Fourth Scientific Report

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Condensed matter analogs of the quantum vacuum gravitational awakening effect

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1 Introduction

This project's aim is to search for condensed matter analogs of the Vacuum Awakening Effect (VAE) [1, 2, 3]. This is an effect predicted in the framework of quantum field theory in curved spacetimes, where the formation of a star may lead to an exponentially fast grow of the vacuum energy density of a quantum field.

This search is justified by the enormous success that such approach have met in contexts such as the phonon analog for Hawking radiation [4] and its experimental confirmation [5]. In particular, the possibility of analog systems in Bose-Einstein condensates have spawn some interest.

We present below a brief review of the Vacuum Awakening Effect, followed by a review of the theory of energy bands in a periodic lattice and finally, we describe an analog for the Effect in a driven Bose-Einstein condensate confined to an optical lattice.

2 Vacuum Awakening

The Vacuum Awakening Effect (VAE) [1, 2, 3], predicts the exponentially fast increase of the vacuum energy density for a scalar field properly coupled to the spacetime curvature. This is induced by the formation of a relativistic massive body, like a static neutron star. This increase ("awakening") will overwhelm the energy density of the star, possibly destabilizing it and controlling its fate.

We consider a globally hyperbolic spacetime $(\mathcal{M}, g_{ab})$ [6]. This means that we can find Cauchy surfaces Σ such that $\mathcal{M}$ is given by the union of the all the points in the future of Σ and all the points in the past of Σ. Global hyperbolicity implies that the spacetime can be foliated by Cauchy surfaces parametrized by t: $\mathcal{M} = \cup_t \Sigma_t$.

Now, for each Σ_t, we find maps (coordinates) $x_i : \Sigma_t \to \mathbb{R}^3$ such that $n^a \nabla_a x^i = 0$, where n^a is the future-directed unit vector normal to Σ_t. After this procedure, we end up with coordinates $x = (t, x^i)$. This allows us to write a general metric

$$ds^2 = g_{ab} dx^a dx^b = N(x)^2 (-dt^2 + h_{ij} dx^i dx^j).$$

The description of the VAE is given by a free scalar field of mass m non-minimally coupled to the Ricci scalar R with coupling ξ:

$$S = -\frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{-g} (\nabla_a \phi \nabla^a \phi + m^2 \phi^2 + \xi R \phi^2),$$

where $g = \det(g_{ab})$. This leads to the Klein-Gordon equation:

$$\left(-\nabla^a \nabla_a + m^2 + \xi R\right) \phi = 0. \tag{1}$$

The background metric is taken to evolve from an asymptotically flat configuration in the past, to an asymptotically static and spherically symmetric configuration in the future, realized for example in the formation of a static relativistic star from collapsed low density matter:

$$ds^2 = \begin{cases} -dt^2 + dx^2 + dy^2 + dz^2 & , \text{ past} \\ -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2 & , \text{ future} \end{cases} ,$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$.

To find the general solution to (1), we define an inner product between solutions (known as the Klein-Gordon inner product),

$$(u, v)_{KG} = i \int_{\Sigma_t} d\Sigma n^a [u^* \nabla_a v - v \nabla_a u^*].$$

There exists a complete set of solutions, u_α, together with their complex conjugate, u^*_α, orthonormal in this inner product:

$$(u_\alpha, u_\beta)_{KG} = -(u^*_\alpha, u^*_\beta)_{KG} = \delta(\alpha, \beta)$$
$$(u_\alpha, u^*_\beta)_{KG} = 0,$$

where $\delta(\alpha, \beta)$ is the delta function for α, β, the labels for each solution, which in the quantum theory will correspond to quantum numbers. We can make an analogy to momentum states ($\mathbf{k}$).

A general solution to (1) will be an expansion in terms of this basis:

$$\phi = \sum_\sigma [a_\sigma u_\sigma + \tilde{a}_\sigma u^*_\sigma] . \tag{2}$$

This can be immediately converted to a quantum field operator $\hat{\mathbf{\Phi}}$, satisfying[1]

$$[\hat{\mathbf{\Phi}}(t, \mathbf{x}), \hat{\mathbf{\Pi}}(t, \mathbf{x}')]_{\Sigma_t} = i\delta^3(\mathbf{x}, \mathbf{x}'),$$

where the conjugate-momentum density is $\Pi = \delta S/\delta\dot{\phi}$. The imposition of this commutation relation will imply, by (2), the usual commutation relations between creation and annihilation operators $[a_\alpha, a^\dagger_\beta] = \delta(\alpha, \beta)$, so that we get an expansion of the field in these operators:

$$\hat{\Phi} = \sum_\sigma \left[\hat{a}_\sigma u_\sigma + \hat{a}^\dagger_\sigma u^*_\sigma \right] .$$

The vacuum state will be defined by $a_\sigma|0\rangle = 0$, and our solutions u_σ, u^*_σ will dictate the behaviour of observables such as the field amplitude and energy-momentum tensor.

[1] from now on we set $\hbar = 1$

In the past, solutions to (1) will take the form of plane waves $u_{\mathbf{k}} \sim e^{i(\omega_{\mathbf{k}} t - \mathbf{k}\cdot x)}$, $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$, properly normalized according to the Klein-Gordon inner product (we are setting momentum as the quantum number). So, in the past, the field can be expanded as

$$\hat{\Phi} = \int d^3k \left[\hat{a}_{\mathbf{k}} u_{\mathbf{k}} + \hat{a}^{\dagger}_{\mathbf{k}} u^{*}_{\mathbf{k}} \right].$$

