Theoretical Questions of Chapter 3

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3.6.1-I.

Consider $s \in \mathbb{S}_3^2$ on $[0,2]$:

$$s(x) = \begin{cases} p(x) & \text{if } x \in [0,1], \\ (2-x)^3 & \text{if } x \in [1,2]. \end{cases}$$

Determine $p \in \mathbb{P}_3$ such that $s(0) = 0$. Is $s(x)$ a natural cubic spline?

解. 设 $p(x) = ax^3 + bx^2 + cx$，根据 $\mathbb{S}_3^2$ 的条件，$p(x)$ 满足 $p(1) = 1, p'(1) = -3, p''(1) = 6$。

$$\begin{cases} a + b + c = 1 \\ 3a + 2b + c = -3 \\ 6a + 2b = 6 \end{cases}$$

解得，$(a,b,c) = (7,-18,12)$，所以 $p(x) = 7x^3 - 18x^2 + 12$。由于 $s''(0) = p''(0) = -36 \neq 0$，所以 $s(x)$ 不是自然三次样条函数。 □

3.6.1-II.

Given $f_i = f(x_i)$ of some scalar function at points $a = x_1 < x_2 < \cdots < x_n = b$, we consider interpolating f on $[a,b]$ with a quadratic spline $s \in \mathbb{S}_2^1$.

(a) Why is an additional condition needed to determine s uniquely?

(b) Define $m_i = s'(x_i)$ and $p_i = s\big|_{[x_i, x_{i+1}]}$. Determine p_i in terms of f_i, f_{i+1} and m_i for $i = 1, 2, ..., n-1$.

(c) Suppose $m_1 = f'(a)$ is given. Show how $m_2, m_3, ..., m_{n-1}$ can be computed.

解.

(a) 一共需要求得 $n-1$ 个区间上的二次函数，共 $3(n-1)$ 个未知系数。定义 $p_i = s\big|_{[x_i, x_{i+1}]}$。
已知 $p_i(x_i) = f_i, p_i(x_{i+1}) = f_{i+1}, i = 1, \cdots, n-1$ 以及 $p_i'(x_{i+1}) = p_{i+1}'(x_{i+1}), i = 1, \cdots, n-2$。
共有 $3n-4$ 个条件，因此要唯一确定 s 还需要一个额外的条件。

(b) 根据条件 $p_i(x_i) = f_i, p_i(x_{i+1}) = f_{i+1}, p_i'(x_i) = m_i$。计算差商表如下：

x_i	f_i		
x_i	f_i	m_i	
x_{i+1}	f_{i+1}	$\frac{f_{i+1}-f_i}{x_{i+1}-x_i}$	$\frac{f_{i+1}-f_i-m_i(x_{i+1}-x_i)}{(x_{i+1}-x_i)^2}$

所以 $p_i(x) = f_i + m_i(x-x_i) + \frac{f_{i+1}-f_i-m_i(x_{i+1}-x_i)}{(x_{i+1}-x_i)^2}(x-x_i)^2$。

(c) 由 (2)，因为 $m_{i+1} = p_i'(x_{i+1})$，则

$$m_{i+1} = p_i'(x_{i+1}) = m_i + \frac{2[f_{i+1} - f_i - m_i(x_{i+1} - x_i)]}{(x_{i+1} - x_i)} = 2\frac{f_{i+1} - f_i}{x_{i+1} - x_i} - m_i$$

所以可以通过 $m_1, f_1, \cdots, f_n$ 递推地求出 $m_2, \cdots, m_n$。

□

3.6.1-III.

Let $s_1(x) = 1 + c(x+1)^3$ where $x \in [-1, 0]$ and $c \in \mathbb{R}$. Determine $s_2(x)$ on $[0, 1]$ such that

$$s(x) = \begin{cases} s_1(x) & \text{if } x \in [-1, 0] \\ s_2(x) & \text{if } x \in [0, 1] \end{cases}$$

is a natural cubic spline on $[-1, 1]$ with knots $-1, 0, 1$. How must c be chosen if one wants $s(1) = -1$?

