Problems of Chapter 9

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Exercise 9.5

Theorem 9.4. The relative error of an approximate solution is bounded by ites relative residual.

$$\frac{1}{\mathrm{cond}(A)}\frac{\|\boldsymbol{r}\|_2}{\|\boldsymbol{b}\|_2} \le \frac{\|\boldsymbol{e}\|_2}{\|\boldsymbol{x}\|_2} \le \mathrm{cond}(A)\frac{\|\boldsymbol{r}\|_2}{\|\boldsymbol{b}\|_2}$$

证明. 根据矩阵范数的定义，$\|A\|_2 = \sup\left\{\frac{\|A\boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2} : \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \ne 0\right\}$，因为 $A\boldsymbol{e} = \boldsymbol{r}, A^{-1}\boldsymbol{b} = \boldsymbol{x}$，所以

$$\|A\|_2\|\boldsymbol{e}\|_2 \ge \|A\boldsymbol{e}\|_2 \ge \|\boldsymbol{r}\|_2, \quad \|A^{-1}\|_2\|\boldsymbol{b}\|_2 \ge \|A^{-1}\boldsymbol{b}\|_2 \ge \|\boldsymbol{x}\|_2,$$

又因为 $\mathrm{cond}(A) = \|A\|_2\|A^{-1}\|_2$，所以，

$$\begin{aligned}
&\|A\|_2\|A^{-2}\|_2\|\boldsymbol{e}\|_2\|\boldsymbol{b}\|_2 \ge \|\boldsymbol{r}\|_2\|\boldsymbol{x}\|_2 \\
\Rightarrow\quad &\mathrm{cond}(A)\|\boldsymbol{e}\|_2\|\boldsymbol{b}\|_2 \ge \|\boldsymbol{r}\|_2\|\boldsymbol{x}\|_2 \\
\Rightarrow\quad &\frac{1}{\mathrm{cond}(A)}\frac{\|\boldsymbol{r}\|_2}{\|\boldsymbol{b}\|_2} \le \frac{\|\boldsymbol{e}\|_2}{\|\boldsymbol{x}\|_2}
\end{aligned}$$

另一边类似，因为 $A\boldsymbol{x} = \boldsymbol{b}, A^{-1}\boldsymbol{r} = \boldsymbol{e}$，所以，

$$\|A\|_2\|\boldsymbol{x}\|_2 \ge \|A\boldsymbol{x}\|_2 \ge \|\boldsymbol{b}\|_2, \quad \|A^{-1}\|_2\|\boldsymbol{r}\|_2 \ge \|A^{-1}\boldsymbol{r}\|_2 \ge \|\boldsymbol{e}\|_2,$$

所以，

$$\begin{aligned}
&\|A\|_2\|A^{-2}\|_2\|\boldsymbol{x}\|_2\|\boldsymbol{r}\|_2 \ge \|\boldsymbol{b}\|_2\|\boldsymbol{e}\|_2 \\
\Rightarrow\quad &\mathrm{cond}(A)\|\boldsymbol{x}\|_2\|\boldsymbol{r}\|_2 \ge \|\boldsymbol{b}\|_2\|\boldsymbol{e}\|_2 \\
\Rightarrow\quad &\frac{\|\boldsymbol{e}\|_2}{\|\boldsymbol{x}\|_2} \le \mathrm{cond}(A)\frac{\|\boldsymbol{r}\|_2}{\|\boldsymbol{b}\|_2}
\end{aligned}$$

□

Exercise 9.8

What are the values of $\mathrm{cond}(A)$, for A in (7.13) for $n = 8$ and $n = 1024$?

