Problems of Chapter 7

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Exercise 7.14

Suppose a grid function $\boldsymbol{g} \to \boldsymbol{X} \to \mathbb{R}$ has $\boldsymbol{X} := \{x_1, x_2, \cdots, x_N\}$, $g_1 = O(h)$, $g_N = O(h)$, and $g_j = O(h^2)$ for all $j = 2, \cdots, N-1$. Show that

$$\|g\|_\infty = O(h), \|g\|_1 = O(h^2), \|g\|_2 = O(h^{\frac{3}{2}}).$$

As the main point of this exercise, the differences in the max-norm, 1-norm, and 2-norm of a grid function often reveal the percentage of components with large magnitude.

证明. 设 $C = O(1)$，满足 $|g_1|, |g_n| \le Ch$，$|g_j| \le Ch^2, \forall j = 2, 3, \cdots, N-1$。

$$\|g\|_\infty = \max_{j=1}^{N} |g_j| \le Ch = O(h).$$

$$\|g\|_1 = h\sum_{j=1}^{N} |g_j| \le h(2Ch + C(N-2)h^2) \le 3Ch^2 = O(h^2).$$

$$\|g\|_2 = (h\sum_{j=1}^{N} |g_j|^2)^{\frac{1}{2}} \le (h(2C^2h^2 + (N-2)C^2h^4))^{\frac{1}{2}} \le 2Ch^{\frac{3}{2}} = O(h^{\frac{3}{2}}).$$

□

Exercise 7.26

Show that the set of eigenvectors (7.26) of A in (7.13) is orthogonal, i.e.,

$$\langle w_i, w_k \rangle = \begin{cases} 0 & i \ne k; \\ \dfrac{m+1}{2} & i = k \end{cases},$$

where $\langle \cdot, \cdot \rangle$ denotes the dot product.

证明. 对于 $i \ne k$，由 Lemma 7.25，有

$$\langle w_i, w_k \rangle = \sum_{j=1}^{m} \sin\frac{ji\pi}{m+1} \sin\frac{jk\pi}{m+1} = -\frac{1}{2}\sum_{j=1}^{m}\left[\cos\frac{j(i+k)\pi}{m+1} - \cos\frac{j(i-k)\pi}{m+1}\right]$$

因为，当 $d \in \mathbb{Z}, d \ne 0$ 时，

$$\sum_{j=1}^{m+1} e^{\iota\frac{jd\pi}{m+1}} = \frac{e^{\iota\frac{d\pi}{m+1}}\left(\left(e^{\iota\frac{d\pi}{m+1}}\right)^{m+1} - 1\right)}{e^{\iota\frac{d\pi}{m+1}} - 1} = 0,$$

取实部，得

$$\sum_{j=1}^{m+1} \cos\frac{jd\pi}{m+1} = 0 \Rightarrow \sum_{j=1}^{m} \cos\frac{jd\pi}{m+1} = -\cos\frac{(m+1)d\pi}{m+1} = (-1)^{d+1}.$$

所以

$$\langle w_i, w_k\rangle = -\frac{1}{2}[(-1)^{i+k-1} - (-1)^{i-k-1}] = 0.$$

对于 $i = k$，

$$\langle w_i, w_i\rangle = \sum_{j=1}^{m} \sin^2 \frac{ji\pi}{m+1} = -\frac{1}{2}\sum_{j=1}^{m}\left[\cos\frac{j(2i)\pi}{m+1} - \cos 0\right] = \frac{1}{2}[m - (-1)^{2i+1}] = \frac{m+1}{2}.$$

□

Exercise 7.36

Show that all elements of the first column of $B_E = A_E^{-1}$ are $O(1)$.

证明. 根据题意，$A_E B_E = I$。

设 B_E 的第一列为 $\beta_0, \beta_1, \cdots, \beta_m, \beta_{m+1}$。

比较两端的第一列得线性方程组

$$\begin{cases} -\beta_0 + \beta_1 = h \\ \beta_0 - 2\beta_1 + \beta_2 = 0 \\ \beta_1 - 2\beta_2 + \beta_3 = 0 \\ \cdots \\ \beta_{m-1} - 2\beta_m + \beta_{m+1} = 0 \\ \beta_{m+1} = 0 \end{cases}$$

将 $\beta_k = (m-k+1)\beta_m$ 代入第一个方程得到 $\beta_m = -h$。

因此 $\beta_k = -(m-k+1)h \leq -(m+1)h = -(1+\frac{1}{m}) = O(1)$。 □

Exercise 7.41

Show that the LTE τ of the FD method in Example 7.40 is $\tau_{i,j} = -\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\Big|_{(x_i, y_i)} + O(h^4)$.