解. 由 $s_1(0) = 1 + c, s_1'(0) = 3c, s_1''(0) = 6c$ 得 $s_2(x) = 1 + c + 3cx + 6cx^2 + kx^3$，

由于是自然三次样条，所以 $s_2''(1) = 6k + 12c = 0$，

所以 $s_2(x) = 1 + c + 3cx + 6cx^2 - 2cx^3$，

代入 $s(1) = 1 + 8c = -1$，得 $c = -\frac{1}{4}$。 □

3.6.1-IV.

Consider $f(x) = \cos\left(\frac{\pi}{2}x\right)$ with $x \in [-1, 1]$.

(a) Determine the natural cubic spline interpolant to f on knots $-1, 0, 1$.

(b) As discussed in the class, natural cubic splines have the minimal total bending energy. Verify this by taking $g(x)$ be (i) the quadratic polynomial that interpolates f at $-1, 0, 1$, and (ii) $f(x)$.

解. (a) 根据条件 $s(-1) = 0, s(0) = 1, s(1) = 0, s''(-1) = s''(1) = 0$ 计算差商表如下：

-1	0		
0	1	1	
1	0	-1	-1

由 $\frac{1}{2}s''(-1) + 2s''(0) + \frac{1}{2}s''(1) = 6f[-1, 0, 1] = -6$，所以 $s''(0) = -3$。所以

$$s'(-1) = f[-1, 0] - \frac{1}{6}(s''(0) + 2s''(-1)) = \frac{3}{2}$$

$$s'(0) = f[0, 1] - \frac{1}{6}(s''(1) + 2s''(0)) = 0$$

$$s'''(-1) = \frac{s''(0) - s''(-1)}{0 - (-1)} = -3$$

$$s'''(0) = \frac{s''(1) - s''(0)}{1 - 0} = 3$$

计算得到

$$s(x) = \begin{cases} -\frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & x \in [-1, 0] \\ \frac{1}{2}x^3 - \frac{3}{2}x^2 + 1 & x \in [0, 1] \end{cases}$$

(b) $s''(x) = \begin{cases} -3x - 3 & x \in [-1, 0] \\ 3x - 3 & x \in [0, 1] \end{cases}$，则 $\int_{-1}^{1} [s''(x)]^2 \, dx = \int_{-1}^{0} (-3x-3)^2 dx + \int_0^1 (3x-3)^2 dx = 6$。

(i) $g(x) = (x+1) - x(x+1) = -x^2 + 1, g''(x) = -2$,

$$\int_{-1}^{1} \left[g''(x)\right]^2 dx = 8 > 6$$

(ii) $f''(x) = \frac{-\pi^2}{4} \cos\left(\frac{\pi}{2}x\right)$,

$$\int_{-1}^{1} \left[f''(x)\right]^2 dx = \frac{\pi^4}{16} \int_{-1}^{1} \cos^2\left(\frac{\pi}{2}x\right) dx = \frac{\pi^4}{16} \approx 6.088 > 6$$

.

□

3.6.1-V.

The quadratic B-spline $B_i^2(x)$.

(a) Derive the same explicit expression of $B_i^2(x)$ as that in the notes from the recursive definition of B-splines and the hat function.

(b) Verify that $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x)$ is continuous at t_i and t_{i+1}.

(c) Show that only one $x^* \in (t_{i-1}, t_{i+1})$ satisfies $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x^*) = 0$. Express x^* in terms of the knots within the interval of support.

(d) Consequently, show $B_i^2(x) \in [0, 1)$.

(e) Plot $B_i^2(x)$ for $t_i = i$.

解.

(a)

$$B_i^1(x) = \begin{cases} \frac{x-t_{i-1}}{t_i-t_{i-1}} & x \in (t_{i-1}, t_i], \\ \frac{t_{i+1}-x}{t_{i+1}-t_i} & x \in (t_i, t_{i+1}], \\ 0 & \text{others.} \end{cases} \qquad B_{i+1}^1(x) = \begin{cases} \frac{x-t_i}{t_{i+1}-t_i} & x \in (t_i, t_{i+1}], \\ \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} & x \in (t_{i+1}, t_{i+2}], \\ 0 & \text{others.} \end{cases}$$

由于 $B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}}B_i^1(x) + \frac{t_{i+2}-x}{t_{i+2}-t_i}B_{i+1}^1(x)$，

当 $x \in (t_{i-1}, t_i]$ 时，

$$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \cdot \frac{x-t_{i-1}}{t_i-t_{i-1}} = \frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})};$$