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

解. 根据 Lemma 7.25，

$$\lambda_k(A) = \frac{4}{h^2}\sin^2\frac{k\pi}{2n}, k = 1, 2, \cdots, n-1$$

因为 $A^T = A$，所以

$$\|A\|_2 = \sqrt{\rho(A^TA)} = \rho(A) = \frac{4}{h^2}\sin\frac{(n-1)\pi}{2n} = \frac{4}{h^2}\cos\frac{\pi}{2n},$$

因为 $\lambda^k(A^{-1}) = \lambda_k(A)^{-1}$，所以 $\|A^{-1}\|_2 = \dfrac{1}{\dfrac{4}{h^2}\sin^2\dfrac{\pi}{2n}}$。所以

$$\mathrm{cond}(A) = \|A\|_2\|A^{-1}\|_2 = \frac{\cos^2\frac{\pi}{2n}}{\sin^2\frac{\pi}{2n}} = \cot^2\frac{\pi}{2n},$$

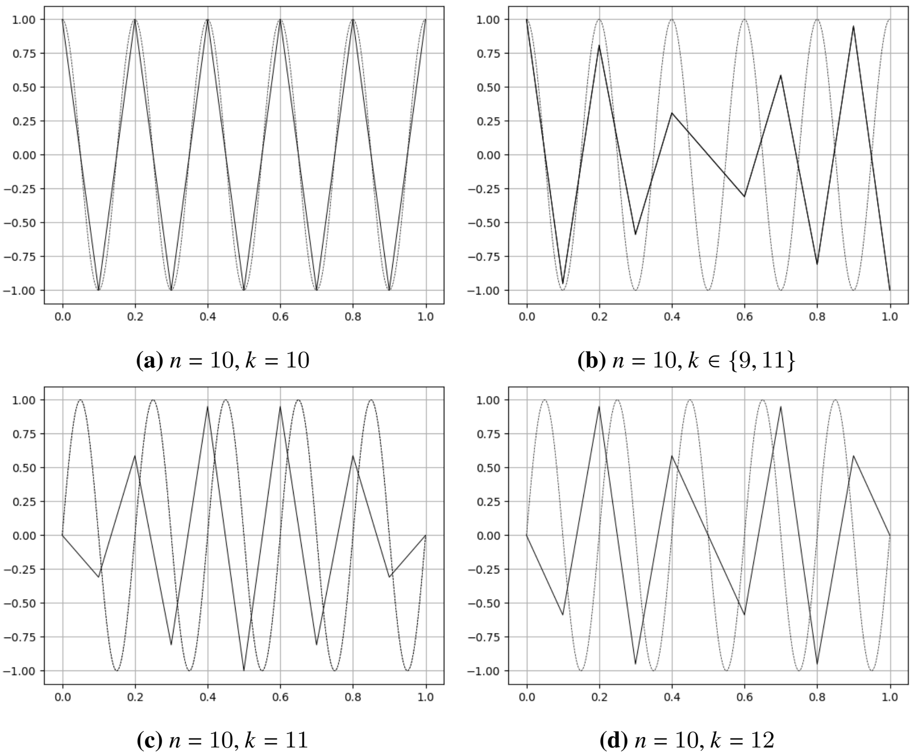
当 $n = 8$ 时，$\mathrm{cond}(A) \approx 25.2741$；当 $n = 1024$ 时，$\mathrm{cond}(A) \approx 424971.2$。

□

Exercise 9.11

For $\Omega = (0, 1)$, plot to show that the maximum wavenumber that is representable on Ω^h is $n_{\max} = \frac{1}{h}$. What if we require that the Fourier mode be 0 at the boundary points?

解. 为了在离散网格 Ω^h 上表示更多的波数，结果如图 (a) 所示，取 $n = 10, h = \frac{1}{n}$。



(a) $n = 10, k = 10$ **(b)** $n = 10, k \in \{9, 11\}$

(c) $n = 10, k = 11$ **(d)** $n = 10, k = 12$

图 1: Exercise 9.11

若取 $k = 11$ 得到的结果与 $k = 9$ 相同，都只能展示 $n-1$ 个半波，如图 (b) 所示。因此可以得到结论，$n_{\max} = \frac{1}{h}$。

若设置边界位置为 0，因为 $\sin(n\pi x_i) \equiv 0, x_i = \frac{i}{n}, i = 0, \ldots, n$，此时最多展示 $n-1$ 个半波，如图 (c) 所示，若取 $k = n+2$，则可以展示 $n-2$ 个半波，如图 (d) 所示。 □

Exercise 9.14

Plot the case of $n = 6$ for Example 9.13.

