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QUANTILE REGRESSION UNDER MISSPECIFICATION, WITH AN APPLICATION TO THE U.S. WAGE STRUCTURE

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Quantile regression (QR) fits a linear model for conditional quantiles just as ordinary least squares (OLS) fits a linear model for conditional means. An attractive feature of OLS is that it gives the minimum mean-squared error linear approximation to the conditional expectation function even when the linear model is misspecified. Empirical research using quantile regression with discrete covariates suggests that QR may have a similar property, but the exact nature of the linear approximation has remained elusive. In this paper, we show that QR minimizes a weighted mean-squared error loss function for specification error. The weighting function is an average density of the dependent variable near the true conditional quantile. The weighted least squares interpretation of QR is used to derive an omitted variables bias formula and a partial quantile regression concept, similar to the relationship between partial regression and OLS. We also present asymptotic theory for the QR process under misspecification of the conditional quantile function. The approximation properties of QR are illustrated using wage data from the U.S. census. These results point to major changes in inequality from 1990 to 2000.

KEYWORDS: Conditional quantile function, best linear predictor, wage inequality, income distribution.

1. INTRODUCTION

THE LINEAR QUANTILE REGRESSION (QR) estimator is an increasingly important empirical tool, allowing researchers to fit parsimonious models to an entire conditional distribution. Part of the appeal of quantile regression derives from a natural parallel with conventional ordinary least squares (OLS) or mean regression. Just as OLS regression coefficients offer convenient summary statistics for conditional expectation functions, quantile regression coefficients can be used to make easily interpreted statements about conditional distributions. Moreover, unlike OLS coefficients, QR estimates capture changes in distribution shape and spread, as well as changes in location.

An especially attractive feature of OLS regression estimates is their robustness and interpretability under misspecification of the conditional expectation function. In addition to consistently estimating a linear conditional expectation function, OLS estimates provide the minimum mean-squared error lin-

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ear approximation to a conditional expectation function of any shape. The approximation properties of OLS have been emphasized by White (1980), Chamberlain (1984), Goldberger (1991), and Angrist and Krueger (1999). The fact that OLS provides a meaningful and well-understood summary statistic for conditional expectations under almost all circumstances undoubtedly contributes to the primacy of OLS regression as an empirical tool. In view of the possibility of interpretation under misspecification, modern theoretical research on regression inference also allows for misspecification of the regression function when deriving limiting distributions (White (1980)).

While QR estimates are as easy to compute as OLS regression coefficients, an important difference between OLS and QR is that most of the theoretical and applied work on QR postulates a correctly specified linear model for conditional quantiles. This raises the question of whether and how QR estimates can be interpreted when the linear model for conditional quantiles is misspecified (for example, QR estimates at different quantiles may imply conditional quantile functions that cross). One interpretation for QR under misspecification is that it provides the best linear predictor for a response variable under asymmetric loss. This interpretation is not very satisfying, however, since prediction under asymmetric loss is typically not the object of interest in empirical work. (An exception is the forecasting literature; see, e.g., Giacomini and Komunjer (2003).) Empirical research on quantile regression with discrete covariates suggests that QR may have an approximation property similar to that of OLS, but the exact nature of the linear approximation has remained an important unresolved question (Chamberlain (1994, p. 181)).

The first contribution of this paper is to show that QR is the best linear approximation to the conditional quantile function using a weighted mean-squared error loss function, much as OLS regression provides a minimum mean-squared error fit to the conditional expectation function. The implied QR weighting function can be used to understand which, if any, parts of the distribution of regressors contribute disproportionately to a particular set of QR estimates. We also show how this approximation property can be used to interpret multivariate QR coefficients as partial regression coefficients and to develop an omitted variables bias formula for QR. A second contribution is to present distribution theory for the QR process that accounts for possible misspecification of the conditional quantile function. The approximation theorems and inference results in the paper are illustrated with an analysis of wage data from recent U.S. censuses.[2] The results show a sharp change in the quantile process of schooling coefficients in the 2000 census, and an increase in conditional inequality in the upper half of the wage distribution from 1990 to 2000.

[2]Quantile regression has been widely used to model changes in the wage distribution, see, e.g., Buchinsky (1994), Abadie (1997), Gosling, Machin, and Meghir (2000), and Autor, Katz, and Kearney (2004).