当 $x \in (t_i, t_{i+1}]$ 时，

$$B_i^2(x) = \frac{x-t_{i-1}}{t_{i+1}-t_{i-1}} \cdot \frac{t_{i+1}-x}{t_{i+1}-t_i} + \frac{t_{i+2}-x}{t_{i+2}-t_i} \cdot \frac{x-t_i}{t_{i+1}-t_i} = \frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)};$$

当 $x \in (t_{i+1}, t_{i+2}]$ 时，

$$B_i^2(x) = \frac{t_{i+2}-x}{t_{i+2}-t_i} \cdot \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}} = \frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}.$$

对于其他情况，$B_i^2(x) = 0$。

(b)

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) = \begin{cases} \frac{2(x-t_{i-1})}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} & x \in (t_{i-1}, t_i] \\ \frac{-2x+(t_{i+1}+t_{i-1})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{-2x+(t_i+t_{i+2})}{(t_{i+2}-t_i)(t_{i+1}-t_i)} & x \in (t_i, t_{i+1}] \\ \frac{-2(t_{i+2}-x)}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} & x \in (t_{i+1}, t_{i+2}] \\ 0 & \text{others.} \end{cases}$$

计算得到

$$\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_i) = \lim_{x\to t_i^-}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) = \frac{2}{t_{i+1}-t_{i-1}}$$

$$\begin{aligned}\lim_{x\to t_i^+}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) &= \frac{-2t_i+(t_{i+1}+t_{i-1})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{-2t_i+(t_i+t_{i+2})}{(t_{i+2}-t_i)(t_{i+1}-t_i)} \\ &= \frac{-2t_i(t_{i+2}-t_i+t_{i+1}-t_{i-1}) + (t_{i+1}+t_{i-1})(t_{i+2}-t_i) + (t_i+t_{i+2})(t_{i+1}-t_{i-1})}{(t_{i+1}-t_{i-1})(t_{i+1}t_{i+2}+t_i^2-t_i(t_{i+1}+t_{i+2}))} \\ &= \frac{-2t_it_{i+2}+2t_i^2-2t_it_{i+1}+2t_it_{i-1}+2t_{i+1}t_{i+2}-2t_it_{i-1}}{(t_{i+1}-t_{i-1})(t_{i+1}t_{i+1}+t_i^2-t_it_{i+1}-t_it_{i+2})} \\ &= \frac{2}{t_{i+1}-t_{i-1}} = \lim_{x\to t_i}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) = \frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_i)\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_{i+1}) = \lim_{x\to t_{i+1}^-}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) &= \frac{-2}{t_{i+2}-t_i}\\
\lim_{x\to t_{i+1}^+}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) &= \frac{-2(t_{i+2}-t_{i+1})}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})}\\
&= \frac{-2}{t_{i+2}-t_i} = \lim_{x\to t_{i+1}}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_{i+1}) = \frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_{i+1})
\end{aligned}
$$

所以 $\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x)$ 在 t_i 和 t_{i+1} 处连续。

(c) 因为 $\lim_{x\to t_{i-1}^+}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) = \frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_{i+2}) = 0, \frac{\mathrm{d}}{\mathrm{d}x}B_i^2(t_i) = \frac{2}{t_{i+1}-t_{i-1}} > 0, \lim_{x\to t_{i+1}}\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) = \frac{-2}{t_{i+2}-t_i} < 0$，$B_i^2(x)$ 是线性函数，所以 (t_{i_1}, t_i) 和 (t_{i+1}, t_{i+2}) 中不存在零点。

当 $x \in [t_i, t_{i+1}]$ 时，

$$
\frac{-2x^* + (t_{i+1}+t_{i-1})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{-2x^* + (t_{i+2}+t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)} = 0 \Rightarrow x^* = \frac{t_{i+1}t_{i+2}-t_{i-1}t_i}{t_{i+2}+t_{i+1}-t_i-t_{i-1}}
$$