解. 绘制 $\sin\frac{3}{2}n\pi x$ 和 $-\sin\frac{1}{2}n\pi x$ 的图像，以及在 $n = 6$ 的离散网格处的点值，可以发现在网格处两个函数重合。两个函数在此网格上都只能描述出 3 个半波。

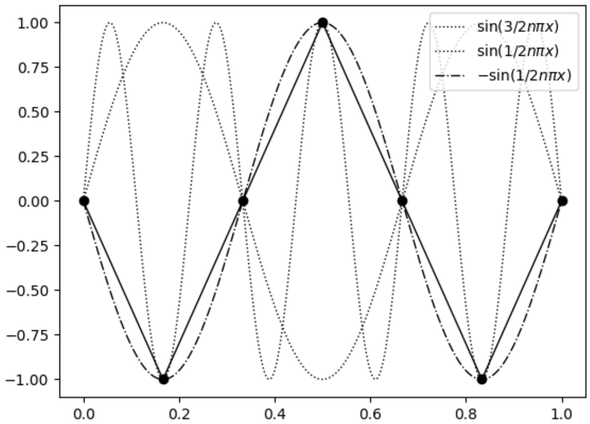


图 **2:** Example 9.13

□

Exercise 9.17

Lemma 9.16 For the linear system (9.7), the weighted Jacobi in Definition 8.9 has the iteration matrix

$$T_\omega = (1-\omega)I + \omega D^{-1}(L+U) = I - \frac{\omega h^2}{2}A,$$

whose eigenvectors are the same as those of A, with the corresponding eigenvalues as

$$\lambda_k(T_\omega) = 1 - 2\omega\sin^2\frac{k\pi}{2n},$$

where $k = 1, 2, \ldots, n-1$.

证明. 根据 Definition 8.9，方程 $\boxed{A\boldsymbol{x} = \boldsymbol{b}}$ 的带权 Jacobi 迭代为

$$\boldsymbol{x}^{(k+1)} = (1-\omega)\boldsymbol{x}^{(k)} + \omega[T_J\boldsymbol{x}^{(k)} + c]$$

其中 $T_J = D^{-1}(L+U)$，$D, -L, -U$ 分别表示 A 的对角、下三角，上三角部分。所以，

$$\begin{aligned}
T_\omega &= (1-\omega)I + \omega D^{-1}(L+U) \\
&= (1-\omega)I + \omega\left(\frac{2}{h^2}I\right)^{-1}(-A+D) \\
&= (1-\omega)I + \frac{\omega h^2}{2}\left(-A + \left(\frac{2}{h^2}I\right)\right) \\
&= (1-\omega)I - \frac{\omega h^2}{2}A + \omega I \\
&= I - \frac{\omega h^2}{2}A.
\end{aligned}$$

A 的特征值为 $\lambda_k(A) = \frac{4}{h^2}\sin^2\frac{k\pi}{2n}$，设对应的特征向量为 v_k， $k = 1, 2, \ldots, n-1$。

$$\begin{aligned} T_\omega v_k &= \left(1 - \frac{\omega h^2}{2}A\right) v_k = v_k - \frac{\omega h^2}{2} A v_k \\ &= v_k - \frac{\omega h^2}{2}\lambda_k v_k = \left(1 - \frac{\omega h^2}{2}\lambda_k\right) v_k \\ &= \left(1 - 2\omega\sin^2\frac{k\pi}{2n}\right) v_k \Rightarrow \lambda_k(T_\omega) = 1 - 2\omega\sin^2\frac{k\pi}{2n} \end{aligned}$$

□

Exercise 9.18

Write a program to reproduce Fig. 2.7 in the book by Briggs et al. [2000]. For $n = 64$, $\omega \in [0, 1]$, verify $\rho(T_\omega) \geq 0.9986$ and hence slow convergence.