However, in the future, the solutions take the (properly normalized) form

$$v_{\omega l \mu} \propto \frac{1}{r} e^{-i\omega t} F_{\omega l}(r) Y_{l\mu}(\theta, \varphi),$$

where $Y_{l\mu}$ are the spherical harmonics, and $F_{\omega l}$ are the radial part. Writing (1) in terms of metric components[2], we end up with an effective potential for F:

$$\left[-f\partial_r \left(f\partial_r \right) + V^{(l)}_{\text{eff}} \right] F_{\omega l} = \omega^2 F_{\omega l}, \tag{3}$$

where

$$V^{(l)}_{\text{eff}} = f(r) \left(m^2 + \frac{l(l+1)}{r^2} + \xi R + \frac{\partial_r f(r)}{r} \right). \tag{4}$$

Notice that negative eigenvalue $\omega^2 = -\Omega^2 < 0$, positive-norm solutions to (3), $w_{\Omega l \mu}$, have $\exp(\pm \Omega t)$ dependence:

$$w_{\Omega l \mu} \sim \left(e^{-\Omega t + i\pi/12} + e^{\Omega t - i\pi/12} \right) \frac{F_{\Omega l}(r)}{r} Y_{l\mu}(\theta, \varphi).$$

These are called tachyonic modes. In the future, the expansion of the field will be:

$$\hat{\Phi} = \sum_{l\mu} \int d\omega \left[\hat{b}_{\omega l \mu} v_{\omega l \mu} + \hat{b}^{\dagger}_{\omega l \mu} v^{*}_{\omega l \mu} \right] + \sum_{\Omega l \mu} \left[\hat{c}_{\Omega l \mu} w_{\Omega l \mu} + \hat{c}^{\dagger}_{\Omega l \mu} w^{*}_{\Omega l \mu} \right].$$

The existence of tachyonic modes implies that part of the plane-wave modes in the asymptotic past eventually undergo an exponential growth, leading to an unbounded increase of the vacuum expectation value of $\hat{\mathbf{\Phi}}^2$.

It is clear that the vacuum in the asymptotic past, given by $a_k|0\rangle_{in} = 0$, will not coincide with the vacuum in the future, in general. This means

[2] We are using here that

$$\begin{aligned}\nabla^{\mu}\nabla_{\mu}\phi &= \frac{1}{\sqrt{|g|}} \partial_{\mu} \left(\sqrt{|g|} g^{\mu\nu} \partial_{\nu} \right) \phi \\ &= -f(r)^{-1} \partial_t^2 \phi + \frac{1}{r^2} \partial_r \left(r^2 f(r) \partial_r \phi \right) - \frac{\hat{L}^2}{r^2} \phi,\end{aligned}$$

where $g^{\mu\nu}$ is the inverse metric and $\hat{L}$ is the angular momentum operator, $L^2 = -(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2})$

that, in the presence of tachyonic modes, the vacuum expectation value of $\hat{\mathbf{\Phi}}^2$, as well as the energy-momentum tensor expectation value, will increase exponentially fast[3]:

$$
{}_{\rm in}\langle 0|\hat{\mathbf{\Phi}}^2|0\rangle_{\rm in} \propto e^{2\Omega t}. \tag{5}
$$

The effect only takes place if (3) has negative eigenvalue solutions, and this depends on the dependence of $V_{\rm eff}$ on the mass of the star, its radius and ξ.

We can take a massless field $m = 0$ with $l = 0$, as these are the values for which the VAE will be triggered most easily. The remaining factors involving R and $f(r)$ can be related to the matter distribution through Einstein's equations.

As shown in [2], taking a uniform density star, there are values of ξ and the mass-radius ratio of the star for which the effect is triggered, and the vacuum energy density becomes dominant over the star density in a time scale as short as milliseconds.

This can be understood as a phase transition in a condensed matter system such as a magnet undergoing spontaneous symmetry breaking - the star's radius R_S acts as the temperature, and as it is varied, the system's behaviour changes significantly: the magnet acquires a permanent magnetization and, analogously, the vacuum of the scalar field is awakened.

3 Periodic Potentials and Energy Bands

Before presenting a concrete condensed matter analog of the VAE, we shall review the description of particles subject to a periodic potential[7], for instance, electrons in a crystal lattice. This systems' dispersion relation is characterized by a band structure, which will be important in the analysis of our analog.

We consider a fixed lattice described by vectors $\mathbf{R}$ in terms of a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$:

$$
\mathbf{R} = \sum n_i \mathbf{a}_i, \quad n_i \in \mathbb{Z}.
$$

This means that translating the lattice by any such $\mathbf{R}$ leaves it the same. It is useful to consider the reciprocal lattice, defined by the vectors

$$
\mathbf{G} = \sum m_i \mathbf{b}_i, \quad m_i \in \mathbb{Z},
$$

where the basis vectors $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ are defined such that

$$
\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij}.
$$

We also define the 1$^{\rm st}$ Brillouin Zone, which is the set of vectors that are closer to $\mathbf{G} = 0$ than to any other reciprocal lattice vector:

$$1^{st}BZ = \{\mathbf{k} : |\mathbf{k}| < |\mathbf{k} - \mathbf{G}|, \forall \mathbf{G} \neq 0\}.$$

Consider a Hamiltonian of the form:

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r}), \tag{6}$$

where $V(\mathbf{r}) = V(\mathbf{r} + \mathbf{R})$ is periodic on the lattice. We wish to obtain the eigenstates of (6).

Theorem 1. *Bloch's Theorem: The eigenstates of a periodic Hamiltonian with periodicity* $\mathbf{R}$ *are given by Bloch Waves*

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r}),$$

where $u_{\mathbf{k}}(\mathbf{r})$ *is periodic in* $\mathbf{R}$.

