Problems of Chapter 7

张志心 计科 2106

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Exercise 7.14

Suppose a grid function $\boldsymbol{g} : \boldsymbol{X} \to \mathbb{R}$ has $\boldsymbol{X} := \{x_1, x_2, \dots, x_N\}$, $g_1 = O(h)$, $g_N = O(h)$, and $g_j = O(h^2)$ for all $j = 2, \dots, N-1$. Show that

$$\|\boldsymbol{g}\|_{L_\infty} = O(h), \|\boldsymbol{g}\|_1 = O(h^2), \|\boldsymbol{g}\|_2 = O(h^{\frac{3}{2}}).$$

As the main point of this exercise, the differences in the max-norm, 1-norm, and 2-norm of a grid function often reveal the percentage of components with large magnitude.

证明. 设 $C = O(1)$，满足 $|g_1|, |g_n| \le Ch$，$|g_j| \le Ch^2, \forall j = 2, 3, \cdots, N-1$。

$$\|\boldsymbol{g}\|_{L_\infty} = \max_{1 \le i \le N} |g_i| \le Ch = O(h) \Rightarrow \|\boldsymbol{g}\|_{L_\infty} = O(h).$$

$$\|\boldsymbol{g}\|_{L_1} = h\sum_{i=1}^{N} |g_i| \le hC(2h + (N-2)h^2) \le 3Ch^2 = O(h^2) \Rightarrow \|\boldsymbol{g}\|_{L_1} = O(h^2).$$

$$\|\boldsymbol{g}\|_{L_2} = \left(h\sum_{i=1}^{N} |g_i|^2\right)^{\frac{1}{2}} \le (hC^2(2h^2 + (N-2)h^4))^{\frac{1}{2}} \le \sqrt{3}Ch^{\frac{3}{2}} = O(h^{\frac{3}{2}}) \Rightarrow \|\boldsymbol{g}\|_2 = O(h^{\frac{3}{2}}).$$

□

Exercise 7.26

Show that the set of eigenvectors (7.26) of A in (7.13) is orthogonal, i.e.,

$$\langle \boldsymbol{w}_i, \boldsymbol{w}_k \rangle = \begin{cases} 0 & i \ne k; \\ \frac{m+1}{2} & i = k, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the dot product.

证明. 对于 $i \ne k$，由 Lemma 7.25，有

$$\langle \boldsymbol{w}_i, \boldsymbol{w}_k \rangle = \sum_{j=1}^{m} \sin\frac{ji\pi}{m+1} \sin\frac{jk\pi}{m+1} = -\frac{1}{2}\sum_{j=1}^{m}\left[\cos\frac{j(i+k)\pi}{m+1} - \cos\frac{j(i-k)\pi}{m+1}\right]$$

因为，当 $d \in \mathbb{Z}, d \ne 0$ 时，

$$\sum_{j=1}^{m+1} e^{\imath\frac{jd\pi}{m+1}} = \frac{e^{\imath\frac{d\pi}{m+1}}\left(\left(e^{\imath\frac{d\pi}{m+1}}\right)^{m+1} - 1\right)}{e^{\imath\frac{d\pi}{m+1}} - 1} = 0,$$

取实部，得

$$\sum_{j=1}^{m+1} \cos\frac{jd\pi}{m+1} = 0 \Rightarrow \sum_{j=1}^{m} \cos\frac{jd\pi}{m+1} = -\cos\frac{(m+1)d\pi}{m+1} = (-1)^{d+1}.$$

所以

$$\langle \boldsymbol{w}_i, \boldsymbol{w}_k \rangle = -\frac{1}{2}[(-1)^{i+k-1} - (-1)^{i-k-1}] = 0.$$

对于 $i = k$,

$$\langle \boldsymbol{w}_i, \boldsymbol{w}_i \rangle = \sum_{j=1}^{m} \sin^2 \frac{ji\pi}{m+1} = -\frac{1}{2}\sum_{j=1}^{m}\left[\cos\frac{j(2i)\pi}{m+1} - \cos 0\right] = \frac{1}{2}[m - (-1)^{2i+1}] = \frac{m+1}{2}.$$

□

Exercise 7.37

Show that all elements of the first column of $B_E = A_E^{-1}$ are $O(1)$.

