2D Taylor-Green Vortex Application

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1 PROBLEM STATEMENT

The two-dimensional Taylor-Green Vortex (2D-TGV) application simplifies the 3D problem [1] using a square domain where vortices analytically decay without generating turbulence. The simplified problem has an analytical time-dependent equilibrium solution. However, this equilibrium solution is not numerically stable [2]. The 2D specification serve as benchmark for testing and validating Navier-Stokes codes.

2 GOVERNING EQUATIONS

We use nitrogen as the working gas, described by the density $\rho(x, y)$, velocity $\mathbf{u}(x, y)$, total energy $E(x, y)$ and pressure $p(x, y)$ fields. The following 2D Navier-Stokes formulation as given by the continuity Eq.(1), momentum Eq.(2) and energy Eq.(3) equations defines the system of PDEs describing the fluid dynamics.

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{u}\rho) \tag{1}$$

$$\frac{\partial (\rho u)_i}{\partial t} = -\nabla \cdot ((\rho u)_i \cdot \mathbf{u} + p) + \mu \nabla^2 u_i \tag{2}$$

$$\begin{aligned} \frac{\partial (\rho E)}{\partial t} =& - \nabla \cdot ((\rho E) \cdot \mathbf{u} + p \cdot \mathbf{u}) \\ &+ \mu(u_x \nabla^2 u_x + u_y \nabla^2 u_y) + \frac{\mu c_p}{Pr} \nabla^2 T \end{aligned} \tag{3}$$

To close the system, we use additional thermodynamic relations between state variables, including the equation of state and the velocity and temperature relations (4).

$$p = \rho R T \quad , \quad u_i = \frac{(\rho u)_i}{\rho} \quad , \quad T = \frac{1}{c_v}\left(\frac{\rho E}{\rho} - \frac{1}{2}\mathbf{u} \cdot \mathbf{u}\right) \tag{4}$$

Finally, for simulation output purposes we add to the equation system the relation for the computation of the fluid vorticity $\boldsymbol{\omega}$ quantity (5).

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u} \tag{5}$$

3 INITIAL CONDITIONS

Flow fields' initial conditions are determined by thermodynamics relations, and geometric and flow parameters:

$$\rho = \frac{p}{RT_0} \tag{6}$$

$$\mathbf{u} = \begin{cases} u_x &= U_0 \sin \frac{x}{L} \cos \frac{y}{L} \\ u_y &= -U_0 \cos \frac{x}{L} \sin \frac{y}{L} \end{cases} \tag{7}$$

$$p = p_0 + \frac{3\rho_0 U_0^2}{16}\left(\cos \frac{2x}{L} + \cos \frac{2y}{L}\right) \tag{8}$$

$$E = c_v T_0 + \frac{1}{2}\mathbf{u} \cdot \mathbf{u} \tag{9}$$

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With initial temperature $T_0 = 273K$, initial pressure $p_0 = \rho_0 R T_0$, gas constant $R = 297 J/(kgK)$, Mach number $Ma = \frac{U_0}{C_{so}} = 0.1$, speed of sound for nitrogen $c_{so} = \sqrt{\gamma R T_0}$, specific heat ratio $\gamma = \frac{c_p}{c_v} = 1.4$, Reynolds number $Re = \frac{\rho_0 U_0 L}{\mu_0}$, dynamic shear viscosity for nitrogen $\mu_0 = 1.67 \cdot 10^{-5} kg/(ms)$ and Prandtl number $Pr = 0.71$.

4 SIMULATION SETUP

We simulate the Taylor-Green vortex decay in nitrogen at a temperature of standard conditions for turbulent (Re = 1600, 2000) decay regimes [3]. We set following configuration parameters for the 2D-TGV simulation:

Table 1: 2D-TGV initial flow parameters

Re	U_0 (m/s)	ρ_0 (kg/m^3)	p_0 ($Pa \equiv J/m^3$)
1600	33, 7	$155,68 \cdot 10^-3$	12622, 80
2000	33, 7	$194,60 \cdot 10^-3$	15778, 49

We use enough time steps to ensure the apparition of numerical instabilities. Table 2 shows the chosen simulation parameters based on geometry and stability conditions.

Table 2: 2D-TGV simulation parameters

Mesh points	Δx_i (m)	β	Δt (s)	nsteps
64^2	$5,0 \cdot 10^-4$	0, 4	$6,0 \cdot 10^-7$	250000
128^2	$2,5 \cdot 10^-4$	0, 4	$3,0 \cdot 10^-7$	500000
400^2	$8,0 \cdot 10^-5$	0, 32	$7,6 \cdot 10^-8$	800000

5 NUMERICAL EVALUATION

We compare Saiph numerical output results with the time-dependent analytical solution given in terms of the vorticity as $\omega = \omega_{init} e^{-\frac{2}{Re}\frac{t}{t_c}}$. We test different spatial accuracies and time-integration methods available in Saiph and use the L2 norm as error measurement for the vorticity. The expression for the L2 norm is given below with ω_A and ω_N analytical and numerical values respectively and N the number of mesh points in each spatial direction,

$$||L2|| = \sqrt{\frac{\sum_{j=0}^{j=N} \sum_{i=0}^{i=N} (\omega_N(i,j) - \omega_A(i,j))^2}{N * N}}$$

Figure 1 reports the error results for two mesh sizes of 64×64 and 128×128 points, respectively, using simulation parameters from Table 2 and the configuration parameters from Table 1 corresponding to the 2D-TGV problem with a Reynolds number of 1600.