The paper is organized as follows. Section 2 introduces assumptions and notation, and presents the main approximation theorems. Section 3 presents inference theory for QR processes under misspecification. Section 4 illustrates QR approximation properties with U.S. census data. Section 5 concludes. The Appendix provides the proofs; variable definitions, data, and programs are available in the on-line supplement (Angrist, Chernozhukov, and Fernández-Val (2006)).

2. INTERPRETING QR UNDER MISSPECIFICATION

2.1. *Notation and Framework*

Given a continuous response variable Y and a $d \times 1$ regressor vector X, we are interested in the conditional quantile function (CQF) of Y given X. The conditional quantile function is defined as

$$Q_\tau(Y|X) := \inf\{y : F_Y(y|X) \geq \tau\},$$

where $F_Y(y|X)$ is the distribution function for Y conditional on X, which is assumed to have conditional density $f_Y(y|X)$. Assuming integrability, the CQF solves the minimization problem

$$Q_\tau(Y|X) \in \arg\min_{q(X)} E\big[\rho_\tau(Y - q(X))\big], \tag{1}$$

where $\rho_\tau(u) = (\tau - \mathbb{1}(u \leq 0))u$ and the minimum is over the set of measurable functions of X; see, e.g., Fox and Rubin (1964). This is a potentially infinite-dimensional problem if covariates are continuous and can be high dimensional even with discrete X. It may nevertheless be possible to capture important features of the CQF using a linear model. This motivates linear quantile regression.

The linear quantile regression (QR) vector solves the population minimization problem

$$\beta(\tau) := \arg\min_{\beta \in \mathbb{R}^d} E[\rho_\tau(Y - X'\beta)], \tag{2}$$

assuming integrability and uniqueness of the solution. Quantile regression was introduced by Koenker and Bassett (1978) as a generalization of median regression. If $q(X)$ is, in fact, linear, the QR minimand will find it, just as when the conditional expectation function is linear, OLS will find it. More generally, QR provides the best linear predictor for Y under the asymmetric loss function, ρ_τ. As noted in the Introduction, however, prediction under asymmetric loss is rarely the object of empirical work. Rather, the conditional quantile function is usually of intrinsic interest. For example, labor economists are often interested in comparisons of conditional deciles as a measure of how the spread of a wage distribution changes conditional on covariates. (See, e.g., Katz and Murphy (1992), Juhn, Murphy, and Pierce (1993), and Buchinsky (1994).) Thus, our first goal is to establish the nature of the approximation to conditional quantiles that QR provides.

2.2. *QR Approximation Properties*

Our principal theoretical result is that the population QR vector minimizes a weighted sum of squared specification errors. This is easiest to show using notation for a quantile-specific specification error and for a quantile-specific residual. For any quantile index $\tau \in (0,1)$, we define the QR specification error as

$$\Delta_\tau(X,\beta) := X'\beta - Q_\tau(Y|X).$$

Similarly, let ϵ_τ be a quantile-specific residual, defined as the deviation of the response variable from the conditional quantile of interest,

$$\epsilon_\tau := Y - Q_\tau(Y|X),$$

with conditional density $f_{\epsilon_\tau}(e|X)$ at $\epsilon_\tau = e$. The following theorem shows that QR is a weighted least squares approximation to the unknown CQF.

THEOREM 1—Approximation Property: *Suppose that* (i) *the conditional density* $f_Y(y|X)$ *exists a.s.*, (ii) $E[Y]$, $E[Q_\tau(Y|X)]$, *and* $E\|X\|$ *are finite*, *and* (iii) $\beta(\tau)$ *uniquely solves* (2). *Then*

$$\beta(\tau) = \arg\min_{\beta\in\mathbb{R}^d} E[w_\tau(X,\beta)\cdot\Delta_\tau^2(X,\beta)],$$

where

$$\begin{aligned} w_\tau(X,\beta) &= \int_0^1 (1-u)\cdot f_{\epsilon_\tau}\big(u\Delta_\tau(X,\beta)|X\big)\,du \\ &= \int_0^1 (1-u)\cdot f_Y\big(u\cdot X'\beta + (1-u)\cdot Q_\tau(Y|X)|X\big)\,du \geq 0. \end{aligned}$$

