Theoretical Questions of Chapter 2

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日期：2023 年 10 月 9 日

2.9.1-I.

For $f \in C^2[x_0, x_1]$ and $x \in (x_0, x_1)$, linear interpolation of f at x_0 and x_1 yields

$$f(x) - p_1(f;x) = \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1).$$

Consider the case $f(x) = \frac{1}{x},\ x_0 = 1,\ x_1 = 2.$

- Determine $\xi(x)$ explicity.
- Extend the domain of ξ continuously from (x_0, x_1) to $[x_0, x_1]$. Find $\max \xi(x)$, $\min \xi(x)$ and $\max f''(\xi(x))$.

解.

- 线性插值结果为 $p_1(f;x) = \frac{1-\frac{1}{2}}{1-2}(x-1) + 1 = -\frac{1}{2}x + \frac{3}{2}$, 因为 $f(x) = \frac{1}{x}$, $f''(x) = \frac{2}{x^3}$，所以

$$f(x) - p_1(f;x) = \frac{1}{x} + \frac{1}{2}x - \frac{3}{2} = \frac{f''(\xi(x))}{2}(x - x_0)(x - x_1) = \xi^{-3}(x)(x-1)(x-2).$$

$$\frac{(x-1)(x-2)}{2x} = \frac{(x-1)(x-2)}{\xi^3(x)}.$$

  所以 $\xi(x) = (2x)^{\frac{1}{3}}$。

- 由于 $\xi(x)$ 在 $[x_0, x_1] = [1, 2]$ 上单调递增，所以

$$\max \xi(x) = \xi(2) = \sqrt[3]{4}, \qquad \min \xi(x) = \xi(1) = \sqrt[3]{2}$$

  由于 $f''(\xi(x)) = 2\left(\sqrt[3]{2x}\right)^{-3} = \frac{1}{x}$，在 $[1, 2]$ 单调递减，所以

$$\max f''(\xi(x)) = \frac{1}{1} = 1.$$

□

2.9.1-II.

Let $\mathbb{P}_m^+$ be the set of all polynomials of degree $\le m$ that are non-negative on the real line,

$$\mathbb{P}_m^+ = \{p : p \in \mathbb{P}_m,\ \forall x \in \mathbb{R},\ p(x) \ge 0\}.$$

Find $p \in \mathbb{P}_{2n}^+$ such that $p(x_i) = f_i$ for $i = 0, 1, \ldots, n$ where $f_i \ge 0$ and x_i are distinct points on $\mathbb{R}$.

解. 设 $p_n(f;x)$ 为在 $x_0, x_1, \cdots, x_n$ 点的插值多项式，满足

$$\forall i = 0, \cdots, n, p_n(f;x_i) = \sqrt{f_i},$$

那么令 $p(x) = p_n^2(f;x)$，则有 $\forall x \in \mathbb{R}, p(x) \ge 0$, 并且对于 $i = 0, \cdots, n,, p(x_i) = f_i$。具体的，

$$p(x) = \left(\sum_{i=0}^{n} \sqrt{f_i} \prod_{j=0, j\ne i}^{n} \frac{x - x_j}{x_i - x_j}\right)^2.$$

□

2.9.1-III.

Cnosider $f(x) = e^x$.

- Prove by induction that

$$\forall t \in \mathbb{R}, \qquad f[t, t+1, \ldots, t+n] = \frac{(e-1)^n}{n!}e^t.$$

- From Corollary 2.22 we know

$$\exists \xi \in (0, n) \text{ s.t. } f[0, 1, \ldots, n] = \frac{1}{n!}f^{(n)}(\xi).$$

Determine ξ from the above two equations. Is ξ located to the left or to the right of the midpoint $n/2$?

解.

- 当 $n = 0$ 时, $f[t] = f(t) = e^t = \frac{(e-1)^n}{n!}e^t$, 归纳法, 若 $f[t, t+1, \cdots, t+n-1] = \frac{(e-1)^{n-1}}{(n-1)!}e^t, \forall t \in \mathbb{R}$, 那么

$$\begin{aligned} f[t, t+1, \cdots, t+n] &= \frac{f[t+1, t+2, \cdots, t+n] - f[t, t+1, \cdots, t+n-1]}{(t+n)-t} \\ &= \frac{\frac{(e-1)^{n-1}}{(n-1)!}e^{t+1} - \frac{(e-1)^{n-1}}{((n-1)!)}e^t}{n} = \frac{(e-1)^n e^t}{n!} \end{aligned}$$

所以结论成立。

- $e^{\xi} = (e-1)^n$，所以 $\xi = n\ln(e-1)$，因为 $\ln(e-1) = 0.5413\ldots > \frac{1}{2}$，所以 ξ 在中点右侧。

□

2.9.1-IV.

