Problems of Chapter 11

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Exercise 11.11

Show that the matrix form of the Crank-Nicolson method for solving the heat equation (11.3) with Dirichlet condition is

$$(I-\frac{k}{2}A)\boldsymbol{U}^{n+1}=(I+\frac{k}{2}A)\boldsymbol{U}^{n}+\boldsymbol{b}^{n},$$

where

$$\boldsymbol{b}^{n}=\frac{r}{2}\begin{bmatrix}g_0(t_n)+g_0(t_{n+1})\\0\\\vdots\\0\\g_1(t_n)+g_1(t_{n+1})\end{bmatrix}$$

解. 根据定义 11.10 得

$$\frac{U_i^{n+1}-U_i^n}{k}=\frac{1}{2}\left(f(U_i^n,t_n)+f(U_i^{n+1},t_{n+1})\right)$$

即

$$U_i^{n+1}-\frac{k}{2}f(U_i^{n+1},t_{n+1})=U_i^n+\frac{k}{2}f(U_i^n,t_n)$$

代入半离散系统 $\boldsymbol{U}'=A\boldsymbol{U}+\boldsymbol{g}$ 得

$$\boldsymbol{U}^{n+1}-\frac{k}{2}(A\boldsymbol{U}^{n+1}+\boldsymbol{g}^{n+1})=\boldsymbol{U}^{n}+\frac{k}{2}(A\boldsymbol{U}^{n}+\boldsymbol{g}^{n}).$$

整理得

$$(I-\frac{k}{2}A)\boldsymbol{U}^{n+1}=(I+\frac{k}{2}A)\boldsymbol{U}^{n}+\frac{k}{2}(\boldsymbol{g}^{n}+\boldsymbol{g}^{n+1}).$$

因为

$$\frac{k}{2}(\boldsymbol{g}^{n}+\boldsymbol{g}^{n+1})=\frac{k\nu}{2h^2}\begin{bmatrix}g_0(t_n)+g_0(t_{n+1})\\0\\\vdots\\0\\g_1(t_n)+g_1(t_{n+1})\end{bmatrix}=\boldsymbol{b}^{n},$$

所以待证式成立。 □

Exercise 11.26

Prove Lemma 11.25 via the stability function of one-step methods.

解. 根据单步法的稳定性函数定义，我们将 θ−method 的半离散格式

$$\boldsymbol{U}^{n+1} = \boldsymbol{U}^n + k(\theta f(U^{n+1}) + (1-\theta) f(U^n))$$

应用于方程 $u' = \lambda u$ 得

$$U^{n+1} = U^n + k(\theta \lambda U^{n+1} + (1-\theta)\lambda U^n).$$

整理得

$$U^{n+1} = \frac{1 + k(1-\theta)\lambda}{1 - k\theta\lambda} U^n.$$

代入 $z = k\lambda$ 得稳定性函数为

$$R(z) = \frac{1 + (1-\theta)z}{1 - \theta z}.$$

要使上述格式稳定，我们需要对一切特征值 λ 均有

$$|R(k\lambda)| \le 1 + O(k).$$

注意到 $-\frac{4\nu}{h^2} \le \lambda < 0$，而当 $\theta \in [\frac{1}{2}, 1]$ 时，$|R(z)| \le 1$ 对一切 $z \le 0$ 恒成立。故此时 θ−method 无条件稳定。当 $\theta \in [0, \frac{1}{2})$ 时，解不等式可得

$$z \ge \frac{-2}{1 - 2\theta}.$$

故 k 需要满足 $\frac{-2}{1-2\theta} \le -\frac{4\nu}{h^2} k$。即 $k \le \frac{h^2}{2(1-2\theta)\nu}$。 □

Exercise 11.41

Show that any grid function in $L^1(h\mathbb{Z})$ can be recovered by a Fourier transform followed by an inverse Fourier transform.

