# Problems of Chapter 5

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日期：2023 年 12 月 20 日

## 1 Theoretical questions

**5.6.1-I Give a detailed proof of Theorem 5.7.**

集合 $C[a,b]$（$[a,b]$ 上全体连续函数）是 $\mathbb{C}$ 上内积空间，内积运算定义为 $\langle u,v\rangle=\int_a^b\rho(t)u(t)\overline{v(t)}\mathrm{d}t$。其中 $\overline{v(t)}$ 是 $v(t)$ 的复共轭，权重函数 $\rho(x)\in C[a,b]$ 满足 $\rho(x)>0,\forall x\in[a,b]$，且 $\|u\|_2=\left(\int_a^b\rho(t)|u(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}$ 是 $C[a,b]$ 在 $\mathbb{R}$ 上的范数。

**证明.**

1. 证明：$C[a,b]$ 是 $\mathbb{C}$ 上线性空间。
   (1) $\mathbb{C}$ 是一个域。加法单位元为 0，乘法单位元为 1；
   (2) 交换律：$\forall \boldsymbol{u},\boldsymbol{v}\in C[a,b],\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$;
   (3) 加法结合律：$\forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in C[a,b],(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$;
   (4) 乘法结合律：$\forall \boldsymbol{u}\in C[a,b],\forall a,b\in\mathbb{C},(ab)\boldsymbol{u}=a(b\boldsymbol{u})$;
   (5) 加法单位元：$0(x)\equiv 0\in C[a,b],\forall \boldsymbol{u}\in C[a,b],\boldsymbol{u}+0=0+\boldsymbol{u}=\boldsymbol{u}$;
   (6) 加法逆元：$\forall \boldsymbol{u}\in C[a,b]$，令 $\boldsymbol{v}(t)=-\boldsymbol{u}(t)\in C[a,b]$, 满足 $\boldsymbol{v}+\boldsymbol{u}=\boldsymbol{u}+\boldsymbol{v}=0$;
   (7) 乘法单位元：$\forall u\in C[a,b],1\boldsymbol{u}(t)=1\times\boldsymbol{u}(t)=\boldsymbol{u}\Rightarrow 1\boldsymbol{u}=\boldsymbol{u}$;
   (7) 分配律：$\forall \boldsymbol{u},\boldsymbol{v}\in C[a,b],\forall a,b\in\mathbb{C},\begin{cases}(a+b)\boldsymbol{u}=a\boldsymbol{u}+b\boldsymbol{u},\\ a(\boldsymbol{u}+\boldsymbol{v})=a\boldsymbol{u}+b\boldsymbol{v}.\end{cases}$
2. 证明：$\langle\cdot,\cdot\rangle$ 是内积运算，$C[a,b]$ 是 $\mathbb{C}$ 上内积空间。
   $\forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in C[a,b],\lambda\in\mathbb{C}$:
   (1) 正定性：$\langle\boldsymbol{u},\boldsymbol{u}\rangle=\int_a^b\rho(t)\boldsymbol{u}(t)\overline{\boldsymbol{u}(t)}\mathrm{d}t=\int_a^b\rho(t)|\boldsymbol{u}(t)|^2\mathrm{d}t\geq 0$;
   (2) 绝对性：若 $\langle\boldsymbol{u},\boldsymbol{u}\rangle=\int_a^b\rho(t)|\boldsymbol{u}(t)|^2\mathrm{d}t=0$，因为 $\rho(t)>0,\boldsymbol{u}(t)\in C[a,b]$，所以 $\boldsymbol{u}(t)\equiv 0$;
   (3) 线性性：$\lambda\langle\boldsymbol{u}+\boldsymbol{v},\boldsymbol{w}\rangle=\int_a^b\rho(t)(\lambda\boldsymbol{u}(t)+\boldsymbol{v}(t))\overline{\boldsymbol{w}(t)}\mathrm{d}t=\lambda\int_a^b\rho(t)\boldsymbol{u}(t)\overline{\boldsymbol{w}(t)}\mathrm{d}t+\int_a^b\rho(t)\boldsymbol{v}(t)\overline{\boldsymbol{w}(t)}\mathrm{d}t=\lambda\langle\boldsymbol{u},\boldsymbol{w}\rangle+\langle\boldsymbol{v},\boldsymbol{w}\rangle$;
   (4) 共轭对称性：$\langle\boldsymbol{v},\boldsymbol{u}\rangle=\int_a^b\rho(t)\boldsymbol{v}(t)\overline{\boldsymbol{u}(t)}\mathrm{d}t=\int_a^b\overline{\rho(t)\boldsymbol{u}(t)\overline{\boldsymbol{v}(t)}}\mathrm{d}t=\overline{\langle\boldsymbol{u},\boldsymbol{v}\rangle}$.
3. 证明：$\|u\|_2$ 是 $C[a,b]$ 在 $\mathbb{R}$ 上范数。
   根据定义，$\|\cdot\|_2$ 为一个映射 $C[a,b]\to\mathbb{R}$: $\|\boldsymbol{u}\|_2=\sqrt{\langle\boldsymbol{u},\boldsymbol{u}\rangle}=\left(\int_a^b\rho(t)|u(t)|^2\mathrm{d}t\right)^{\frac{1}{2}},\forall\boldsymbol{u}\in C[a,b]$。

