Theoretical Questions of Chapter 1

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1.8.1-I.

Consider the bisection method starting with the ini- tial interval $[1.5, 3.5]$. In the following questions "the interval"refers to the bisection interval whose width changes across different loops.

- What is the width of the interval at the n-th step?
- What is the supremum of the distance between the root r and the midpoint of the interval?

解. 二分法每次让搜索区间减半，因此 $b_n - a_n = 2^{-n}(b_0 - a_0) = 2^{-(n-1)}$；由于 $x^* \in [a_n, b_n]$，所以 $|x^* - \frac{a_n + b_n}{2}| \le \frac{1}{2}|b_n - a_n| = 2^{-n}$。 □

1.8.1-II.

In using the bisection algorithm with its initial interval as $[a_0, b_0]$ with $a_0 > 0$, we want to determine the root with its relative error no greater than ϵ. Prove that this goal of accuracy is guaranteed by the following choice of the number of steps,

$$n \ge \frac{\log(b_0 - a_0) - \log\epsilon - \log a_0}{\log 2} - 1.$$

证明. 设根为 $r > 0, r \in [a_0, b_0]$，由于 $\epsilon \ge \frac{|\frac{a_n+b_n}{2} - r|}{r}$。而 $|\frac{a_n+b_n}{2} - r| \le \frac{b_0-a_0}{2^{n+1}}$，若使

$$\frac{|\frac{a_n+b_n}{2} - r|}{r} \le \frac{b_0 - a_0}{2^{n+1}a_0} \le \epsilon,$$

则结论可满足，所以

$$2^{n+1} \ge \frac{b_0 - a_0}{a_0\epsilon} \Rightarrow n \ge \log_2 \frac{b_0 - a_0}{a_0\epsilon} - 1 = \frac{\log(b_0 - a_0) - \log\epsilon - \log a_0}{\log 2} - 1.$$

□

1.8.1-III.

Perform four iterations of Newton's method for the polynomial equation $p(x) = 4x^3 - 2x^2 + 3 = 0$ with the starting point $x_0 = -1$. Use a hand calculator and organize results of the iterations in a table.

解. $p'(x) = 12x^2 - 4x, p(x) = 4x^3 - 2x^2 + 3$，根据牛顿法，$x_{n+1} = x_n - \frac{p(x_n)}{p'(x_n)}$。

Iteration	x_n	$p(x_n)$	$p'(x_n)$
0	-1	-3	16
1	-0.8125	-0.465820	11.1719
2	-0.770804	-0.020138	10.2129
3	-0.768832	-0.000044	10.1686
4	-0.768828	2e-10	10.1685

□

1.8.1-IV.

Consider a variation of Newton's method in which only the derivative at $x0$ is used,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)}.$$

Find C and s such that $e_{n+1} = Ce_n^s$, where en is the error of Newton's method at step n, s is a constant, and C may depend on x_n, the true solution α, and the derivative of the function f.

解. 设根为 r，由变种牛顿迭代的公式可的得

$$e_{n+1} = |x_{n+1} - r| = |x_n - r - \frac{f(x_n)}{f'(x_0)}| = |x_n - r|\left|1 - \frac{f(x_n)}{f'(x_0)(x_n - r)}\right|$$

所以 $s = 1, C = \left|1 - \dfrac{f(x_n) - f(r)}{f'(x_0)(x_n - r)}\right|$. □

1.8.1-V.

Within $(-\frac{\pi}{2}, \frac{\pi}{2})$, will the iteration $x_{n+1} = \tan^{-1} x_n$ converge?

解. 收敛，理由如下：由于 $(\tan^{-1} x - x)' = \frac{1}{1+x^2} - 1 < 0$，所以 $\tan^{-1} x - x$ 单调递减，又因为 $\tan^{-1} 0 - 0 = 0$，所以当 $x < 0$ 时,$\tan^{-1} x > x$，当 $x > 0$ 时候，$\tan^{-1} x < x$。因此，当 $x_0 = 0$ 时，显然 $x_n \equiv 0$，因此迭代收敛；当 $x \in (-\frac{\pi}{2}, 0)$ 时，$x_n < x_{n+1} < 0$，根据单调收敛定理，迭代收敛；同理，当 $x \in (0, \frac{\pi}{2})$，$x_n > x_{n+1} > 0$，根据单调收敛定理，迭代收敛。□

1.8.1-VI.

Let $p \geq 1$. What is the value of the following continued fraction?

$$x = \frac{1}{p + \frac{1}{p + \frac{1}{p + \cdots}}}.$$

Prove that the sequence of values converges. (Hint: this can be interpreted as $x = \lim_{n\to+\infty} x_n$, where $x_1 = \dfrac{1}{p}, x_2 = \dfrac{1}{p + \frac{1}{p}}, \cdots$,and so forth. Formulate x as a fixed point of some function.)