Consider $f(0) = 5,\ f(1) = 3,\ f(3) = 5,\ f(4) = 12$.

- Use the Newton's formula to obtain $p_3(f; x)$;
- The data suggests that f has a minimum in $x \in (1, 3)$. Find an approximate value for the location $x_{\min}$ of the minimum.

解.

- 根据 Newton's formula 计算差商表如下：

0	5			
1	3	-2		
2	5	1	1	
3	12	7	2	$\frac{1}{4}$

所以 $p_3(f; x) = 5 - 2x + x(x-1) + \frac{1}{4}x(x-1)(x-3) = \frac{1}{4}x^3 - \frac{9}{4}x + 5$。

- $p_3'(f; x) = \frac{3}{4}x^2 - \frac{9}{4} = 0 \Rightarrow x = \sqrt{3}$，此时 $f_3''(f; \sqrt{3}) = \frac{2\sqrt{3}}{2} \geq 0$，所以 $x_min = \sqrt{3} \approx 1.732$。

□

2.9.1-V.

Consider $f(x) = x^7$.

- Compute $f[0, 1, 1, 1, 2, 2]$.
- We konw that this devided difference is expressible in terms of the 5th derivative of f evaluated at some $\xi \in (0, 2)$. Determine ξ.

解.

- $f(0) = 0, f(1) = 1, f'(1) = -1, \frac{f''(1)}{2} = 21, f(2) = 128, f'(2) = 448,$。

根据 Newton's formula 计算差商表如下：

0	0					
1	1	1				
1	1	7	6			
1	1	7	21	15		
2	128	127	120	99	42	
2	128	448	321	201	102	30

所以 $f[0,1,1,1,2,2]=30$。

- $\dfrac{f^{(5)}(x)}{5!}=\dfrac{2520x^2}{120}=21x^2$，所以 $21\xi^2=30, \xi=\sqrt{\dfrac{10}{7}}$。

□

2.9.1-VI.

f is a function on $[0,3]$ for which one knows that

$$f(0)=1, \quad f(1)=2, \quad f'(1)=-1, \quad f(3)=f'(3)=0.$$

- Estimate $f(2)$ using Hermite's interpolation.
- Estimate the maximum possible error of the above answer if one konws, in addition, that $f\in C^5[0,3]$ and $|f^{(5)}(x)|\le M$ on $[0,3]$. Express the answer in terms of M.

解. 差商表计算如下：

0	1				
1	2	1			
1	2	-1	-2		
3	0	-1	0	$\frac{2}{3}$	
3	0	0	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{5}{36}$

因此 $f(x)=1+x-2x(x-1)+\dfrac{2}{3}x(x-1)^2-\dfrac{5}{36}x(x-1)^2(x-3)=\dfrac{36+147x-155x^2+49x^3-5x^4}{36}$。

$f(2)=\dfrac{11}{18}$。

根据讲义 Thm 2.37，$f(x)-p_4(f;x)=\dfrac{f^{(5)}(\xi)}{5!}x(x-1)^2(x-3)^2$。

因为 $f\in C^5$ 且 $|f^{(5)}(x)|\le M$，且

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x(x-1)^2(x-3)\right]^2=(x-1)(x-3)(5x^2-12x+3)$$

令导数为 0 得到 $x=1,3,\frac{6\pm\sqrt{21}}{5}$，分别代入得 $x=\frac{6+\sqrt{21}}{5}$ 时取得最大绝对值 $\frac{48(102+7\sqrt{21})}{3125}$。

因此 $|f(x)-p_4(f;x)|\le\dfrac{204+14\sqrt{21}}{15625}M$。

□

2.9.1-VII.