(d) 根据 (c) 可以得到，

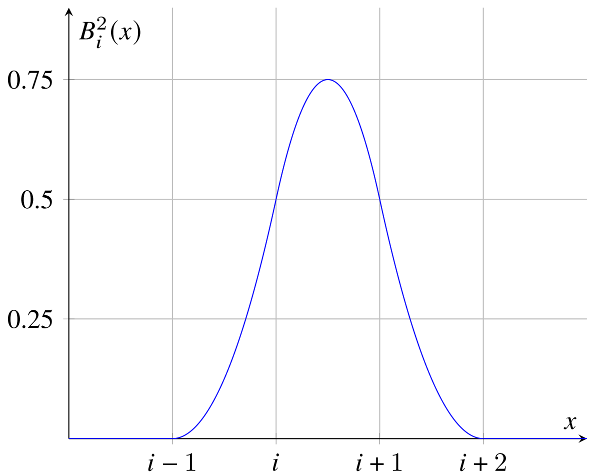
$$
\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) > 0, & x \in (t_{i-1}, x^*)\\
\frac{\mathrm{d}}{\mathrm{d}x}B_i^2(x) < 0, & x \in (x^*, t_{i+2})
\end{cases}
$$

$$
\begin{aligned}
B_i^2(x^*) &= \frac{-2x^* + (t_{i+1}+t_{i-1})}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{-2x^* + (t_i+t_{i+2})}{(t_{i+2}-t_i)(t_{i+1}-t_i)}\\
&= \frac{(t_{i+1}-t_{i-1})(t_{i+2}-t_{i-1})(t_{i+1}-t_{i-1})(t_{i+1}-t_i)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)(t_{i+2}+t_{i+1}-t_i-t_{i-1})^2} + \frac{(t_{i+2}-t_{i-1})(t_{i+2}-t_i)(t_{i+1}-t_i)(t_{i+2}-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)(t_{i+2}+t_{i+1}-t_i-t_{i-1})^2}\\
&= \frac{(t_{i+2}-t_{i-1})(t_{i+2}+t_{i+1}-t_i-t_{i-1})}{(t_{i+2}+t_{i+1}-t_i-t_{i-1})^2}\\
&= \frac{t_{i+2}-t_{i-1}}{t_{i+2}+t_{i+1}-t_i-t_{i-1}} < 1
\end{aligned}
$$

所以 $B_i^2(x) \in [0,1)$。

(e)

$$
B_i^2(x) = \begin{cases}
\frac{(x-i+1)^2}{2} & x \in [i-1, i]\\
-x^2 + (2i+1)x - i^2 - i + \frac{1}{2} & x \in [i, i+1]\\
\frac{(i+2-x)^2}{2} & x \in [i+1, i+2]
\end{cases}
$$



□

3.6.1-VI.

Verify Theorem 3.32 algebraically for the case of $n = 2$, i.e.

$$(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = B_i^2(x).$$

解. 当 $x \in (t_{i-1}, t_i]$ 时,

$$\begin{aligned}
&[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 \\
=&\frac{\frac{\frac{(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+1}-2x)}{t_{i+2}-t_{i+1}} - \frac{(t_{i+1}-t_i)(t_{i+1}-t_i-2x)}{t_{i+1}-t_i}}{t_{i+2}-t_i} - \frac{\frac{(t_{i+1}-t_i)(t_{i+1}-t_i-2x)}{t_{i+1}-t_i} - \frac{(t_i-x)^2}{t_i-t_{i-1}}}{t_{i+2}-t_i}}{t_{i+2}-t_{i-1}} \\
=&\frac{1-\frac{t_{i+1}+t_i-2x-\frac{(t_i-x)^2}{t_i-t_{i-1}}}{t_{i+1}-t_{i-1}}}{t_{i+2}-t_{i-1}} = \frac{(2x-t_{i-1}-t_i)(t_i-t_{i-1})-(t_i-x)^2}{(t_{i+2}-t_{i-1})(t_{i+1}-t_{i-1})(t_i-t_{i-1})} \\
=&\frac{(x-t_{i-1})^2}{(t_{i+1}-t_{i-1})(t_i-t_{i-1})} \cdot \frac{1}{t_{i+2}-t_{i-1}} = \frac{B_i^2(x)}{t_{i+2}-t_{i-1}}
\end{aligned}$$