PROOF: The proof proceeds by demonstrating the equivalence of the two objective functions. We have that $\beta(\tau) = \arg\min_{\beta\in\mathbb{R}^d} E[\rho_\tau(\epsilon_\tau - \Delta_\tau(X,\beta))]$ or, equivalently, since $E[\rho_\tau(\epsilon_\tau)]$ does not depend on β and is finite by condition (ii),

$$\beta(\tau) = \arg\min_{\beta\in\mathbb{R}^d}\big\{E\big[\rho_\tau(\epsilon_\tau - \Delta_\tau(X,\beta))\big] - E[\rho_\tau(\epsilon_\tau)]\big\}. \tag{3}$$

By definition of ρ_τ and the law of iterated expectations, it follows further that

$$\beta(\tau) = \arg\min_{\beta\in\mathbb{R}^d}\big\{E[\mathcal{A}(X,\beta)] - E[\mathcal{B}(X,\beta)]\big\},$$

where

$$\mathcal{A}(X,\beta) = E\big[\big(\mathbb{1}\{\epsilon_\tau < \Delta_\tau(X,\beta)\} - \tau\big)\Delta_\tau(X,\beta)|X\big],$$
$$\mathcal{B}(X,\beta) = E\big[\big(\mathbb{1}\{\epsilon_\tau < \Delta_\tau(X,\beta)\} - \mathbb{1}\{\epsilon_\tau < 0\}\big)\epsilon_\tau|X\big].$$

The conclusion of the theorem can then be obtained by showing that

$$\mathcal{A}(X,\beta) = \left(\int_0^1 f_{\epsilon_\tau}\big(u\Delta_\tau(X,\beta)|X\big)\,du\right)\cdot \Delta_\tau^2(X,\beta), \tag{4}$$

$$\mathcal{B}(X,\beta) = \left(\int_0^1 uf_{\epsilon_\tau}\big(u\Delta_\tau(X,\beta)|X\big)\,du\right)\cdot \Delta_\tau^2(X,\beta), \tag{5}$$

establishing that both components are density-weighted quadratic specification errors.

Consider $\mathcal{A}(X,\beta)$ first. Observe that

$$\begin{aligned}\mathcal{A}(X,\beta) &= \big[F_{\epsilon_\tau}\big(\Delta_\tau(X,\beta)|X\big) - F_{\epsilon_\tau}(0|X)\big]\Delta_\tau(X,\beta)\\ &= \left(\int_0^1 f_{\epsilon_\tau}\big(u\Delta_\tau(X,\beta)|X\big)\Delta_\tau(X,\beta)\,du\right)\Delta_\tau(X,\beta),\end{aligned} \tag{6}$$

where the first statement follows by the definition of conditional expectation and noting that $E[\mathbb{1}\{\epsilon_\tau \le 0\}|X] = F_{\epsilon_\tau}(0|X) = \tau$, and the second statement follows from the fundamental theorem of calculus (for Lebesgue integrals). This verifies (4). Turning to $\mathcal{B}(X,\beta)$, suppose first that $\Delta_\tau(X,\beta) > 0$. Then, setting $u_\tau = \epsilon_\tau/\Delta_\tau(X,\beta)$, we have

$$\begin{aligned}\mathcal{B}(X,\beta) &= E\big[\mathbb{1}\big\{\epsilon_\tau \in [0,\Delta_\tau(X,\beta)]\big\}\cdot\epsilon_\tau\big|X\big]\\ &= E\big[\mathbb{1}\{u_\tau \in [0,1]\}\cdot u_\tau\cdot\Delta_\tau(X,\beta)|X\big]\\ &= \left(\int_0^1 uf_{u_\tau}(u|X)\,du\right)\Delta_\tau(X,\beta)\\ &= \left(\int_0^1 uf_{\epsilon_\tau}\big(u\Delta_\tau(X,\beta)|X\big)\Delta_\tau(X,\beta)\,du\right)\cdot\Delta_\tau(X,\beta),\end{aligned} \tag{7}$$

which verifies (5). A similar argument shows that (5) also holds if $\Delta_\tau(X,\beta) < 0$. Finally, if $\Delta_\tau(X,\beta) = 0$, then $\mathcal{B}(X,\beta) = 0$, so that (5) holds in this case too. *Q.E.D.*

Theorem 1 states that the population QR coefficient vector $\beta(\tau)$ minimizes the expected weighted mean-squared approximation error, i.e., the square of the difference between the true CQF and a linear approximation, with weighting function $w_\tau(X, \beta)$.[3] The weights are given by the average density of the response variable over a line from the point of approximation, $X'\beta$, to the true conditional quantile, $Q_\tau(Y|X)$. Premultiplication by the term $(1-u)$ in the integral results in more weight being applied at points on the line closer to the true CQF.