Proof. Let $T_{\mathbf{R}}$ be the translation operator which shifts the system by $\mathbf{R}$. If $H(\mathbf{r}) = H(\mathbf{r} + \mathbf{R})$, then $[T_{\mathbf{R}}, H] = 0$, therefore they share eigenstates ψ:

$$T_{\mathbf{R}}\psi(\mathbf{r}) = \psi(\mathbf{r} + \mathbf{R}) = c(\mathbf{R})\psi(\mathbf{r}). \tag{7}$$

Note that we can write $c(\mathbf{R}) = e^{i\theta(\mathbf{R})}$. $c(\mathbf{R})$ has unit modulus because we require $|\psi(\mathbf{r})|^2 = |\psi(\mathbf{r} + \mathbf{R})|^2$. Furthermore, as $T_{\mathbf{R}}T_{\mathbf{S}}\psi(\mathbf{r}) = T_{\mathbf{R}+\mathbf{S}}\psi(\mathbf{r})$, $\theta(\mathbf{R})$ must be linear, thus we can write $c(\mathbf{R}) = e^{i\mathbf{k}\cdot\mathbf{R}}$, for some $\mathbf{k}$. Defining $u_{\mathbf{k}}(\mathbf{r}) = e^{-i\mathbf{k}\cdot\mathbf{r}}\psi(\mathbf{r})$

$$\begin{aligned} u_{\mathbf{k}}(\mathbf{r} + \mathbf{R}) &= e^{-i\mathbf{k}\cdot\mathbf{r}} e^{-i\mathbf{k}\cdot\mathbf{R}} \psi(\mathbf{r} + \mathbf{R}) = e^{-i\mathbf{k}\cdot\mathbf{r}} c(\mathbf{R})^{-1} \psi(\mathbf{r} + \mathbf{R}) \\ &= e^{-i\mathbf{k}\cdot\mathbf{r}} \psi(\mathbf{r}) = u_{\mathbf{k}}(\mathbf{r}). \end{aligned}$$

Therefore $\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r})$, for u periodic in $\mathbf{R}$ □

Some observations follow. First, we shall refer to the vector $\hbar\mathbf{k}$ as the quasi-momentum. It is not an eigenvalue of momentum!

Next, note that $\mathbf{k}$ can be restricted by imposing boundary conditions:

$$\psi(\mathbf{r} + \sum N_i \mathbf{e}_i) = \psi(\mathbf{r}) \Rightarrow e^{i\mathbf{k}\cdot \sum N_i \mathbf{e}_i} = 1 \Rightarrow \mathbf{k} \cdot \mathbf{e}_i = \frac{2\pi}{N_i} m_i \Rightarrow \mathbf{k} = \sum_i \frac{m_i}{N_i} \mathbf{b}_i.$$

We can always restrict $\mathbf{k}$ to assume values in the 1^{st} Brillouin Zone. For any $\psi_{\mathbf{k}}$ with $\mathbf{k}$ outside of the 1BZ, we can always find a reciprocal lattice vector $\mathbf{G}$ such that $\mathbf{k}' = \mathbf{k} - \mathbf{G}$ belongs to the unit cell. As $e^{i\mathbf{G}\cdot\mathbf{R}} = 1$,

$$\psi_{\mathbf{k}'}(\mathbf{r}) = e^{-i\mathbf{k}'\cdot\mathbf{R}} \psi_{\mathbf{k}'}(\mathbf{r} + \mathbf{R}) = e^{-i(\mathbf{k}-\mathbf{G})\cdot\mathbf{R}} \psi_{\mathbf{k}'}(\mathbf{r} + \mathbf{R}) = e^{-i\mathbf{k}\cdot\mathbf{R}} \psi_{\mathbf{k}'}(\mathbf{r} + \mathbf{R}),$$

hence $\psi_{\mathbf{k}}$ and $\psi_{\mathbf{k}'}$ are the same eigenstate (see eq. (7)).

Finally notice that, for a given quasi-momentum, there can be many solutions to the Schrödinger Equation:

$$H\psi_{\mathbf{k}} = \epsilon(\mathbf{k})\psi_{\mathbf{k}},$$

which can be regarded as an eigenvalue problem for $u(\mathbf{r})$ with boundary conditions $u(\mathbf{r}) = u(\mathbf{r}+\mathbf{R})$:

$$H_{\mathbf{k}}u_{\mathbf{k}}(\mathbf{r}) = \epsilon(\mathbf{k})u_{\mathbf{k}}(\mathbf{r}),$$

where $H_{\mathbf{k}} = e^{-i\mathbf{k}\cdot\mathbf{r}}He^{i\mathbf{k}\cdot\mathbf{r}}$.

It will have solutions of the form $u_{n,\mathbf{k}}$ with eigenvalues $\epsilon_n(\mathbf{k})$. This way, we can label each eigenstate as $\psi_{n,\mathbf{k}}$. The boundary condition generally implies that n are discrete. The eigenvalues $\epsilon_n(\mathbf{k})$ are called energy bands.

As $\psi_{\mathbf{k}}$ are periodic in $\mathbf{G}$ ($\psi_{\mathbf{k}+\mathbf{G}} = \psi_{\mathbf{k}}$), we must have

$$\epsilon_n(\mathbf{k}+\mathbf{G}) = \epsilon_n(\mathbf{k}),$$

that is, the energy bands are continuous periodic functions of $\mathbf{k}$. This implies that they have a maximum and a minimum, so that they are indeed "bands".

As an example [8], consider a one-dimensional potential

$$V(x) = V_0\cos(2\pi x/a).$$

The underlying lattice has periodicity $R = a$, so that the reciprocal lattice has $G = 2\pi/a$. Therefore, the 1st Brillouin Zone consists of points in the reciprocal space between $-\pi/a$ and π/a. Solving the Schrödinger equation using perturbation theory, it is possible to calculate the energy bands displayed in part in Fig. 1.

4 The driven Bose-Einstein condensate

Now we are ready to describe the driven Bose-Einstein condensate that, as we will see, is a good analog for the VAE. Our system of interest will be a collection of N cold atoms trapped in an oscillating optical lattice[9].

