Problems of Chapter 10.6.5-10.6.10

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Exercise 10.196

Prove Lemma 10.195 for the scalar case.

解. 考虑方程 $u' = f(u,t)$。记 $u(t_n) = u_n = u, f(u(t_n),t_n) = f$,

$$\begin{aligned}
u(t_{n+1}) = u(t_n+k) &= u + ku' + \frac{k^2}{2}u'' + \frac{k^3}{6}u''' + \frac{k^4}{24}u'''' + O(k^5)\\
&= u + kf + \frac{k^2}{2}(f_t + f_u f) + \frac{k^3}{6}(f_u^2 f + f_{uu}f^2 + f_u f_t + 2f_{tu}f + f_{tt})\\
&+ \frac{k^4}{24}(5f_{ut}f_u f + 4f_{uu}f_u f^2 + f_u^2 f_t + f_u^3 f + 3f_{uut}f^2 + f_{uuu}f^3 + 3f_{uu}f_t f + 3f_{ut}f_t + f_u f_{tt} + 3f_{utt}f + f_{ttt})
\end{aligned}$$

$$\begin{aligned}
y_1 &= f(u_n,t_n) = f,\\
y_2 &= f(u_n + \frac{k}{2}y_1, t_n + \frac{k}{2})\\
&= f + \frac{k}{2}ff_u + \frac{k}{2}f_t + \frac{k^2}{8}f^2 f_{uu} + \frac{k^2}{4}ff_{ut} + \frac{k^2}{8}f_{tt} + \frac{k^3}{48}f^3 f_{uuu} + \frac{k^3}{16}f^2 f_{uut} + \frac{k^3}{16}ff_{utt} + \frac{k^3}{48}f_{ttt} + O(k^4),
\end{aligned}$$

$$\begin{aligned}
y_3 &= f(u_n + \frac{k}{2}y_2, t_n + \frac{k}{2})\\
&= (f + \frac{k}{2}f_t + \frac{k^2}{8}f_{tt} + \frac{k^3}{48}f_{ttt})\\
&+ y_2(\frac{k}{2}f_u + \frac{k^2}{4}f_{ut} + \frac{k^3}{16}f_{utt}) + y_2^2(\frac{k^2}{8}f_{uu} + \frac{k^3}{16}f_{uut}) + y_2^3(\frac{k^3}{48}f_{uuu}) + O(k^4)\\
&= (f + \frac{k}{2}f_t + \frac{k^2}{8}f_{tt} + \frac{k^3}{48}f_{ttt})\\
&+ (f + \frac{k}{2}ff_u + \frac{k}{2}f_t + \frac{k^2}{8}f^2 f_{uu} + \frac{k^2}{4}ff_{ut} + \frac{k^2}{8}f_{tt})(\frac{k}{2}f_u + \frac{k^2}{4}f_{ut} + \frac{k^3}{16}f_{utt})\\
&+ (f + \frac{k}{2}ff_u + \frac{k}{2}f_t)^2(\frac{k^2}{8}f_{uu} + \frac{k^3}{16}f_{uut})\\
&+ f^3(\frac{k^3}{48}f_{uuu}) + O(k^4)\\
&= f + k(\frac{1}{2}f_t + \frac{1}{2}f_u f) + k^2(\frac{1}{8}f_{tt} + \frac{1}{4}f_u^2 f + \frac{1}{4}f_{ut}f + \frac{1}{8}f_{uu}f^2 + \frac{1}{4}f_u f_t)\\
&+ k^3(\frac{1}{48}f_{ttt} + \frac{1}{16}f_{utt} + \frac{1}{4}f_{ut}f_u f + \frac{1}{8}f_{ut}f_t + \frac{3}{16}f_{uu}f_u f^2 + \frac{1}{16}f_{tt}f_u + \frac{1}{16}f_{uut}f^2 + \frac{1}{8}f_{uu}f_t + \frac{1}{48}f_{uuu}f^3) + O(k^4)
\end{aligned}$$

