Computing Method - Chapter3

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Problems：2,7,9

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给出数据

x	-1.00	-0.75	-0.50	-0.25	0	0.25	0.50	0.75	1.00
y	-0.2209	0.3295	0.8826	1.4392	2.0003	2.5645	3.1334	3.7061	4.2836

用一次、二次、三次多项式及最小二乘原理拟合这些数据，并写出正规方程组.

解： 令 $\boldsymbol{x}_1=[x_1,\cdots,x_9]^T,\boldsymbol{x}_2=[x_1^2,\cdots,x_9^2]^T,\boldsymbol{x}_3=[x_1^3,\cdots,x_9^3]^T,\boldsymbol{v}=[1,\cdots,1]^T,\boldsymbol{y}=[y_1,\cdots,y_9]^T\in\mathbb{R}^9$.

(1) **一次拟合**

设拟合多项式为 $y=ax+b$， 那么正规方程组为：

$$\begin{cases}\boldsymbol{x}_1^T(\boldsymbol{y}-a\boldsymbol{x}_1-b\boldsymbol{v})=\boldsymbol{0}\\ \boldsymbol{v}^T(\boldsymbol{y}-a\boldsymbol{x}_1-b\boldsymbol{v})=\boldsymbol{0}.\end{cases}\Rightarrow\begin{cases}\boldsymbol{x}_1^T\boldsymbol{x}_1a+\boldsymbol{x}_1^T\boldsymbol{v}b=\boldsymbol{x}_1^T\boldsymbol{y}\\ \boldsymbol{v}^T\boldsymbol{x}_1a+\boldsymbol{v}^T\boldsymbol{v}b=\boldsymbol{v}^T\boldsymbol{y}\end{cases}$$

解得：$[a,b]=[2.25164667,2.01314444]$. 所以 $y_1^*=2.25165x+2.01314$.

(2) **二次拟合**

设拟合多项式为 $y=ax^2+bx+c$， 那么正规方程组为：

$$\begin{cases}\boldsymbol{x}_1^T(\boldsymbol{y}-a\boldsymbol{x}_2-b\boldsymbol{x}_1-c\boldsymbol{v})=\boldsymbol{0}\\ \boldsymbol{x}_2^T(\boldsymbol{y}-a\boldsymbol{x}_2-b\boldsymbol{x}_1-c\boldsymbol{v})=\boldsymbol{v}(0)\\ \boldsymbol{v}^T(\boldsymbol{y}-a\boldsymbol{x}_2-b\boldsymbol{x}_1-c\boldsymbol{v})=\boldsymbol{0}.\end{cases}\Rightarrow\begin{cases}\boldsymbol{x}_1^T\boldsymbol{x}_2a+\boldsymbol{x}_1^T\boldsymbol{x}_1b+\boldsymbol{x}_1^T\boldsymbol{v}c=\boldsymbol{x}_1^T\boldsymbol{y}\\ \boldsymbol{x}_2^T\boldsymbol{x}_2a+\boldsymbol{x}_2^T\boldsymbol{x}_1b+\boldsymbol{x}_2^T\boldsymbol{v}c=\boldsymbol{x}_2^T\boldsymbol{y}\\ \boldsymbol{v}^T\boldsymbol{x}_2a+\boldsymbol{v}^T\boldsymbol{x}_1b+\boldsymbol{v}^T\boldsymbol{v}c=\boldsymbol{v}^T\boldsymbol{y}\end{cases}$$

解得：$[a,b,c]=[0.03130563,2.25164667,2.00010043]$， 所以 $y_2^*=3.13056e-2x^2+2.25165x+2.00010$.

(3) **三次拟合**

设拟合多项式为 $y=ax^3+bx^2+cx+d$， 那么正规方程组为：

$$\begin{cases}\boldsymbol{x}_1^T(\boldsymbol{y}-a\boldsymbol{x}_3-b\boldsymbol{x}_2-c\boldsymbol{x}_1-d\boldsymbol{v})=\boldsymbol{0}\\ \boldsymbol{x}_2^T(\boldsymbol{y}-a\boldsymbol{x}_3-b\boldsymbol{x}_2-c\boldsymbol{x}_1-d\boldsymbol{v})=\boldsymbol{v}(0)\\ \boldsymbol{x}_3^T(\boldsymbol{y}-a\boldsymbol{x}_3-b\boldsymbol{x}_2-c\boldsymbol{x}_1-d\boldsymbol{v})=\boldsymbol{v}(0)\\ \boldsymbol{v}^T(\boldsymbol{y}-a\boldsymbol{x}_3-b\boldsymbol{x}_2-c\boldsymbol{x}_1-d\boldsymbol{v})=\boldsymbol{0}.\end{cases}\Rightarrow\begin{cases}\boldsymbol{x}_1^T\boldsymbol{x}_3a+\boldsymbol{x}_1^T\boldsymbol{x}_2b+\boldsymbol{x}_1^T\boldsymbol{x}_1c+\boldsymbol{x}_1^T\boldsymbol{v}d=\boldsymbol{x}_1^T\boldsymbol{y}\\ \boldsymbol{x}_2^T\boldsymbol{x}_3a+\boldsymbol{x}_2^T\boldsymbol{x}_2b+\boldsymbol{x}_2^T\boldsymbol{x}_1c+\boldsymbol{x}_2^T\boldsymbol{v}d=\boldsymbol{x}_2^T\boldsymbol{y}\\ \boldsymbol{x}_3^T\boldsymbol{x}_3a+\boldsymbol{x}_3^T\boldsymbol{x}_2b+\boldsymbol{x}_3^T\boldsymbol{x}_1c+\boldsymbol{x}_3^T\boldsymbol{v}d=\boldsymbol{x}_3^T\boldsymbol{y}\\ \boldsymbol{v}^T\boldsymbol{x}_3a+\boldsymbol{v}^T\boldsymbol{x}_2b+\boldsymbol{v}^T\boldsymbol{x}_1c+\boldsymbol{v}^T\boldsymbol{v}d=\boldsymbol{v}^T\boldsymbol{y}\end{cases}$$