Define foward difference by

$$\Delta f(x) = f(x+h) - f(x), \qquad \Delta^{k+1} f(x) = \Delta\Delta^k f(x) = \Delta^k f(x+h) - \Delta^k f(x).$$

and backward difference by

$$\nabla f(x) = f(x) - f(x-h), \qquad \nabla^{k+1} f(x) = \nabla\nabla^k f(x) = \nabla^k f(x) - \nabla^k f(x-h).$$

Prove

$$\Delta^k f(x) = k! h^k f[x_0, x_1, \ldots, x_k],$$
$$\nabla^k f(x) = k! h^k f[x_0, x_{-1}, \ldots, x_{-k}],$$

where $x_j = x + jh$.

证明.

$$\Delta f(x) = f(x+h) - f(x) = hf[x, x+h]$$
$$\nabla f(x) = f(x) - f(x-h) = hf[x, x-h]$$

设结论对 $k-1$ 成立，即

$$\Delta^{k-1} f(x) = (k-1)! h^{k-1} f[x_0, x_1, \ldots, x_{k-1}]$$
$$\nabla^{k-1} f(x) = (k-1)! h^{k-1} f[x_0, x_{-1}, \ldots, x_{-(k-1)}]$$

则

$$\begin{aligned}
\Delta^k f(x) =& (k-1)! h^{k-1} f[x_1, \cdots, x_k] - (k-1)! h^{k-1} f[x_0, \cdots, x_{k-1}] \\
=& (k-1)! h^{k-1} (hk f[x_0, x_1, \cdots, x_k]) \\
=& k! h^k f[x_0, x_1, \cdots, x_k]. \\
\nabla^k f(x) =& (k-1)! h^{k-1} f[x_{-1}, \cdots, x_{-k}] - (k-1)! h^{k-1} f[x_0, \cdots, x_{-(k-1)}] \\
=& (k-1)! h^{k-1} (hk f[x_0, x_{-1}, \cdots, x_{-k}]) \\
=& k! h^k f[x_0, x_{-1}, \cdots, x_{-k}].
\end{aligned}$$

所以结论成立。 □

2.9.1-VIII.

Assume f is differentiable at x_0. Prove

$$\frac{\partial}{\partial x_0} f[x_0, x_1, \ldots, x_n] = f[x_0, x_0, x_1, \cdots, x_n] \tag{1}$$

What about the partial derivate with respect to one of the other variables?

解. 归纳，对 $n=0$，有 $\frac{\partial}{\partial x_0} f[x_0] = \frac{\partial}{\partial x_0} f(x_0) = f[x_0, x_0]$。

设结论对 $n=k$ 成立，即 $\frac{\partial}{\partial x_0} f[x_0, x_1, \cdots, x_{k-1}] = f[x_0, x_0, x_1, \cdots, x_{k-1}]$，则

$$
\begin{aligned}
&\frac{\partial}{\partial x_0} f[x_0, x_1, \cdots, x_k] \\
=&\frac{\partial}{\partial x_0} \frac{f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots, x_{k-1}]}{x_k - x_0} \\
=&\frac{-f[x_0, x_0, x_1, \cdots, x_{k-1}](x_k - x_0) + (f[x_1, x_2, \cdots, x_k] - f[x_0, x_1, \cdots, x_{k-1}])}{(x_k - x_0)^2} \\
=&-\frac{f[x_0, x_0, x_1, \cdots, x_{k-1}]}{x_k - x_0} + \frac{f[x_0, x_1, \cdots, x_k]}{x_k - x_0} \\
=&f[x_0, x_0, x_1, \ldots, x_k].
\end{aligned}
$$

□

2.9.1-IX. A min-max problem

For $n \in \mathbb{N}^+$, determine

$$\min \max_{x \in [a,b]} |a_0 x^n + a_1 x^{n-1} + \ldots + a_n|.$$

where $a_0 \neq 0$ is fixed and the minimum is taken over all $a_i \in \mathbb{R},\ i = 1, 2, \cdots, n$.