证明.

$$A_E B_E = \frac{1}{h^2}\begin{bmatrix} -h & h & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 0 & h^2 \end{bmatrix} B_E = I.$$

设 B_E 的第一列为 $\beta_0, \beta_1, \ldots, \beta_m, \beta_{m+1}$。根据上式得到:

$$\begin{cases} -\beta_0 + \beta_1 = h \\ \beta_{i-1} - 2\beta_i + \beta_{i+1} = 0, 1 \le i \le m \\ \beta_{m+1} = 0 \end{cases}$$

根据 $\beta_{i-1} - 2\beta_i + \beta_{i+1} = 0, 1 \le i \le m$,得到 $\{\beta_i\}_{i=1}^{n}$ 是等差数列。又因为 $\beta_{m+1} = 0$,所以 $\beta_i = (m+1-i)\beta_m, 1 \le i \le m+1$, 所以 $\beta_m = \beta_0 - \beta_1 = -h$。

因此 $\beta_i = -(m+1-i)h \le -(m+1)h = -(1+\frac{1}{m}) = O(1)$。 □

Exercise 7.42

Show that the LTE τ of the FD method in Example 7.41 is

$$\tau_{i,j} = -\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\bigg|_{(x_i, y_i)} + O(h^4).$$

证明. 根据 LTE 公式,

$$\begin{aligned}\tau_{i,j} = &-\frac{u(x_{i-1}, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j-1}) + u(x_i, y_{j+1}) - 4u(x_i, y_j)}{h^2} \\ &+ \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j).\end{aligned} \tag{1}$$

将 u 在 (x_i, y_j) 处关于 x 和 y 泰勒展开到前 6 阶，

$$u(x_i, y_j) = \left(u - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} - \frac{h^5}{120}\frac{\partial^5 u}{\partial x^5} + \frac{h^6}{720}\frac{\partial^6 u}{\partial x^6}\right)\bigg|_{(x_i,y_j)} + o(h^6)$$

$$u(x_{i+1}, y_j) = \left(u + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} + \frac{h^5}{120}\frac{\partial^5 u}{\partial x^5} + \frac{h^6}{720}\frac{\partial^6 u}{\partial x^6}\right)\bigg|_{(x_i,y_j)} + o(h^6)$$

$$u(x_i, y_{j-1}) = \left(u - h\frac{\partial u}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u}{\partial y^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial y^3} + \frac{h^4}{24}\frac{\partial^4 y}{\partial x^4} - \frac{h^5}{120}\frac{\partial^5 u}{\partial y^5} + \frac{h^6}{720}\frac{\partial^6 u}{\partial y^6}\right)\bigg|_{(x_i,y_j)} + o(h^6)$$

$$u(x_i, y_{j+1}) = \left(u + h\frac{\partial u}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u}{\partial y^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial y^3} + \frac{h^4}{24}\frac{\partial^4 y}{\partial x^4} + \frac{h^5}{120}\frac{\partial^5 u}{\partial y^5} + \frac{h^6}{720}\frac{\partial^6 u}{\partial y^6}\right)\bigg|_{(x_i,y_j)} + o(h^6)$$

代入(1)，整理得，

$$\tau_{i,j} = -\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\bigg|_{(x_i,y_i)} - \frac{1}{360}h^4\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right)\bigg|_{(x_i,y_i)} + o(h^4).$$

所以，$\tau_{i,j} = -\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\bigg|_{(x_i,y_i)} + O(h^4)$. □

Exercise 7.62

Show that, in Example 7.61, the LTE at an irregular equation-discretization point is $O(h)$ while the LTE at a regualr equation-discretization point is $O(h^2)$.

证明. 对于正则点，同 Exercise 7.42，有

$$\begin{aligned}\tau_{i,j} =& -\frac{u(x_{i-1}, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j-1}) + u(x_i, y_{j+1}) - 4u(x_i, y_j)}{h^2} \\ &+ \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j) \\ =& -\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\bigg|_{(x_i,y_j)} + O(h^4)\end{aligned}$$

对于非正则点，可能的情况有：(1) x 轴方向两侧有格点不可达；(2) y 轴方向两侧有格点不可达；(3) x,y 轴方向两侧均有格点不可达。值得一提的是，沿一个坐标轴方向，尽管有可能其两侧格点都不可达，但是对于这种情况，只需要加密网格，就可以转化为只有一侧不可达的情况，所以在此我们只证明非正则点沿坐标轴方向有一侧的 stencil 不可达的情况。

