Problems of Chapter 4

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1 Theoretical questions

4.4.1-I.

Convert the decimal integer 477 to a normalized FPN with $\beta = 2$.

解. $477 = (111011101)_2 = (1.11011101)_2 \times 2^8$ □

4.4.1-II.

Convert the decimal fraction 3/5 to a normalized FPN with $\beta = 2$.

解. $\frac{3}{5} = (0.\dot{1}00\dot{1})_2 = (1.\dot{0}01\dot{1})_2 \times 2^{-1}$ □

4.4.1-III.

Let $x = \beta^e, e \in \mathbb{Z}, L < e < U$ be a normalized FPN in $\mathbb{F}$ and $x_L, x_R \in \mathbb{F}$ are the two normalized FPNs adjacent to x such that $x_L < x < x_R$. Prove $x_R - x = \beta(x - x_L)$.

证明. 根据题意 $x = 1.0 \times \beta^e$，$x_R = (1 + \beta^{-p+1}) \times \beta^e, x_L = \left[\sum_{i=0}^{p-1} (\beta - 1)\beta^{-i}\right] \times \beta^{e-1} = (1 - \beta^{-p}) \times \beta^e$。所以 $x_R - x = \beta^{e-p+1}, x - x_L = \beta^{e-p}$，$x_R - x = \beta(x - x_L)$。 □

4.4.1-IV.

By reusing your result if II, find out the two normalized FPNs adjacent to $x = 3/5$ under the IEEE 754 single-precision protocol. What is $\mathrm{fl}(x)$ and the relative roundoff error?

解. 在 IEEE 754 单精度浮点数系统下，尾数为 23 位。所以:

$$x_L = (1.00110011001100110011001)_2 \times 2^{-1}, x_R = (1.00110011001100110011010)_2 \times 2^{-1}$$

显然 $x_R - x < x - x_L$，所以 $\mathrm{fl}(x) = x_R$，相对误差 $E_{rel}(x_R) = \frac{|x_R - x|}{|x|} = \frac{(0.\dot{0}11\dot{0})_2 \times 2^{-23}}{3/5} = \frac{2}{3} \times 2^{-23}$。 □

4.4.1-V.

If the IEEE 754 single-precision protocol did not round off numbers to the nearest, but simply dropped excess bits, what would the unit roundoff be?

解. 设 $x=(m_x+\beta^{1-p}-\epsilon)\times\beta^n$，其中 $0<\epsilon<\beta^{-p+1}$，则 $\mathrm{fl}(x)=m_x\times\beta^n$。

$$\delta=\left|\frac{\mathrm{fl}(x)-x}{x}\right|=\frac{\beta^{1-p}-\epsilon}{m_x+\beta^{1-p}-\epsilon}\to\frac{\beta^{1-p}}{m_x+\beta^{1-p}}(\epsilon\to 0^+)$$

又因为 $1\le m_x<\beta$，所以有 $\delta<\beta^{1-p}=\epsilon_M(=2^{-23})$ 成立，所以舍入单位 $\epsilon_u=\epsilon_M$。 □

4.4.1-VI.

How many bits of precision are lost in the subtraction $1-\cos x$ when $x=\frac{1}{4}$?

解. $\cos\frac{1}{4}=0.9689\cdots, 1-\cos\frac{1}{4}=0.0310\cdots\in(2^{-6},2^{-5})$，因此损失了 6 位的精度。 □

4.4.1-VII.

Suggest at least two ways to compute $1-\cos x$ to avoid catastrophic cancellation caused by subtraction.

解. 为了避免减法运算带来的巨量消失，考虑将减法转化为别的四则运算。

1. $1-\cos x=\begin{cases}\mathrm{fl}(1-\mathrm{fl}(\cos x)) & \cos x<0\\ \mathrm{fl}\left(\dfrac{\mathrm{fl}(\mathrm{fl}(\sin x))^2}{\mathrm{fl}(1+\mathrm{fl}(\cos x))}\right) & \cos\ge 0\end{cases}$
2. $1-\cos x=2\mathrm{fl}(\sin(\frac{x}{2}))^2$
3. $1-\cos x=\mathrm{fl}[\mathrm{fl}(\frac{x^2}{2!})-\mathrm{fl}(\frac{x^4}{4!})+\mathrm{fl}(\frac{x^6}{6!})-\cdots]=\mathrm{fl}\left[\sum_{k=1}^{M}\mathrm{fl}\left((-1)^{k+1}\frac{x^{2k}}{(2k)!}\right)\right]$，这里 M 可以取一个足够大的整数，使得计算的相对误差在可接受范围内。

□

4.4.1-VIII.