当 $x \in (t_i, t_{i+1}]$ 时,

$$\begin{aligned}
&[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 \\
=&\frac{\frac{\frac{(t_{i+2}-t_{i+1})(t_{i+2}-t_{i+1}-2x)}{t_{i+2}-t_{i+1}} - \frac{(t_{i+1}-x)^2}{t_{i+1}-t_i}}{t_{i+2}-t_i}}{t_{i+2}-t_{i-1}} = \frac{\frac{-x^2+2t_ix-t_it_{i+1}-t_it_{i+2}+t_{i+1}t_{i+2}}{(t_{i+2}-t_i)(t_{i+1}-t_i)} - \frac{(t_{i+1}-x)^2}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})}}{t_{i+2}-t_{i-1}} \\
=&\left[\frac{(x-t_{i-1})(t_{i+1}-x)}{(t_{i+1}-t_{i-1})(t_{i+1}-t_i)} + \frac{(t_{i+2}-x)(x-t_i)}{(t_{i+2}-t_i)(t_{i+1}-t_i)}\right] \cdot \frac{1}{t_{i+2}-t_{i-1}} = \frac{B_i^2(x)}{t_{i+2}-t_{i-1}}
\end{aligned}$$

当 $x \in (t_{i+1}, t_{i+2}]$ 时,

$$\begin{aligned}
&[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 \\
=&\frac{(t_{i+2}-x)^2}{(t_{i+2}-t_{i+1})(t_{i+2}-t_i)(t_{i+2}-t_{i-1})} \\
=&\frac{(t_{i+2}-x)^2}{(t_{i+2}-t_i)(t_{i+2}-t_{i+1})} \cdot \frac{1}{t_{i+2}-t_{i-1}} = \frac{B_i^2(x)}{t_{i+2}-t_{i-1}}
\end{aligned}$$

当 $x \in (-\infty, t_{i-1}]$ 时, $[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = \frac{1-1}{t_{i+2}-t_{i-1}} = 0 = \frac{B_i^2(x)}{t_{i+2}-t_{i-1}}$.

当 $x \in (t_{i+2}, -\infty)$ 时 $[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = 0 = \frac{B_i^2(x)}{t_{i+2}-t_{i-1}}$.

所以 $(t_{i+2} - t_{i-1})[t_{i-1}, t_i, t_{i+1}, t_{i+2}](t-x)_+^2 = B_i^2(x)$。 □

3.6.1-VII.

Scaled integral of B-splines.

Deduce from the Theorem on deriviates of B-splines that the scaled integral of a B-spline $B_i^n(x)$ over its support is independent of its index i even if the spacing of the knots is not uniform.

证明. 由

$$\frac{\mathrm{d}}{\mathrm{d}x} B_i^{n+1}(x) = \frac{(n+1)B_i^n(x)}{t_{i+n}-t_{i-1}} - \frac{(n+1)B_{i+1}^n(x)}{t_{i+n+1}-t_i}$$

$$\begin{aligned}
\int_{t_{i-1}}^{t_{i+n}} B_i^n(x)\mathrm{d}x &= \frac{t_{i+n}-t_{i-1}}{n+1}\left(B_i^{n+1}(t_{i+n}) - B_i^{n+1}(t_{i-1}) + \frac{n+1}{t_{i+n+1}-t_i}\int_{t_{i-1}}^{t_{i+n}} B_{i+1}^n(x)\mathrm{d}x\right)\\
&= \frac{t_{i+n}-t_{i-1}}{n+1}\left(B_i^{n+1}(t_{i+n}) - B_i^{n+1}(t_{i-1})\right)\\
&\quad + \frac{t_{i+n}-t_{i-1}}{t_{i+n+1}-t_i}\cdot\frac{t_{i+n+1}-t_i}{n+1}\left(B_{i+1}^{n+1}(t_{i+n}) - B_{i+2}^{n+1}(t_{i-1}) + \frac{n+1}{t_{i+n+2}-t_{i+1}}\int_{t_{i-1}}^{t_{i+n}} B_{i+2}^n(x)\mathrm{d}x\right)\\
&= \frac{t_{i+n}-t_{i-1}}{n+1}\left(B_i^{n+1}(t_{i+n}) + B_{i+1}^{n+1}(t_{i+n}) + \frac{n+1}{t_{i+n+2}-t_{i+1}}\int_{t_{i-1}}^{t_{i+n}} B_{i+2}^n(x)\mathrm{d}x\right)\\
&= \cdots\\
&= \frac{t_{i+n}-t_{i-1}}{n+1}\left(B_i^{n+1}(t_{i+n}) + B_{i+1}^{n+1}(t_{i+n}) + \cdots + B_{i+n}^{n+1}(t_{i+n})\right) + \frac{n+1}{t_{i+2n+1}-t_{i+n}}\int_{t_{i-1}}^{t_{i+n}} B_{i+n+1}^n(x)\mathrm{d}x\\
&= \frac{t_{i+n}-t_{i-1}}{n+1}
\end{aligned}$$