We refer to the function $w_\tau(X, \beta)$ as defining *importance weights*, since this function determines the importance the QR minimand gives to points in the support of X for a given distribution of X.[4] In addition to the importance weights, the probability distribution of X also determines the ultimate weight given to different values of X in the least squares problem. To see this, note that we can also write the QR minimand as

$$\beta(\tau) = \arg\min_{\beta \in \mathbb{R}^d} \int \Delta_\tau^2(x, \beta) w_\tau(x, \beta)\, d\Pi(x),$$

where $\Pi(x)$ is the distribution function of X with associated probability mass or density function $\pi(x)$. Thus, the overall weight varies in the distribution of X according to

$$w_\tau(x, \beta) \cdot \pi(x).$$

A natural question is what determines the shape of the importance weights. This can be understood using the following approximation. When Y has a smooth conditional density, we have, for β in the neighborhood of $\beta(\tau)$,

$$\begin{aligned} &w_\tau(X, \beta) = 1/2 \cdot f_Y\big(Q_\tau(Y|X)|X\big) + \varrho_\tau(X), \\ &|\varrho_\tau(X)| \le 1/6 \cdot |\Delta_\tau(X, \beta)| \cdot \bar{f}'(X). \end{aligned} \tag{8}$$

Here, $\varrho_\tau(X)$ is a remainder term and the density $f_Y(y|X)$ is assumed to have a first derivative in y bounded in absolute value by $\bar{f}'(X)$ a.s.[5] Hence in many cases the *density weights* $1/2 \cdot f_Y(Q_\tau(Y|X)|X)$ are the primary determinants of the importance weights, a point we illustrate in Section 4. It is also of interest to note that $f_Y(Q_\tau(Y|X)|X)$ is constant across X in location models and inversely proportional to the conditional standard deviation in location–scale models.[6]

[3]Note that if we define $\beta(\tau)$ via (3), then integrability of Y is not required in Theorem 1.

[4]This terminology should not be confused with similar terminology from Bayesian statistics.

[5]The remainder term $\varrho_\tau(X) = w_\tau(X, \beta) - (1/2) \cdot f_{\epsilon_\tau}(0|X)$ is bounded as $|\varrho_\tau(X)| = |\int_0^1 (1-u)(f_{\epsilon_\tau}(u \cdot \Delta_\tau(X, \beta)|X) - f_{\epsilon_\tau}(0|X))\, du| \le |\Delta_\tau(X, \beta)| \cdot \bar{f}'(X) \cdot \int_0^1 (1-u) \cdot u\, du = (1/6) \cdot |\Delta_\tau(X, \beta)| \cdot \bar{f}'(X)$.

[6]A location–scale model is any model of the form $Y = \mu(X) + \sigma(X) \cdot e$, where e is independent of X. The location model results from setting $\sigma(X) = \sigma$.

Quantile regression has a second approximation property closely related to the first. This second property is particularly well suited to the development of a partial regression decomposition and the derivation of an omitted variables bias formula for QR.

THEOREM 2—Iterative Approximation Property: *Suppose that* (i) *the conditional density* $f_Y(y|X)$ *exists and is bounded a.s.*, (ii) $E[Y]$, $E[Q_\tau(Y|X)^2]$, *and* $E\|X\|^2$ *are finite, and* (iii) $\beta(\tau)$ *uniquely solves* (2). *Then* $\bar{\beta}(\tau) = \beta(\tau)$ *uniquely solves the equation*

$$\bar{\beta}(\tau) = \arg\min_{\beta\in\mathbb{R}^d} E\big[\bar{w}_\tau(X, \bar{\beta}(\tau)) \cdot \Delta_\tau^2(X, \beta)\big], \tag{9}$$

where

$$\begin{aligned} \bar{w}_\tau(X, \bar{\beta}(\tau)) &= \frac{1}{2}\int_0^1 f_{\epsilon_\tau}\big(u \cdot \Delta_\tau(X, \bar{\beta}(\tau))|X\big)\, du \\ &= \frac{1}{2}\int_0^1 f_Y\big(u \cdot X'\bar{\beta}(\tau) + (1-u) \cdot Q_\tau(Y|X)|X\big)\, du \geq 0. \end{aligned}$$