解. 当 $n = 64, \omega \in (0, 1)$ 时，根据图上的单调性，可以得知

$$\rho(T_\omega) = \lambda_{\max} > 1 - 2\sin^2\frac{\pi}{2n} \approx 0.9988$$

所以收敛速度很缓慢。

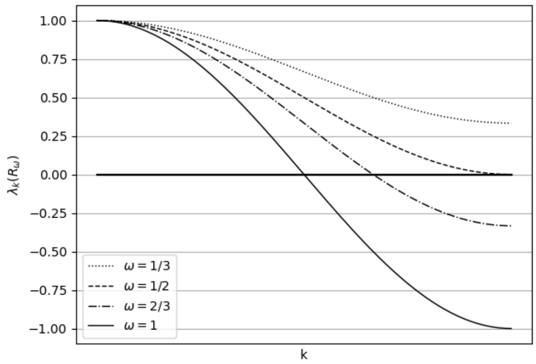


图 **3:** Reproduce Fig. 2.7

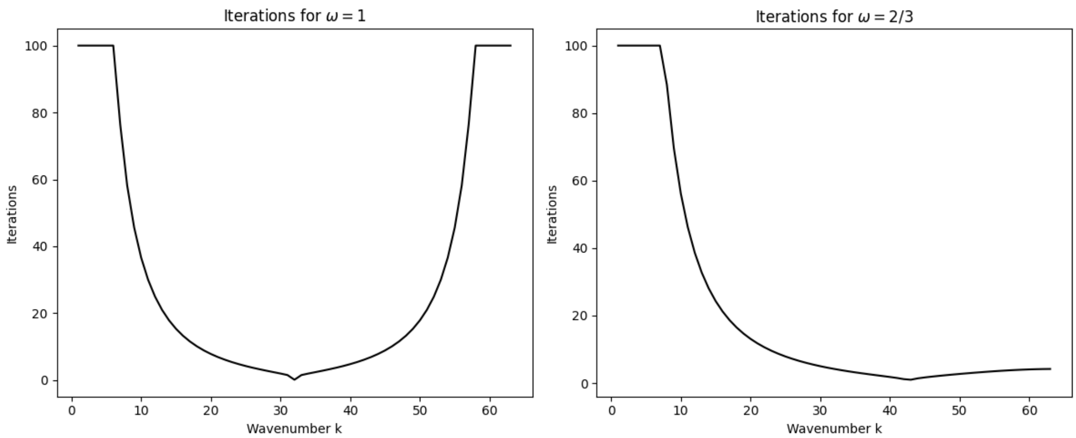
```
X = np.linspace(0, 6, 1000)
W = [1/3, 1/2, 2/3, 1]
Y = []
for w in W:
    Y.append([1-2*w*m.sin(x*pi/2/6)**2 for x in X])
plt.plot(X, Y[0], linewidth=1, label="$\omega=$1/3", linestyle=':' , color = 'black')
plt.plot(X, Y[1], linewidth=1, label="$\omega=$1/2", linestyle='--', color = 'black')
plt.plot(X, Y[2], linewidth=1, label="$\omega=$2/3", linestyle='-.', color = 'black')
plt.plot(X, Y[3], linewidth=1, label="$\omega=$1"  , linestyle='-' , color = 'black')
plt.grid(True)
plt.ylabel("$\lambda_k(R_{\omega})$")
plt.xlabel("k")
plt.xticks([])
plt.plot(X, [0]*1000, linewidth=1.6, color='black')
plt.legend(loc = "lower left")
```

□

Exercise 9.21

Write a program to reproduce Figure 2.8 in the book by Briggs et al. [2000], verifying that regular Jacobi is only good for damping modes $16 \le k \le 48$. In contrast, for $\omega = \frac{2}{3}$, the modes $16 \le k < 64$ are all damped out quickly.

解. 如下图所示，不加权 Jacobi 方法的迭代次数在 $16 \le k \le 48$ 时比较小，在 $k < 16, k > 48$ 的时候比较大，而加权 Jacobi 迭代（取 $\omega = \frac{2}{3}$）在 $16 \le k \le 64$ 的迭代次数都比较小。




```
import math as m
n = 64
X = [i for i in range(1, n)]
Y1 = []
Y2 = []
def lambda_k(k, o):
    return 1-2*o*m.sin(k*pi/2/64)**2

def get_n(k, o):
    return min(100, (m.log(0.01) / m.log(abs(lambda_k(k, o)))))

for i in X:
    Y1.append(get_n(i, 1))
    Y2.append(get_n(i, 2/3))

plt.figure(figsize=(12, 5))

plt.subplot(1, 2, 1)
plt.plot(X, Y1, color='black', label='$\omega=1$')
plt.xlabel("Wavenumber k")
plt.ylabel("Iterations")
plt.title('Iterations for $\omega=1$')
plt.subplot(1, 2, 2)
plt.plot(X, Y2, color='black', label='$\omega=2/3$')
plt.xlabel("Wavenumber k")
```

```
plt.ylabel("Iterations")
plt.title('Iterations for $\omega=2/3$')
plt.tight_layout()
```