What are the condition numbers of the following functions? Where are they large?

- $(x-1)^\alpha$
- $\ln x$
- e^x
- $\arccos x$

解.

- $\mathrm{Cond}_f(x)=\begin{cases}\left|\dfrac{x\mathrm{d}(x-1)^{\alpha-1}}{(x-1)^\alpha}\right|=\left|\dfrac{\alpha x}{x-1}\right| & \alpha\ne 0\\ 0 & \alpha=0\end{cases}$，$\lim_{x\to 1}\mathrm{Cond}_f(x)=\infty(\alpha\ne 0)$；
- $\mathrm{Cond}_f(x)=\left|\dfrac{x\cdot\frac{1}{x}}{\ln x}\right|=\dfrac{1}{|\ln x|}$，$\lim_{x\to 1}\mathrm{Cond}_f(x)=\infty$；
- $\mathrm{Cond}_f(x)=\left|\dfrac{x\cdot e^x}{e^x}\right|=|x|$，$\lim_{x\to\infty}\mathrm{Cond}_f(x)=\infty$；
- $\mathrm{Cond}_f(x)=\left|\dfrac{x\cdot\frac{-1}{\sqrt{1-x^2}}}{\arccos x}\right|=\left|\dfrac{x}{\sqrt{1-x^2}\arccos x}\right|$，$\lim_{x\to\pm 1}\mathrm{Cond}_f(x)=\infty$。

□

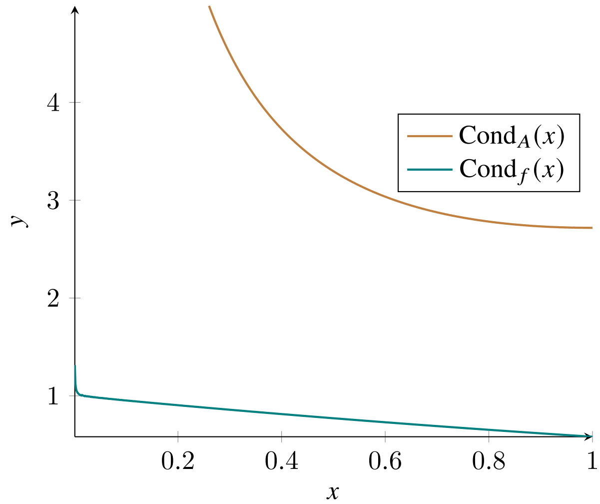
4.4.1-IX.

Consider the function $f(x) = 1 - e^{-x}$ for $x \in [0, 1]$.

- Show that $\text{cond}_f(x) \leq 1$ for $x \in [0, 1]$.
- Let A be the algorithm that evaluates $f(x)$ for the machine nuumber $x \in F$. Assume that the exponential function is computed with relative error within machine roundoff. Estimate $\text{cond}_A(x)$ for $x \in [0, 1]$.
- Plot $\text{cond}_f(x)$ and the estimated upper bound of $\text{cond}_A(x)$ as a function of x on $[0, 1]$. Discuss your results.

解.