□

### 5.6.1-II

Consider the Chebyshev polynomials of the first kind.

(a) Show that they are orthogonal on $[-1, 1]$ with respect to the inner product in Theorem 5.7 with the weight function $\rho(x) = \frac{1}{\sqrt{1-x^2}}$.

(b) Normalize the first three Chebyshev polynomials to arrive at an orthogonal system.

**解.**

(a)

$$\begin{aligned}
\langle T_n, T_m\rangle &= \int_{-1}^{1} \rho(t)T_n(t)T_m(t)\mathrm{d}t = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cos(n \arccos t)\cos(m \arccos t)\mathrm{d}t \\
&= -\int_{-1}^{1} \cos(n \arccos t)\cos(m \arccos t)\mathrm{d} \arccos t \\
&= \int_0^{\pi} \cos(n\theta)\cos(m\theta)\mathrm{d}\theta = \int_0^{\pi} \frac{\cos[(n-m)\theta] + \cos[(n+m)\theta]}{2}\mathrm{d}\theta \\
&= \begin{cases} \left(\dfrac{\sin[(n-m)\theta]}{2(n-m)} + \dfrac{\sin[(n+m)\theta]}{2(n+m)}\right)\Bigg|_0^{\pi} & (n \neq m) \\ \left(\dfrac{\theta}{2} + \dfrac{\sin 2n\theta}{4n}\right)\Bigg|_0^{\pi} & (n = m \neq 0) \\ \theta\Big|_0^{\pi} & (n = m = 0) \end{cases} = \begin{cases} 0 & (n \neq m) \\ \frac{\pi}{2} & (n = m \neq 0) \\ \pi & (n = m = 0) \end{cases}
\end{aligned}$$

所以 $\{T_n\}$ 正交。

(b) $\|T_n\|_2^2 = \langle T_n, T_n\rangle = \begin{cases} \frac{\pi}{2} & n \neq 0 \\ \pi & n = 0 \end{cases}$,

$T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1$,

所以对于 $\{T_0, T_1, T_2\}$，标准正交基为 $\left\{\sqrt{\frac{1}{\pi}}, \sqrt{\frac{2}{\pi}}x, \sqrt{\frac{2}{\pi}}(2x^2 - 1)\right\}$。

□

### 5.6.1-III 连续函数的最小平方估计

Least-square approximation of a continuous function. Approximate the circular arc given by the equation $y(x) = \sqrt{1-x^2}$ for $x \in [1, 1]$ by a quadratic polynomial with respect to the inner product in Theorem 5.7.

(a) $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ with Fourier expansion,

(b) $\rho(x) = \frac{1}{\sqrt{1-x^2}}$ with normal equations.