□

Exercise 9.35

Show that, for $\nu_1 = \nu_2 = 1$, the computational cost of an FMG cycle is less than $\frac{2}{(1-2^{-D})^2}$WU. Give upper bounds as tight as possible for computational costs of an FMG cycle for $D = 1, 2, 3$.

证明. 在 (FMG-3) 这一步，需要 V-cycles，根据 Lemma 9.33，设最密网格为 2^m，宽度为 h，设当前位于网格 $\Omega^{2^k h}(k = 0, 1, \ldots, m)$ 上，那么在忽略网格内转移后，计算成本为（$\nu_1 = \nu_2 = 1$）

$$2\text{WU}(2^{-kD} + 2^{-(k+1)D} + \cdots + 2^{-mD})$$

考虑在每一个网格上执行 (FMG-3) 这一步的总计算成本，

$$2\text{WU}\sum_{i=0}^{m}\sum_{j=i}^{m}2^{-jD} = 2\text{WU}\sum_{i=0}^{m}\frac{2^{-iD}-2^{-(m+1)D}}{1-2^{-D}} < 2\text{WU}\sum_{i=0}^{m}\frac{2^{-iD}}{1-2^{-D}} = \frac{2}{1-2^{-D}}\text{WU}\sum_{i=0}^{m}2^{-iD} < \frac{2}{(1-2^{-D})^2}\text{WU}$$

当 $D = 1, 2, 3$ 时，分别有上界 8WU, $\frac{32}{9}$WU, $\frac{128}{49}$WU。 □

Exercise 9.41

Rewrite (9.32) as

$$TG\begin{bmatrix} \mathbf{w}_k \\ \mathbf{w}_{k'} \end{bmatrix} = \begin{bmatrix} \lambda_k^{\nu_1+\nu_2}s_k & \lambda_k^{\nu_1}\lambda_{k'}^{\nu_2}s_k \\ \lambda_{k'}^{\nu_1}\lambda_k^{\nu_2} & \lambda_{k'}^{\nu_1+\nu_2}c_k \end{bmatrix}\begin{bmatrix} \mathbf{w}_k \\ \mathbf{w}_{k'} \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}\begin{bmatrix} \mathbf{w}_k \\ \mathbf{w}_{k'} \end{bmatrix}$$

Explain why the magnitude of all four c_i's are small. Deduce the main conclusion $\rho(TG) \approx 0.1$ by reproducing the plots in Figure 9.4 of the damping coefficients of two-grid correction with weighted Jacobi for $n = 64$ and $\omega = \frac{2}{3}$. The horizontal axis represents the wavenumber k. Repeat the plots for $n = 128$ to show the independence of $\rho(TG) \approx 0.1$ from the grid size.

