Problems of Chapter 10.3.1-10.3.3

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Exercise 10.99

Compute the first five coefficients C_j's of the trapezoidal rule and the midpoint rule from Examples 10.85 and 10.87.

解.

(1) Trapezoidal rule:

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \frac{k}{2}\left(f(\boldsymbol{u}^n, t_n) + f(\boldsymbol{u}^{n+1}, t_{n+1})\right), s = 1, \alpha = \{-1, 1\}, \beta = \{\frac{1}{2}, \frac{1}{2}\}.$$

$C_0 = \alpha_0 + \alpha_1 = 0, \quad C_1 = -\beta_0 + \alpha_1 - \beta_1 = 0, \quad C_2 = \frac{1}{2}\alpha_1 - \beta_1 = 0,$

$C_3 = \frac{1}{6}\alpha_1 - \frac{1}{2}\beta_1 = -\frac{1}{12}, \quad C_4 = \frac{1}{24}\alpha_1 - \frac{1}{6}\beta_1 = -\frac{1}{24}.$

(2) Midpoint rule:

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^{n-1} + 2kf(\boldsymbol{u}^n, t_n), s = 2, \alpha = \{-1, 0, 1\}, \beta = \{0, 2, 0\}.$$

$C_0 = \alpha_0 + \alpha_1 + \alpha_2 = 0, \quad C_1 = -\beta_0 + \alpha_1 - \beta_1 + 2\alpha_2 = 0,$

$C_2 = \frac{1}{2}\alpha_1 - \beta_1 + 2\alpha_2 = 0, \quad C_3 = \frac{1}{6}\alpha_1 - \frac{1}{2}\beta_1 + \frac{4}{3}\alpha_2 = \frac{1}{3},$

$C_4 = \frac{1}{24}\alpha_1 - \frac{1}{6}\beta_1 + \frac{2}{3}\alpha_2 = \frac{1}{3}.$

□

Exercise 10.101

Express conditions of $\|\mathcal{L}\boldsymbol{u}(t_n)\| = O(k^3)$ using characteristic polynomials.

解. $\|\mathcal{L}\boldsymbol{u}(t_n)\| = O(k^3)$ 要求 $C_0 = C_1 = C_2 = 0$，即

$$\begin{aligned}
C_0 &= \sum_{j=0}^{s} \alpha_j = \rho(1) = 0, \\
C_1 &= \sum_{j=0}^{s} j\alpha_j - \sum_{j=0}^{s} \beta_j = \rho'(1) - \sigma(1) = 0, \\
C_2 &= \sum_{j=0}^{s} \frac{j^2}{2}\alpha_j - \sum_{j=0}^{s} j\beta_j \\
&= \frac{1}{2}\sum_{j=0}^{s} j(j-1)\alpha_j + \frac{1}{2}\sum_{j=0}^{s} j\alpha_j - \sum_{j=0}^{s} j\beta_j \\
&= \frac{1}{2}\rho''(1) + \frac{1}{2}\rho'(1) - \sigma'(1) = 0
\end{aligned}$$

□

Exercise 10.102

Derive coefficients of LMMs shown below by the method of undetermined coefficients and a programming language with symbolic computation such as Matlab.

解. (1) Adams-Bashforth formulas: $\alpha_s = 1, \alpha_{s-1} = -1, \alpha_{s-2} = \cdots = \alpha_0 = 0$,

Adams-Bashforth formulas in Definition 10.83

s	p	β_s	β_{s-1}	β_{s-2}	β_{s-3}	β_{s-4}
1	1	0	1			
2	2	0	$\frac{3}{2}$	$-\frac{1}{2}$		
3	3	0	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$	
4	4	0	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{9}{24}$

- $s = 1, p = 1, C_0 = C_1 = 0 \Rightarrow \alpha_1 - \beta_0 = 0 \Rightarrow \beta_0 = 1.$
- $s = 2, p = 2, C_0 = C_1 = C_2 = 0 \Rightarrow \begin{cases} (\alpha_1 + 2\alpha_2) - (\beta_0 + \beta_1) = 0 \\ (\frac{1}{2}\alpha_1 + 2\alpha_2) - \beta_1 = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

- $s = 3, p = 3, C_0 = C_1 = C_2 = C_3 = 0 \Rightarrow \begin{cases} (2\alpha_2 + 3\alpha_3) - (\beta_0 + \beta_1 + \beta_2) = 0 \\ (2\alpha_2 + \frac{9}{2}\alpha_3) - (\beta_1 + 2\beta_2) = 0 \\ (\frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3) - (\frac{1}{2}\beta_1 + 2\beta_2) = 0 \end{cases}.$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5}{2} \\ \frac{19}{6} \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{23}{12} \\ -\frac{16}{12} \\ \frac{5}{12} \end{bmatrix}.$$