**解.**

(a)

$$\begin{aligned}
\langle y, T_0^*\rangle &= \langle y, \sqrt{\frac{1}{\pi}}\rangle = \sqrt{\frac{1}{\pi}} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\sqrt{1-t^2}\mathrm{d}t = 2\sqrt{\frac{1}{\pi}} \\
\langle y, T_1^*\rangle &= \langle y, \sqrt{\frac{2}{\pi}}x\rangle = \sqrt{\frac{2}{\pi}} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\sqrt{1-t^2}t\mathrm{d}t = 0 \\
\langle y, T_2^*\rangle &= \langle y, \sqrt{\frac{2}{\pi}}(2x^2-1)\rangle = \sqrt{\frac{2}{\pi}} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}}\sqrt{1-t^2}2t^2 - 1\mathrm{d}t = -\frac{2}{3}\sqrt{\frac{2}{\pi}}
\end{aligned}$$

所以 $\hat{y}(x) = 2\sqrt{\frac{1}{\pi}} \cdot \sqrt{\frac{1}{\pi}} - \frac{2}{3}\sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{2}{\pi}}(2x^2 - 1) = \dfrac{10 - 8x^2}{3\pi}$。

(b) $\langle 1, 1\rangle = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = \arccos x\Big|_{-1}^{1} = \pi$,

$\langle 1, x\rangle = \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\mathrm{d}x = 0$。

$\langle x, x\rangle = \langle 1, x^2\rangle = \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x \overset{x=\cos\theta}{=} -\int_0^{\pi} \frac{\cos^2\theta}{\sin\theta}\mathrm{d}\cos\theta = \int_0^{\pi} \cos^2\theta\mathrm{d}\theta = \frac{\pi}{2}$,
$\langle x, x^2\rangle = \int_{-1}^{1} \frac{x^3}{\sqrt{1-x^2}}\mathrm{d}x = 0$,
$\langle x^2, x^2\rangle = \int_{-1}^{1} \frac{x^4}{\sqrt{1-x^2}}\mathrm{d}x \overset{x=\cos\theta}{=} -\int_0^{\pi} \frac{\cos^4\theta}{\sin\theta}\mathrm{d}\cos\theta = \int_0^{\pi} \cos^4\theta\mathrm{d}\theta = \frac{3\pi}{8}$。

$$G(1, x, x^2) = \begin{bmatrix} \langle 1,1\rangle & \langle 1,x\rangle & \langle 1,x^2\rangle \\ \langle x,1\rangle & \langle x,x\rangle & \langle x,x^2\rangle \\ \langle x^2,1\rangle & \langle x^2,x\rangle & \langle x^2,x^2\rangle \end{bmatrix} = \begin{bmatrix} \pi & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & \frac{3\pi}{8} \end{bmatrix}$$

$\left[\langle y, 1\rangle, \langle y, x\rangle, \langle y, x^2\rangle\right]^{\mathrm{T}} = \left[\int_{-1}^{1} \mathrm{d}x, \int_{-1}^{1} x\mathrm{d}x, \int_{-1}^{1} x^2\mathrm{d}x\right]^{\mathrm{T}} = \left[2, 0, \frac{2}{3}\right]^{\mathrm{T}}$。

$$\begin{bmatrix} \pi & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & 0 \\ \frac{\pi}{2} & 0 & \frac{3\pi}{8} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ \frac{2}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{10}{3\pi} \\ 0 \\ -\frac{8}{3\pi} \end{bmatrix}$$

所以 $\hat{y}(x) = \dfrac{10 - 8x^2}{3\pi}$。

□

**5.6.1-IV 用正交多项式进行离散最小二乘估计**

Discrete least square via orthonormal polynomials. Consider the example on the table of sales record in Example 5.49.

(a) Starting from the independent list $(1, x, x^2)$, construct orthonormal polynomials by the Gram-Schmidt process using

$$\langle u(t), v(t)\rangle = \sum_{i=1}^{N} \rho(t_i)u(t_i)v(t_i).$$

as the inner product with $N = 12$ and $\rho(x) = 1$.

(b) Find the best approximation $\hat{\varphi} = \sum_{i=0}^{2} a_i x^i$ such that $\|y - \hat{\varphi}\| \le \|y - \sum_{i=0}^{2} b_i x^i\|$ for all $b_i \in \mathbb{R}$. Verify that $\hat{\varphi}$ is that same as that of the example on the table of sales record in the notes.

(c) Suppose there are other tables of sales record in the same format as that in the example. Values of $N$ and $x_i$'s are the same, but the values of $y_i$'s are different. Which of the above calculations can be reused? Which cannot be reused? What advantage of orthonormal polynomials over normal equations does this reuse imple?

