Computing Method - Chapter10-2

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Problems：4,6,7

10-4

求 $f(x) = x^3 - 3x - 1 = 0$ 在 $x_0 = 2$ 附近的实根，准确到小数点后第 4 位，讨论其收敛性.
(1) 用迭代法；
(2) 用 Newton 法；
(3) 用弦位法.

解:

(1) 迭代法：$x_{n+1} = (3x+1)^{\frac{1}{3}}$，

```
x = 2
x1 = (1+3*x)**(1/3)
n = 1
print(x, x1)

while abs(x - x1) > 5e-5:
    _ = x1
    x1 = (1+3*x1)**(1/3)
    x = _
    n = n + 1
    print(x1)

print(x1, n)
```

7 次迭代后，得到答案 $\hat{\boldsymbol{x}} = 1.8794$. 对于 $\varphi(x) = (3x+1)^{\frac{1}{3}}$，$\varphi([1,2]) \in [1,2]$，且 $\forall x \in [1,2], |\varphi'(x)| \le L < 1, \varphi'(2) \approx 0.27$，所以该迭代法收敛.

(2) Newton 法：$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n - 1}{3x_n^2 - 3}$，

```
x = 2
x_1 = x - (x**3-3*x-1)/(3*x**2 - 3)
n = 1
print(x, x1)
while abs(x - x1) > 5e-5:
    _ = x1
    x1 = x1 - (x1**3-3*x1-1)/(3*x1**2 - 3)
    x = _
    n = n + 1
    print(x1)
print(x1, n)
```

2 次迭代后得到 $\hat{\boldsymbol{x}} = 1.8794$. 根据 **10-7** 该方法二阶收敛.

(3) 弦位法：$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n) - f(x_{n-1})}(x_n - x_{n-1}) = x_n - \frac{x_n^3 - 3x_n - 1}{x_n^2 + x_n x_{n-1} + x_{n-1}^2 - 3}$，取 $x_0 = 2, x_1 = 1.5$.

```
x0 = 2
x1 = 1.5
n = 1
```

```python
print(x0, x1)
while abs(x0 - x1) > 5e-5:
    _ = x1
    x1 = x1 - (x1**3-3*x1-1)/(x1**2+x0*x1+x0**2-3)
    x0 = _
    n = n + 1
    print(x1)
print(x1, n)
```

6 次迭代后得到 $\hat{\boldsymbol{x}} = 1.8794$. 因为 $f(x)$ 在 x^* 附近有连续的二阶微商，且 $f'(x^*) \neq 0$，当所取的初始值充分接近 x^* 时，该方法收敛，收敛阶接近 $p = \frac{\sqrt{5}+1}{2}$.

□

10-6

用牛顿法求 $x^3 - c = 0$ 的根，写出迭代公式和收敛阶.

解: $f(x) = x^3 - c, f'(x) = 3x^2, x_{n+1} = x_n - \frac{x^3-c}{3x_n^2} = \frac{2}{3}x_n + \frac{c}{3x_n^2}$. 该方法是二阶收敛的，设 $\varepsilon_n = x_n - c^{\frac{1}{3}}$，所以

$$\begin{aligned}
\varepsilon_{n+1} &= \frac{2}{3}x_n + \frac{c}{3x_n^2} - c^{\frac{1}{3}} \\
&= x_n - c^{\frac{1}{3}} - \frac{1}{3}x_n + \frac{c}{3x_n^2} \\
&= x_n - c^{\frac{1}{3}} - \frac{x_n^3 - c}{3x_n^2} \\
&= \left(x_n - c^{\frac{1}{3}}\right)\left(1 - \frac{x_n^2 + x_n c^{\frac{1}{3}} + c^{\frac{2}{3}}}{3x_n^2}\right) \\
&= \left(x_n - c^{\frac{1}{3}}\right)\frac{2x_n^2 - x_n c^{\frac{1}{3}} - c^{\frac{2}{3}}}{3x_n^2} \\
&= \left(x_n - c^{\frac{1}{3}}\right)^2 \frac{2x_n + c^{\frac{1}{3}}}{3x_n^2} \\
&= \varepsilon_n^2 \left(\frac{2x_n + c^{\frac{1}{3}}}{3x_n^2}\right)
\end{aligned}$$

所以 $\frac{\varepsilon_{n+1}}{\varepsilon_n^2} = \frac{2x_n + c^{\frac{1}{3}}}{3x_n^2} \to \frac{3c^{\frac{1}{3}}}{3c^{\frac{2}{3}}} = c^{-\frac{1}{3}}$，所以收敛阶为 2.