证明. 根据 LTE 公式，

$$\begin{aligned}\tau_{i,j} = &-\frac{u(x_{i-1}, y_j) - 2u(x_i, y_j) + u(x_{i+1}, y_j)}{h^2} - \frac{u(x_i, y_{j-1}) - 2u(x_i, y_j) + u(x_i, y_{j+1})}{h^2} \\ &+ \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j).\end{aligned}$$

将 u 在 (x_i, y_j) 处关于 x 和 y 泰勒展开到前 6 阶，

$$\begin{aligned}
u(x_i, y_j) &= \left(u - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} - \frac{h^5}{120}\frac{\partial^5 u}{\partial x^5} + \frac{h^6}{720}\frac{\partial^6 u}{\partial x^6}\right)\Bigg|_{(x_i, y_j)} + o(h^6) \\
u(x_{i+1}, y_j) &= \left(u + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} + \frac{h^5}{120}\frac{\partial^5 u}{\partial x^5} + \frac{h^6}{720}\frac{\partial^6 u}{\partial x^6}\right)\Bigg|_{(x_i, y_j)} + o(h^6) \\
u(x_i, y_{j-1}) &= \left(u - h\frac{\partial u}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u}{\partial y^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial y^3} + \frac{h^4}{24}\frac{\partial^4 y}{\partial x^4} - \frac{h^5}{120}\frac{\partial^5 u}{\partial y^5} + \frac{h^6}{720}\frac{\partial^6 u}{\partial y^6}\right)\Bigg|_{(x_i, y_j)} + o(h^6) \\
u(x_i, y_{j+1}) &= \left(u + h\frac{\partial u}{\partial y} + \frac{h^2}{2}\frac{\partial^2 u}{\partial y^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial y^3} + \frac{h^4}{24}\frac{\partial^4 y}{\partial x^4} + \frac{h^5}{120}\frac{\partial^5 u}{\partial y^5} + \frac{h^6}{720}\frac{\partial^6 u}{\partial y^6}\right)\Bigg|_{(x_i, y_j)} + o(h^6)
\end{aligned}$$

代入，整理得，

$$\tau_{i,j} = -\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right) - \frac{1}{360}h^4\left(\frac{\partial^6 u}{\partial x^6} + \frac{\partial^6 u}{\partial y^6}\right) + o(h^4).$$

所以，$\tau_{i,j} = -\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\Big|_{(x_i,y_i)} + O(h^4).$ □

Exercise 7.61

Show that, in Example 7.60, the LTE at an irregular equation-discretization point is $O(h)$ while the LTE at a regualr equation-discretization point is $O(h^2)$.

证明. 根据 LTE 公式，在正则点处，若它的 Stencil 都是正则点，则有

$$\tau_{i,j} = -\frac{u(x_{i-1},y_j) - 2u(x_i,y_j) + u(x_{i+1},y_j) - u(x_i,y_{j-1}) - 2u(x_i,y_j) + u(x_i,y_{j+1})}{h^2} - \frac{\partial^2 u}{\partial x^2}(x_i,y_j) + \frac{\partial^2 u}{\partial y^2}(x_i,y_j).$$

这和规则区域的误差相同，都为 $-\frac{1}{12}h^2\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)\Big|_{(x_i,y_j)} + O(h^4)$。