**解.**

(a) $\{u_0, u_1, u_2\} = \{1, x, x^2\}$。

$\langle u_0, u_0\rangle = \langle 1, 1\rangle = \sum\limits_{i=1}^{12} 1 = 12 \Rightarrow u_0^* = \frac{u_0}{\|u_0\|_2} = \frac{1}{2\sqrt{3}}$;

$\langle u_0^*, u_1\rangle = \frac{1}{2\sqrt{3}} \sum\limits_{i=1}^{12} x_i = \frac{78}{2\sqrt{3}}$,

$v_1 = u_1 - \langle u_0^*, u_1\rangle u_0^* = x - \frac{13}{2}$, $\langle v_1, v_1\rangle = \sum\limits_{i=1}^{12}(x_i - \frac{13}{2})^2 = 143 \Rightarrow u_1^* = \frac{v_1}{\|u_1\|_2} = \frac{x - \frac{13}{2}}{\sqrt{143}}$;

$\langle u_0^*, u_2\rangle = \frac{1}{2\sqrt{3}} \sum\limits_{i=1}^{12} x_i^2 = \frac{650}{2\sqrt{3}}$, $\langle u_1^*, u_2\rangle = \frac{1}{\sqrt{143}} \sum\limits_{i=1}^{12}(x_i - \frac{13}{2})x_i^2 = \frac{1859}{\sqrt{143}}$,

$v_2 = u_2 - \langle u_0^*, u_2\rangle u_0^* - \langle u_1^*, u_2\rangle u_1^* = x^2 - 13x + \frac{91}{3}$, $\langle v_2, v_2\rangle = \sum\limits_{i=1}^{12}(x_i^2 - 13x_i + \frac{91}{3})^2 = \frac{4004}{3}$

$\Rightarrow u_2^* = \frac{v_2}{\|u_2\|_2} = \sqrt{\frac{3}{4004}}(x^2 - 13x + \frac{91}{3})$。

所以，一组标准正交基为 $\left\{\frac{1}{2\sqrt{3}}, \frac{2x-13}{2\sqrt{143}}, \sqrt{\frac{3}{4004}}(x^2 - 13x + \frac{91}{3})\right\}$。

(b) $\langle u_0^*, y\rangle = \sum_{i=1}^{12} \frac{y_i}{2\sqrt{3}} = 277\sqrt{3}$，$\langle u_1^*, y\rangle = \sum_{i=1}^{12} \frac{y_i(2x_i-13)}{2\sqrt{143}} = \frac{589}{\sqrt{143}}$，

$\langle u_2^*, y\rangle = \sum_{i=1}^{12} \sqrt{\frac{3}{4004}} y_i(x_i^2 - 13x_i + \frac{91}{3}) = 12068\sqrt{\frac{3}{4004}}$，

所以 $\hat{\varphi} = \langle u_0^*, y\rangle u_0^* + \langle u_1^*, y\rangle u_1^* + \langle u_2^*, y\rangle u_2^* = 9.04196x^2 - 113.427x + 386.000$，，与给出的结果一致。

(c) 标准正交基 $\{u_0^*, u_1^*, u_2^*\}$ 不需要重新计算，需要重新计算 $\langle u_0^*, y\rangle, \langle u_1^*, y\rangle, \langle u_2^*, y\rangle$ 需要重新计算。

若需要基于 $n$ 个给定点值拟合 $p$ 次多项式，有 $q$ 个需要拟合的多项式（使用 $\{x_i\}$ 相同的数据），

若使用标准正交基的方法，首先使用 $O(np^2)$ 次运算得出标准正交基 $\{u_i^*\}_{i=0}^p$，然后对于每组数据，需要 $O(np)$ 次运算得到 $\{\langle u_i^*, y\rangle\}_{i=0}^p$。总共需要的时间复杂度为 $O(np^2 + qnp)$。

若使用正则方程组的方法，首先需要 $O(np^2)$ 次运算得到 $G(1, x, \cdots, x^p)$，然后对于不同的 $y$，需要用 $O(np)$ 次运算得到 $\{\langle x^i, y\rangle\}_{i=0}^p$，最后还需要 $O(p^3)$ 次运算求解方程组。总共需要的时间复杂度为 $O(np^2 + q(np + p^3))$。

可以发现，当 $p$ 较大的时候，后者运算复杂度远高于前者。

□

**5.6.1-V Prove Theorem 5.60 and Lemma 5.61.**

The pseudo-inverse in Definition 5.59 has the following properties.