PROOF: The proof proceeds by showing that

$$\beta(\tau) = \arg\min_{\beta\in\mathbb{R}^d} E[\rho_\tau(Y - X'\beta)] \tag{10}$$

is equivalent to the fixed point $\bar{\beta}(\tau)$ that uniquely solves

$$\bar{\beta}(\tau) = \arg\min_{\beta\in\mathbb{R}^d} E\big[\bar{w}_\tau(X, \bar{\beta}(\tau)) \cdot \Delta_\tau^2(X, \beta)\big], \tag{11}$$

where the former and the latter objective functions are finite by conditions (i) and (ii).

By convexity of (11) in β, any fixed point $\beta = \bar{\beta}(\tau)$ solves the first-order condition

$$\mathcal{F}(\beta) := 2 \cdot E[\bar{w}_\tau(X, \beta)\Delta_\tau(X, \beta)X] = 0.$$

By convexity of (10) in β, the quantile regression vector $\beta = \beta(\tau)$ solves the first-order condition

$$\mathcal{D}(\beta) := E[\mathcal{D}(X, \beta)] = 0,$$

where

$$\mathcal{D}(X, \beta) := E\big[\big(\mathbb{1}\{\epsilon_\tau < \Delta_\tau(X, \beta)\} - \tau\big)X\big|X\big].$$

An argument similar to that used to establish equation (6) yields

$$\begin{aligned}\mathcal{D}(X,\beta) &= \big(F_{\epsilon_\tau}\big(\Delta_\tau(X,\beta)|X\big) - F_{\epsilon_\tau}(0|X)\big)\cdot X\\ &= \left(\int_0^1 f_{\epsilon_\tau}\big(u\Delta_\tau(X,\beta)|X\big)\,du\right)\cdot \Delta_\tau(X,\beta)\cdot X\\ &= 2\cdot \bar{w}_\tau(X,\beta)\cdot \Delta_\tau(X,\beta)\cdot X,\end{aligned}$$

where we also use the definition of $\bar{w}_\tau(X;\beta)$. The functions $\mathcal{F}(\beta)$ and $\mathcal{D}(\beta)$ are therefore identical. Since $\beta = \beta(\tau)$ uniquely satisfies $\mathcal{D}(\beta) = 0$, it also uniquely satisfies $\mathcal{F}(\beta) = 0$. As a result, $\beta = \beta(\tau) = \bar{\beta}(\tau)$ is the unique solution to both (10) and (11). *Q.E.D.*

The point of Theorem 2 is that QR solves a weighted least squares approximation problem

$$\beta(\tau) = \arg\min_{\beta\in\mathbb{R}^d} E[\bar{w}_\tau(X)\cdot \Delta_\tau^2(X,\beta)], \tag{12}$$

where the weight $\bar{w}_\tau(X) = \bar{w}_\tau(X,\beta(\tau))$ is a function of X only (in Theorem 1, the weight depended on the coefficient β). Theorem 2 also shows that the QR coefficient is the unique fixed point to an iterated minimum distance approximation. As in Theorem 1, the weighting function $\bar{w}_\tau(X,\beta(\tau))$ is related to the conditional density of the dependent variable. In particular, when the response variable has a smooth conditional density around the relevant quantile, we have, by a Taylor approximation,

$$\begin{aligned}&\bar{w}_\tau(X,\beta(\tau)) = 1/2\cdot f_Y\big(Q_\tau(Y|X)|X\big) + \bar{\varrho}_\tau(X),\\ &|\bar{\varrho}_\tau(X)| \le 1/4\cdot \big|\Delta_\tau(X,\beta(\tau))\big|\cdot \bar{f}'(X),\end{aligned}$$

where $\bar{\varrho}_\tau(X)$ is a remainder term and the density $f_Y(y|X)$ is assumed to have a first derivative in y bounded in absolute value by $\bar{f}'(X)$ a.s. When either $\Delta_\tau(X,\beta(\tau))$ or $\bar{f}'(X)$ is small, we then have

$$\bar{w}_\tau(X,\beta(\tau)) \approx w_\tau(X,\beta(\tau)) \approx 1/2\cdot f_Y\big(Q_\tau(Y|X)|X\big).$$

The approximate weighting function is therefore the same as derived using Theorem 1.

