cdot 10^-5$	0, 32	$7, 6 \cdot 10^-8$	800000

5 SAIPH CODE

The application code for the 2D-TGV application using Saiph is available at https://github.com/EulerStokes/CFD-benchmarkApp/blob/master/2D/TaylorGreenVortex/2DTGV-Vort.saiph.

6 NUMERICAL EVALUATION

We use the 2D-TGV application to test Saiph numerical methods; we assess their accuracy and stability performances for different mesh refinements and compare numerical results against analytical ones.

We compare Saiph numerical output results with the time-dependent analytical solution given in terms of the vorticity as $\omega = \omega_{init} e^{-\frac{2}{Re}\frac{t}{t_c}}$. We test different spatial accuracies and time-integration methods available in Saiph and use the L2 norm metric, normalised with the number of mesh points, as the error measurement for the vorticity. The expression for the metric is given below with ω_A and ω_N analytical and numerical values respectively and N the number of mesh points in each spatial direction,

$$||L2|| = \sqrt{\frac{\sum_{j=0}^{j=N} \sum_{i=0}^{i=N} (\omega_N(i,j) - \omega_A(i,j))^2}{N * N}}$$

Figure 1 reports the error results for two mesh sizes of 64×64 and 128×128 points, respectively, using simulation parameters from Table 2 and the configuration parameters from Table 1 corresponding to the 2D-TGV problem with a Reynolds number of 1600.

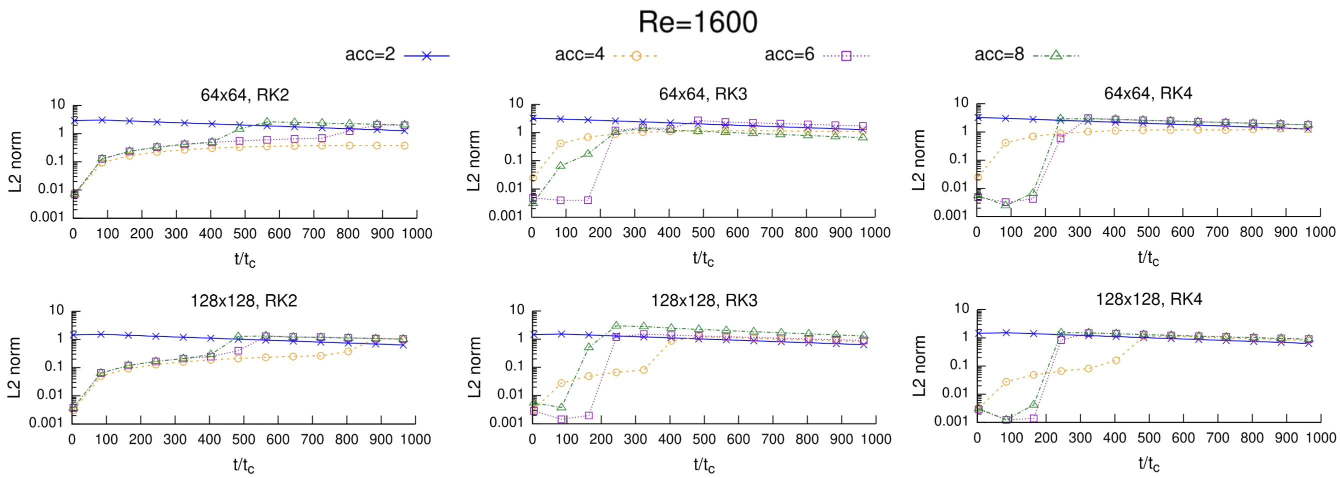

Figure 1: Saiph 2D-TGV vorticity L2 norm, using different orders of accuracy and time integration methods

We do not report Euler's results in Figure 1 since their values are above the unit, out of the competitive range. Similarly, a second-order spatial accuracy does not meet a sufficiently lower error. In contrast, higher spatial accuracy combined with Runge Kutta integration methods preserves the symmetry of the solution for, at least, the first non-dimensional time stamp for both mesh dimensions. Overall, the finer-grained the mesh and higher the time accuracy, the lower the errors and longer the preservation of the symmetry solution. In contrast, a spatial accuracy increase does not imply better and more stable results; using a spatial order of 6 presents better results than those obtained with the spatial eight-order accuracy.

To refine the results, we ran the 2D-TGV application using a finer mesh of 400×400 points, combining the higher-order numerical methods and using an increased Reynolds number of 2000. Figure 2 shows the vorticity contours of the Saiph simulation output, and Figure 3 reports the errors obtained over the same quantity. Results show how around $t/t_c = 200$, the numerical solution becomes unstable and a similar error is obtained for the different accuracies. Saiph numerical accuracy and stability results are comparable with those found in the literature [2].

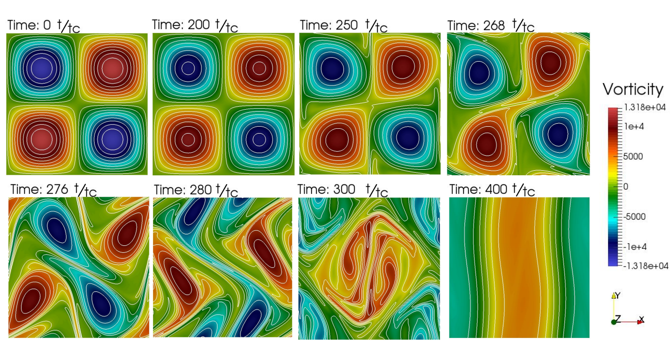

Figure 2: Saiph 2D-TGV vorticity contours at different time-steps using a RK3 time-integration method in a 400×400 **mesh at** $Re = 2000$

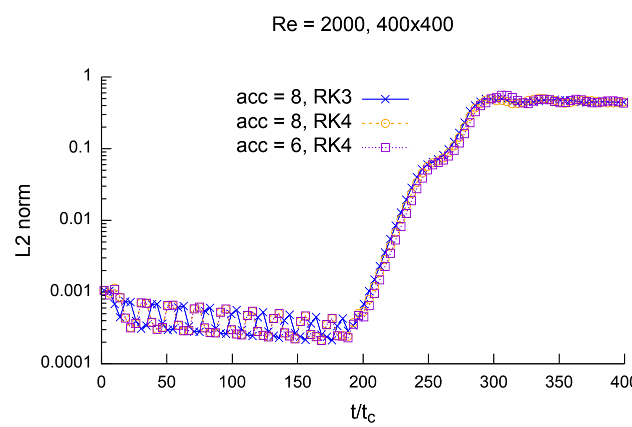

Figure 3: Saiph 2D-TGV vorticity error, using high-order methods

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