(PDI-1) $AA^+$ maps all columns of A to themselves: $AA^+A = A$;

(PDI-2) $A^+$ acts like a weak inverse: $A^+AA^+ = A^+$;

(PDI-3) Both $AA^+$ and $A^+A$ are Hermitian: $(AA^+)^* = AA^+, (A^+A)^* = A^+A$.

If A has linearly independent columns, then $A^*A$ is invertible and we have

$$A^+ = (A^*A)^{-1}A^*,$$

which is then a *left inverse*, i.e. $A^+A = I$. Similarly, if $A$ has linearly independent rows, then

$$A^+ = A^*(AA^*)^{-1},$$

which is then a *right inverse*, i.e. $AA^+ = I$.

**证明.** 设 $A$ 的奇异值分解是 $A = U\Sigma V^*, U = (u_1, \cdots, u_m), V = (v_1, \cdots, v_n)$，$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right]_{m\times n}$。

则 $A^+ = V\Sigma^+U^*$, $\Sigma^+ = \left[\begin{array}{ccc|c} \sigma_1^{-1} & & & \\ & \ddots & & 0 \\ & & \sigma_r^{-1} & \\ \hline & 0 & & 0 \end{array}\right]_{n\times m}$。$\Sigma\Sigma^+ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}_{m\times m}$, $\Sigma\Sigma^+\Sigma = \Sigma$, $\Sigma^+\Sigma = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}$, $\Sigma^+\Sigma\Sigma^+ = \Sigma^+$。

(PDI-1) $AA^+A = (U\Sigma V^*)(V\Sigma^+U^*)(U\Sigma V^*) = U\Sigma\Sigma^+\Sigma V^* = U\Sigma V^* = A$;

(PDI-2) $A^+AA^+ = (V\Sigma^+U^*)(U\Sigma V^*)(V\Sigma^+U^*) = V\Sigma^+\Sigma\Sigma^+U^* = V\Sigma^+U^* = A^+$;

(PDI-3) $(AA^+)^* = ((U\Sigma V^*)(V\Sigma^+U^*))^* = (U\Sigma\Sigma^+U^*)^* = U(\Sigma\Sigma^+)U^* = U\Sigma\Sigma^+U^* = AA^+$,

$(A^+A)^* = ((V\Sigma^+U^*)(U\Sigma V^*))^* = V(\Sigma^+\Sigma^*)V^* = V\Sigma^+\Sigma V^* = A^+A$。

$(A^*A)^{-1}A^* = ((U\Sigma V^*)^*(U\Sigma V^*))^{-1}(U\Sigma V^*)^* = (V\Sigma^*\Sigma V^*)^{-1}V\Sigma^*U$，因为 $A$ 列满秩，所以 $r = n$，

$$\Sigma = \left[\begin{array}{ccc} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ \hline & 0 & \end{array}\right]_{m\times n}, \Sigma^*\Sigma = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix}, (\Sigma^*\Sigma)^{-1}\Sigma^* = \Sigma^+.$$

所以 $(A^*A)^{-1}A^* = V(\Sigma^*\Sigma)^{-1}V^*V\Sigma^*U^* = V(\Sigma^*\Sigma)^{-1}\Sigma^*U^* = V\Sigma^+U^* = A^+ \Rightarrow A^+A = I$。
$A^*(AA^*)^{-1} = A^*((U\Sigma V^*)(U\Sigma V^*)^*)^{-1} = A^*(U\Sigma\Sigma^*U^*)^{-1}$，因为 $A$ 行满秩，所以 $r = m$，

$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_n & \end{array}\right]_{m\times n}, \Sigma\Sigma^* = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_m^2 \end{bmatrix}, \Sigma^*(\Sigma\Sigma^*)^{-1} = \Sigma^+,$$

所以 $A^*(AA^*)^{-1} = (U\Sigma V^*)^*U(\Sigma\Sigma^*)^{-1}U^* = V\Sigma^*(\Sigma\Sigma^*)^{-1}U^* = V\Sigma^+U^* = A^+ \Rightarrow AA^+ = I$。 □

