Problems of Chapter 10.3.4-10.3.6

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Exercise 10.118

Prove Theorem 10.117 by induction.

Theorem 10.117 Let θ_n be the solution to the homogeneous linear difference equation

$$\theta_{n+s} + \sum_{i=0}^{s-1} \alpha_i \theta_{n+i} = 0.$$

with constant coefficients α_i's and the initial values

$$\begin{bmatrix} \theta_0 & \theta_{-1} & \ddots & \theta_{-s+2} & \theta_{-s+1} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & \ddots & 0 & 0 \end{bmatrix}^T.$$

Then the inhomogeneous equation

$$y_{n+s} + \sum_{i=0}^{s-1} \alpha_i y_{n+i} = \psi_{n+s}$$

with initial values $y_0, y_1, \cdots, y_{s-1}$ is uniquely solved by

$$y_n = \sum_{i=0}^{s-1} \theta_{n-i} \tilde{y}_i + \sum_{i=s}^{n} \theta_{n-i} \psi_i$$

where

$$\begin{bmatrix} \tilde{y}_{s-1} \\ \tilde{y}_{s-2} \\ \tilde{y}_{s-3} \\ \ddots \\ \tilde{y}_1 \\ \tilde{y}_0 \end{bmatrix} = \begin{bmatrix} 1 & \theta_1 & \theta_2 & \cdots & \theta_{s-2} & \theta_{s-1} \\ 0 & 1 & \theta_1 & \cdots & \theta_{s-3} & \theta_{s-2} \\ 0 & 1 & 1 & \cdots & \theta_{s-3} & \theta_{s-2} \\ \ddots & \ddots & \ddots & & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 1 & \theta_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} y_{s-1} \\ y_{s-2} \\ y_{s-3} \\ \ddots \\ y_1 \\ y_0 \end{bmatrix}$$

证明. 当 $n \le s$ 时，结论显然成立。下面设结论对 $n = s+1, s+2, \cdots, n+s-1$ 均成立，则

$$\begin{aligned} y_{n+s} &= -\sum_{j=0}^{s-1} \alpha_i y_{n+i} + \psi_{n+s} \\ &= -\sum_{i=0}^{s-1} \alpha_i \left(\sum_{j=0}^{s-1} \theta_{n+i-j} \tilde{y}_j + \sum_{j=s}^{n+i} \theta_{n+i-j} \psi_j \right) + \psi_{n+s} \\ &= \sum_{j=0}^{s-1} \theta_{n-j} \tilde{y}_j + \sum_{j=s}^{n+s-1} \theta_{n-j} \psi_j + \psi_{n+s} \\ &= \sum_{j=0}^{s-1} \theta_{n-j} \tilde{y}_j + \sum_{j=s}^{n+s} \theta_{n-j} \psi_j . \end{aligned}$$

□

Exercise 10.123

Prove Lemma 10.122 by the approach similar with that for Lemma 10.120, i.e., by considering the particular IVP Problems

$$u'(t) = f(t) = 0, u(0) = 1;$$

$$u'(t) = f(t) = 1, u(0) = 0.$$

Lemma 10.122 A convergent LMM is consistent.

证明. 对 IVP $u'(t) = 0, u(0) = 1$，由收敛性，

$$\forall T > 0, \lim_{k\to 0, Nk=T} u^N = \lim_{k\to 0, Nk=T} \frac{1}{\alpha_s} \sum_{j=0}^{s-1} -\alpha_j u^{N-s+j} = 1.$$

所以 $\frac{1}{\alpha_s} \sum_{j=0}^{s-1} -\alpha_j = 1, \sum_{j=0}^{s} \alpha_j = 1$. 对 IVP $u'(t) = 1, u(0) = 0$，由收敛性，

$$\forall T > 0, \lim_{k\to 0, Nk=T} u^N = \frac{1}{\alpha_s} \left(-\sum_{j=0}^{s-1} \alpha_j u^{N-s+j} + k \sum_{j=0}^{s} \beta_j \right) = T.$$

所以 $\frac{1}{\alpha_s} \left(-\sum_{j=0}^{s-1} \alpha_j k(N-s+j) + k \sum_{j=0}^{s} \beta_j \right) = kN.$

所以 $k \sum_{j=0}^{s} \alpha_j (N-s+j) = k \sum_{j=0}^{s} \beta_j \Rightarrow \sum_{j=0}^{s} \alpha_j (N-s) + \sum_{j=0}^{s} j\alpha_j = \sum_{j=0}^{s} \beta_j$，

因为 $\sum_{j=0}^{s} \alpha_j = 0$，所以 $\sum_{j=0}^{s} j\alpha_j = \sum_{j=0}^{s} \beta_j$。 □

Exercise 10.138

Write a program to reproduce the RAS plots in Figures 10.4 and 10.5.

解.