2.3. *Partial Quantile Regression and Omitted Variables Bias*

Partial quantile regression is defined with regard to a partition of the regressor vector X into a variable X_1 and the remaining variables X_2, along with the corresponding partition of QR coefficients $\beta(\tau)$ into $\beta_1(\tau)$ and $\beta_2(\tau)$. We can now decompose $Q_\tau(Y|X)$ and X_1 using orthogonal projections onto X_2 weighted by $\bar{w}_\tau(X) := \bar{w}(X, \beta(\tau))$ defined in Theorem 2:

$$\begin{aligned} &Q_\tau(Y|X) = X_2'\pi_Q + q_\tau(Y|X), \quad \text{where} \\ &E\big[\bar{w}_\tau(X)\cdot X_2 \cdot q_\tau(Y|X)\big] = 0, \\ &X_1 = X_2'\pi_1 + V_1, \quad \text{where} \quad E[\bar{w}_\tau(X)\cdot X_2 \cdot V_1] = 0. \end{aligned}$$

In this decomposition, $q_\tau(Y|X)$ and V_1 are residuals created by a weighted linear projection of $Q_\tau(Y|X)$ and X_1 on X_2, respectively, using $\bar{w}_\tau(X)$ as the weight.[7] Standard least squares algebra then gives

$$\beta_1(\tau) = \arg\min_{\beta_1} E\big[\bar{w}_\tau(X)\big(q_\tau(Y|X) - V_1\beta_1\big)^2\big]$$

and also $\beta_1(\tau) = \arg\min_{\beta_1} E[\bar{w}_\tau(X)(Q_\tau(Y|X) - V_1\beta_1)^2]$. This shows that $\beta_1(\tau)$ is a partial quantile regression coefficient in the sense that it can be obtained from a weighted least squares regression of $Q_\tau(Y|X)$ on X_1, once we have partialled out the effect of X_2. Both the first-step and second-step regressions are weighted by $\bar{w}_\tau(X)$.

We can similarly derive an omitted variables bias formula for QR. In particular, suppose we are interested in a quantile regression with explanatory variables $X = [X_1', X_2']'$, but X_2 is not available, e.g., a measure of ability or family background in a wage equation. We run QR on X_1 only, obtaining the coefficient vector $\gamma_1(\tau) = \arg\min_{\gamma_1} E[\rho_\tau(Y - X_1'\gamma_1)]$. The long regression coefficient vectors are given by $(\beta_1(\tau)', \beta_2(\tau)')' = \arg\min_{\beta_1,\beta_2} E[\rho_\tau(Y - X_1'\beta_1 - X_2'\beta_2)]$. Then

$$\gamma_1(\tau) = \beta_1(\tau) + \big(E[\tilde{w}_\tau(X)\cdot X_1X_1']\big)^{-1}E[\tilde{w}_\tau(X)\cdot X_1R_\tau(X)],$$

where $R_\tau(X) := Q_\tau(Y|X) - X_1'\beta_1(\tau)$, $\tilde{w}_\tau(X) := \int_0^1 f_{\epsilon_\tau}(u\cdot\Delta_\tau(X,\gamma_1(\tau))|X)\,du/2$, $\Delta_\tau(X,\gamma_1) := X_1'\gamma_1 - Q_\tau(Y|X)$, and $\epsilon_\tau := Y - Q_\tau(Y|X)$.[8] Here $R_\tau(X)$ is the part of the CQF not explained by the linear function of X_1 in the long QR. If the CQF is linear, then $R_\tau(X) = X_2'\beta_2(\tau)$. The proof of this result is similar to the derivation above and therefore is omitted.

As with OLS short and long calculations, the omitted variables formula in this case shows the short QR coefficients to be equal to the corresponding long QR coefficients plus the coefficients in weighted projections of omitted effects on included variables. Although the parallel with OLS seems clear, there are two complications in the QR case. First, the effect of omitted variables appears through the remainder term, $R_\tau(X)$. In practice, it seems reasonable to think

[7]Thus, $\pi_Q = E[\bar{w}_\tau(X)X_2X_2']^{-1}E[\bar{w}_\tau(X)X_2Q_\tau(Y|X)]$ and $\pi_1 = E[\bar{w}_\tau(X)X_2X_2']^{-1}\times E[\bar{w}_\tau(X)X_2X_1]$.