- $s = 4, p = 4, C_0 = C_1 = C_2 = C_3 = C_4 = 0 \Rightarrow \begin{cases} (3\alpha_3 + 4\alpha_4) - (\beta_0 + \beta_1 + \beta_2 + \beta_3) = 0 \\ (\frac{9}{2}\alpha_3 + 8\alpha_4) - (\beta_1 + 2\beta_2 + 3\beta_3) = 0 \\ (\frac{9}{2}\alpha_3 + \frac{32}{3}\alpha_4) - (\frac{1}{2}\beta_1 + 2\beta_2 + \frac{9}{2}\beta_3) = 0 \\ (\frac{27}{8}\alpha_3 + \frac{32}{3}\alpha_4) - (\frac{1}{6}\beta_1 + \frac{4}{3}\beta_2 + \frac{9}{2}\beta_3) = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & \\ \frac{9}{2} & 2 & \frac{1}{2} & 0 \\ \frac{9}{2} & \frac{4}{3} & \frac{1}{6} & 0 \end{bmatrix} \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{7}{2} \\ \frac{37}{6} \\ \frac{175}{24} \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{55}{24} \\ -\frac{59}{24} \\ \frac{37}{24} \\ -\frac{9}{24} \end{bmatrix}.$$

(2) Adams-Moulton formulas: $\alpha_s = 1, \alpha_{s-1} = -1, \alpha_{s-2} = \cdots = \alpha_0 = 0$,

Adams-Moulton formulas in Definition 10.83

s	p	β_s	β_{s-1}	β_{s-2}	β_{s-3}	β_{s-4}
1	1	1				
1	2	$\frac{1}{2}$	$\frac{1}{2}$			
2	3	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$		
3	4	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	
4	5	$\frac{251}{720}$	$\frac{646}{720}$	$-\frac{264}{720}$	$\frac{106}{720}$	$-\frac{19}{720}$

- $s = 1, p = 1, C_0 = C_1 = 0 \Rightarrow \alpha_1 - \beta_1 = 0 \Rightarrow \beta_1 = 1$.
- $s = 1, p = 2, C_0 = C_1 = C_2 = 0 \Rightarrow \begin{cases} -(\beta_0 + \beta_1) = 0 \\ \frac{1}{2}\alpha_1 - \beta_1 = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

- $s = 2, p = 3, C_0 = C_1 = C_2 = C_3 = 0 \Rightarrow \begin{cases} (\alpha_1 + 2\alpha_2) - (\beta_0 + \beta_1 + \beta_2) = 0 \\ (\frac{1}{2}\alpha_1 + 2\alpha_2) - (\beta_1 + 2\beta_2) = 0 \\ (\frac{1}{6}j\alpha_1 + \frac{4}{3}\alpha_2) - (\frac{1}{2}\beta_1 + 2\beta_2) = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \\ \frac{7}{6} \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{5}{12} \\ \frac{8}{12} \\ -\frac{1}{12} \end{bmatrix}.$$

- $s = 3, p = 4, C_0 = C_1 = C_2 = C_3 = C_4 = 0 \Rightarrow \begin{cases} (2\alpha_2 + 3\alpha_3) - (\beta_0 + \beta_1 + \beta_2 + \beta_3) = 0 \\ (2\alpha_2 + \frac{9}{2}\alpha_3) - (\beta_1 + 2\beta_2 + 3\alpha_3) = 0 \\ (\frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3) - (\frac{1}{2}\beta_1 + 2\beta_2 + \frac{9}{2}\beta_3) = 0 \\ (\frac{2}{3}\alpha_2 + \frac{27}{8}\alpha_3) - (\frac{1}{6}\beta_1 + \frac{4}{3}\beta_2 + \frac{9}{2}\beta_3) = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & \\ \frac{9}{2} & 2 & \frac{1}{2} & 0 \\ \frac{9}{2} & \frac{4}{3} & \frac{1}{6} & 0 \end{bmatrix} \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5}{2} \\ \frac{19}{6} \\ \frac{65}{24} \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{9}{24} \\ -\frac{19}{24} \\ -\frac{5}{24} \\ \frac{1}{24} \end{bmatrix}.$$

