Computing Method - Programming 4

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1 问题

应用科学计算的方法，求下列问题的数值近似解：
设双曲线 $C_1 : xy = 4$ 及椭圆 $C_2 : x^2 + 4y^2 = 4$，求在曲线 C_1, C_2 各找一个点 P_1, P_2，使得 $|P_1P_2|$ 的距离最小，即

$$|P_1P_2| = \min_{Q_1 \in C_1, Q_2 \in C_2} |Q_1Q_2|$$

2 算法

设 $P_1\left(x_1, \frac{4}{x_1}\right), P_2(2\cos x_2, \sin x_2)$，则目标函数：

$$f(x) = (x_1 - 2\cos x_2)^2 + \left(\frac{4}{x_1} - \sin x_2\right)^2.$$

求偏导和二阶导矩阵，得

$$\nabla f(x) = \left[2(x_1 - 2\cos x_2) - \frac{8}{x_1^2}\left(\frac{4}{x_1} - \sin x_2\right), 4x_1 \sin x_2 - 3\sin 2x_2 - \frac{8\cos x_2}{x_1}\right]^T$$

$$\nabla^2 f(x) = \begin{bmatrix} 2 + \frac{96}{x_1^4} - \frac{16\sin x_2}{x_1^3} & 4\sin x_2 + \frac{8\cos x_2}{x_1^2} \\ 4\sin x_2 + \frac{8\cos x_2}{x_1^2} & 4x_1\cos x_2 - 6\cos 2x_2 + \frac{8\sin x_2}{x_1} \end{bmatrix}$$

我们采用纯 Newton 法（不带步长因子搜索）求解该问题，算法流程如下（取 $\alpha = 1$）：

1. 输入 $x^{(0)} \in \mathbb{R}^2, 0 \le \varepsilon < 1$
2. 对于 $k = 0, 1, \ldots$，循环：
   (a) $p_k \leftarrow -\left[\nabla^2 f\left(x^{(k)}\right)\right]^{-1} \nabla f\left(x^{(k)}\right)$.
   (b) $x^{(k+1)} \leftarrow x^{(k)} + \alpha p_k$.
   (c) 如果 $\left\|\nabla f\left(x^{(k+1)}\right)\right\| \le \varepsilon$，退出循环；否则，继续执行.
3. 输出 $x^{(k+1)}$.

3 程序

```
x0 = [0.5; 0.5];
X = Newton(x0,1e-6);
X

function P = Newton(X, eps)
    while norm(Diff(X)) > eps
        p = - inv(Hess(X)) * Diff(X);
        X = X + p;
    end
    P = X;
end
```

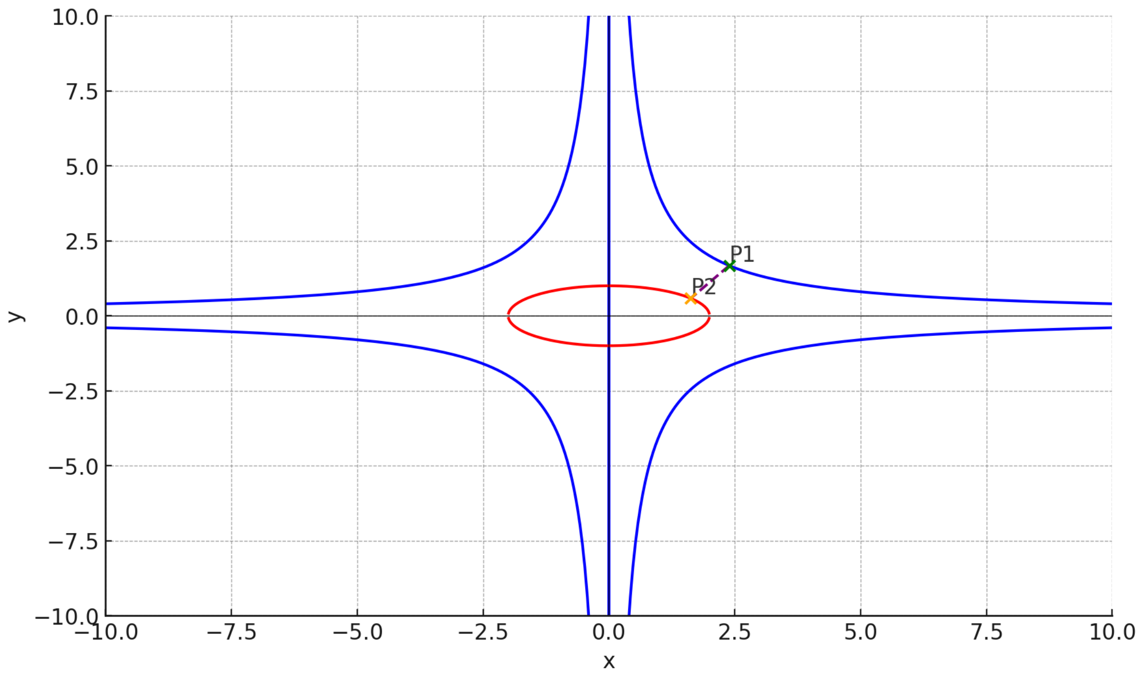
```
function P = Diff(x)
    x1 = x(1);
    x2 = x(2);
    y1 = 2*(x1-2*cos(x2))-8*(4/x1-sin(x2))/(x1^2);
    y2 = 4*x1*sin(x2)-3*sin(2*x2)-8*cos(x2)/x1;
    P = [y1; y2];
end

function P = Hess(x)
    x1 = x(1);
    x2 = x(2);
    y11 = 2 + 96/(x1^4)-16*sin(x2)/(x1^3);
    y12 = 4*sin(x2)+8*cos(x2)/(x1^2);
    y21 = y12;
    y22 = 4*x1*cos(x2)-6*cos(2*x2)+8*sin(x2)/x1;
    P = [y11, y12; y21, y22];
end
```

4 数据与结果

我们以 $(x_1, x_2) = (0.5, 0.5)$，作为初始猜测执行 Newton 法，最后得到答案为 $(x_1, x_2) = (2.3910, 6.9036)$.

5 结论



将求解出的 P_1, P_2 标在图像上，可以看出，求解结果正确．根据图的对称性，将 P_1, P_2 沿坐标轴对称后的点对同样满足条件．