[8]Note that the weights in this case depend on how the regressor vector is partitioned.

of this as being approximated by the omitted linear part, $X_2'\beta_2(\tau)$. Second, the regression of omitted variables on included variables is weighted by $\tilde{w}_\tau(X)$, whereas for OLS it is unweighted.[9]

3. SAMPLING PROPERTIES OF QR UNDER MISSPECIFICATION

Paralleling the interest in robust inference methods for OLS, it is also of interest to know how specification error affects inference for QR. In this case, inference under misspecification means the use of distribution theory for quantile regressions in large samples without imposing the restriction that the CQF is linear. Although not consistent for the true nonlinear CQF, quantile regression consistently estimates the approximations to the CQF given in Theorems 1 and 2. We would therefore like to quantify the sampling uncertainty in estimates of these approximations. This question can be compactly and exhaustively addressed by obtaining the large sample distribution of the sample quantile regression process, which is defined by considering all or many sample quantile regressions at the same time.

The entire QR process is of interest here because we would like to either test global hypotheses about (approximations to) conditional distributions or make comparisons across different quantiles. Therefore, our interest is not confined to a specific quantile. The second motivation for studying the QR process comes from the fact that formal statistical comparisons across quantiles, often of interest in empirical work, require the construction of simultaneous (joint) confidence regions. Process methods provide a natural and simple way to construct these regions.

The QR process $\hat{\beta}(\cdot)$ is formally defined as

$$\hat{\beta}(\tau) \in \arg\min_{\beta\in\mathbb{R}^d} n^{-1}\sum_{i=1}^{n}\rho_\tau(Y_i - X_i'\beta),\tag{13}$$
$$\tau \in \mathcal{T} := \text{a closed subset of } [\epsilon, 1-\epsilon] \quad \text{for} \quad \epsilon > 0.$$

Gutenbrunner and Jurečková (1992) and Koenker and Xiao (2002) previously focused on QR process inference in correctly specified models, whereas earlier treatments of specification error discussed only pointwise inference for a single quantile coefficient (Hahn (1997) and Kim and White (2002)). As it turns out, the empirical results in the next section show misspecification has a larger effect on process inference than on pointwise inference. Our main theoretical result on inference is as follows:

[9]The formula obtained above can be used to determine the bias from measurement error in regressors by treating the measurement error as the omitted variable. This suggests that classical measurement error is likely to generate an attenuation bias in QR as well as OLS estimates. We thank Arthur Lewbel for pointing this out.

THEOREM 3: *Suppose that* (i) $(Y_i, X_i, i \leq n)$ *are independent and identically distributed on the probability space* $(\Omega, \mathcal{F}, P)$ *for each* n, (ii) *the conditional density* $f_Y(y|X = x)$ *exists, and is bounded and uniformly continuous in* y, *uniformly in* x *over the support of* X, (iii) $J(\tau) := E[f_Y(X'\beta(\tau)|X)XX']$ *is positive definite for all* $\tau \in \mathcal{T}$, *and* (iv) $E\|X\|^{2+\varepsilon} < \infty$ *for some* $\varepsilon > 0$. *Then the quantile regression process is uniformly consistent,* $\sup_{\tau\in\mathcal{T}} \|\hat{\beta}(\tau) - \beta(\tau)\| = o_p(1)$, *and* $J(\cdot)\sqrt{n}(\hat{\beta}(\cdot) - \beta(\cdot))$ *converges in distribution to a zero mean Gaussian process* $z(\cdot)$, *where* $z(\cdot)$ *is defined by its covariance function* $\Sigma(\tau, \tau') := E[z(\tau)z(\tau')']$ *with*

$$\Sigma(\tau, \tau') = E\big[\big(\tau - \mathbb{1}\{Y < X'\beta(\tau)\}\big)\big(\tau' - \mathbb{1}\{Y < X'\beta(\tau')\}\big)XX'\big]. \tag{14}$$

If the model is correctly specified, i.e., $Q_\tau(Y|X) = X'\beta(\tau)$ *a.s., then* $\Sigma(\tau, \tau')$ *simplifies to*

$$\Sigma_0(\tau, \tau') := [\min(\tau, \tau') - \tau\tau'] \cdot E[XX']. \tag{15}$$

