Computing Method - Chapter9-1

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Problems：Chapter9-1,2,Chapter8-10

1

用幂法计算矩阵

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 5 & 1 \\ 2 & 1 & 6 \end{bmatrix}$$

的最大特征值和相应的特征向量.

解:

```python
import numpy as np
def solve(A, x, eps, N):
    k = 1
    mu = 0
    y = np.zeros(np.shape(x))
    while (1):
        l = x[0]
        for y in x:
            if abs(y) > abs(l):
                l = y
        y = x / l
        x = A @ y
        print("k =", k, "y_k =", y, "x_(k+1) =", x)
        if abs(l - mu) < eps:
            break
        else :
            k = k + 1
            mu = l
        if k == N:
            print ("Time Limit Exceeded!")
            break
    return [l, y]

A = np.array([[4,2,2],[2,5,1],[2,1,6]])
x = np.array([1,1,1])

print(solve(A, x, 5e-5, 100))
```

求解过程如下：

```
k = 1 y_k = [1. 1. 1.] x_(k+1) = [8. 8. 9.]
k = 2 y_k = [0.88888889 0.88888889 1.        ] x_(k+1) = [7.33333333 7.22222222 8.66666667]
k = 3 y_k = [0.84615385 0.83333333 1.        ] x_(k+1) = [7.05128205 6.85897436 8.52564103]
k = 4 y_k = [0.82706767 0.80451128 1.        ] x_(k+1) = [6.91729323 6.67669173 8.45864662]
k = 5 y_k = [0.81777778 0.78933333 1.        ] x_(k+1) = [6.84977778 6.58222222 8.42488889]
k = 6 y_k = [0.81304073 0.78128297 1.        ] x_(k+1) = [6.81472885 6.53249631 8.40736442]
k = 7 y_k = [0.81056661 0.77699693 1.        ] x_(k+1) = [6.79626027 6.50611784 8.39813014]
```

```
k = 8 y_k = [0.80925875 0.77471029 1.        ] x_(k+1) = [6.78645557 6.49206895 8.39322779]
k = 9 y_k = [0.80856325 0.77348895 1.        ] x_(k+1) = [6.78123092 6.48457126 8.39061546]
k = 10 y_k = [0.80819231 0.77283619 1.        ] x_(k+1) = [6.77844163 6.48056556 8.38922081]
k = 11 y_k = [0.80799418 0.77248718 1.        ] x_(k+1) = [6.7769511  6.47842429 8.38847555]
k = 12 y_k = [0.80788828 0.77230055 1.        ] x_(k+1) = [6.77615423 6.47727932 8.38807712]
k = 13 y_k = [0.80783166 0.77220074 1.        ] x_(k+1) = [6.7757281  6.47666699 8.38786405]
k = 14 y_k = [0.80780137 0.77214735 1.        ] x_(k+1) = [6.77550019 6.47633949 8.3877501 ]
k = 15 y_k = [0.80778518 0.77211879 1.        ] x_(k+1) = [6.77537829 6.47616433 8.38768915]
k = 16 y_k = [0.80777651 0.77210352 1.        ] x_(k+1) = [6.7753131  6.47607063 8.38765655]
k = 17 y_k = [0.80777188 0.77209535 1.        ] x_(k+1) = [6.77527822 6.47602052 8.38763911]
[8.38765654798032, array([0.80777188, 0.77209535, 1.        ])]
```

所以 $\lambda = 8.3876565, \boldsymbol{v} = [0.80777188, 0.77209535, 1.]^T$.

□

2

求下面三对角矩阵

$$A = \begin{bmatrix} 0 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

的特征值.

解:

```python
A = np.array([[1,5,0,0,0,0],[1,1,4,0,0,0],[0,1,1,3,0,0],[0,0,1,1,2,0],[0,0,0,1,1,1],
[0,0,0,0,1,1]])

def QR(A):
    for i in range(1000):
        Q, R = np.linalg.qr(A)
        A = R @ Q
    eigenvalues = np.diag(A)
    return eigenvalues

print(QR(A) - np.array([1,1,1,1,1,1]))
```

使用 QR 法计算 $A + I$ 的特征值，然后分别减去 1. 最后计算结果为： $[3.32425743\ 1.88917588\ -3.32425743\ 0.61670659\ -1.88917588\ -0.61670659]$

□

10

设 λ 是 n 阶矩阵 A 的一个特征值，试证存在 k，

$$|\lambda - a_{kk}| \leq \sum_{j \neq k} |a_{kj}| \equiv R_k$$

从而所有特征值在下面 Gerschgoring 圆盘的并集中

$$G_k = \{\lambda : |\lambda - a_{kk}| \le R_k \quad k = 1, 2, \cdots, n\} \quad k = 1, 2, \cdots, n$$

证明: 对于 A 的任意特征值 λ，设 $\boldsymbol{x}$ 为其对应的特征向量．那么 $A\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow \sum_{j=1}^{n} a_{ij}x_j = \lambda x_{i(\forall i=1,2,\ldots,n)}$．令 $k = \arg\max_k |x_k|$，那么有

$$\begin{aligned}
\sum_{j=1}^{k} a_{kj}x_j = \lambda x_k \Rightarrow x_k(\lambda - a_{kk}) = \sum_{j\ne k} a_{kj}x_j \\
\Rightarrow |x_k||\lambda - a_{kk}| = \left|\sum_{j\ne k} a_{kj}x_j\right| \le \sum_{j\ne k} |a_{kj}||x_j| \le |x_k| \sum_{j\ne k} |a_{kj}| \\
\Rightarrow |\lambda - a_{kk}| \le \sum_{j\ne k} |a_{kj}| = R_k
\end{aligned}$$

□