对于 x 轴方向，设正方向格点不可达，边界与其 x 方向格线相交于 $(x_i + \theta h, y_i), -1 \le \theta \le 1$。可以计算该方向对 LTE 的贡献如下：

$$\begin{aligned}& -\frac{u(x_i + \theta h, y_j) - \theta u(x_i - h, y_j) - (1+\theta)u(x_i, y_j)}{\frac{1}{2}\theta(1+\theta)h^2} + \frac{\partial^2 u}{\partial x^2}\bigg|_{(x_i,y_j)} \\ =& \left(-\frac{\theta\left(u - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + O(h^4)\right) + \left(u + \theta h\frac{\partial u}{\partial x} + \frac{\theta^2 h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\theta^3 h^3}{6}\frac{\partial^3 u}{\partial x^3} + O(h^4)\right) - (1+\theta)u}{\frac{1}{2}\theta(1+\theta)h^2} + \frac{\partial^2 u}{\partial x^2}\right)\Bigg|_{(x_i,y_j)} \\ =& \frac{1-\theta}{3}h\frac{\partial^3 u}{\partial x^3}\bigg|_{(x_i,y_j)} + O(h^2) = O(h)\end{aligned}$$

同理，y 轴方向有不可达点对 LTE 的贡献也是 $O(h)$。

若两个坐标轴方向都有不可达点，LTE 即为两个方向对 LTE 对贡献的和，也是 $O(h)$。

综上，正则点 LTE 为 $O(h^2)$；非正则点 LTE 为 $O(h)$。 □

Exercise 7.64

Prove Theorem 7.63 by choosing a function ψ to which Lemma 7.58 applies.

> Suppose that, in the notation of Theorem 7.59, the set X_Ω of equation-discretization points can be partitioned as
>
> $$X_\Omega = X_1 \cup X_2, X_1 \cap X_2 = \emptyset,$$
>
> the nonnegative function : $\phi : X \to \mathbb{R}$ satisfies
>
> $$\forall P \in X_1, L_h\phi_P \le -C_1 < 0;$$
>
> $$\forall P \in X_2, L_h\phi_P \le -C_2 < 0,$$
>
> and the LTE of (7.79) satisfy
>
> $$\forall P \in X_1, |T_P| < T_1;$$
>
> $$\forall P \in X_2, |T_P| < T_2.$$
>
> Then the solution error $E_p := U_P - u(P)$ of the FD method (7.79) is bounded by
>
> $$\forall P \in X, |E_P| \le \left(\max_{Q \in X_{\partial\Omega}} \phi(Q)\right) \max\left\{\frac{T_1}{C_1}, \frac{T_2}{C_2}\right\}$$

证明. 令 $T_{max} = \max\left\{\frac{T_1}{C_1}, \frac{T_2}{C_2}\right\}$，定义

$$\psi_P := E_P + T_{max}\phi_P.$$

当 $P \in X_1$ 时，

$$L_h\psi_P \le -T_P - T_{max}C_1 \le -T_P - \frac{T_1}{C_1}C_1 = -T_P - T_1 \le 0,$$

同理，当 $P \in X_2$ 时，

$$L_h\psi_P \le 0.$$

进一步地，因为 $\phi_P \ge 0$，所以 $\max_{P \in X}\psi_P \ge 0$。且 $\forall Q \in X_{\partial\Omega}, E_Q = 0$，根据 Lemma 7.58 得，

$$\begin{aligned} E_P &\le \max_{P \in X}(E_P + T_{max}\phi_P) \\ &\le \max_{Q \in X_{\partial\Omega}} (E_Q + T_{max}\phi_Q) \\ &= T_{max} \max_{Q \in X_{\partial\Omega}} (\phi_Q). \end{aligned}$$

因此 $E_P \le T_m \max_{Q \in X_{\partial\Omega}}$

对 $\psi_P = -E_P + T_m\phi_P$ 作同样处理，则可证明 $-E_P \le T_{max} \max_{Q \in X_{\partial\Omega}} (\phi_Q)$。

所以，

$$\forall P \in X, |E_P| \le \left(\max_{Q \in X_{\partial\Omega}} \phi(Q)\right) \max\left\{\frac{T_1}{C_1}, \frac{T_2}{C_2}\right\}.$$

□

参考文献

[1] 张庆海. “Notes on Numerical Analysis and Numerical Methods for Differential Equations”. In: (2024).