An optical lattice is formed by the interference of two counterpropagating laser beams with wavelength λ_L and a relative phase of θ. The electric field of the laser induces an effective potential on the atoms by means of the Stark effect. Ignoring interactions between atoms and the confining potential that constrains them to one dimension, the resulting Hamiltonian is:

$$H(t) = \frac{k_x^2}{2m} + V\cos^2\left(k_0 x + \frac{\theta(t)}{2}\right),$$

where $k_0 = 2\pi/\lambda_L$. The phase can be time-periodically modulated as $\theta(t) = f\cos(\omega t)$.

Using that

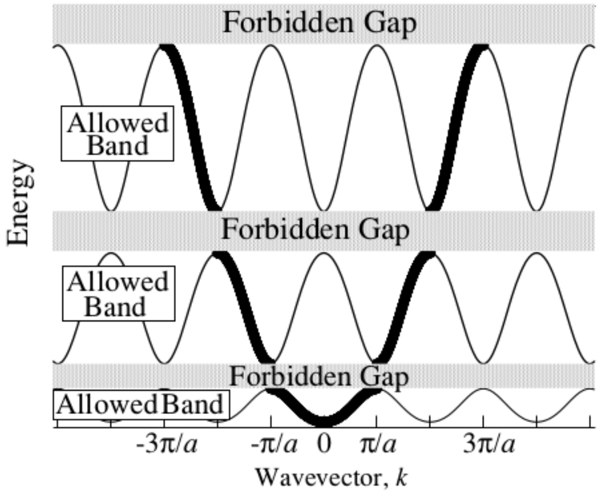

Figure 1: Energy band structure for the cosine potential. This figure displays, from bottom to top, the bands $\epsilon_1(k), \epsilon_2(k), \epsilon_3(k)$, showing forbidden energy values ("gaps") for which there are no eigenstates. The bold line shows that the dispersion is nearly parabolic, with gaps opening up at the Brillouin Zone boundaries. Reproduced from [8].

$$\cos^2 x = \frac{1}{4}(e^{2ix} + e^{-2ix} + 2).$$

and the Jacobi-Anger expansion

$$e^{iz\cos\theta} = J_0(z) + 2\sum_{n=1}^{\infty} i^n J_n(z)\cos(n\theta),$$

where J_n are Bessel functions of the first kind, we arrive at

$$e^{2ix}e^{2ia\cos\theta} = e^{2ix}(J_0(2a) + 2\sum i^n J_n(2a)\cos(n\theta)),$$

and therefore, retaining the expansion up to $n = 1$:

$$\cos^2(x + a\cos\theta) = \frac{1}{2} + \frac{1}{4}\Bigg(e^{2ix}\Big(J_0(2a) + 2iJ_1(2a)\cos\theta\Big)$$
$$+e^{-2ix}\Big(J_0(-2a) + 2iJ_1(-2a)\cos\theta\Big)\Bigg),$$

but $J_0(-x) = J_0(x), J_1(-x) = -J_1(x)$, so

$$\begin{aligned}\cos^2(x + a\cos\theta) &= \frac{1}{2} + \frac{1}{4}\Big(J_0(2a)2\cos(2x) + 2iJ_1(2a)\cos\theta(e^{2ix} - e^{-2ix})\Big)\\ &= \frac{1}{2} + \frac{J_0(2a)}{2}\cos(2x) + i^2 J_1(2a)\cos\theta\sin(2x),\end{aligned}$$

and with the identity $\cos(2x) = 2\cos^2 x - 1$ we end up with

$$\cos^2(x + a\cos\theta) = \frac{1}{2} - \frac{J_0(2a)}{2} + J_0(2a)\cos^2(x) - J_1(2a)\sin(2x)\cos\theta.$$

Finally, substituting $x \to k_0 x, a \to \frac{f}{2}, \theta \to \omega t$, the second term of the Hamiltonian can be written as:

$$V\cos^2\left(k_0 x + \frac{f\cos(\omega t)}{2}\right) = \underbrace{VJ_0(f)\cos^2(k_0 x)}_{V_0(x)} - \underbrace{VJ_1(f)\sin(2k_0 x)\cos(\omega t)}_{V(x,t)},$$

up to a constant. Therefore, our Hamiltonian becomes

$$H = \frac{k_x^2}{2m} + V_0(x) + V(x,t) = H_0 + V(x,t).$$

T he term H_0 has the structure of a periodic Hamiltonian ($V_0(x + n\lambda_L/2) = V_0(x), n \in \mathbb{Z}$), giving rise to a band structure $\epsilon_n(k)$. To deal with the time-dependent perturbation $V(x,t)$, we consider a 2-state approximation in which the Hilbert space is restricted to the two lowest energy Bloch states $\{|1,k\rangle, |2,k\rangle\}$, such that

$$\langle n,k|H_0|n,k\rangle = \epsilon_n(k),$$

and the cross terms are 0. In this basis, the matrix elements of the perturbation is

$$\begin{aligned}
\langle 2,k|V(x,t)|1,k\rangle &= \int dx dx' \langle 2,k|x\rangle\langle x|V(x,t)|x'\rangle\langle x'|1,k\rangle \\
&= \int dx u_{2,k}^*(x) V(x,t) u_{1,k}(x) \\
&= -VJ_1(f)\cos(\omega t)\int dx u_{2,k}^*(x) u_{1,k}(x)\sin(2k_0 x)
\end{aligned}$$

and

$$\langle i,k|V(x,t)|i,k\rangle = -VJ_1(f)\cos(\omega t)\int dx |u_{i,k}|^2 \sin(2k_0 x) = 0,$$

because $|\psi_{i,k}(x)|^2 = |u_{i,k}(x)|^2$ is even and sin is odd. Defining

$$\Omega(k) \equiv -VJ_1(f)\int dx u_{2,k}^*(x) u_{1,k}(x)\sin(2k_0 x),$$

the Hamiltonian is

$$H = \begin{pmatrix} \epsilon_2(k) & 2\Omega\cos(\omega t) \\ 2\Omega^*\cos(\omega t) & \epsilon_1(k) \end{pmatrix}.$$

