Computing Method - Chapter2

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Problems：Exercise-2,9,10,14

Exercise-2

导出三弯矩方程组，并对 $s'(x_0), s'(x_n)$ 给定进行讨论.

解: 记 $M_i = s''(x_i), i = 0, ..., n$， 将 $s(x)$ 在 x_i 处展开得

$$s(x) = y_i + s'(x_i)(x - x_i) + \frac{M_i}{2}(x - x_i)^2 + \frac{s'''(x_i)}{6}(x - x_i)^3, x \in [x_i, x_{i+1}]$$

对其求二阶导，得 $s''(x) = M_i + s'''(x_i)(x - x_i)$，取 $x = x_{i+1}$ 得 $s'''(x_i) = \frac{M_{i+1} - M_i}{x_{i+1} - x_i}$， 代回上式得

$$y_{i+1} - y_i = s'(x_i)(x_{i+1} - x_i) + \left(\frac{M_i}{2} + \frac{M_{i+1} - M_i}{6}\right)(x_{i+1} - x_i)^2$$

$$\Rightarrow s'(x_i) = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{1}{6}(M_{i+1} + 2M_i)(x_{i+1} - x_i)$$

同理可得

$$s'(x_i) = \frac{y_{i-1} - y_i}{x_{i-1} - x_i} - \frac{1}{6}(M_{i-1} + 2M_i)(x_{i-1} - x_i)$$

设 $f[x_{i-1}, x_i, x_{i+1}] = \frac{\frac{y_{i-1} - y_i}{x_{i-1} - x_i} - \frac{y_{i+1} - y_i}{x_{i+1} - x_i}}{x_{i+1} - x_{i-1}}, \mu_i = \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}, \lambda_i = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}, i = 1, ..., n - 1$. 所以有

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}],$$

若给定条件，$s'(x_0) = \beta_1, s'(x_n) = \beta_2$，那么有

$$\beta_1 = \frac{y_1 - y_0}{x_1 - x_0} - \frac{1}{6}(M_1 + 2M_0)(x_1 - x_0),$$

$$\beta_2 = \frac{y_{n-1} - y_n}{x_{n-1} - x_n} - \frac{1}{6}(M_{n-1} + 2M_n)(x_{n-1} - x_n)$$

最后得到方程组

$$\begin{bmatrix} 1 & 2 & & & & & \\ & 1 & 2 & 1 & & & \\ & & 1 & 2 & 1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 2 & 1 \\ & & & & & & 1 & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-2} \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} 6\frac{y_1 - y_0}{(x_1 - x_0)^2} - 6\frac{\beta_1}{x_1 - x_0} \\ 6f[x_0, x_1, x_2] \\ 6f[x_1, x_2, x_3] \\ \vdots \\ 6f[x_{n-3}, x_{n-2}, x_{n-1}] \\ 6f[x_{n-2}, x_{n-1}, x_n] \\ 6\frac{y_{n-1} - y_n}{(x_{n-1} - x_n)^2} - 6\frac{\beta_2}{x_{n-1} - x_n} \end{bmatrix}$$

□

9

利用(2.37)式和 Newton 插值公式作两点三次 Hermite 插值多项式， 并求前一题的解.

解:

$$
\begin{array}{c|cccc}
x_0 & y_0 & & & \\
x_0 & y_0 & y_0' & & \\
x_1 & y_1 & f[x_0,x_1] & \frac{y_0'-f[x_0,x_1]}{x_0-x_1} & \\
x_1 & y_1 & y_1' & \frac{y_1'-f[x_0,x_1]}{x_1-x_0} & \frac{y_1'+y_0'-2f[x_0,x_1]}{(x_1-x_0)^2}
\end{array}
$$

所以，

$$f(x)=\frac{y_1'+y_0'-2f[x_0,x_1]}{(x_1-x_0)^2}(x-x_0)^2(x-x_1)+\frac{y_0'-f[x_0,x_1]}{x_0-x_1}(x-x_0)^2+y_0'(x-x_0)+y_0$$

代入 $y_0=1,y_0'=\frac{1}{2},y_1=2,y_1'=\frac{1}{2},x_0=0,x_1=1$，得

$$f(x)=-x^3+\frac{3}{2}x^2+\frac{1}{2}x+1$$

□

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给定数组

x	75	76	77	78	79	80
y	2.768	2.833	2.903	2.979	3.062	3.153

(1) 作一分段线性插值函数；
(2) 取第二类边界条件，作三次样条插值多项式；
(3) 用两种插值函数分别计算 $x=75.5$ 和 $x=78.3$ 的函数值.