- $s = 4, p = 5, C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0 \Rightarrow \begin{cases} (3\alpha_3 + 4\alpha_4) - (\beta_0 + \beta_1 + \beta_2 + \beta_3 + \beta_4) = 0 \\ (\frac{9}{2}\alpha_3 + 8\alpha_4) - (\beta_1 + 2\beta_2 + 3\beta_3 + 4\beta_4) = 0 \\ (\frac{29}{2}\alpha_3 + \frac{32}{3}\alpha_4) - (\frac{1}{2}\beta_1 + 2\beta_2 + \frac{9}{2}\beta_3 + 8\beta_4) = 0 \\ (\frac{27}{8}\alpha_3 + \frac{32}{3}\alpha_4) - (\frac{1}{6}\beta_1 + \frac{4}{3}\beta_2 + \frac{9}{2}\beta_3 + 8\beta_4) = 0 \\ (\frac{81}{4}0\alpha_3 + \frac{128}{15}\alpha_4) - (\frac{1}{24}\beta_1 + \frac{3}{2}\beta_2 + \frac{27}{8}\beta_3 + \frac{32}{3}\beta_4) = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 8 & \frac{9}{2} & 2 & \frac{1}{2} & 0 \\ \frac{32}{3} & \frac{9}{2} & \frac{4}{3} & \frac{1}{6} & 0 \\ \frac{32}{3} & \frac{27}{8} & \frac{3}{2} & \frac{1}{24} & 0 \end{bmatrix} \begin{bmatrix} \beta_4 \\ \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{7}{2} \\ \frac{37}{6} \\ \frac{175}{24} \\ \frac{781}{120} \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_4 \\ \beta_3 \\ \beta_2 \\ \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} \frac{251}{720} \\ \frac{646}{720} \\ -\frac{264}{720} \\ \frac{106}{720} \\ -\frac{19}{720} \end{bmatrix}$$

(3) Backward differentiation formulas： $\beta_{s-1} = \cdots = \beta_0 = 0$，指定 $\alpha_s = 1$，

Backward differentiation formulas in Definition 10.88

s	p	α_s	α_{s-1}	α_{s-2}	α_{s-3}	α_{s-4}	β_s
1	1	1	-1				1
2	2	1	$-\frac{4}{3}$	$\frac{1}{3}$			$\frac{2}{3}$
3	3	1	$-\frac{18}{11}$	$\frac{9}{11}$	$-\frac{2}{11}$		$\frac{6}{11}$
4	4	1	$-\frac{48}{25}$	$\frac{36}{25}$	$-\frac{16}{25}$	$\frac{3}{25}$	$\frac{12}{25}$

- $s = 1, p = 1, C_0 = C_1 = 0 \Rightarrow \begin{cases} \alpha_0 + \alpha_1 = 0 \\ \alpha_1 - \beta_1 = 0 \end{cases} \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} \beta_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

- $s = 2, p = 2, C_0 = C_1 = C_2 = 0 \Rightarrow \begin{cases} \alpha_0 + \alpha_1 + \alpha_2 = 0 \\ (\alpha_1 + 2\alpha_2) - \beta_2 = 0 \\ (\frac{1}{2}\alpha_1 - 2\alpha_2) - 2\beta_2 = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 1 \\ -\frac{1}{2} & 0 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} \\ \frac{1}{3} \\ \frac{3}{2} \end{bmatrix}.$$

- $s = 3, p = 3, C_0 = C_1 = C_2 = C_3 = 0 \Rightarrow \begin{cases} \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0 \\ (\alpha_1 + 2\alpha_2 + 3\alpha_3) - \beta_3 = 0 \\ (\frac{1}{2}\alpha_1 + 2\alpha_2 + \frac{9}{2}\alpha_3) - 3\beta_3 = 0 \\ (\frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3) - \frac{9}{2}\beta_3 = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} -1 & -1 & -1 & 0 \\ -2 & -1 & 0 & 1 \\ -2 & -\frac{1}{2} & 0 & 3 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ \frac{9}{2} \\ \frac{9}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} -\frac{18}{11} \\ \frac{9}{11} \\ -\frac{2}{11} \\ \frac{6}{11} \end{bmatrix}.$$

- $s = 4, p = 4, C_0 = C_1 = C_2 = C_4 = 0 \Rightarrow \begin{cases} \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \\ (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4) - \beta_4 = 0 \\ (\frac{1}{2}\alpha_1 + 2\alpha_2 + \frac{9}{2}\alpha_3 + 8\alpha_4) - 4\beta_4 = 0 \\ (\frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2 + \frac{9}{2}\alpha_3 + \frac{32}{3}\alpha_4) - 8\beta_4 = 0 \\ (\frac{1}{24} + \frac{2}{3}\alpha_2 + \frac{27}{8} + \frac{32}{3}\alpha_4) - \frac{32}{3}\beta_4 = 0 \end{cases}$

$$\Rightarrow \begin{bmatrix} -1 & -1 & -1 & -1 & 0 \\ -3 & -2 & -1 & 0 & 1 \\ -\frac{9}{2} & -2 & -\frac{1}{2} & 0 & 4 \\ -\frac{9}{2} & -\frac{4}{3} & -\frac{1}{6} & 0 & 8 \\ -\frac{27}{8} & -\frac{2}{3} & -\frac{1}{24} & 0 & \frac{32}{3} \end{bmatrix} \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 8 \\ \frac{32}{3} \\ \frac{32}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} \alpha_3 \\ \alpha_2 \\ \alpha_1 \\ \alpha_0 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} -\frac{48}{25} \\ \frac{36}{25} \\ -\frac{16}{25} \\ \frac{3}{25} \\ \frac{12}{25} \end{bmatrix}.$$

□

Exercise 10.107

For the third-order BDF in Definition 10.88 and Exercise 10.102, derive its characteristic polynomials and apply Theorem 10.105 to verify that the order of accuracy is indeed 3.