解. 因为 $\lambda_k, \lambda_{k'}, s_k, c_k \in [0,1]$，而 $\nu_1, \nu_2 \in \mathbb{N}^*$，所以，可以通过增大 ν_1, ν_2 让 c_1, c_2, c_3, c_4 变得很小。

```
n = 64
def lambda_k(k, o=2/3):
    return 1-2*o*m.sin(k*pi/2/n)**2
def c(k):
    return m.cos(k*pi/2/n)**2
def s(k):
    return m.sin(k*pi/2/n)**2
K1 = range(1, (n//2)+2)
K2 = range((n//2), n+1)
args = [[0, 0], [0, 2], [1, 1], [2, 0], [2, 2], [4, 0]]
plt.figure(figsize=(15, 10))

def rho(matrix):
```

```
    eigenvalues, _ = np.linalg.eig(matrix)
    max_eigenvalue_abs = np.max(np.abs(eigenvalues))
    return max_eigenvalue_abs

for i, [nu1, nu2] in enumerate(args, start=1):
    plt.subplot(2, 3, i)
    C1 = []
    C2 = []
    C3 = []
    C4 = []
    R = []
    for k in K1:
        k_ = n - k
        C1.append(lambda_k(k)**(nu1+nu2)*s(k))
        C3.append(lambda_k(k_)**nu1*lambda_k(k)**nu2*c(k))
    for k in K2:
        k_ = n - k
        C2.append(lambda_k(k_)**nu1*lambda_k(k)**nu2*s(k_))
        C4.append(lambda_k(k)**(nu1+nu2)*c(k_))
    id1 = 0
    id2 = len(C2) - 1
    for k in K1:
        k_ = n - k
        c1 = C1[id1]
        c2 = C2[id2]
        c3 = C3[id1]
        c4 = C4[id2]
        id1 = id1 + 1
        id2 = id2 - 1
        R.append(rho([[c1, c2], [c3, c4]]))
    plt.plot(K1, R,  color = 'red', label = 'rho')
    plt.plot(K1, C1, color = 'black', linestyle = '-', linewidth = 1, label = 'c1')
    plt.plot(K2, C2, color = 'black', linestyle = '-.', linewidth = 1, label = 'c2')
    plt.plot(K1, C3, color = 'black', linestyle = ':', linewidth = 1, label = 'c3')
    plt.plot(K2, C4, color = 'black', linestyle = '-', linewidth = 2, label = 'c4')
    plt.title("$v_1 = {}, v_2 = {}$".format(nu1, nu2))
    plt.legend()
```

在上述代码中，还计算了相应的 $\rho(A)$ 的值，通过比较 $n = 64, n = 128$ 的图像可以发现 $\rho(A) \approx 0.1$ 且与网格大小没有关系。