解. 设 $\boldsymbol{U} \in L^1(h\mathbb{Z}), U(hj) = U_j$。则由 Lemma 11.39 得

$$\begin{aligned}
&(\mathcal{F}^{-1}\mathcal{F})(U)_n \\
=&\mathcal{F}^{-1}(\mathcal{F}U)_n \\
=&\mathcal{F}^{-1}\Big(\frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} e^{-\mathrm{i}mh\xi} U_m h\Big)_n \\
=&\frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathrm{i}nh\xi} \sum_{m\in\mathbb{Z}} e^{-\mathrm{i}mh\xi} U_m h \mathrm{d}\xi \\
=&\frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \sum_{m\in\mathbb{Z}} e^{\mathrm{i}(n-m)h\xi} U_m h \mathrm{d}\xi \\
=&\frac{1}{2\pi} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} U_n h \mathrm{d}\xi \\
=&U_n.
\end{aligned}$$

□

Exercise 11.48

Prove Lemma 11.25 via Von-Neumann analysis. What can you say after comparing this proof with that for Exercise 11.26?

解. 对迭代式

$$-\theta r U_{j-1}^{n+1}+(1+2\theta r)U_j^{n+1}-\theta r U_{j+1}^{n+1}=(1-\theta)rU_{j-1}^n+[1-2(1-\theta)r]U_j^n+(1-\theta)rU_{j+1}^n.$$

两端同时 Fourier 变换，即代入 (11.48) 式得

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\left(-\theta r e^{\mathrm{i}(j-1)h\xi}+(1+2\theta r)e^{\mathrm{i}jh\xi}-\theta r e^{\mathrm{i}(j+1)h\xi}\right)\hat{U}^{n+1}(\xi)\mathrm{d}\xi\\=&\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}}\left((1-\theta)r e^{\mathrm{i}(j-1)h\xi}+[1-2(1-\theta)r]e^{\mathrm{i}jh\xi}+(1-\theta)r e^{\mathrm{i}(j+1)h\xi}\right)\hat{U}^{n}(\xi)\mathrm{d}\xi.\end{aligned}$$

设 $\hat{U}^{n+1}=\hat{U}^n(\xi)g(h\xi)$，则

$$g(h\xi)=\frac{(1-\theta)r e^{\mathrm{i}(j-1)h\xi}+[1-2(1-\theta)r]e^{\mathrm{i}jh\xi}+(1-\theta)r e^{\mathrm{i}(j+1)h\xi}}{-\theta r e^{\mathrm{i}(j-1)h\xi}+(1+2\theta r)e^{\mathrm{i}jh\xi}-\theta r e^{\mathrm{i}(j+1)h\xi}}=\frac{2(1-\theta)r\cos(h\xi)+1-2(1-\theta)r}{-2\theta r\cos(h\xi)+1+2\theta r}.$$

$\theta-$method 稳定需要 $|g(h\xi)|\le 1+O(k),\forall\xi\in[-\frac{\pi}{h},\frac{\pi}{h}]$。当 $\theta\in[\frac{1}{2},1]$ 时不等式恒成立，故该方法无条件稳定。当 $\theta\in[0,\frac{1}{2})$ 时，解得 $r\le\dfrac{1}{(1-2\theta)(1-\cos(h\xi))}$。因为 $-1\le\cos(h\xi)\le 1$，所以 $r\le\dfrac{1}{2(1-2\theta)}$。即 $k\le\dfrac{h^2}{2(1-2\theta)\nu}$。 □

Exercise 11.78

Show that the Beam-Warming method is second-order accurate both in time and in space.