证明. 根据 **Exercise 10.102**，三阶 BDF 公式为

$$\boldsymbol{u}^{n+3} - \frac{18}{11}\boldsymbol{u}^{n+2} + \frac{9}{11}\boldsymbol{u}^{n+1} - \frac{2}{11}\boldsymbol{u}^{n} = \frac{6}{11}kf(\boldsymbol{u}^{n+3}, t_{n+3}).$$

特征多项式为 $\rho(\zeta) = \zeta^3 - \frac{18}{11}\zeta^2 + \frac{9}{11}\zeta - \frac{2}{11}, \sigma(\zeta) = \frac{6}{11}\zeta^3$.

$$\begin{aligned}
\frac{\rho(z)}{\sigma(z)} &= \frac{z^3 - \frac{18}{11}z^2 + \frac{9}{11}z - \frac{2}{11}}{\frac{6}{11}z^3} \\
&= \frac{(z-1)^3 + \frac{15}{11}(z-1)^2 + \frac{6}{11}(z-1)}{\frac{6}{11}(z-1+1)^3} \\
&= \frac{11}{6}\left((z-1)^3 + \frac{15}{11}(z-1)^2 + \frac{6}{11}(z-1)\right)(1 - 3(z-1) + 6(z-1)^2 - 10(z-1)^3 + O(z^4)) \\
&= (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \frac{1}{2}(z-1)^4 + O((z-1)^5) \\
&= \log(z) - \frac{1}{4}(z-1)^4 + O((z-1)^5).
\end{aligned}$$

所以，三阶 BDF 的精度为 3。 □

Exercise 10.108

Prove that an s-step LMM has order of accuracy p if and only if, when applied to an ODE $u_t = q(t)$, it gives exact results whenever q is a polynomial of degree $< p$, but not whenever q is a polynomial of degree p. Assume arbitrary continuous initial condition u_0 and exact numerical initial data $v_0, \cdots, v_{s-1}$.

证明.

”$\Rightarrow$” :

因为 LMM 的精度为为 p，所以 $U^N = u(T) + \sum_{n=p+1}^{\infty} C_n k^n u^n(t_N)$.

因为 $\deg(q) < p, u = \int q(t)\mathrm{d}t$, 所以 $\deg(u) \le p$, 所以 $\forall n \ge p+1, u^n \equiv 0$.

所以 $\boldsymbol{U}^N = \boldsymbol{u}(T)$，即 LMM 对次数小于 p 的多项式 p 精确。因为 $C_{p+1} \neq 0$，取 $q(t) = t^P, u_0 = 0$，则 $u(t) = \frac{1}{p+1}t^{p+1}, U^N = u(T) + C_{p+1}k^{p+1} \cdot p! \neq u(T)$.

”$\Leftarrow$” :

反证法。若 LMM 的精度为 $p' < p$，则同必要性的证明可以知道 LMM 的解对 p' 次多项式 $q(t) = t^{p'}$ 不精确，矛盾！所以 LMM 的精度不小于 p；若 LMM 的精度大于 p，则由必要性的证明可知，对于任意 p 次多项式，LMM 的解都精确，矛盾！所以 LMM 的精度为 p。 □

Exercise 10.112

Show that

$$p_M(z) = z^s + \sum_{j=0}^{s-1} \alpha_j z^j$$

if the characteristic polynomial of

$$M = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{s-2} & -\alpha_{s-1} \end{bmatrix} \in \mathbb{C}^{s\times s}.$$

证明.

$$\mathrm{ch}f(M) = \det(zI - M) = \det \begin{bmatrix} z & -1 & & & \\ & z & -1 & & \\ & & \ddots & \ddots & \\ & & & z & -1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_{s-2} & z + \alpha_{s-1} \end{bmatrix}$$

$$= z^{s-1}(z + \alpha_{s-1}) + z^{s-2}\alpha_{s-2} + z^{s-3}\alpha_3 + \cdots + \alpha_0 = p_M(z).$$

注：关于 $zI - M$ 的行列式，可以观察到 $\prod_{i=1}^{s}(zI - M)_{i,p_i}$，其中 $\{p_i\}$ 是 $1, \ldots, s$ 的排列，只有 s 种情况非 0，考虑将其求和即可得到上式。 □

参考文献

[1] 张庆海. “Notes on Numerical Analysis and Numerical Methods for Differential Equations”. In: (2024).