- $\text{Cond}_f(x) = \left|\frac{xe^{-x}}{1-e^{-x}}\right| = \frac{x}{1-e^{x}}, \forall x \in [0, 1]$，因为 $(\frac{x}{1-e^{-x}})' = \frac{(1-x-e^{-x})e^{-x}}{(1-e^{-x})^2} \leq 0 (x \in [0, 1])$，并且在 $x = 0$ 处 $\lim_{x\to 0} \text{Cond}_f(x) = 1$，所以 $\text{Cond}_f(x) \leq 1$, for $x \in [0, 1]$;
- $f_A(x) = \text{fl}(1 - \text{fl}(e^{-x})) = (1 - e^{-x}(1 + \delta_1))(1 + \delta_2) \simeq (1 - e^{-x})(1 + \delta_2 + \delta_1 \frac{e^{-x}}{1-e^{-x}})$,
因此有 $f_A(x) = f(x)(1 + \delta(x)), |\delta(x)| = \left|\delta_2 + \delta_1 \frac{e^{-x}}{1-e^{-x}}\right| \leq |\frac{\delta_2+(\delta_1-\delta_2)e^{-x}}{1-e^{-x}}| \leq \frac{\epsilon_u}{1-e^{-x}}$,
$\text{Cond}_A(x) \leq \frac{1}{(1-e^{-x})\text{Cond}_f(x)} = \frac{1}{xe^{-x}}$;
- 如下图所示，$\text{Cond}_f(x)$, upper bound of $\text{Cond}_A(x)$ 均在 $x \to 0$ 最大，且 $\text{Cond}_A > \text{Cond}_f$。



□

4.4.1-X.

Prove Lemma 4.68. Base on the 2-norm, the condition number of a nonsingular square matrix A is

$$\text{cond}A := \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_{max}}{\sigma_{min}}.$$

where σ_{max} and σ_{min} are respectively the largest and the smallest singular values of A. If A is also normal, we have,

$$\text{cond}_2 A = \frac{\lambda_{max}}{\lambda_{min}}$$

where λ_{max} and λ_{min} are eigenvalues of A with the largest and the smallest moduli, respectively. Furthermore, if A is unitary, we have $\text{cond}_2 A = 1$.

证明.

$$\|A\|_2^2 = \sup_{\|x\|_2=1} \|Ax\|_2^2 = \sup_{\|x\|_2=1} (Ax)^T(Ax) = \sup_{\|x\|_2=1} x^TA^TAx = \sup_{\|x\|_2=1} x^TP^T\Lambda Px = \sup_{\|Px\|_2=1} (Px)^T\Lambda(Px) = \lambda_{max}$$

$$\|A^{-1}\|_2^2 = \sup_{\|x\|_2=1} \|A^{-1}x\|_2^2 = \cdots = \sup_{\|Px\|_2=1} (Px)^T\Lambda^{-1}(Px) = \lambda_{min}^{-1}$$

$$\Rightarrow \mathrm{Cond}(A) = \|A\|_2\|A^{-1}\|_2 = \sqrt{\frac{\lambda_{max}}{\lambda_{min}}} = \frac{\sigma_{max}}{\sigma_{min}}$$

□

4.4.1-XI.

The math problem of root finding for a polynomial

$$q(x) = \sum_{i=0}^{n} a_i x^i, \qquad a_n = 1, a_0 \neq 0, a_i \in \mathbb{R}$$

can be considered as a vector function $f : \mathbb{R}^n \to \mathbb{C}$:

$$r = f(a_0, a_1, \cdots, a_{n-1}).$$

Derive the componentwise condition number of f based on the 1-norm. For the Wilkinson example, compute your condition number, and compare your result with that in the Wilkinson Example. What does the comparison tell you?

解. $\mathrm{Cond}_f(\mathrm{a}) = \max_j \left|\frac{a_j \frac{\partial f}{\partial a_j}}{f(a)}\right|$,

$\frac{\partial f}{\partial a_j}(\mathrm{a}) = \lim_{\epsilon\to 0} \frac{-\epsilon \frac{r^j}{q'(r)}}{\epsilon} = -\frac{r^j}{q'(r)}, r = f(a)$，即 $q(x)$ 的根。

所以 $\mathrm{Cond}_f(\mathrm{a}) = \max_j \left|\frac{a_j r^{j-1}}{q'(r)}\right|$,

对于 Wilkinson 例子即 $q(x) = \prod_{i=1}^{p}(x-i)$，当 p 足够大时，对于 $q(x)$ 最大根 p，

$$\mathrm{Cond}_f(\mathrm{a}) = \max_j \left|\frac{a_j p^{j-1}}{(p-1)!}\right| = \frac{\max_j |a_j p^{j-1}|}{(p-1)!}$$

其中 a_j 是 $\prod_{i=1}^{p}(x-i)$ 的 j 次项系数，注意到 $|a_{p-1}| = \frac{p(p-1)}{2}$，因此 $\mathrm{Cond}_f(x)$ 至少和 $\frac{p^p}{2(p-2)!}$ 同阶。所以 $n\to\infty$ 的时候，$\mathrm{Cond}_f(\mathrm{a}) \to \infty$。可以看出求得高阶多项式的数值根是很困难的。 □

4.4.1-XII.