使用 python 绘制，代码如下：

```
import matplotlib.pyplot as plt
import numpy as np
import math as m


def PLOT(f, leg, l=None, r=None, u = None, d = None, st = '-'):
  t = np.linspace(0, 2*m.pi, 5000)
  z = np.exp(1j*t)
  c = f(z)
  if(l != None):
      plt.xlim(left = l, right = r)
  if(u != None):
      plt.ylim(d, u)
```

```
  plt.plot(c.real, c.imag, linewidth = 0.7, color = 'black', label= leg, linestyle = st)
  plt.legend()

def bashforth1(z):
  return z-1
def bashforth2(z):
    return z*(z-1)/(3/2*z-1/2)
def bashforth3(z):
    return z*z*(z-1)/(23/12*z*z-16/12*z+5/12)
def bashforth4(z):
    return z*z*z*(z-1)/((55*z**3-59*z**2+37*z-9)/24)

plt.figure(figsize=(6, 5))
PLOT(bashforth1, "p=1", -2.1, 0.1, -1.1, 1.1, st = ':')
PLOT(bashforth2, "p=2", st= "--")
PLOT(bashforth3, "p=3")
plt.show()
plt.figure(figsize=(6, 5))
PLOT(bashforth4, "p=4", -1.1, 1.1, -1.1, 1.1)
plt.show()

def moulton3(z):
    return z*(z-1)/(5*z**2+8*z-1)*12
def moulton4(z):
    return z*z*(z-1)/(9*z**3+19*z**2-5*z+1)*24
def moulton5(z):
    return z*z*z*(z-1)/(251*z**4+646*z**3-264*z**2+106*z-19)*720

plt.figure(figsize=(6, 5))
PLOT(moulton3, "p=3", -7, 1, -3.5, 3.5,st  = ":")
PLOT(moulton4, "p=4", st = "--")
PLOT(moulton5, "p=5")
plt.show()

def backward1(z):
    return (z-1)/z
def backward2(z):
    return (3*z**2-4*z+1)/(2*z**2)
```

```
def backward3(z):
    return (11*z**3-18*z**2+9*z-2)/(6*z**3)
def backward4(z):
    return (25*z**4-48*z**3+36*z**2-16*z+3)/(12*z**4)

plt.figure(figsize=(6, 5))
PLOT(backward1, "p=1", -4, 13, -7, 7, st = ":")
PLOT(backward2, "p=2", st = "--")
PLOT(backward3, "p=3", st= "-.")
PLOT(backward4, "p=4")
plt.show()
```

□

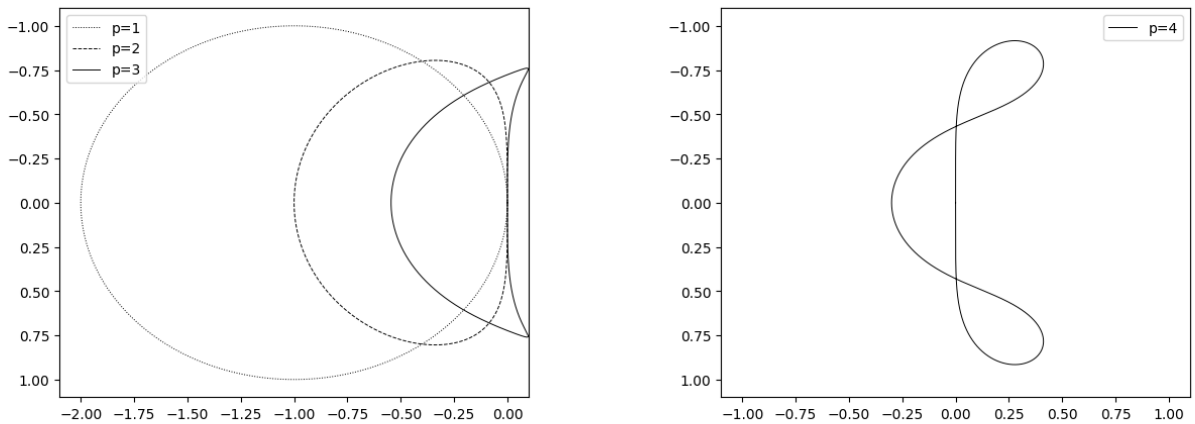

图 **1:** The bounded RAS of Adams-Bashforth formu- las for $p = 1, 2, 3$ (left) and $p = 4$ (right).

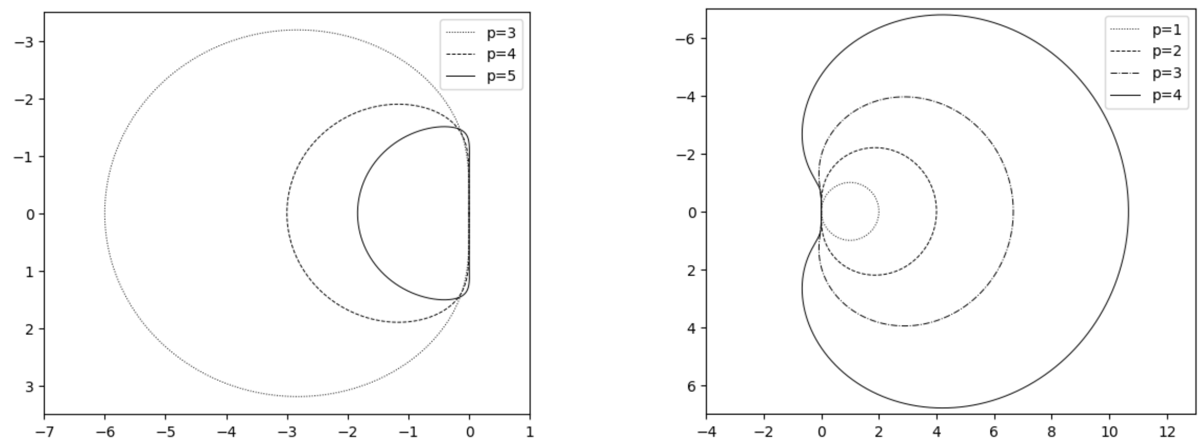

图 **2:** The bounded RASs of Adams-Moulton for- mulas with $p = 3, 4, 5$ (left) and the unbounded RASs of backward differentiation formulas with $p = 1, 2, 3, 4$ (right)

参考文献

[1] 张庆海. “Notes on Numerical Analysis and Numerical Methods for Differential Equations”. In: (2024).