$$
\begin{aligned}
y_4 &= f(u_n + ky_3, t_n + k) \\
&= (f + kf_t + \frac{k^2}{2}f_{tt} + \frac{k^3}{6}f_{ttt}) \\
&+ y_3(kf_u + k^2 f_{ut} + \frac{k^3}{2}f_{utt}) + y_3^2(\frac{k^2}{2}f_{uu} + \frac{k^3}{2}f_{uut}) + y_3^3(\frac{k^3}{6}f_{uuu}) + O(k^4) \\
&= (f + kf_t + \frac{k^2}{2}f_{tt} + \frac{k^3}{6}f_{ttt}) \\
&+ (f + k(\frac{1}{2}f_t + \frac{1}{2}f_u f) + k^2(\frac{1}{8}f_{tt} + \frac{1}{4}f_u^2 f + \frac{1}{4}f_{ut}f + \frac{1}{8}f_{uu}f^2 + \frac{1}{4}f_u f_t))(kf_u + k^2 f_{ut} + \frac{k^3}{2}f_{utt}) \\
&+ (f + k(\frac{1}{2}f_t + \frac{1}{2}f_u f))^2(\frac{k^2}{2}f_{uu} + \frac{k^3}{2}f_{uut}) \\
&+ f^3(\frac{k^3}{6}f_{uuu}) + O(k^4) \\
&= f + k(f_t + f_u f) + k^2(\frac{1}{2}f_{tt} + f_{ut} + \frac{1}{2}f_u f_t + \frac{1}{2}f_u^2 f + f_{uu}f^2) \\
&+ k^3(\frac{1}{6}f_{ttt} + \frac{1}{2}f_{ut}f_t + \frac{3}{4}f_{ut}f_u f + \frac{1}{2}f_{utt}f + \frac{1}{8}f_{tt}f_u + \frac{1}{4}f_u^3 f + \frac{5}{8}f_{uu}f_u f^2 + \frac{1}{4}f_u^2 f_t + \frac{1}{2}f_{uut}f^2 + \frac{1}{2}f_{uu}f_t f + \frac{1}{6}f_{uuu}f^3) + O(k^4)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}u(t_n) &= u(t_{n+1}) - u(t_n) - k\Phi(u(t_n), t_n; k) \\
&= k(f + \frac{k}{2}(f_t + f_u f) + \frac{k^2}{6}(f_u^2 f + f_{uu}f^2 + f_u f_t + 2f_{tu}f + f_{tt}) \\
&+ \frac{k^3}{24}(5f_{ut}f_u f + 4f_{uu}f_u f^2 + f_u^2 f_t + f_u^3 f + 3f_{uut}f^2 + f_{uuu}f^3 + 3f_{uu}f_t f + 3f_{ut}f_t + f_u f_{tt} + 3f_{utt}f + f_{ttt}) \\
&- \frac{1}{6}(y_1 + 2y_2 + 2y_3 + y_4)) \\
&= k \cdot O(k^4) = O(k^5).
\end{aligned}
$$

□

Exercise 10.202

Show that the calssical fourth-order RK method has its stability function as

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4.$$

解. 根据 Definition 10.152，我们有

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & & & \\ \frac{1}{2} & 0 & & \\ 0 & \frac{1}{2} & 0 & \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
\boldsymbol{b} &= \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}^T, \\
\boldsymbol{c} &= \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^T.
\end{aligned}
$$

因此

$$
\begin{aligned}
R(z) =&1+z\boldsymbol{b}^T(I-zA)^{-1}1 \\
=&1+z\begin{bmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}\end{bmatrix}^T\begin{bmatrix}1 & & & \\ -\frac{z}{2} & 1 & & \\ 0 & -\frac{z}{2} & 1 & \\ 0 & 0 & -z & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \\
=&z\begin{bmatrix}\frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}\end{bmatrix}^T\begin{bmatrix}1 & & & \\ \frac{z}{2} & 1 & & \\ \frac{z^2}{4} & \frac{z}{2} & 1 & \\ \frac{z^3}{4} & \frac{z^2}{2} & z & 1\end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \\
=&1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}.
\end{aligned}
$$

□

Exercise 10.206

Define $S_s := \{z : |R_s(z)| \le 1\}$ where $s = 1, 2, 3, 4$ and R_s is the stability function of the $s-$ stage, sth-order ERK method. Show that

$$S_1 \subset S_2 \subset S_3.$$

Does this hold for ERK methods with a higher stage? Why?