解. 我们只证明 $a\ge 0$ 时的情形，$a<0$ 同理。

$$\begin{aligned}&\tau(x,t)\\=&u(x_j,t_n+k)-u(x_j,t_n)-\frac{\mu}{2}(3u(x_j,t_n)-4u(x_{j-1},t_n)+u(x_{j-2},t_n))+\frac{\mu^2}{2}(u(x_j,t_n)-2u(x_{j-1},t_n)+u(x_j,t_n))\\=&u+ku_t+\frac{k^2}{2}u_{tt}-u-\frac{\mu}{2}(3u-4(u+u_x+\frac{h^2}{2}u_{xx})+\\&(u-2hu_x+2h^2u_{xx}))+\frac{\mu^2}{2}(u-2(u-hu_x+\frac{h^2}{2}u_{xx})+(u-2hu_x+2h^2u_{xx}))+O(k^3+h^3)\\=&ku_t+\frac{k^2}{2}u_{tt}+\mu hu_x+\frac{\mu^2}{2}u_{xx}+O(h^3+k^3)\\=&-aku_x+\frac{a^2k^2}{2}u_{xx}+aku_x+\frac{a^2k^2}{2}u_{xx}+O(h^3+k^3)\\=&a^2k^2u_{xx}+O(h^3+k^3)\end{aligned}$$

因此 Beam-Warming 方法具有二阶时空精度。 □

Exercise 11.79

Show that the Beam-Warming methods (11.86) and (11.87) are stable for $\mu\in[0,2]$ and $\mu\in[-2,0]$, respectively. Reproduce the plot in Figure 11.6.

解. 只证明 $a \geq 0$ 的情形，$a < 0$ 同理。当 $a \geq 0$ 时，$\mu = \frac{ak}{h} \geq 0$。Beam-Warming 方法的半离散系统为

$$U'(t) = BU(t), B = -\frac{k\mu}{2}\begin{bmatrix} 3 & & & \dots & 1 & -4 \\ -4 & 3 & & \dots & & 1 \\ 1 & -4 & 3 & \dots & & \\ & \ddots & \ddots & \ddots & & \\ & \dots & 1 & -4 & 3 & \\ & \dots & & 1 & -4 & 3 \end{bmatrix} + \frac{k\mu^2}{2}\begin{bmatrix} 1 & & & \dots & 1 & -2 \\ -2 & 1 & & \dots & & 1 \\ 1 & -2 & 1 & \dots & & \\ & \ddots & \ddots & \ddots & & \\ & \dots & 1 & -2 & 1 & \\ & \dots & & 1 & -2 & 1 \end{bmatrix}$$

因为

$$Bw^p = \left(-\frac{k\mu}{2}(3 - 4e^{-2\pi iph} + e^{-4\pi iph}) + \frac{k\mu^2}{2}(1 - 2e^{-2\pi iph} + e^{-4\pi iph})\right) w^p.$$

所以 $z_p = k\lambda_p = e^{-2\pi iph}\left[(\mu^2 - 2\mu)(cos(2\pi ph) - 1) - i\mu \sin(2\pi ph)\right]$。当 $\mu \in [0, 2]$ 时，

$$\begin{aligned} 1 + z_p =& e^{-2\pi iph}[(\mu^2 - 2\mu)(\cos(2\pi ph) - 1) - \mathrm{i}\mu \sin(2\pi ph) + e^{2\pi iph}] \\ =& e^{-2\pi iph}[(\mu - 1)^2 \cos(2\pi ph) + \mu(2 - \mu) - \mathrm{i}(\mu - 1)\sin(2\pi ph)] \\ =& e^{-2\pi iph}[\eta^2 \cos(2\pi ph) + 1 - \eta^2 - i\eta \sin(2\pi ph)]. \end{aligned}$$

这里 $\eta = \mu - 1 \in [-1, 1]$。取模长平方，得

$$\begin{aligned} |1 + z_p|^2 =& (\eta^2 \cos(2\pi ph) + 1 - \eta^2)^2 + (\eta \sin(2\pi ph))^2 \\ =& \eta^4 \cos^2(2\pi ph) + 2\eta^2(1 - \eta)\cos(2\pi ph) + (1 - \eta^2)^2 + \eta^2(1 - \cos^2(2\pi ph)) \\ =& \eta^2(\eta^2 - 1)(\cos(2\pi ph) - 1)^2 \leq 1. \end{aligned}$$