解. 令 $t = \frac{x - \frac{a+b}{2}}{\frac{b-a}{2}} = \frac{2x - (a+b)}{b-a}$，则 $x = \frac{b-a}{2} t + \frac{a+b}{2}$。

$$
\begin{aligned}
p(x) =& a_0 x^n + a_1 x^{n-1} + \cdots + a_n \\
=& \sum_{j=0}^{n} a_j \sum_{k=0}^{j} \binom{n-j}{k} \left(\frac{b-a}{2} t\right)^k \left(\frac{a+b}{2}\right)^{n-j-k}
\end{aligned}
$$

记上式为 $Q(t)$，其 t^n 的系数为 $a_0 \left(\frac{b-a}{2}\right)^n$。

所以根据讲义推论 2.47 得

$$
\begin{aligned}
&\min \max_{x \in [a,b]} |a_0 x^n + a_1 x^{n-1} + \cdots + a_n| \\
=&\min \max_{t \in [-1,1]} Q(t) \\
=&|a_0| \left(\frac{b-a}{2}\right)^n \frac{1}{2^{n-1}} = \frac{|a_0|(b-a)^n}{2^{2n-1}}
\end{aligned}
$$

□

2.9.1-X. Imitate the proof of Chebyshev's Theorem

Express the Chebyshev polynomial of degree $n \in \mathbb{N}$ as a polynomial T_n and change its domain from $[-1, 1]$ to $\mathbb{R}$. For a fixed $a > 1$, define $\mathbb{P}_n^a := \{p \in \mathbb{P}_n : p(a) = 1\}$ and a polynomial $\hat{p}_n(x) \in \mathbb{P}_n^a$,

$$\hat{p}_n(x) := \frac{T_n(x)}{T_n(a)}.$$

Prove

$$\forall p \in \mathbb{P}_n^a, \qquad \|\hat{p}_n\|_\infty \leq \|p\|_\infty$$

where the max-norm of a function $f : \mathbb{R} \to \mathbb{R}$ is defined as $\|f\|_\infty = \max_{x \in [-1,1]} |f(x)|$.

证明. 反证，设结论不成立。因为 $T_n(x)$ 的最值是 ± 1，所以 $\|\hat{p}_n\|_\infty = \left|\frac{1}{T_n(a)}\right|$。

令 $Q(x) = T_n(x) - p(x)T_n(a)$，则 $Q(a) = T_n(a) - p(a)T_n(a) = 0$，即 a 为 Q 的一个零点。

设 $x'_k, k = 0, \cdots, n$ 为 $T_n(x)$ 的 $n+1$ 个极值点。

又因为 $Q(x'_k) = T_n(x'_k) - p(x'_k)T_n(a), |p(x'_k)T_n(a)| < 1, |T_n(x'_k)| = 1$，
所以对所有奇数 k，$Q(x'_k) < 0$；对所有偶数 k，$Q(x'_k) > 0$，
所以 Q 在 $(x'_0, x'_1), (x'_1, x'_2), \ldots, (x'_{k-1}, x'_k)$ 上分别有至少一个零点。Q 有至少 $n+1$ 个零点。
但 Q 的次数至多为 n，所以 $Q(x) = 0, p(x) = \dfrac{T_n(x)}{T_n(a)}$，矛盾！故原结论成立。 □

2.9.1-XI.

Prove Lemma 2.50: The Bernstein base polynomials $b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$ satisfy

$$\forall k = 0, 1, \ldots, n, \forall t \in (0,1), \quad b_{n,k}(t) > 0$$

$$\sum_{k=0}^{n} b_{n,k}(t) = 1$$

$$\sum_{k=0}^{n} k b_{n,k}(t) = nt$$

$$\sum_{k=0}^{n} (k - nt)^2 b_{n,k}(t) = nt(1-t)$$

证明.

1. 对任意 $k = 0, 1, \ldots, n$ 和 $t \in (0,1)$，$t^k > 0, (1-t)^k > 0, \dbinom{n}{k} > 0$，所以 $b_{n,k}(t) = \dbinom{n}{k} t^k (1-t)^{n-k} > 0$。
2. 设 $X_t \sim B(n; t)$，则 $P(X_t = k) = b_{n,k}(t)$，$\sum_{k=0}^{n} P(X_t = k) = 1$，所以 $\sum_{k=0}^{n} b_{n,k}(t) = 1$。
3. $\sum_{k=0}^{n} k b_{n,k}(t) = E(X_t) = nt$
4. $\sum_{k=0}^{n} (k - nt)^2 b_{n,k}(t) = D(X_t) = nt(1-t)$。

□