We do not report Euler's results in Figure 1 since their values are above the unit, out of the competitive range. Similarly, a second-order spatial accuracy does not meet a sufficiently lower error. In contrast, higher spatial accuracy combined with Runge Kutta integration methods preserves the symmetry of the solution for, at least,

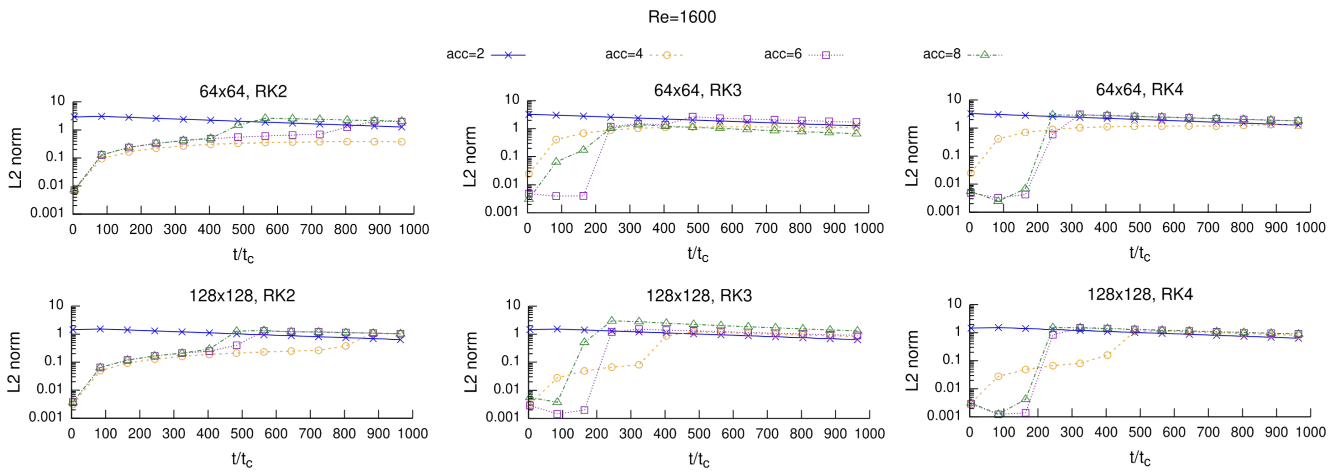

Figure 1: Saiph 2D-TGV vorticity L2 norm, using different orders of accuracy and time integration methods

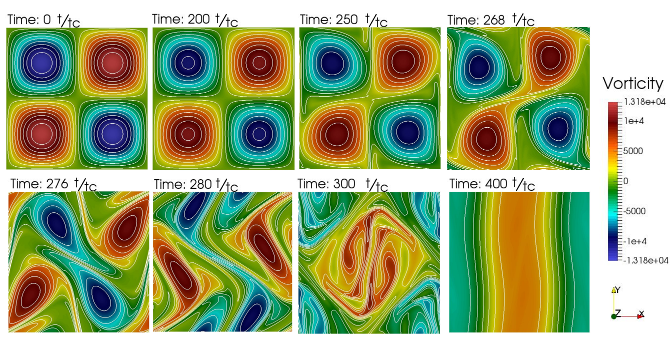

Figure 2: Saiph 2D-TGV vorticity contours at different time-steps using a RK3 time-integration method in a 400×400 **mesh at** $Re = 2000$

the first non-dimensional time stamp for both mesh dimensions. Overall, the finer-grained the mesh and higher the time accuracy, the lower the errors and longer the preservation of the symmetry solution. In contrast, a spatial accuracy increase does not imply better and more stable results; using a spatial order of 6 presents better results than those obtained with the spatial eight-order accuracy.

To refine the results, we ran the 2D-TGV application using a finer mesh of 400×400 points, combining the higher-order numerical methods and using an increased Reynolds number of 2000. Figure 2 shows the vorticity contours of the Saiph simulation output, and Figure 3 reports the errors obtained over the same quantity. Results show how around $t/t_c = 200$, the numerical solution becomes unstable and a similar error is obtained for the different accuracies. Saiph numerical accuracy and stability results are comparable with those found in the literature [2].

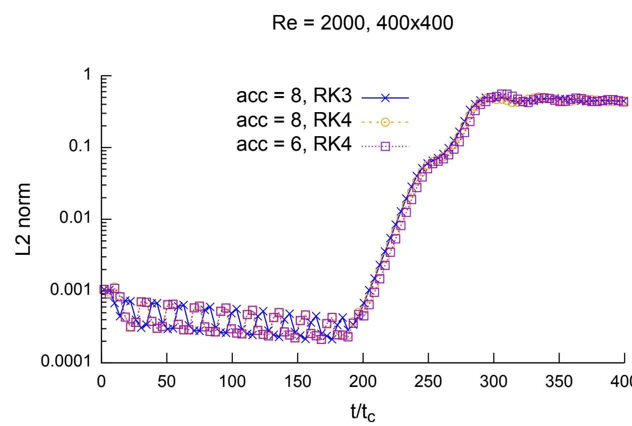

Figure 3: Saiph 2D-TGV vorticity error, using high-order methods

REFERENCES

[1] Marc E Brachet, Daniel I Meiron, Steven A Orszag, BG Nickel, Rudolf H Morf, and Uriel Frisch. 1983. Small-scale structure of the Taylor–Green vortex. *Journal of Fluid Mechanics* 130 (1983), 411–452.

[2] Nidhi Sharma and Sengupta. 2018. Non-linear instability analysis of the two-dimensional Navier-Stokes equation: The Taylor-Green vortex problem. *Physics of Fluids* 30 (05 2018), 054105. https://doi.org/10.1063/1.5024765

[3] Ivan A Shirokov and Tatiana G Elizarova. 2014. Simulation of laminar–turbulent transition in compressible Taylor–Green flow basing on quasi-gas dynamic equations. *Journal of turbulence* 15, 10 (2014), 707–730.