解. 问题等价于求解 $x = \lim_{x\to+\infty} x_n$，其中，$x_1 = 0, x_{n+1} = \dfrac{1}{p + x_n}(n \geq 1)$。又因为 $(\dfrac{1}{p+x})' = \dfrac{-1}{(p+x)^2} \leq \dfrac{1}{p^2} < 1(x \geq 0)$，根据压缩映射的性质，$\{x_n\}$ 收敛于 $x = \dfrac{1}{p+x}$，所以 $x = \dfrac{-p + \sqrt{p^2+4}}{2}$（舍去负数值）。所以该连分数的值为 $\dfrac{-p + \sqrt{p^2+4}}{2}$。□

1.8.1-VII.

What happens in problem II if $a_0 < 0 < b_0$? Derive an inequality of the number of steps similar to that in II. In this case, is the relative error still an appropriate measure?

解. 由 II 可知，$\dfrac{|\frac{a_n+b_n}{2} - r|}{|r|} \leq \frac{b_0 - a_0}{2^{n+1}|r|}$，因此有 $n \geq \log_2 \dfrac{b_0 - a_0}{a_0\epsilon} - 1 = \dfrac{\log(b_0 - a_0) - \log\epsilon - \log|r|}{\log 2} - 1$，然而由于 $|r|$ 可以无限小，所以相对误差可以无限大，特别的，当 $r = 0$ 时，相对误差无意义，因此相对误差不再是可估计的。□

1.8.1-VIII.

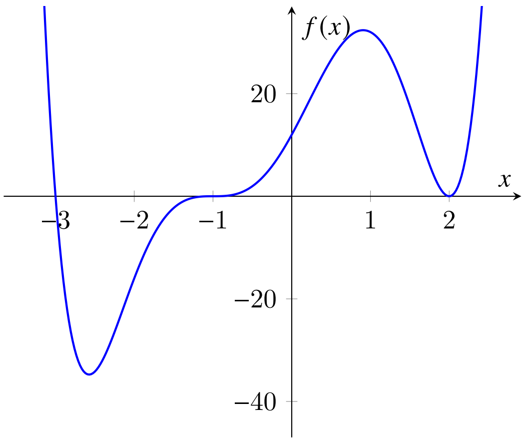
(*) Consider solving $f(x) = 0 (f \in C^{k+1})$ by Newton's method with the starting point x_0 close to a root of multiplicity k. Note that α is a zero of multiplicity k of the function f iff

$$f^{(k)}(\alpha) \neq 0; \forall i < k, f^{(i)}(\alpha) = 0.$$

- How can a multiple zero be detected by examining the behavior of the points $(x_n, f(x_n))$?
- Prove that if r is a zero of multiplicity k of the function f, then quadratic convergence in Newton's iteration will be restored by making this modification:

$$x_{n+1} = x_n - k\frac{f(x_n)}{f'(x_n)}.$$

解. 函数在重根处的一阶导为 0，而在单根处的一阶导不为 0。可以通过观察函数图像在零点处的斜率来判断是否为重根。例如 $f(x) = (x+3)(x+1)^3(x-2)^2$ ，根据下图可以很容易判断出，$x = -1, 2$ 为重根，而 $x = -3$ 为单根。



□

对于求解 k 重根的牛顿迭代法，

设 $f(x) = g(x)(x-x^*)^k, g(x^*) \neq 0, f \in C^{k+1} \Rightarrow g \in C^1$，那么

$$x_{n+1} = x_n - \frac{kg(x_n)(x_n - x^*)^k}{kg(x_n)(x_n - x^*)^{k-1} + g'(x_n)(x_n - x^*)^k} = x_n - \frac{kg(x_n)(x_n - x^*)}{kg(x_n) + g'(x_n)(x_n - x^*)}.$$

所以，

$$\begin{aligned}
x_{n+1} - x^* &= x_n - x^* - \frac{kg(x_n)(x_n - x^*)}{kg(x_n) + g'(x_n)(x_n - x^*)} \\
&= (x_n - x^*)\left(1 - \frac{kg(x_n)}{kg(x_n) + g'(x_n)(x_n - x^*)}\right) \\
&= (x_n - x^*)\left(\frac{g'(x_n)(x_n - x^*)}{kg(x_n) + g'(x_n)(x_n - x^*)}\right) \\
&= (x_n - x^*)^2\left(\frac{g'(x_n)}{kg(x_n) + g'(x_n)(x_n - x^*)}\right).
\end{aligned}$$

所以 $|x_{n+1} - x^*| = |x_n - x^*|^2 \left|\dfrac{1}{k\dfrac{g(x_n)}{g'(x_n)} + (x_n - x^*)}\right|$,

当 $|x_0 - x^*| < \left|\dfrac{k \min_{x \in \mathcal{D}_f} g(x_n)}{\max_{x \in \mathcal{D}_f} g'(x_n)}\right| - \epsilon$ 时，该迭代方法二阶收敛。