Suppose the division of two FPNs is calculated in a register of precision $2p$. Give an example that contradicts the conclusion of the model of machine arithmetic.

解. 取 FPN 系统 $\beta = 2, p = 2, L = -1, U = 1$，考虑 $a = 1, b = \frac{3}{2}$ 两个数在该系统中，则 $\mathrm{fl}(\frac{a}{b}) = \mathrm{fl}((1.010)_2 \times 2^{-1}) = (1.0)_2 \times 2^{-1}$，相对误差为 $\frac{-0.5+\frac{2}{3}}{\frac{2}{3}} = 2^{-2} = \epsilon_u$，矛盾。 □

4.4.1-XIII.

If the bisection method is used in single precision FPNs of IEEE 754 starting with the interval $[128, 129]$, can we compute the root with absolute accuracy $< 10^6$? Why?

解. 在单精度系统下 $\epsilon_u = \frac{1}{2}\epsilon_M = 2^{-24}$，根的绝对误差最大为 $129 \times 2^{-24} \approx 7.69 \times 10^{-6} > 10^{-6}$。不能。 □

4.4.1-XIV.

In fitting a curve by cubic splines, one gets inaccurate results when the distance between two adjacent points is much smaller than those of other adjacent pairs. Use the condition number of a matrix to explain this phenomenon.

解. 对于在 $[x_i, x_{i+1}]$ 上的样条函数 $s(x)$，求解时需根据 $s(x_i), s(x_{i+1}), s'(x_i), s'(x_{i+1})$ 的值列出方程进行求解，方程的系数矩阵如下：

$$\begin{bmatrix} x_i^3 & x_i^2 & x_i & 1 \\ x_{i+1}^3 & x_{i+1}^2 & x_{i+1} & 1 \\ 3x_i^2 & 2x_i & 1 & 0 \\ 3x_{i+1}^2 & 2x_{i+1} & 1 & 0 \end{bmatrix}$$

在 $|x_{i+1} - x_i|$ 很小的情况下，上述矩阵的 1,2 两行和 3,4 两行的值十分接近，因此上述矩阵是一个病态矩阵，其最小特征值接近于 0，所以求解该样条函数的准确性很差。