# 2 Programming assignments

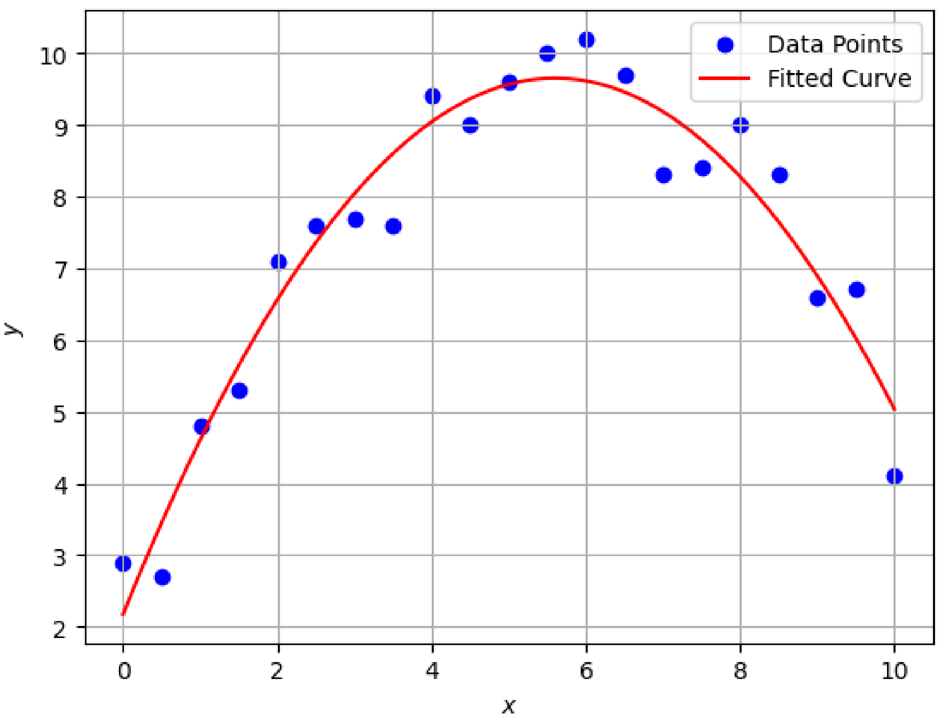
使用 `make all + make run` 运行得到所有结果。

## 2.1 A.

使用正则方程组计算结果为：$2.17572 + 2.67041x - 0.238444x^2$。其中

$$G = \begin{bmatrix} 21 & 105 & 717.5 \\ 105 & 717.5 & 5512.5 \\ 717.5 & 5512.5 & 45166.6 \end{bmatrix}$$

二条件数 $\kappa(G) = \|G\|_2\|G^{-1}\|_2 = 18981.08307675254$。



### 2.2 B.

使用正交分解法计算结果为：$2.17572+2.67041x-0.238444x^2$。其中

$$R_1=\begin{bmatrix}-4.58258 & -22.9129 & -156.571\\ 0 & 13.8744 & 138.744\\ 0 & 0 & -37.4438\end{bmatrix}$$

二条件数 $\kappa(R_1)=137.77147945973786$

矩阵的条件数反映了方程组的稳定性，进而反映了算法的稳定性。因此，正交分解的稳定性比正则化方程组要好。

### 2.3 C.

1. 证明 $\forall j=0,1,\cdots,n,\sum_{k=0}^{n}\dfrac{\alpha_k}{j+k+1}=\int_0^1 f(x)x^j\mathrm{d}x, f(x)=\frac{1}{1+x}$, 的解为如下形式：
$$\forall j=0,1,\cdots,n,\alpha_j=\beta_j\ln 2+\gamma_j,\beta_j,\gamma_j\in\mathbb{Q}.$$

**证明.** 根据题意，$G=\{\langle x^j,x^k\rangle\}_{(n+1)\times(n+1)}=\{\frac{1}{j+k+1}\}_{(n+1)\times(n+1)}\in\mathbb{Q}^{(n+1)\times(n+1)}, j,k=0,\cdots n$, 方程组的右端项 $r_j:=\int_0^1\dfrac{x^j}{1+x}\mathrm{d}x=(-1)^j\ln 2+\sum\limits_{i=1}^{j}(-1)^{i+j}\frac{1}{i}\in\mathbb{Q}+\mathbb{Q}\ln 2$。因为 $G\in\mathbb{Q}^{(n+1)*(n+1)}$，Hilbert 矩阵可逆，所以 $G^{-1}\in\mathbb{Q}^{(n+1)\times(n+1)}$。所以 $[\alpha_0,\cdots,\alpha_n]^T=G^{-1}[r_0,\cdots,r_n]^T\in\mathbb{Q}+\mathbb{Q}\ln 2$。 □

