

Notes on Algorithms for Constraint Satisfaction Problems ¹

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Backtracking Search

Figure 1 presents the pseudocode for backtracking search (BT). Note that backtracking search is simply depth-first search with backtracking on the search tree of partial assignments as its nodes. The input is a (partial) assignment A and a list of unassigned variables U . The output is either *failure* or a (complete) assignment. We denote the domain of variable X as $D(X)$. The algorithm is initially called with A the empty assignment and U the list of all the variables of the CSP.

```
BT( $A, U$ )
  if  $A$  is complete then
    return  $A$ 
  end if
  Remove a variable  $X$  from  $U$ 
  for all values  $x \in D(X)$  do
    if  $X = x$  is consistent with  $A$  according to the constraints then
      Add  $X = x$  to  $A$ 
       $result \leftarrow BT(A, U)$ 
      if  $result \neq failure$  then
        return  $result$ 
      end if
      Remove  $X = x$  from  $A$ 
    end if
  end for
  return failure
```

Figure 1: Pseudocode for backtracking search.

Backtracking Search with Forward Checking

Figure 2 presents the pseudocode for backtracking search with forward checking (BT+FC). The algorithm uses an extra input D that corresponds to the current domains of the variables. Initially, D corresponds to the set of original domains.

¹These notes are primarily based on the recitation notes of Kimberle Koile and the slides of Tomás Lozano-Perez from previous terms, as well as the book by Russell and Norvig [2003].

```
BT+FC( $A, U, D$ )
if  $A$  is complete then
    return  $A$ 
end if
Remove a variable  $X$  from  $U$ 
for all values  $x \in D(X)$  do
    if  $X = x$  is consistent with  $A$  according to the constraints then
        Add  $X = x$  to  $A$ 
         $D' \leftarrow D$  (Save the current domains)
        for all  $Y \in U$  (i.e.,  $Y$  an unassigned variable),  $Y - - - X$  (i.e.,  $Y$  a neighbor of  $X$  in the
        constrained graph) do
            Remove values for  $Y$  from  $D'(Y)$  that are inconsistent with  $A$ 
        end for
        if for all  $Y \in U, Y - - - X$ , we have  $D'(Y)$  not empty then
             $result \leftarrow$  BT+FC( $A, U, D'$ )
            if  $result \neq failure$  then
                return  $result$ 
            end if
        end if
        Remove  $X = x$  from  $A$ 
    end if
end for
return failure
```

Figure 2: Pseudocode for backtracking search with forward checking.

Constraint Propagation

Figure 3 presents the pseudocode for the arc consistency algorithm (AC), the most basic form of constraint propagation.² The basic idea of the algorithm is to just check that all the arcs in the constraint graph are consistent. If not, it eliminates values from the domain of the variable that make the arc inconsistent, and insert to the queue arcs that are affected by that change in domain values. We denote that there is an arc from variable X to variable Y in the constraint graph by $X \rightarrow Y$.

Time Complexity (Optional)

The running time of the algorithm is $O(ed^3)$ where e is the number of constraints and d is the number of values for each variable (i.e., the size of the domain). The running time analysis is as follows. Every arc is inserted into the queue Q at most d times, since d is the number of values for each variable and the algorithm eliminates at least one value from the domain of some variable at each round (otherwise, if no value is removed for the arcs in the queue Q , then Q becomes empty and the algorithm stops). Therefore, the maximum number of iterations before Q becomes empty is $O(ed)$. Checking that an arc is consistent at each iteration takes $O(d^2)$. Therefore, the total

²In the CSP literature, this particular version is actually named AC-3, because this was the third version given by its inventor in Mackworth [1977].

```
AC
Add all arcs in the constraint graph to  $Q$ 
while  $Q$  is not empty do
  Remove first arc  $X \rightarrow Y$  from  $Q$ 
  if REMOVE-INCONSISTENT-VALUES( $X, Y$ ) then
    for all variables  $Z \neq Y$  such that  $Z \rightarrow X$  do
      Add  $Z \rightarrow X$  to  $Q$  (if not there already)
    end for
  end if
end while
REMOVE-INCONSISTENT-VALUES( $X, Y$ )
for all values  $x \in D(X)$  do
  if there does not exist a value  $y \in D(Y)$  such that  $X = x, Y = y$  is consistent with the
  constraint between  $X$  and  $Y$  then
    Delete  $x$  from  $D(X)$ 
  end if
end for
if deleted values then
  return yes
else
  return no
end if
```

Figure 3: Pseudocode for the constraint propagation algorithm for arc consistency.

running time $O((ed)(d^2))$.

You should contrast this worst-case running time, which is linear in n , to that of backtracking search which is $O(d^n)$, which is exponential in n . Hence, we can apply constraint propagation to try to reduce the size of the domains before we do backtracking since it does not take that long to run. Alternatively, you can run constraint propagation at every partial assignment node of the search tree. Usually in this case, however, constraint propagation is run only when there is only one value left for the variable just extended.

Alternative Search Formulation and Algorithms for CSPs (Optional)

An alternative search formulation to solve CSPs is by first constructing a *state-space graph*. The nodes in the graph correspond to complete assignments. There is an edge between two nodes if their respective assignments differ by the value assignment of only one variable. This graph looks just like the graph for map problems we saw for basic and optimal search. The goal nodes correspond to solutions for the CSP. We start from some arbitrary node (i.e., some complete assignment) and try to reach a goal node in the graph in a minimum number of steps.

References

- Alan K. Mackworth. Consistency in networks of relations. *Artificial Intelligence*, 8(1):99–118, February 1977.
- Stuart J. Russell and Peter Norvig. *Artificial Intelligence: A Modern Approach*. Pearson Education, second edition, 2003.