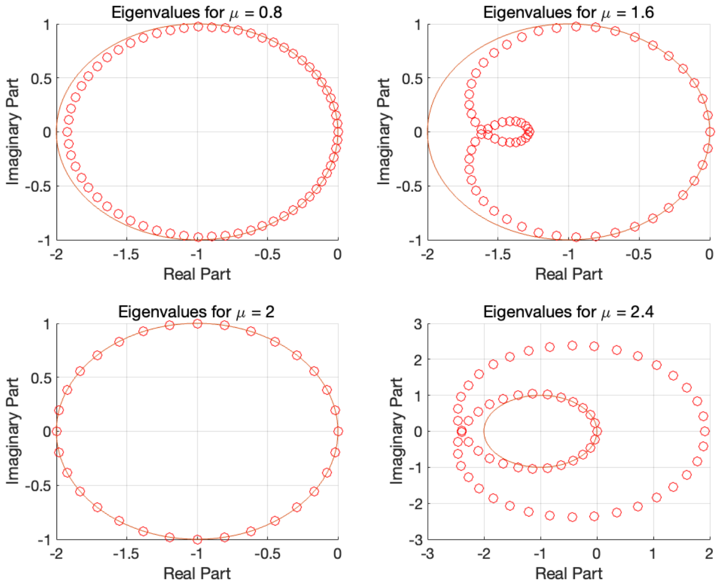
所以 Warm-Beamer 算法在 $\mu \in [0, 2]$ 时绝对稳定。

```
mu_values = [0.8, 1.6, 2.0, 2.4];
m = 64;
theta = linspace(0, 2*pi, 1000);
C = exp(1i * theta) - 1;

figure;

for k = 1:length(mu_values)
    mu = mu_values(k);
    eigens = zeros(1, m+1);
    for I = 0:m
        eigens(I+1) = -mu/2 * (3 - 4*exp(-2i*pi*I/m) + exp(-4i*pi*I/m)) + mu^2/2 * (1 - 2*exp
            (-2i*pi*I/m) + exp(-4i*pi*I/m));
    end
    subplot(2, 2, k);
    scatter(real(eigens), imag(eigens), 'r');
    hold on;
    plot(real(C), imag(C));
    hold off;
    grid on;
    xlabel('Real Part');
    ylabel('Imaginary Part');
```

```
    title(['Eigenvalues for \mu = ', num2str(mu)]);
end
```



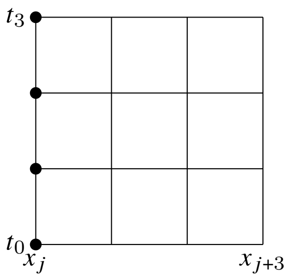
□

Exercise 11.82

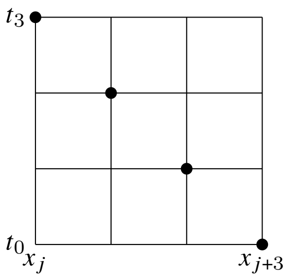
Plot the numerical domains of dependence of the grid point (x_j, t_3) for the upwind method with $a < 0$ and $\mu = 0, 1, 2$.

解.