所以 $\frac{1}{t_{i+n}-t_{i-1}}\int_{t_{i-1}}^{t_{i+n}} B_i^n(x)\mathrm{d}x = \frac{1}{n+1}$ 与 i 无关。 □

3.6.1-VIII.

Symmetric Polynomials.

We have a theorem on expressing complete symmetric polynomials as divided difference of monomials.

(a) Verify this theorem for $m = 4$ and $n = 2$ by working out the table of divided difference and comparing the result to the definition of complete symmetric polynomials.

(b) Prove this theorem by the lemma on the recursive relation on complete symmetric polynomials.

证明.

(a) x^4 在 x_i, x_{i+1}, x_{i+2} 的差商表计算如下：

x_i	x_i^4		
x_{i+1}	x_{i+1}^4	$(x_{i+1}^2+x_i^2)(x_{i+1}+x_i)$	
x_{i+2}	x_{i+2}^4	$(x_{i+2}^2+x_{i+1}^2)(x_{i+2}+x_{i+1})$	$\frac{(x_{i+2}^2+x_{i+1}^2)(x_{i+2}+x_{i+1})-(x_{i+1}^2+x_i^2)(x_{i+1}+x_i)}{x_{i+2}-x_i}$

$$\begin{aligned}
[x_i, x_{i+1}, x_{i+2}]x^4 &= \frac{(x_{i+2}^3 + x_{i+2}^2 x_{i+1} + x_{i+2}x_{i+1}^2 + x_{i+1}^3) - (x_{i+1}^3 + x_{i+1}^2 x_i + x_{i+1}x_i^2 + x_i^3)}{x_{i+2}-x_i}\\
&= x_{i+2}^2 + x_{i+2}x_i + x_i^2 + x_{i+1}x_{i+2} + x_{i+1}x_i + x_{i+1}^2 = \tau_2(x_i, x_{i+1}, x_{i+2})
\end{aligned}$$

(b) $n = 0$ 时，$\tau_m(x_i) = [x_i]x^m = x_i^m$，设结论对 $n(< m)$ 时成立，则

$$\tau_{m-n} = [x_i, \cdots, x_{i+n}]x^m, \tau_m(x_{i+1}, \cdots, x_{i+n+1}) = [x_{i+1}, \cdots, x_{i+n+1}]x^m,$$

$$\tau_{m-n-1}(x_i, \cdots, x_{i+n+1}) = \frac{\tau_{m-n}(x_{i+1}, \cdots, x_{i+n+1}) - \tau_{m-n}(x_i, \cdots, x_{i+n})}{x_{i+n+1} - x_{i+n}} = [x_i, \cdots, x_{i+n+1}]x^m$$

$$\Leftrightarrow (x_{i+n+1} - x_i)\tau_{m-n-1}(x_i, \cdots, x_{i+n+1}) = \tau_{m-n}(x_{i+1}, \cdots, x_{i+n+1}) - \tau_{m-n}(x_i, \cdots, x_{i+n})$$

$$\Leftrightarrow \tau_{m-n}(x_i, \cdots, x_{i+m}) + x_{i+n+1}\tau_{m-n-1}(x_i, \cdots, x_{i+n+1}) = \tau_{m-n}(x_i, \cdots, x_{i+n+1}) + x_i\tau_{m-n-1}(x_i, \cdots, x_{i+n+1})$$

因为 LHS = RHS = $\tau_{m-n}(x_i, \cdots, x_{i+n+1})$，所以 $\tau_{m-n-1}(x_i, \cdots, x_{i+n+1}) = [x_i, \cdots, x_{i+n+1}]x^m$ 成立。

根据归纳假设，

$$\forall m \in \mathbb{N}^+, \forall i \in \mathbb{N}, \forall n = 0, 1, \cdots, m, \tau_{m-n}(x_i, \cdots, x_{i+n}) = [x_i, \cdots, x_{i+n}]x^m.$$

□