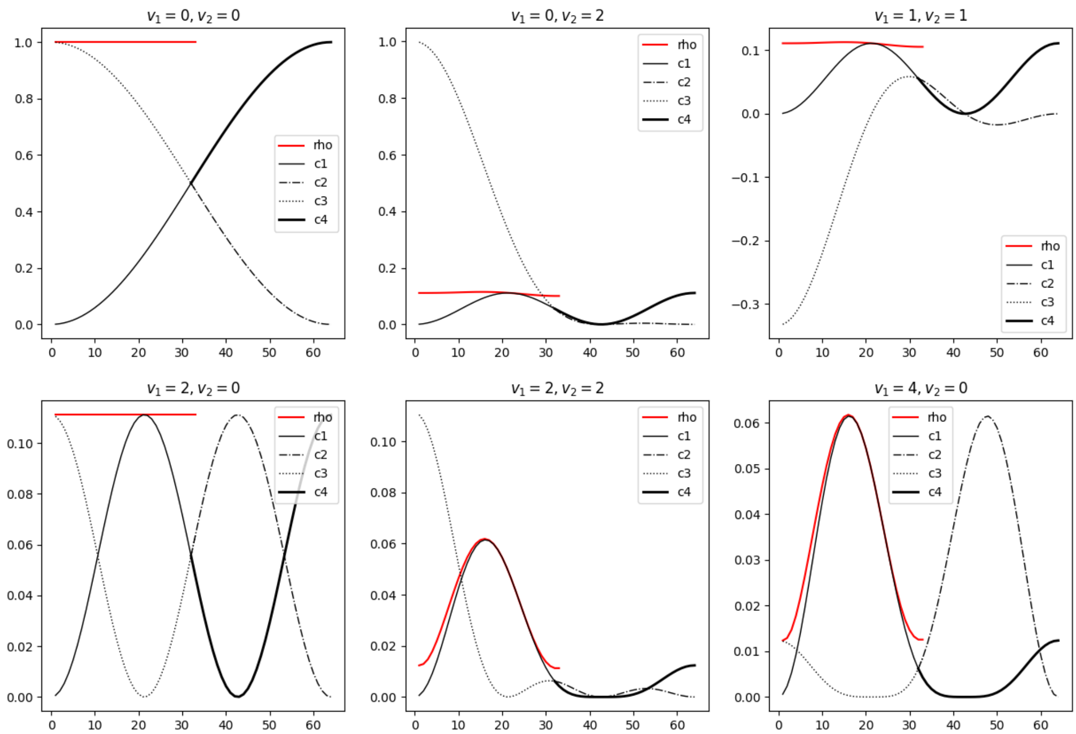


图 **4:** Reproduce of figures of $n = 64$

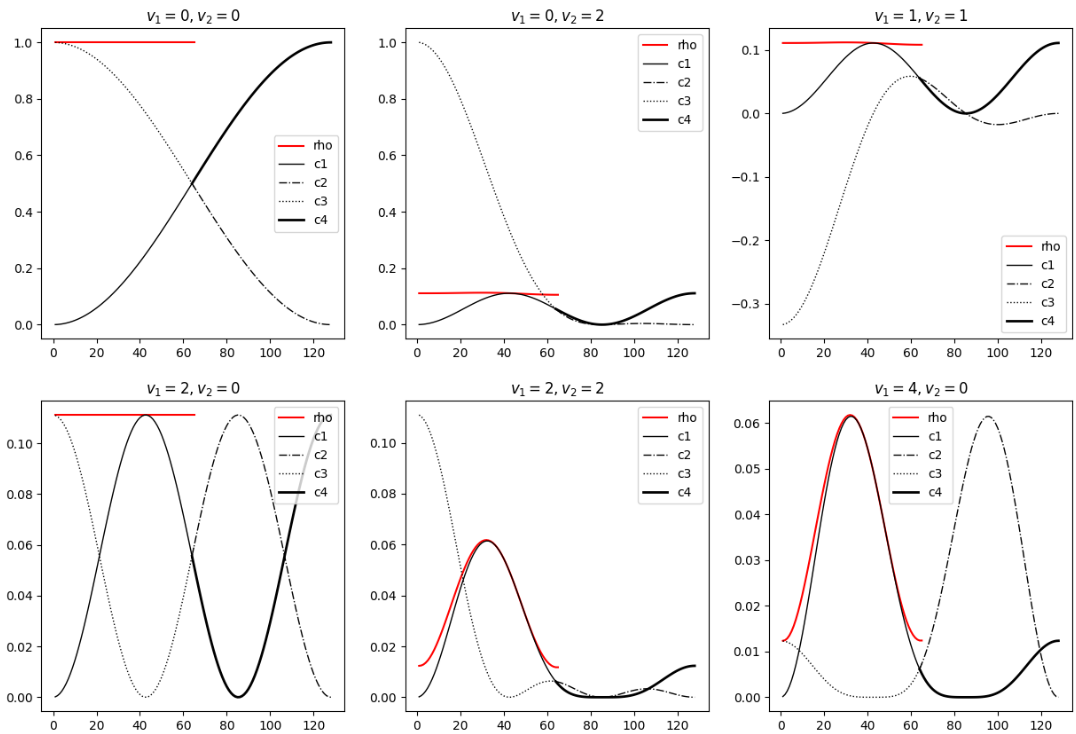


图 **5:** Reproduce of figures of $n = 128$

□

Exercise 9.45

Lemma 9.44. The full-weighting operator satisfies

$$\dim \mathcal{R}(I_h^{2h}) = \frac{n}{2} - 1, \quad \dim \mathcal{N}(I_h^{2h}) = \frac{n}{2}.$$

证明. 全加权算子 I_h^{2h} 为

$$I_h^{2h} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 & & & & & \\ & & 1 & 2 & 1 & & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & 1 & 2 & 1 \end{bmatrix}$$

所以 I_h^{2h} 满秩，

$$\dim \mathcal{R}(I_h^{2h}) = \operatorname{rank}(I_h^{2h}) = \frac{n}{2} - 1$$

根据线性映射基本定理，

$$n - 1 = \dim \mathbb{R}^{n-1} = \dim \mathcal{R}(I_h^{2h}) + \dim \mathcal{N}(I_h^{2h}).$$

所以，$\dim \mathcal{N}(I_h^{2h}) = (n-1) - \mathcal{R}(I_h^{2h}) = (n-1) - \left(\frac{n}{2} - 1\right) = \frac{n}{2}$ □

参考文献

[1] 张庆海. “Notes on Numerical Analysis and Numerical Methods for Differential Equations”. In: (2024).