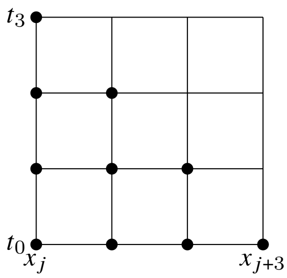
□



$\mu = 0$



$\mu = -1$



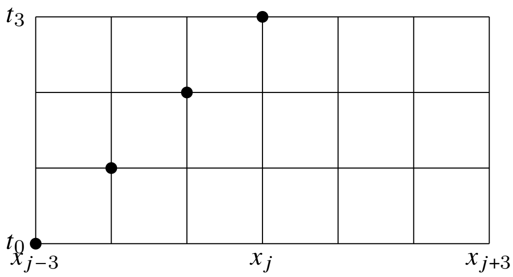
$\mu = -2$

Exercise 11.83

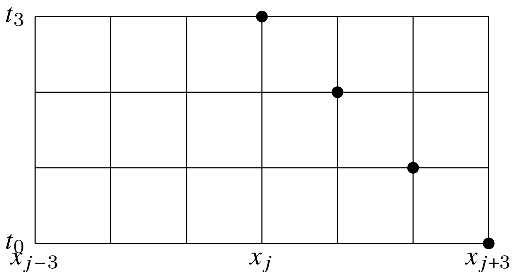
Plot the numerical domains of dependence of the grid point (x_j, t_3) for the Lax-Wendroff method with $\mu = +1, -1$.

解.

□



$\mu = +1$



$\mu = -1$

Exercise 11.92

Show that the modified equation of the leapfrog method is also (11.96). However, if one more term of higher-order derivative had been retained, the modified equation of the leapfrog method would have been

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} = \epsilon_f v_{xxxxx}$$

while that of the Lax-Wendroff method would have been

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} = \epsilon_w v_{xxxx}.$$

解. 先考虑 leapfrog 方法

$$\frac{U_j^{n+1} - U_j^{n-1}}{2k} = -\frac{a}{2h}(U_{j+1}^n - U_{j-1}^n).$$

将 U_j^n 替换为 $v(x,t)$ 得

$$\frac{v(x,t+k) - v(x,t-k)}{2k} = -\frac{a}{2h}(v(x+h,t) - v(x-h,t)).$$

在 (x,t) 处泰勒展开至五阶，得

$$v_t + \frac{k^2}{6}v_{ttt} + \frac{k^4}{120}v_{ttttt} = -a\left(v_x + \frac{h^2}{6}v_{xxx} + \frac{h^4}{120}v_{xxxxx}\right).$$

即（假设 $k = O(h)$）

$$v_t + av_x = -\frac{1}{6}(k^2 v_{ttt} + ah^2 v_{xxx}) - \frac{1}{120}(k^4 v_{ttttt} + ah^4 v_{xxxxx}) + O(h^6).$$

因为

$$\begin{aligned}
v_{ttttt} =& -a^5 v_{xxxxx} + O(h^2),\\
v_{ttt} =& -av_{xtt} - \frac{1}{6}(k^2 v_{ttttt} + ah^2 v_{xxxtt}) + O(h^4)\\
=& a^2 v_{xxt} + \frac{a}{6}(k^2 v_{xtttt} + ah^2 v_{xxxxt}) - \frac{1}{6}(k^2 a^5 v_{xxxxx} + h^2 a^3 v_{xxxxx}) + O(h^4)\\
=& -a^3 v_{xxx} - \frac{a^2}{6}(k^2 v_{xxttt} + ah^2 v_{xxxxx}) + \frac{a}{6}(k^2 a^4 v_{xxxxx} - h^2 a^2 v_{xxxxx}) - \frac{1}{6}(k^2 a^5 - h^2 a^3)v_{xxxxx} + O(h^4)\\
=& -a^3 v_{xxx} + \frac{1}{2}(k^2 a^5 - h^2 a^3)v_{xxxxx} + O(h^4),
\end{aligned}$$

所以

$$v_t + av_x + \frac{k^2}{6}\left(-a^3 v_{xxx} + \frac{1}{2}(k^2 a^5 - h^2 a^3)v_{xxxxx}\right) - \frac{k^4}{120}a^5 v_{xxxxx} + \frac{ah^2}{6}v_{xxx} + \frac{ah^4}{120}v_{xxxxx} + O(h^6) = 0.$$

代入 $\mu = \dfrac{ak}{h}$ 整理得

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} + \frac{ah^4}{120}(1-10\mu^2+9\mu^4)v_{xxxxx} + O(h^6) = 0.$$