解. 要证 $S_1 \subset S_2$，即证对任意 $z \in \mathbb{C}$，若 $|1+z| \le 1$，则 $|1+z+\frac{z^2}{2}| \le 1$。令 $w = 1+z$，则 $|w| \le 1$，而

$$|1+z+\frac{z^2}{2}| = |w+\frac{(w-1)^2}{2}| = |\frac{w^2+1}{2}| \le \frac{|w|^2+1}{2} \le 1.$$

因此 $S_1 \subset S_2$。

要证 $S_2 \subset S_3$，即证对任意 $z \in \mathbb{C}$，若 $|1+z+\frac{z^2}{2}| \le 1$，则 $|1+z+\frac{z^2}{2}+\frac{z^3}{6}| \le 1$。令 $w = 1+z+\frac{z^2}{2}$，则 $|w| \le 1$，$z = \dfrac{-1+\sqrt{2w-1}}{2}$。

$$
\begin{aligned}
&|1+z+\frac{z^2}{2}+\frac{z^3}{6}| \\
=&|w+\frac{(-1+\sqrt{2w-1})^3}{48}| \\
=&|\frac{21w+1+(w+1)\sqrt{2w-1}}{24}|
\end{aligned}
$$

令

$$f(\theta) = |21e^{i\theta}+1+(e^{i\theta}+1)\sqrt{2e^{i\theta}-1}|$$

则

$$f'(\theta) = \frac{3ie^{i\theta}(7\sqrt{2e^{i\theta}-1}+e^{i\theta})(e^{i\theta}(21+\sqrt{2e^{i\theta}-1})+\sqrt{2e^{i\theta}-1})}{\sqrt{2e^{i\theta}-1}|21e^{i\theta}+1+(e^{i\theta}+1)\sqrt{2e^{i\theta}-1}}$$

注意到 $f'(0) = 0, f'(\pi) = 0, f(0) > f(\pi)$，且 f 是有界函数，因此 f 的极值点也是最值点就是 $f(0) = 24$。因此 $|1+z+\frac{z^2}{2}+\frac{z^3}{6}| \le 1$，$S_2 \subset S_3$。

对更大的 p，上述结论不再成立。例如 $S_3 \subset S_4$ 的反例有：$z = 0.00353523 - 0.776261i$, $|1+z+\frac{z^2}{2}+\frac{z^3}{6}| = 0.983133 < 1$, $|1+z+\frac{z^2}{2}+\frac{z^3}{6}+\frac{z^4}{24}| = 1.00419 > 1$。 □

Exercise 10.212

Prove that an A-stable RK method with its stability function as a rational polynomial $R(z) = \frac{P(z)}{Q(z)}$ is L-stable if only if $\deg Q(z) > \deg P(z)$.

解. 设 $P(z) = \sum_{j=0}^{m} p_j z^j, Q(z) = \sum_{j=0}^{n} q_j z^j$，则

$$\exists R > 0, \text{ s.t. } \forall z \geq R, \frac{1}{2} \leq \frac{|p(z)|}{|p_m||z|^m} \leq 2, \frac{1}{2} \leq \frac{|q(z)|}{|q_n||z|^n} \leq 2. \tag{1}$$

根据 Definition 10.139，RK method L 稳定当且仅当 $\lim_{z\to\infty} |R(z)| = \lim_{z\to\infty} \frac{|P(z)|}{|Q(z)|} = 0$。

若 $m < n$，则

$$\lim_{|z|\to\infty} \frac{|P(z)|}{|Q(z)|} = \lim_{|z|\to\infty} \frac{2|p_m||z|^m}{\frac{1}{2}|q_n||z|^n} = 0. \tag{2}$$

若 $m \geq n$，则

$$\lim_{|z|\to\infty} \frac{|P(z)|}{|Q(z)|} = \lim_{|z|\to\infty} \frac{\frac{1}{2}|p_m||z|^m}{2|q_n||z|^n} > 0. \tag{3}$$

因此 RK method L 稳定，当且仅当 $m < n$，即 $\deg Q > \deg P$。 □

Exercise 10.215

Show that if an A-stable RK method with a nonsingular RK matrix A satisfies

$$a_{i,1} = b_1, i = 1, \ldots, s,$$

then it is L-stable.