□

2 Programming assignments

2.1 A

计算以下三个函数在 $[0.99, 1.01]$ 上 101 个等距点上的函数值并作图。注意这三个函数在理论上是完全相同的。观察结果并解释原因。

代码实现：见 `Problem-A.cpp`

运行结果：见 `res/Output_A.csv`

将三个函数的图像绘制如下：

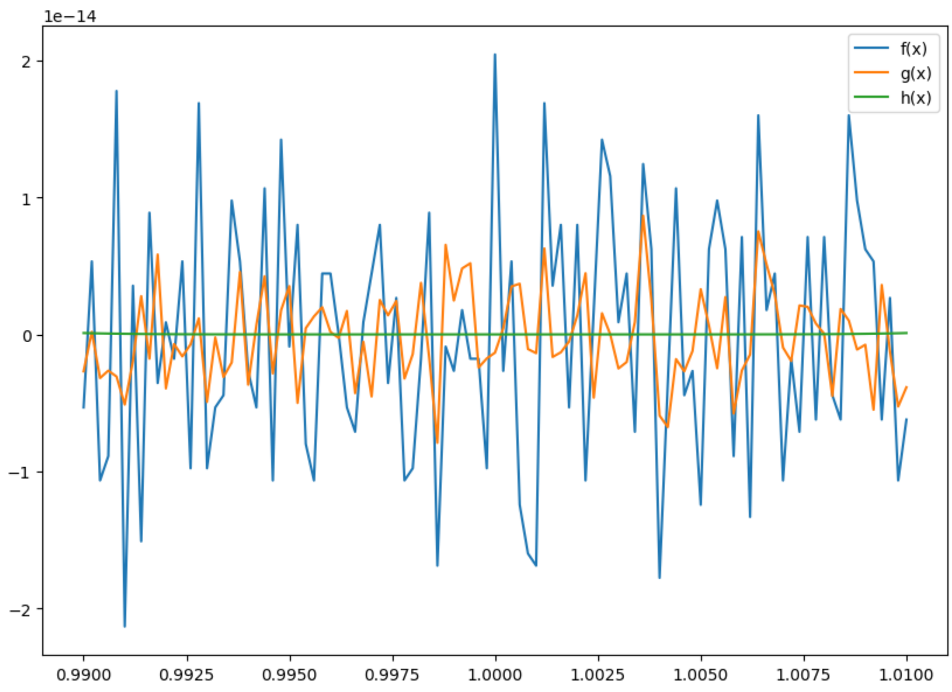


图 1: $f(x), g(x), h(x)$ 对比图

$$f(x) = x^7 - 8x^7 + 28x^6 - 56x^5 + 70x^4 - 56x^3 + 28x^2 - 8x + 1$$

$$g(x) = (((((((x-8)x+28)x-56)x+70)x-56)x+28)x-8)x+1$$

$$h(x) = (x-1)^8$$

由图像，h 的精确度比 f,g 显著高。f 是振荡最厉害的，最不精确。

三个函数的条件数都是 $|\frac{8x}{x-1}|$，在 $x \to 1$ 的条件数区域正无穷，在 1 附近计算的函数值容易不精确。

然而 $h(x)$ 的精确度比另外两个高很多，这是因为 $f(x), g(x)$ 的计算方法的最后一步，都是进行 $\phi(x)$+1，其中 $\phi(x) = f(x) - 1$ 非常接近 -1（$f(x)$ 的绝对值在 $[10^{-30}, 10^{-16}]$ 之间数量级接近甚至小于 ϵ_u），因此进行最后一次计算的时候，将会导致巨量消失，导致有效位被舍入。而 $h(x)$ 只在最开始进行了一次减法运算 $x-1$ 只会损失一部分精度，且 $x-1$ 的数量级高于 ϵ_u，之后 $h(x)$ 只进行乘法运算，精度损失很小。而 $f(x)$ 和 $g(x)$ 则进行了大量的加减运算，相近的数运算带来了很多的精度损失。

$g(x)$ 利用相比 $f(x)$，使用了秦九韶法，大大减少了乘除法的计算次数，提高了精确度。

2.2 B

代码实现：见 `Problem-B.cpp`

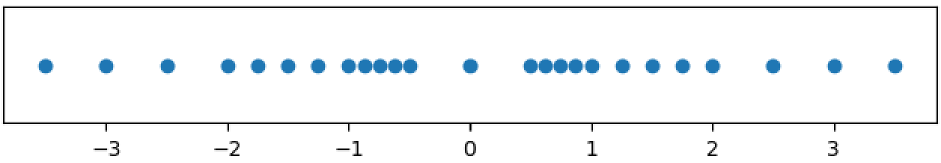
考虑正则 FPN 系统 $\mathbb{F}: \beta = 2, p = 3, L = -1, U = 1$。

1. $\mathrm{UFL}(\mathbb{F}) = \beta^L = 0.5$

   $\mathrm{OFL}(\mathbb{F}) = \beta^U(\beta - \beta^{1-p}) = 3.5$；
2. $\mathbb{F}$ 中所有浮点数一共 25 个：

```
-3.5,-3,-2.5,-2,-1.75,-1.5,-1.25,-1,-0.875,-0.75,-0.625,
-0.5,0,0.5,0.625,0.75,0.875,1,1.25,1.5,1.75,2,2.5,3,3.5
```

   与讲义中关于 $\#\mathbb{F}$ 的推论相符合：$\#\mathbb{F} = 2^p(U - L + 1) + 1 = 8 \times 3 + 1 = 25$。
3. 在数轴上画出 $\mathbb{F}$ 如下：



4. $\mathbb{F}$ 中所有非正则浮点数一共 6 个：

```
-0.375,-0.25,-0.125,0.125,0.25,0.375
```

5. 在数轴上画出扩展的 $\mathbb{F}$ 如下：