因此，保留到三阶的方程为

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} = 0,$$

保留到五阶的方程为

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} + \frac{ah^4}{120}(1-10\mu^2+9\mu^4)v_{xxxxx} = 0.$$

再考虑 Lax-Wendroff 方法

$$U_j^{n+1} - U_j^n = -\frac{\mu}{2}(U_{j+1}^n - U_{j-1}^n) + \frac{\mu^2}{2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n).$$

将 U_j^n 替换为 $v(x,t)$ 得

$$v(x,t+k) - v(x,t) = -\frac{\mu}{2}(v(x+h,t) - v(x-h,t)) + \frac{\mu^2}{2}(v(x+h,t) - 2v(x,t) + v(x-h,t)).$$

在 (x,t) 处泰勒展开至四阶，得

$$kv_t + \frac{k^2}{2}v_{tt} + \frac{k^3}{6}v_{ttt} + \frac{k^4}{24}v_{tttt} + O(h^5) = -\mu\left(hv_x + \frac{h^3}{6}v_{xxx} + O(h^5)\right) + \mu^3(\frac{h^2}{2}v_{xx} + \frac{h^4}{24}v_{xxxx} + O(h^6)).$$

因为

$$\begin{aligned}
v_{tttt} =& a^4 v_{xxxx} + O(h),\\
v_{ttt} =& -av_{xtt} - \frac{k}{2}v_{tttt} + \frac{\mu ah}{2}v_{xxtt} + O(h^2)\\
=& -a\left(-av_{xxt} - \frac{k}{2}v_{xttt} + \frac{\mu ah}{2}v_{xxxt}\right) - \frac{k}{2}a^4v_{xxxx} + \frac{\mu a^3 h}{2}v_{xxxx} + O(h^2)\\
=& a^2\left(-av_{xxx} - \frac{k}{2}v_{xxtt} + \frac{\mu ah}{2}v_{xxxx}\right) - \frac{k}{2}a^4v_{xxxx} + \frac{\mu a^3 h}{2}v_{xxxx} + O(h^2)\\
=& -a^3v_{xxx} + O(h^2),\\
v_{tt} =& -av_{xt} - \frac{k}{2}v_{ttt} + \frac{\mu ah}{2}v_{xxt} - \frac{k^2}{6}v_{tttt} - \frac{ah^2}{6}v_{xxxt} + O(h^3)\\
=& -a\left(-av_{xx} - \frac{k}{2}v_{xtt} + \frac{\mu ah}{2}v_{xxx} - \frac{k^2}{6}v_{xttt} - \frac{ah^2}{6}v_{xxxx}\right) + \frac{ka^3}{2}v_{xxx} - \frac{\mu a^2 h}{2}v_{xxx} - \frac{k^2a^4}{6}v_{xxxx} - \frac{a^2h^2}{6}v_{xxxx} + O(h^3)\\
=& a^2v_{xx} + \frac{a^2h^2}{3}(1-\mu^2)v_{xxxx} + O(h^3),
\end{aligned}$$

代入泰勒展开式，整理得

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} + \frac{ah^3}{8}(\mu-\mu^3)v_{xxxx} + O(h^4) = 0.$$

因此保留到四阶的方程为

$$v_t + av_x + \frac{ah^2}{6}(1-\mu^2)v_{xxx} + \frac{ah^3}{8}(\mu-\mu^3)v_{xxxx} = 0.$$

□

Exercise 11.93

Show that the modified equation of the Beam-Warming method aplied to the advection equation (11.56) with $a \geq 0$ is

$$v_t + av_x + \frac{ah^2}{6}(-2+3\mu-\mu^2)v_{xxx} = 0.$$

Thus we have

$$C_p(\xi) = a + \frac{ah^2}{6}(\mu-1)(\mu-2)\xi^2,$$
$$C_g(\xi) = a + \frac{ah^2}{2}(\mu-1)(\mu-2)\xi^2.$$

How do these facts answer Question (e) of Example 11.87?