解. $a_{i,1} = b_1, i = 1, \ldots, s$ 等价于 $A\boldsymbol{e}_1 = \boldsymbol{b}$。此时，

$$\begin{aligned}
&\lim_{|z|\to\infty} R(z) \\
=&\lim_{|z|\to\infty} 1 + z\boldsymbol{b}^T(I - zA)^{-1}1 \\
=&1 + \lim_{|z|\to\infty} \boldsymbol{b}^T(\frac{1}{z} - A)^{-1}1 \\
=&1 - \boldsymbol{b}^T A^{-1}1 \\
=&1 - \boldsymbol{b}^T A^{-1}\frac{A\boldsymbol{e}_1}{b_1} \\
=&1 - \boldsymbol{b}^T\frac{\boldsymbol{e}_1}{b_1} \\
=&1 - \frac{b_1}{b_1} \\
=&0 .
\end{aligned}$$

□

Exercise 10.220

Show that the collocation method
$$\begin{array}{c|ccc} \frac{4-\sqrt{6}}{10} & \frac{88-7\sqrt{6}}{360} & \frac{296-169\sqrt{6}}{1800} & \frac{-2+3\sqrt{6}}{225} \\ \frac{4+\sqrt{6}}{10} & \frac{296+169\sqrt{6}}{1800} & \frac{88+7\sqrt{6}}{360} & \frac{-2-3\sqrt{6}}{225} \\ 1 & \frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9} \\ \hline & \frac{16-\sqrt{6}}{36} & \frac{16+\sqrt{6}}{36} & \frac{1}{9} \end{array}$$
is 5th-order accurate and L-stable.

解.
$$\begin{aligned} q_r(x) =&(x-\frac{4-\sqrt{6}}{10})(x+\frac{4+\sqrt{6}}{10})(x-1)=x^3-\frac{9}{5}x^2+\frac{9}{10}x-\frac{1}{10}, \\ \int_0^1 q_r(x)\mathrm{d}x =&(\frac{1}{4}x^4-\frac{3}{5}x^3+\frac{9}{20}x^2-\frac{1}{10}x)\Big|_0^1=0, \\ \int_0^1 xq_r(x)\mathrm{d}x =&(\frac{1}{5}x^5-\frac{9}{20}x^4+\frac{3}{10}x^3-\frac{1}{20}x^2)\Big|_0^1=0, \\ \int_0^1 x^2q_r(x)\mathrm{d}x =&(\frac{1}{6}x^6-\frac{9}{25}x^5+\frac{9}{40}x^4-\frac{1}{30}x^3)\Big|_0^1=-\frac{1}{600} \end{aligned}$$
所以 $r=2$。由 Theorem 10.183 得组合方法的精度为 $3+2=5$。因为 $\det(A)=\frac{1}{60}\neq 0$，所以 A 非奇异。且因为 $a_{s,j}=b_j, j=1,2,3$，所以组合方法是刚性稳定的。 □

Exercise 10.229

Rewrite the implicit midpoint method
$$\boldsymbol{U}^{n+1}=\boldsymbol{U}^n+k\boldsymbol{f}(\frac{\boldsymbol{U}^n+\boldsymbol{U}^{n+1}}{2},t_n+\frac{k}{2})$$
in the standard form and derive its Butcher tableau. Show that it is B-stable.