The time dependence of this Hamiltonian can be removed by a "rotating wave approximation". First, we use an unitary transformation:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\omega t} \end{pmatrix},$$

so the states transform as

$$\psi \to U\psi,$$

and the Schrödinger equation becomes

$$\begin{aligned} i\frac{d\psi}{dt} &= H\psi \to i\frac{d(U\psi))}{dt} = HU\psi \\ \Rightarrow iU\frac{d\psi}{dt} &+ i\dot{U}\psi = HU\psi \\ \Rightarrow i\frac{d\psi}{dt} &= U^\dagger HU\psi - iU^\dagger\dot{U}\psi, \end{aligned}$$

so that the transformed Hamiltonian under this unitary transformation is

$$\begin{aligned} H' &= U^\dagger HU - iU^\dagger\dot{U} \\ &= \begin{pmatrix} \epsilon_2 & \Omega(e^{2i\omega t}+1) \\ \Omega^*(1+e^{-2i\omega t}) & \epsilon_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}. \end{aligned}$$

The rotating wave approximation consists in assuming that the relevant time scales involved are much larger that the inverse frequency, $t \gg \omega^{-1}$, so that the $e^{2i\omega t}$ terms are rapidly oscillating. This means that we can substitute these terms by their average (0):

$$H = \begin{pmatrix} \epsilon_2 & \Omega \\ \Omega^* & \epsilon_1 + \omega \end{pmatrix}$$

and find the eigenvalues (Fig. 2).

$$\epsilon_\pm(k) = \frac{\epsilon_1(k) + \epsilon_2(k) + \omega}{2} \pm \sqrt{\frac{(\epsilon_2(k) - \epsilon_1(k) - \omega)^2}{4} + |\Omega|^2}. \tag{8}$$

As Fig. 2 shows, for atoms in the $|+\rangle$ state, the system displays a ferromagnetic phase transition when the shaking amplitude assumes a critical amplitude f_c, and the dispersion relation passes from having a single minimum at zero quasi-momentum to double minima at quasi-momentum $\pm k^*$. We call this a ferromagnetic transition because the quasi-momentum plays

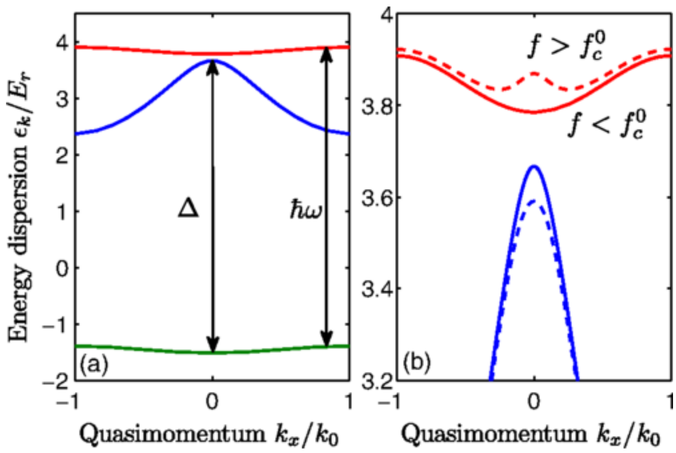

Figure 2: (a) Green: ϵ_1, blue: ϵ_2, red: $\epsilon_1 + \omega$. For a small shaking amplitude $f \ll 1$, $\Omega \sim 0$ so that $\epsilon_-(k) \sim \epsilon_2(k)$ and $\epsilon_+(k) \sim \epsilon_1(k) + \omega$ (see (8)). (b) red: ϵ_+, blue: ϵ_-. Solid is before and dashed is after the phase transition. The energy is given in units of the lattice recoil energy, $E_r = k_0^2/(2m)$. We consider a situation where $\hbar\omega \gtrapprox \Delta$, $\Delta = \epsilon_2(0) - \epsilon_1(0)$. Reproduced from [10]

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the role of the magnetization, assuming a positive or negative value $\pm k^*$ across a critical shaking amplitude (which plays the role of the temperature).

As the shaking amplitude increases, some of the atoms initially in the $\epsilon_1(k)$ band end up in the $\epsilon_+(k)$ band (from now on we refer to it as ϵ_k) [10]. Then, the shaking amplitude can be adjusted to trigger a phase transition.

Thus we found that single atoms in a periodically driven optical lattice can display a phase transition. Now we want to consider a Bose-Einstein condensate, which consists of many atoms loaded into the lattice, subject not just to the potential $V(x,t)$, but also to a two-body interaction $U(x_1 - x_2)$. In momentum space, the full Hamiltonian is [11]:

$$H = \sum_k \epsilon_k a_k^\dagger a_k + \frac{1}{2L} \sum_{k,q',p} U_k a_{k+p}^\dagger a_{k'-p}^\dagger a_k a_{k'},$$

where L, the system's length, comes from the Fourier transform of the two-body interaction, $a_k^\dagger$ creates an atom with quasi-momentum k and ϵ is the single atom potential described above: $\epsilon_k \approx \alpha k^2 + \beta k^4$, $\beta > 0$ and $\alpha > 0$ for $f < f_c$, $\alpha < 0$ for $f > f_c$.

From now on, we set $\epsilon_0 = 0$, as we are only interested in energy differences.