若 x 轴方向有一个非正则点，不妨设非正则点在正方向。设其坐标为 $(x_i + \theta h, y_i)$。

则 x 方向对 LTE 的贡献为

$$\begin{aligned}
&-\frac{\theta u(x_i - h, y_j) - (1+\theta)u(x_i,y_j) + u(x_i+\theta h, y_j)}{\frac{1}{2}\theta(1+\theta)h^2} + \frac{\partial^2 u}{\partial x^2}(x_i,y_j)\\
=&\left(-\frac{\theta(u - hu_x + \frac{h^2}{2}u_{xx} + \frac{h^3}{6}u_{xxx} + O(h^4)) - (1+\theta)u + (u + \theta h u_x + \frac{\theta^2 h^2}{2}u_{xx} + \frac{\theta^3 h^3}{6}u_{xxx} + O(h^4))}{\frac{1}{2}\theta(1+\theta)h^2} + u_{xx}\right)\Bigg|_{(x_i,y_j)}\\
=&\frac{1-\theta}{3}hu_{xxx}(x_i,y_j) + O(h^2)
\end{aligned}$$

同理可以证明，当 y 轴方向有非正则点，其 LTE 也为 $O(h)$。特别地，如果 P 点附近边界即 x,y 方向都有非正则点，则 LTE 的表达式为

$$\tau_P = \left(\frac{1-\theta}{3}hu_{xxx} + \frac{1-\alpha}{3}hu_{yyy}\right)\Big|_P.$$

综上，若 Stencil 都为正则点，则 LTE 为 $O(h^2)$；否则，LTE 为 $O(h)$。 □

Exercise 7.63

Prove Theorem 7.62 by choosing a function ψ to which Lemma 7.57 applies.

Suppose that, in the notation of Theorem 7.59, the set $\boldsymbol{X}_\Omega$ of equation-discretization points can be partitioned as

$$\boldsymbol{X}_\Omega = \boldsymbol{X}_1 \cup \boldsymbol{X}_2, \boldsymbol{X}_1 \cap \boldsymbol{X}_2 = \emptyset,$$

the nonnegative function : $\phi : \boldsymbol{X} \to \mathbb{R}$ satisfies

$$\forall P \in \boldsymbol{X}_1, L_h\phi_P \leq -C_1 < 0;$$

$$\forall P \in \boldsymbol{X}_2, L_h\phi_P \leq -C_2 < 0,$$

and the LTE of (7.79) satisfy

$$\forall P \in \boldsymbol{X}_1, |T_P| < T_1;$$

$$\forall P \in \boldsymbol{X}_2, |T_P| < T_2.$$

Then the solution error $E_p := U_P - u(P)$ of the FD method (7.79) is bounded by

$$\forall P \in \boldsymbol{X}, |E_P| \leq \left(\max_{Q \in \boldsymbol{X}_{\partial\Omega} \phi(Q)}\right) \max\left\{\frac{T_1}{C_1}, \frac{T_2}{C_2}\right\}$$

证明. 定义

$$\psi : \boldsymbol{X} \to \mathbb{R}, \psi_P = E_P + T_m\phi_P.$$

其中 $T_m = \max\{\frac{T_1}{C_1}, \frac{T_2}{C_2}\}$。则当 $P \in \boldsymbol{X}_1$ 时，

$$L_h\psi_P = L_h(E_P + T_m\phi_P) \leq -T_P - \frac{T_1}{C_1}C_1 \leq 0,$$

同理 $L_h\psi_P \leq 0, P \in \boldsymbol{X}_2$。

因此 $L_h\psi_P \leq 0, P \in \boldsymbol{X}$。

又因为 $\max_{P\in\boldsymbol{X}} \phi_P \geq 0$，所以 $\max_{P\in\boldsymbol{X}} \phi_P \geq 0$。再由 $E_Q\big|_{\boldsymbol{X}_{\partial\Omega}} = 0$，结合 Lemma 7.57 得，

$$E_P \leq \max_{P\in\boldsymbol{X}}(E_P + T_m\phi_P) \leq \max_{Q\in\boldsymbol{X}_{\partial\Omega}} E_Q + T_m\phi_Q = T_m \max_{Q\in\boldsymbol{X}_{\partial\Omega}} (\phi_Q).$$

因此 $E_P \leq T_m \max_{Q\in\boldsymbol{X}_{\partial\Omega}}$

对 $\psi_P = -E_P + T_m\phi_P$ 作同样处理，则可证明 $-E_P \leq T_m \max_{Q\in\boldsymbol{X}_{\partial\Omega}}$。 □

参考文献

[1] 张庆海. “Notes on Numerical Analysis and Numerical Methods for Differential Equations”. In: (2023).