解. Beam-Warming 方法的半离散格式

$$U_j^{n+1} = U_j^n - \frac{\mu}{2}(3U_j^n - 4U_{j-1}^n + U_{j-2}^n) + \frac{\mu^2}{2}(U_j^n - 2U_{j-1}^n + U_{j-2}^n)$$

中 U_j^n 替换为 $v(x,t)$ 得

$$v(x,t+k) - v(x,t) = -\frac{\mu}{2}(3v(x,t) - 4v(x-h,t) + v(x-2h,t)) + \frac{\mu^2}{2}(v(x,t) - 2v(x-h,t) + v(x-2h,t)).$$

在 (x,t) 处泰勒展开至三阶，得

$$kv_t + \frac{k^2}{2}v_{tt} + \frac{k^3}{6}v_{ttttt} + O(k^4) = -\frac{\mu}{2}\left(hv_x - \frac{2}{3}v_{xxx}\right) + \frac{\mu^2}{2}(h^2v_{xx} - h^3v_{xxx}).$$

因为

$$v_{ttt} = -a^3v_{xxx} + O(h),$$

$$\begin{aligned}
v_{tt} =& -av_{xt} - \frac{k}{2}v_{ttt} + \frac{\mu ah}{2}v_{xxt} + O(h^2)\\
=& -a\left(-av_{xx} - \frac{k}{2}v_{xtt} + \frac{\mu ah}{2}v_{xxx}\right) + \frac{k}{2}a^3v_{xxx} - \frac{\mu a^2h}{v}_{xxx} + O(h^2)\\
=& a^2v_{xx} + O(h^2),
\end{aligned}$$

代入泰勒展开式得

$$v_t + av_x + \frac{1}{6}ah^2(-2+3\mu-\mu^2)v_{xxx} + O(h^3) = 0.$$

故保留到三阶的方程为

$$v_t + av_x + \frac{1}{6}ah^2(-2+3\mu-\mu^2)v_{xxx} = 0.$$

因此

$$\begin{aligned}
\omega(\xi) =& a\xi + \frac{ah^2}{6}(2-3\mu+\mu^2)\xi^3 = a\xi + \frac{ah^2}{6}(\mu-1)(\mu-2)\xi^3,\\
C_p(\xi) =& \frac{\omega(\xi)}{\xi} = a + \frac{ah^2}{6}(\mu-1)(\mu-2)\xi^2,\\
C_g(\xi) =& \frac{\mathrm{d}\omega(\xi)}{\mathrm{d}\xi} = a + \frac{ah^2}{2}(\mu-1)(\mu-2)\xi^2.
\end{aligned}$$

当 $0 < \mu < 1$ 时，$|C_p(\xi)| > |a|$。在 Example 11.87 中，$\mu = 0.8$，因此 Beam-Warming 方法求出的解的峰值更高，且峰值出现的时间比真解要早。 □

Exercise 11.94

What if $\mu = 1$? Discuss this case for both Lax-Wendroff and leapfrog methods to answer Question (f) of Example 11.87.

解. 当 $\mu = 1$ 时，Lax-Wendroff 方法的半离散格式退化为 $U_j^{n+1} = U_{j-1}^n$。此时，显然所有格点处的解均为精确的：$U_j^n = U_{j-n}^0 = \eta(jh - nak) = \eta((j-n)h)$。leapfrog 方法的半离散格式退化为 $U_j^{n+1} = U_j^{n-1} - U_{j+1}^n + U_{j-1}^n$。我们归纳证明所有格点处的解均为精确的。$U_j^0 = \eta(jh)$ 的值由初值条件直接给出，是精确的；假设 U_j^n 为精确的，即 $U_j^n = \eta((j-n)h)$，则 $U_j^{n+1} = U_j^{n-1} - U_{j+1}^n + U_{j-1}^n = \eta((j-n+1)h) - \eta((j+1-n)h) + \eta((j-1-n)h) = \eta((j-1-n)h)$ 是精确解。故归纳成立。综上，$\mu = 1$ 时 Lax-Wendroff 和 leapfrog 方法都可以得到精确解。 □

Exercise 11.96

Apply the von Neumann analysis to the Lax-Friedrichs method to derive its amplification factor as

$$g(\xi h) = \cos(\xi h) - \mu \mathrm{i} \sin(\xi h).$$

For which values of μ would the method be stable?