If we make an approximation that U is a point-like potential:

$$U(x) = g\delta(x) \Rightarrow U_k = g\int dx = gL,$$

$$\begin{aligned}
H &= \sum_k \epsilon_k a_k^\dagger a_k + \frac{g}{2}\sum_{k,k',p} a_{k+p}^\dagger a_{k'-p}^\dagger a_k a_{k'} \\
&= \sum_{k>0} \epsilon_k (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \frac{g}{2} a_0^\dagger a_0^\dagger a_0 a_0 + \frac{g}{2}\sum_{k\neq 0} a_{k+p}^\dagger a_{k'-p}^\dagger a_k a_{k'}.
\end{aligned}$$

Assuming that the ground state is occupied by a macroscopic number of particles, we can make a Bogoliubov approximation, replacing a_0 by a number,

$$\begin{aligned}
a_0|N_{k=0} = N\rangle = \sqrt{N}|N_{k=0} = N-1\rangle &\approx \sqrt{N}|N_{k=0} = N\rangle \\
\Rightarrow a_0 = \sqrt{N}, a_0^\dagger &= \sqrt{N}.
\end{aligned}$$

Moreover, if we neglect any terms in the Hamiltonian with $k \neq 0$, the ground state energy is

$$E_0 = \frac{gN^2}{2}.$$

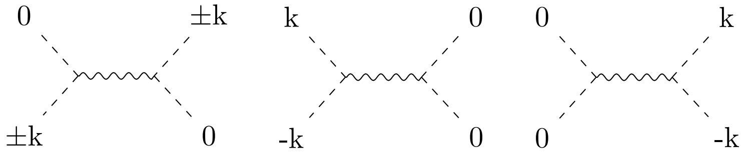
The chemical potential, which is the energy needed to add one particle to the condensate, is given by

$$\mu = \frac{\partial E_0}{\partial N} = gN.$$

Under the Bogoliubov approximation, the Hamiltonian is

$$\begin{aligned}
H &= \sum_{k>0} \epsilon_k (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \frac{gN^2}{2} + \frac{g}{2}\sum_{k\neq 0} a_{k+p}^\dagger a_{k'-p}^\dagger a_k a_{k'} \\
&= \sum_{k>0} \epsilon_k (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \frac{\mu N}{2} + \frac{g}{2}\sum_{k\neq 0} a_{k+p}^\dagger a_{k'-p}^\dagger a_k a_{k'}.
\end{aligned}$$

For calculating the third term under the Bogoliubov approximation, we note that the 4 non-zero diagrams that involve operators a_0 and $a_0^\dagger$ are (time going to the right, $k > 0$):



So we end up with

$$
\begin{aligned}
H =& \frac{1}{2}\mu N \\
&+ \sum_{k>0} \left[\epsilon_k (a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \mu (a_k^\dagger a_{-k}^\dagger + a_k a_{-k} + a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) \right] \\
=& \frac{1}{2}\mu N + \sum_{k>0} (\epsilon_k + \mu)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \sum_{k>0} \mu (a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) \\
=& \frac{1}{2} N\mu + \sum_{k>0} \begin{pmatrix} a_k^\dagger & a_{-k} \end{pmatrix} \begin{pmatrix} \epsilon_k + \mu & \mu \\ \mu & \epsilon_k + \mu \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} - \sum_{k>0} (\mu + \epsilon_k),
\end{aligned}
$$

where the last term comes from the commutation relation $[a_{-k}^\dagger, a_{-k}] = -1$.

Now, if we were to diagonalize this Hamiltonian using a standard Bogoliubov transformation

$$
\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix} \to \begin{pmatrix} \sinh\theta_k & \cosh\theta_k \\ \cosh\theta_k & \sinh\theta_k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}
$$

we would find that

$$
\begin{aligned}
& \begin{pmatrix} \sinh\theta_k & \cosh\theta_k \\ \cosh\theta_k & \sinh\theta_k \end{pmatrix} \begin{pmatrix} \epsilon_k + \mu & \mu \\ \mu & \epsilon_k + \mu \end{pmatrix} \begin{pmatrix} \sinh\theta_k & \cosh\theta_k \\ \cosh\theta_k & \sinh\theta_k \end{pmatrix} \\
=& \begin{pmatrix} (\epsilon_k + \mu)\cosh 2\theta_k + \mu \sinh 2\theta_k & (\epsilon_k + \mu)\sinh 2\theta_k + \mu \cosh 2\theta_k \\ (\epsilon_k + \mu)\sinh 2\theta_k + \mu \cosh 2\theta_k & (\epsilon_k + \mu)\cosh 2\theta_k + \mu \sinh 2\theta_k \end{pmatrix}
\end{aligned}
$$

and this is only diagonalized by $\tanh 2\theta_k = \frac{-\mu}{\epsilon_k + \mu}$.

But this is impossible because for $f > f_c$, $\epsilon_k < 0$ near k^* and there exists no θ_k such that $\tanh 2\theta_k < -1$. So no such transformation can exist. Instead, we introduce new operators ("inflatons")

$$
\begin{pmatrix} \iota_k^\dagger \\ \iota_{-k} \end{pmatrix} = \begin{pmatrix} u_k & \nu_k \\ \nu_k & u_k \end{pmatrix} \begin{pmatrix} a_k^\dagger \\ a_{-k} \end{pmatrix} \tag{9}
$$

with inverse

$$
\begin{pmatrix} a_k^\dagger \\ a_{-k} \end{pmatrix} = \begin{pmatrix} u_k & -\nu_k \\ -\nu_k & u_k \end{pmatrix} \begin{pmatrix} \iota_k^\dagger \\ \iota_{-k} \end{pmatrix},
$$

such that $u_k^2 - \nu_k^2 = 1$. These coefficients can be written as

$$
\begin{cases} u_k^2 = \frac{1}{2}(\beta_k + 1) \\ \nu_k^2 = \frac{1}{2}(\beta_k - 1) \end{cases}.
$$