解. 令 $\boldsymbol{y}_1=\boldsymbol{f}(\frac{\boldsymbol{U}^n+\boldsymbol{U}^{n+1}}{2},t_n+\frac{k}{2})$。将 $\boldsymbol{U}^{n+1}$ 的公式代入自身，可得
$$\begin{aligned} \boldsymbol{U}^{n+1} =&\boldsymbol{U}^n+k\boldsymbol{f}(\frac{\boldsymbol{U}^n+\boldsymbol{U}^{n+1}}{2},t_n+\frac{k}{2}) \\ =&\boldsymbol{U}^n+k\boldsymbol{f}(\frac{\boldsymbol{U}^n+\boldsymbol{U}^n+k\boldsymbol{f}(\frac{\boldsymbol{U}^n+\boldsymbol{U}^{n+1}}{2},t_n+\frac{k}{2})}{2},t_n+\frac{k}{2}) \\ =&\boldsymbol{U}^n+k\boldsymbol{f}(\boldsymbol{U}^n+\frac{k}{2}\boldsymbol{f}(\frac{\boldsymbol{U}^n+\boldsymbol{U}^{n+1}}{2},t_n+\frac{k}{2}),t_n+\frac{k}{2}) \\ =&\boldsymbol{U}^n+k\boldsymbol{f}(\boldsymbol{U}^n+\frac{k}{2}\boldsymbol{y}_1,t_n+\frac{k}{2}). \end{aligned}$$
因此 $\boldsymbol{y}_1=\boldsymbol{f}(\boldsymbol{U}^n+\frac{k}{2}\boldsymbol{y}_1,t_n+\frac{k}{2})$。标准形式为
$$\begin{cases} \boldsymbol{y}_1 =\boldsymbol{f}(\boldsymbol{U}^n+\frac{k}{2}\boldsymbol{y}_1) \\ \boldsymbol{U}^{n+1} =\boldsymbol{U}^n+k\boldsymbol{y}_1 \end{cases}$$
下面证明 B-稳定性。设初值问题 $\boldsymbol{u}'=\boldsymbol{f}(\boldsymbol{u},t)$ 是压缩的，且 $\boldsymbol{U}^n,\boldsymbol{V}^n$ 为两个数值解，记 $\boldsymbol{e}^n=\boldsymbol{U}^n-\boldsymbol{V}^n$，则
$$\begin{aligned} &\langle\frac{\boldsymbol{e}^n+\boldsymbol{e}^{n+1}}{2},\boldsymbol{e}^{n+1}\rangle \\ =&\langle\frac{\boldsymbol{e}^n+\boldsymbol{e}^{n+1}}{2},\boldsymbol{e}^n+k(\boldsymbol{f}(\frac{\boldsymbol{U}^n+\boldsymbol{U}^{n+1}}{2},t_n+\frac{k}{2})-\boldsymbol{f}(\frac{\boldsymbol{V}^n+\boldsymbol{V}^{n+1}}{2},t_n+\frac{k}{2}))\rangle \end{aligned}$$

因为 $\boldsymbol{f}$ 收缩，而 $\frac{\boldsymbol{e}^n+\boldsymbol{e}^{n+1}}{2}=\frac{\boldsymbol{U}^n+\boldsymbol{U}^{n+1}}{2}-\frac{\boldsymbol{V}^n+\boldsymbol{V}^{n+1}}{2}$，所以根据 Definition 10.224 有 $\langle\frac{\boldsymbol{e}^n+\boldsymbol{e}^{n+1}}{2},\boldsymbol{f}(\frac{\boldsymbol{U}^n+\boldsymbol{U}^{n+1}}{2},t_n+\frac{k}{2})-\boldsymbol{f}(\frac{\boldsymbol{V}^n+\boldsymbol{V}^{n+1}}{2},t_n+\frac{k}{2})\rangle\leq 0$。故 $\langle\frac{\boldsymbol{e}^n+\boldsymbol{e}^{n+1}}{2},\boldsymbol{e}^{n+1}\rangle\leq\langle\frac{\boldsymbol{e}^n+\boldsymbol{e}^{n+1}}{2},\boldsymbol{e}^n\rangle$。所以 $\|\boldsymbol{e}^{n+1}\|\leq\|\boldsymbol{e}^n\|$，隐式中点法 B-稳定。 □

参考文献

[1] 张庆海. “Notes on Numerical Analysis and Numerical Methods for Differential Equations”. In: (2024).