解. Lax-Friedrichs 方法的半离散格式为

$$U_j^{n+1} = \frac{1}{2}(U_{j+1}^n + U_{j-1}^n) - \frac{\mu}{2}(U_{j+1}^n - U_{j-1}^n).$$

两端同时作 Fourier 变换得

$$\frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathrm{i}jh\xi} \hat{U}^{n+1}(\xi) \mathrm{d}\xi = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left[\frac{1}{2}(e^{\mathrm{i}(j+1)h\xi} + e^{\mathrm{i}(j-1)h\xi}) - \frac{\mu}{2}(e^{\mathrm{i}(j+1)h\xi} - e^{\mathrm{i}(j-1)h\xi}) \right] \hat{U}^n(\xi) \mathrm{d}\xi.$$

右端可化简为

$$\frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathrm{i}jh\xi} (\cos h\xi - \mu \mathrm{i} \sin h\xi) \hat{U}^n(\xi) \mathrm{d}\xi.$$

因此 $g(\xi) = \cos h\xi - \mu \mathrm{i} \sin h\xi$。由 $|g(\xi)| \le 1$ 得

$$\cos^2 h\xi + \mu^2 \sin^2 h\xi \le 1$$

即

$$1 + (\mu^2 - 1) \sin^2 h\xi \le 1.$$

因为 $0 \le \sin^2 h\xi \le 1$，所以要使上式对任意 ξ 成立，需要 $|\mu| \le 1$。即 $|\mu| \le 1$ 时方法稳定。 □

Exercise 11.97

Apply the von Neumann analysis to the Lax-Wendroff method to derive its amplification factor as

$$g(\xi h) = 1 - 2\mu^2 \sin^2 \frac{\xi h}{2} - \mathrm{i}\mu \sin(\xi h).$$

For which values of μ would the method be stable?

解. Lax-Wendroff 方法的半离散格式为

$$U_j^{n+1} = U_j^n - \frac{\mu}{2}(U_{j+1}^n - U_{j-1}^n) + \frac{\mu^2}{2}(U_{j+1}^n - 2U_j^n + U_{j-1}^n).$$

两端同时作 Fourier 变换得

$$\frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathrm{i}jh\xi}\hat{U}^{n+1}(\xi)\mathrm{d}\xi = \frac{1}{\sqrt{2\pi}}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{\mathrm{i}jh\xi}\left(1-\mu\mathrm{i}\sin h\xi - 2\mu^2\sin^2\frac{h\xi}{2}\right)\hat{U}^{n}(\xi)\mathrm{d}\xi.$$

因此

$$g(\xi) = 1-\mu\mathrm{i}\sin h\xi - 2\mu^2\sin^2\frac{h\xi}{2}.$$

由 $|g(\xi)| \le 1$ 得

$$(1-2\mu^2\sin^2\frac{h\xi}{2})^2 + \mu^2\sin^2 h\xi \le 1,$$

即

$$(1-\mu^2(1-\cos h\xi))^2 + \mu^2\sin^2 h\xi \le 1,$$

即

$$(\mu^4-\mu^2)(1-\cos h\xi)^2 \le 0.$$

故 $|\mu| \le 1$。即 $|\mu| \le 1$ 时方法稳定。 □

参考文献

[1] 张庆海. “Notes on Numerical Analysis and Numerical Methods for Differential Equations”. In: (2024).