□

10-7

试证明：

(1) Newton 法在单根附近二阶收敛；

(2) Newton 法在重根附近一阶收敛.

证明: 对于方程 $f(x) = 0$，使用 Newton 法 $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$，设根为 α：

(1) 在根处 Taylor 展开得到：（ξ 在 α 到 x_n 之间）

$$f(\alpha) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{(\alpha - x_n)^2}{2}f''(\xi)$$

因为 $f(\alpha) = 0$，所以 $-\alpha = -x_n + \frac{f(x_n)}{f'(x_n)} + \frac{(\alpha - x_n)^2}{2}\frac{f''(\xi)}{f'(x_n)}$，所以

$$x_{n+1}-\alpha=\frac{1}{2}(x_n-\alpha)^2\frac{f''(\xi)}{f'(x_n)}$$

因为 f' 连续，且由于 α 是单根，所以 $f'(\alpha)\neq 0$，所以 $\exists\delta>0,\text{s.t. }\forall x\in[\alpha-\delta,\alpha+\delta],f'(x)\neq 0$．定义

$$M=\frac{\max_{x\in[\alpha-\delta,\alpha+\delta]}|f''(x)|}{2\min_{x\in[\alpha-\delta,\alpha+\delta]}|f'(x)|}$$

若选择足够接近 α 的 x_0，满足以下条件： $1,|x_0-\alpha|<\delta$; $2,M|x_0-\alpha|<1$． 那么

$$x_{n+1}-\alpha\leq M(x_n-\alpha^2)\Rightarrow|x_n-\alpha|\leq\frac{1}{M}(M|x_0-\alpha|)^{2^n}$$

所以 $\{x_n\}$ 二阶收敛到 α.

(2) 对于 k 重根 α，设 $f(x)=g(x)(x-\alpha)^k$， 那么$g(\alpha)\neq 0,g\in C^1$，那么

$$\begin{aligned}x_{n+1}&=x_n-\frac{g(x_n)(x_n-\alpha)^k}{kg(x_n)(x_n-\alpha)^{k-1}+g'(x_n)(x_n-\alpha)^k}\\&=x_n-\frac{g(x_n)(x_n-\alpha)}{kg(x_n)+g'(x_n)(x_n-\alpha)}\end{aligned}$$

所以，

$$\begin{aligned}x_{n+1}-\alpha&=x_n-\alpha-\frac{g(x_n)(x_n-\alpha)}{kg(x_n)+g'(x_n)(x_n-\alpha)}\\&=(x_n-\alpha)\left(1-\frac{g(x_n)}{kg(x_n)+g'(x_n)(x_n-\alpha)}\right)\\&=(x_n-\alpha)\left[\frac{(k-1)g(x_n)+g'(x_n)(x_n-\alpha)}{kg(x_n)+g'(x_n)(x_n-\alpha)}\right]\\&=(x_n-\alpha)\left[\frac{1}{1+\frac{g(x_n)}{(k-1)g(x_n)+g'(x_n)(x_n-\alpha)}}\right]\end{aligned}$$

当 $|x_0-\alpha|<\left|\frac{(k-1)\min_{x\in D_f}|g(x)|-\varepsilon}{\max_{x\in D_f}|g'(x)|}\right|,\left((k-1)\min_{x\in D_f}|g(x)|>\varepsilon>0\right)$ 时,

$$\frac{x_{n+1}-\alpha}{x_n-\alpha}<M=\frac{1}{1+\frac{\max_{x\in D_f}|g(x_n)|}{\varepsilon}}<1$$

所以 $\{x_n\}$ 一阶收敛到 α.

□