The commutation relations obeyed by the inflatons are $[\iota_q, \iota_k] = [\iota_q^\dagger, \iota_k^\dagger] = 0$ and $[\iota_q, \iota_k^\dagger] = \delta_{qk}$ (it is a canonical transformation). Applying this transformation to the Hamiltonian:

$$
\begin{pmatrix} u_k & -\nu_k \\ -\nu_k & u_k \end{pmatrix}
\begin{pmatrix} \epsilon_k+\mu & \mu \\ \mu & \epsilon_k+\mu \end{pmatrix}
\begin{pmatrix} u_k & -\nu_k \\ -\nu_k & u_k \end{pmatrix}
$$
$$
= \begin{pmatrix} (\epsilon_k+\mu)(u_k^2+\nu_k^2)-2\mu\nu_k u_k & -2(\epsilon_k+\mu)u_k\nu_k+\mu(\nu_k^2+u_k^2) \\ -2(\epsilon_k+\mu)u_k\nu_k+\mu(\nu_k^2+u_k^2) & (\epsilon_k+\mu)(u_k^2+\nu_k^2)-2\mu\nu_k u_k \end{pmatrix}.
$$

Now, as $u_k^2+\nu_k^2=\beta_k$ and $2u_k\nu_k=\sqrt{\beta_k^2-1}$, we arrive at

$$
\begin{pmatrix} (\epsilon_k+\mu)\beta_k-\mu\sqrt{\beta_k^2-1} & -(\epsilon_k+\mu)\sqrt{\beta_k^2-1}+\mu\beta_k \\ -(\epsilon_k+\mu)\sqrt{\beta_k^2-1}+\mu\beta_k & (\epsilon_k+\mu)\beta_k-\mu\sqrt{\beta_k^2-1} \end{pmatrix}.
$$

If we choose

$$
\beta_k=\frac{\mu}{\sqrt{-\epsilon_k^2-2\epsilon_k\mu}}
$$

and define the growth rate (which is positive for $\epsilon_k<0$, see Fig. 3).

$$
\lambda_k=\sqrt{-\epsilon_k^2-2\epsilon_k\mu}=\sqrt{-\epsilon_k(\epsilon_k+2\mu)},
$$

then $\beta_k=\frac{\mu}{\lambda_k}$ and $\sqrt{\beta_k^2-1}=\sqrt{\frac{\mu^2}{\lambda_k^2}-1}=\frac{1}{\lambda_k}(\epsilon_k+\mu)$, so

$$
\begin{aligned}
H &= \frac{1}{2}N\mu+\sum_{k>0}\begin{pmatrix} \iota_k & \iota_{-k}^\dagger \end{pmatrix}\begin{pmatrix} 0 & \frac{\mu^2-(\epsilon_k+\mu)^2}{\lambda_k} \\ \frac{\mu^2-(\epsilon_k+\mu)^2}{\lambda_k} & 0 \end{pmatrix}\begin{pmatrix} \iota_k^\dagger \\ \iota_{-k} \end{pmatrix}-\sum_{k>0}(\mu+\epsilon_k) \\
&= \frac{1}{2}N\mu+\sum_{k>0}\lambda_k(\iota_k^\dagger\iota_{-k}^\dagger+\iota_k\iota_{-k})-\sum_{k>0}(\mu+\epsilon_k). \qquad (10)
\end{aligned}
$$

The dynamics of the inflaton are given by Heisenberg equation:

$$
\begin{aligned}
\frac{d\iota_{\pm k}}{dt} &= i[H,\iota_{\pm k}]=-i\lambda_k\iota_{\mp k}^\dagger, \\
\frac{d\iota_{\pm k}^\dagger}{dt} &= i[H,\iota_{\pm k}^\dagger]=i\lambda_k\iota_{\mp k},
\end{aligned}
$$

which has solutions

$$
\begin{aligned}
\iota_{\pm k}(t) &= \iota_{\pm k}(0)\cosh(\lambda_k t)-\iota_{\mp k}^\dagger(0)i\sinh(\lambda_k t), \\
\iota_{\pm k}^\dagger(t) &= \iota_{\pm k}^\dagger(0)\cosh(\lambda_k t)+\iota_{\mp k}(0)i\sinh(\lambda_t).
\end{aligned}
$$

The inflaton population in a given quasi-momentum state can be calculated as

$$
m_{\pm k}(t)=\langle\iota_{\pm k}^\dagger(t)\iota_{\pm k}(t)\rangle=m_{\pm k}(0)\cosh^2\lambda_k t+m_{\mp k}(0)\sinh^2\lambda_k t+\sinh^2\lambda_k t,
$$

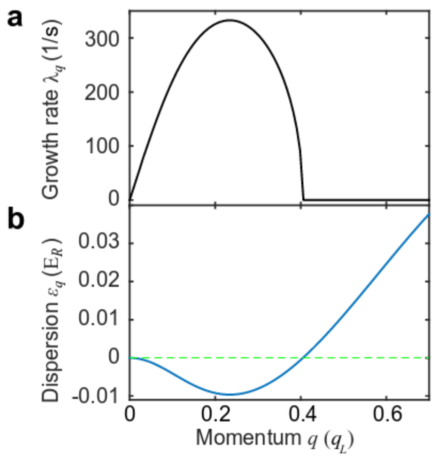


Figure 3: Reproduced from [10]

where the last term comes from the commutation relation $[\iota_k, \iota_k^\dagger] = 1$.

This can be further simplified if we define the total population in $\pm k$, $M_k = m_k + m_{-k}$:

$$\begin{aligned} M_k(t) &= M_k(0)(\cosh^2 \lambda_k t + \sinh^2 \lambda_k t) + 2\sinh^2 \lambda_k t \\ \Rightarrow [M_k(t) + 1] &= [M_k(0) + 1] \cosh 2\lambda_k t, \end{aligned}$$

where we used that $\lambda_k = \lambda_{-k}$ and the identity $\cosh 2x = \sinh^2 x + \cosh^2 x = 2\sinh^2 x + 1$.