解得：$[a,b,c,d]=[2.08484848e-3,3.13056277e-02,2.25010909,2.00010043]$， 所以 $y_3^*=2.08485e-3x^3+3.13056e-2x^2+2.25011x+2.00010$.

□

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设 $f(x),\varphi_1(x),\varphi_2(x),\cdots,\varphi_p(x)\in C_{[a,b]}$， 且 (3.29) 式成立，试求 $\varphi(x)=\alpha_1\varphi_1(x)+\cdots\alpha_p\varphi_p(x)$ 达到

$$\min_{\alpha_1,\cdots,\alpha_p}(f-\varphi,f-\varphi)$$

之参数 $\alpha_1,\alpha_2,\cdots,\alpha_p$.

解： 令 $g(a_1,\cdots a_p)=(f-\varphi,f-\varphi)=\int_a^b\left[f(x)-\sum_i^p a_i\varphi_i(x)\right]^2\mathrm{d}x$，对 $a_i(i=1,\cdots p)$ 求偏导得到：

$$\begin{aligned}\frac{\partial g}{\partial a_i} &= -2\int_a^b f(x)\varphi_i(x)\mathrm{d}x + 2\sum_{j\neq i,j=1}^{p} a_j\int_a^b \varphi_i(x)\varphi_j(x)\mathrm{d}x + 2a_i\int_a^b \varphi_i^2(x)\mathrm{d}x \\ &= -2\int_a^b f(x)\varphi_i(x)\mathrm{d}x + 2a_i\sigma_i.\end{aligned}$$

令 $\frac{\partial g}{\partial a_i} = 0, (i = 1, \cdots p)$ 得到，

$$a_i = \frac{\int_a^b f(x)\varphi_i(x)\mathrm{d}x}{\sigma_i}, i = 1, \cdots p.$$

□

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试证明如下给出的多项式是正交多项式系.

$$\begin{gathered} P_0(x) = 1, P_1(x) = x - \alpha_0 \\ P_{k+1}(x) = (x-\alpha_k)P_k(x) - \beta_{k-1}P_{k-1}(x), \\ \alpha_k = \frac{(xP_k, P_k)}{(P_k, P_k)}, \beta_k = \frac{(xP_k, P_{k+1})}{(P_k, P_k)} = \frac{(P_{k+1}, P_{k+1})}{(P_k, P_k)}, k = 0, 1, 2, \cdots \end{gathered}$$

证明： 设 $(f, p) = \int_a^b w(x)f(x)g(x)\mathrm{d}x$，

$$((x-\alpha_k)P_k, P_k) = (xP_k, P_k) - \frac{(xP_k, P_k)}{(P_k, P_k)}(P_k, P_k) = 0,$$

因为 $P_1 = (x-\alpha_0)P_0$，所以 $(P_0, P_1) = 0$. 下用数学归纳法证明，设当 $n \le k (k \ge 1)$ 时，满足 $P_0, P_1, ..., P_n$ 是正交多项式系，对于 $P_{k+1}(x) = (x-\alpha_k)P_k(x) - \beta_{k-1}P_{k-1}(x)$，

$$\begin{aligned}(P_{k+1}, P_k) &= ((x-\alpha_k)P_k, P_k) - \beta_{k-1}(P_{k-1}, P_k) = 0, \\ (P_{k+1}, P_{k-1}) &= ((x-\alpha_k)P_k, P_{k-1}) - \beta_{k-1}(P_{k-1}, P_{k-1}) \\ &= (xP_k, P_{k-1}) - \alpha_k(P_k, P_{k-1}) - \frac{(xP_{k-1}, P_k)}{(P_{k-1}, P_{k-1})}(P_{k-1}, P_{k-1}) \\ &= (xP_k, P_{k-1}) - (P_{k-1}, xP_k) \\ &= \int_a^b w(x)xP_kP_{k-1}\mathrm{d}x - \int_a^b w(x)P_{k-1}xP_k\mathrm{d}x = 0,\end{aligned}$$

对于 $j = 0, \cdots, k-2$，

$$\begin{aligned}(P_{k+1}, P_j) &= ((x-\alpha_k)P_k, P_j) - \beta_{k-1}(P_{k-1}, P_j) \\ &= (xP_k, P_j) - \alpha_k(P_k, P_j) \\ &= (xP_k, P_j) = (P_k, xP_j) \\ &= (P_k, P_{j+1} + \beta_{j-1}P_{j-1} + \alpha_jP_j) \\ &= (P_k, P_{j+1}) + \beta_{j-1}(P_k, P_{j-1}) + \alpha_j(P_k, P_j) \\ &= 0\end{aligned}$$

所以，$P_0, \cdots, P_{n+1}$ 也是正交多项式系. 因此题中给出的多项式是正交多项式系.

□