2. 使用高斯消元分别求解 $\beta_j,\gamma_j,j=0,\cdots,n$，$n$ 取遍 $1,2,\cdots,6$。输出结果见 `data/resC.txt`。
3. 计算 $\alpha_j=\beta_j\ln 2+\gamma_j$ 得到结果如下：

| $\beta\ln(2)+\gamma$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ |
|---|---|---|---|---|---|---|
| $\alpha_0$ | 0.931472 | 0.986039 | 0.997279 | 0.999483 | 0.999904 | 0.999982 |
| $\alpha_1$ | -0.476649 | -0.80405 | -0.938929 | -0.983022 | -0.995633 | -0.998938 |
| $\alpha_2$ | | 0.3274 | 0.664599 | 0.863017 | 0.951295 | 0.984346 |
| $\alpha_3$ | | | -0.224799 | -0.533448 | -0.768857 | -0.901063 |
| $\alpha_4$ | | | | 0.154325 | 0.41916 | 0.667044 |
| $\alpha_5$ | | | | | -0.105934 | -0.324072 |
| $\alpha_6$ | | | | | | 0.0727128 |

横向观察，可以看出 $a_0,a_1,a_2,a_3$ 都分别收敛到了 $(-1)^j$，说明该计算有较好的收敛性质。

4. 直接带入 $\ln 2$ 的估计值计算 $\alpha_j$ 的值的结果如下：

| | $n=1$ | $n=2$ | $n=3$ | $n=4$ | $n=5$ | $n=6$ |
|---|---|---|---|---|---|---|
| $\alpha_0$ | 0.9315 | 0.98625 | 0.998733 | 1.00908 | 1.0617 | 1.39125 |
| $\alpha_1$ | -0.4767 | -0.8052 | -0.955 | -1.162 | -2.7405 | -16.5816 |
| $\alpha_2$ | | 0.3285 | 0.703 | 1.6345 | 12.684 | 151.095 |
| $\alpha_3$ | | | -0.249667 | -1.69867 | -31.164 | -584.808 |
| $\alpha_4$ | | | | 0.7245 | 33.873 | 1071.96 |
| $\alpha_5$ | | | | | -13.2594 | -926.772 |
| $\alpha_6$ | | | | | | 304.504 |

可以看出并没有收敛，这是因为 Hilbert 矩阵的条件数过大（当 $n\geq 3$ 之后），方程组不稳定，因此计算误差很大。另外使用 $\ln(2)$ 的近似值代替 $\ln(2)$ 也带来了较大的误差。

| Hilbert Matrices | Condition Number |
|---|---|
| $H_1$ | 19.281544936250604 |
| $H_2$ | 524.1012332900597 |
| $H_3$ | 15571.82395090834 |
| $H_4$ | 551998.6957924255 |
| $H_5$ | 7711391.7485829005 |
| $H_6$ | 6894039.332923305 |

5. 使用 Tikhonov Regulation 得到的结果如下:

| | $n = 1$ | $n = 2$ | $n = 3$ | $n = 4$ | $n = 5$ | $n = 6$ |
|---|---|---|---|---|---|---|
| $\alpha_0$ | 0.9315 | 0.986248 | 0.998416 | 0.994862 | 0.995676 | 0.997988 |
| $\alpha_1$ | -0.4767 | -0.805191 | -0.951424 | -0.893607 | -0.908099 | -0.938697 |
| $\alpha_2$ | | 0.328491 | 0.694393 | 0.471413 | 0.523036 | 0.582433 |
| $\alpha_3$ | | | -0.244072 | 0.0631045 | 0.0247755 | 0.0801941 |
| $\alpha_4$ | | | | -0.139109 | -0.182563 | -0.250542 |
| $\alpha_5$ | | | | | 0.0449797 | -0.165638 |
| $\alpha_6$ | | | | | | 0.196227 |

横向观察，可以看出至少 $a_0, a_1$ 都恢复了收敛性。