So we see that this observable grows exponentially in time with a rate $2\lambda_k$. However, we are interested in the atom population (not the inflaton population). Defining the atom population in $\pm k$, $N_k = \langle a_k^\dagger a_k \rangle + \langle a_{-k}^\dagger a_{-k} \rangle$, we find:

$$N_k + 1 = \frac{\mu}{\lambda_k}(M_k + 1) - \frac{\mu + \epsilon_k}{\lambda_k}\Big(\langle \iota_k \iota_{-k} + \iota_k^\dagger \iota_{-k}^\dagger \rangle\Big).$$

But the second term in the right hand side commutes with the Hamiltonian, so that it is constant in time. Assuming that there are no inflatons in the beginning of the phase transition, the initial value of the population is

$$N_k(0) + 1 = \frac{\mu}{\lambda_k}(M_k(0) + 1),$$

and the time evolution will be given by

$$N_k(t) + 1 = (N_k(0) + 1)\cosh 2\lambda_k t. \tag{11}$$

That is, the excited population increases exponentially. Note that only modes with $\epsilon_k < 0$ undergo the exponential growth.

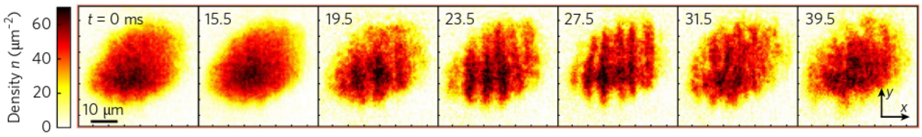


Figure 4: *In situ* images of the condensate, showing the formation of density waves of quasi-momentum k^* as the shaking amplitude is ramped up.

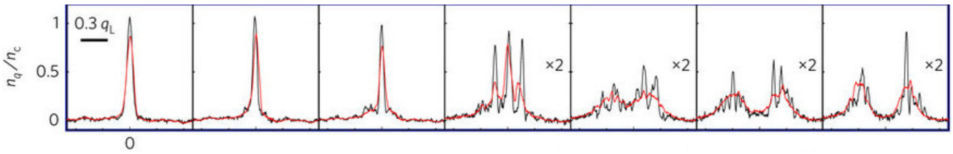


Figure 5: Time-of-flight experiments portraying the evolution of the quasi-momentum distribution N_q with time. N_c is the averaged peak density in the unshaken condensate and $q_L = \frac{\pi\hbar}{d}$, where d is the lattice period. The average over repeated experiments is given in red.

In reference [9], an experiment is made in which 30,000 Cesium atoms are placed into an optical lattice. The shaking amplitude is increased until the phase transition at time 0, and is interrupted after 20ms. Then, the system is probed using *in situ* imaging to record the atomic density and time-of-flight[3] to probe the momentum distribution of the condensate. The results are exhibited in figures 4 and 5. Beyond the critical shaking amplitude, the condensate minimum transitions to the two minima at $\pm k^*$, and the condensate evolve according to the inflaton dynamics.

In agreement with the discussion above, Figure 6 shows that the fractional population of the excited states, $\Delta N = \sum_k N_{k\neq 0}$, increases exponentially up to a limit where the model is no longer valid, as we need to consider the depletion from the ground state.

To see that the driven Bose-Einstein condensate is an analog for the VAE, we compare equations (5) and (11). We see that both the vacuum expectation value of $\hat{\Phi}^2$ and the atom population in the minima $\pm k^*$ present an exponential growth in time with a rate 2Ω and $2\lambda_{k^*}$, respectivelly. In this analog system the population plays the role of the vacuum energy density, and the star's mass-radius ratio's analog is the shaking amplitude of the optical lattice.

[3]time-of-flight measures quasi-momentum by releasing the atoms from the trap potential and measuring the time it takes for them to reach some distance, using absorption imaging, thus obtaining the momentum. Bloch states released from the trap evolve to momentum states $\hbar(k \pm 2na)$, which allow the quasi-momentum to be reconstructed from momentum.

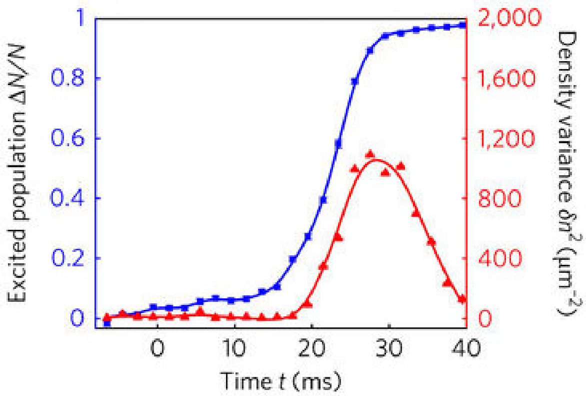


Figure 6: Shown in blue, the fractional population of the excited states which grows exponentially in time. In red, the density variance $\delta n^2 = \int dk \langle \Delta n_k^2 \rangle / N$

.

5 Conclusion

Having obtained a condensed matter analog of the VAE, it remains to explore this analogy further. Both systems display the same exponential growth behaviour after a phase transition, and the underlying reason for this behaviour also seems to be the same in both cases, namely, inflationary dynamics as described by the Hamiltonian (10). Indeed, this Hamiltonian has a form very similar to the one described in [3].

Moreover, in the original VAE, it is clear that the exponential increase will eventually halt as the system stabilizes and the result is known: particle creation. What would be the consequences of stabilization in the analog system?

Finally, the necessity of introducing the "unstable" bogoliubov transformation (9) which leads to inflationary dynamics can arise in many other cases of condensed matter systems (whenever there is an instability in the potential), so it remains to explore other potential examples of analogs for the VAE.

The study chronogram in table 1 was followed:

Table 1: **Chronogram**

	Gen. Relativity	QFT	QFT in CM
2016 (2^{nd} half)	(done)		
2017 (1^{st} half)	(done)	(done)	(done)
2017 (2^{nd} half)		(done)	(done)
2018 (1^{st} half)			(done)

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