解:

(1) $I_5(x)=\sum_{i=1}^{5}y_il_i(x)$，其中，

$$l_0(x)=\begin{cases}76-x & x\in[75,76]\\ 0 & \text{otherwise}\end{cases},$$

$$l_j(x)=\begin{cases}x-j-74 & x\in[74+j,75+j]\\ 76+j-x & x\in[75+j,76+j],\\ 0 & \text{otherwise}\end{cases}\quad (j=1,2,3,4)$$

$$l_5(x)=\begin{cases}80-x & x\in[79,80]\\ 0 & \text{otherwise}\end{cases}$$

(2) 取两端点处一阶导值为 0，根据(2.62)得到方程组

$$\begin{cases}m_0=m_5=0\\ \frac{1}{2}m_0+2m_1+\frac{1}{2}m_2=0.015\\ \frac{1}{2}m_1+2m_2+\frac{1}{2}m_3=0.018\\ \frac{1}{2}m_2+2m_3+\frac{1}{2}m_4=0.014\\ \frac{1}{2}m_3+2m_4+\frac{1}{2}m_5=0.016\end{cases}$$

解得，$m_0=0,m_1=0.0058,m_2=0.0067,m_3=0.0036,m_4=0.0071,m_5=0$， 那么样条在区间 $[x_{i-1},x_i],i=1,2,3,4,5$ 上的表达式为，

$$s(x)=(1+2x-2x_{i-1})(x-x_i)^2y_{i-1}+(1-2x+2x_i)(x-x_{i-1})^2y_i+$$
$$(x-x_{i-1})(x-x_i)^2m_{i-1}+(x-x_i)(x-x_{i-1})^2m_i$$

(3)

$$I_5(75.5)=2.768l_0(75.5)+2.833l_1(75.5)=2.8005$$
$$I_5(78.3)=2.979l_3(78.3)+3.062l_4(78.3)=3.0039$$

$$s(75.5)=(1+2x-2\times 75)(x-76)^2\times 2.768+(1-2x+2\times 76)(x-75)^2\times 2.833+(x-76)(x-75)^2\times 0.0058=2.799$$
$$s(78.3)=(1+2x-2\times 78)(x-79)^2\times 2.979+(1-2x+2\times 79)(x-78)^2\times 3.062+$$
$$(x-78)(x-79)^2\times 0.0036+(x-79)(x-78)^2 0.0071=3.0034$$

□

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称 n 阶方阵 $\boldsymbol{A}=(a_{ij})$ 具有严格对角优势，若

$$|a_{ii}|>\sum_{j=1,j\neq i}^{n}|a_{ij}|,i=1,2,\cdots,n$$

(1) 证明：具有严格对角优势的方阵必可逆；
(2) 证明：方程组 （2.62） 存在唯一解.

证明:

(1) 设矩阵 A 行严格对角占优，如果 A 奇异，那么存在 $\boldsymbol{x}\neq\mathbf{0},A\boldsymbol{x}=\mathbf{0}$. 所以，

$$\sum_{j=1}^{n}a_{ij}x_j=0,i=1,2,...,n$$

令 $i_0=\arg\max_{1\le i\le n}|x_i|$. 则，

$$|a_{i_0i_0}x_{i_0}|=\left|-\sum_{j=1,j\neq i_0}^{n}a_{i_0,j}x_j\right|\le\sum_{j=1,j\neq i_0}^{n}|a_{i_0,j}||x_j|\le|x_{i_0}|\sum_{j=1,j\neq i_0}^{n}|a_{i_0,j}|$$

所以 $|x_{i_0}|\left(|a_{i_0,i_0}|-\sum_{j=1,j\neq i_0}^{n}|a_{i_0,j}|\right)\le 0\Rightarrow|a_{i_0,i_0}|-\sum_{j=1,j\neq i_0}^{n}|a_{i_0,j}|\le 0$， 这与 A 对角占优的假设相矛盾，因此 A 必可逆.
(2) 方程组(2.62)按行严格对角占优，根据(1)，其矩阵必然可逆，所以此方程组必然存在唯一解.

□