The proof of this theorem, in the Appendix, proceeds by establishing the uniform consistency of the QR process $\tau \mapsto \hat{\beta}(\tau)$. Then, noting that the class of functions $(\tau, \beta) \mapsto (\tau - \mathbb{1}(Y < X'\beta))X$ is Donsker, the estimating equation for the QR process, $n^{-1/2}\sum_{i=1}^n[\tau - \mathbb{1}(Y_i \leq X_i\hat{\beta}(\tau))]X_i = o_p(1)$, is expanded in $\hat{\beta}(\tau)$ around $\beta(\tau)$ as $J(\tau)\sqrt{n}(\hat{\beta}(\tau) - \beta(\tau)) = n^{-1/2}\sum_{i=1}^n[\mathbb{1}(Y_i \leq X_i\beta(\tau)) - \tau]X_i + o_p(1)$ uniformly in $\tau \in \mathcal{T}$. The conclusion of the theorem follows by the central limit theorem for empirical processes indexed by Donsker classes of functions (Van der Vaart and Wellner (1996)).

Theorem 3 establishes joint asymptotic normality for the entire QR process. The theorem allows for misspecification and imposes little structure on the underlying conditional quantile function (e.g., smoothness of $Q_\tau(Y|X)$ in X, required for a fully nonparametric approach, is not needed here). The result in the theorem states that the limiting distribution of the QR process (and of any single QR coefficient) will, in general, be affected by misspecification. In particular, the covariance function that describes the limiting distribution is generally different from the covariance function that arises under correct specification. The simplest corollary of Theorem 1 is that any finite collection of $\sqrt{n}(\hat{\beta}(\tau_k) - \beta(\tau_k))$, $k = 1, 2, \ldots$, is asymptotically jointly normal, with asymptotic covariance between the kth and lth subsets equal to $J(\tau_k)^{-1}\Sigma(\tau_k, \tau_l)J(\tau_l)^{-1}$. For a single quantile this gives the formula $J(\tau)^{-1}\Sigma(\tau, \tau)J(\tau)^{-1}$, previously derived by Hahn (1997).

Inference on the QR process is useful for testing basic hypotheses of the form

$$R(\tau)'\beta(\tau) = r(\tau) \quad \text{for all } \tau \in \mathcal{T}. \tag{16}$$

For example, we may be interested in whether a variable or a subset of variables $j \in \{k + 1, \ldots, d\}$ enters models for all conditional quantiles with zero coefficients, i.e., whether $\beta_j(\tau) = 0$ for all $\tau \in \mathcal{T}$ and $j \in \{k+1, \ldots, d\}$. This corresponds to $R(\tau)' = [\, 0_{(d-k)\times k} \quad I_{d-k}\,]$ and $r(\tau) = 0_{d-k}$. Similarly, we may want to construct simultaneous (uniform) confidence intervals for linear functions of parameters

$$R(\tau)'\beta(\tau) - r(\tau) \quad \text{for all } \tau \in \mathcal{T}.$$

Theorem 3 has direct consequences for these sorts of confidence intervals and hypothesis tests because it implies that $(EXX')^{-1}\Sigma(\tau, \tau') \neq [\min(\tau, \tau') - \tau\tau'] \cdot I_d$. That is, the covariance function under misspecification is not proportional to the covariance function of the standard d-dimensional Brownian bridge that arises in the correctly specified case. Hence, unlike in the correctly specified case, the critical values for confidence regions and tests are not distribution-free and cannot be obtained from standard tabulations based on the Brownian bridge. However, the following corollaries facilitate both testing and the construction of confidence intervals under misspecification:

COROLLARY 1: *Define* $V(\tau) := R(\tau)'J(\tau)^{-1}\Sigma(\tau,\tau)J(\tau)^{-1}R(\tau)$ *and* $|x| := \max_j |x_j|$. *Under the conditions of Theorem* 3, *the Kolmogorov statistic* $\mathcal{K}_n := \sup_{\tau\in\mathcal{T}} |V(\tau)^{-1/2}\sqrt{n}(R(\tau)'\hat{\beta}(\tau) - r(\tau))|$ *for testing* (16) *converges in distribution to a random variable* $\mathcal{K} := \sup_{\tau\in\mathcal{T}} |V(\tau)^{-1/2}R(\tau)'J(\tau)^{-1}z(\tau)|$ *with an absolutely continuous distribution. The result is not affected by replacing* $J(\tau)$ *and* $\Sigma(\tau,\tau)$ *with estimates that are consistent uniformly in* $\tau \in \mathcal{T}$.

Thus, Kolmogorov-type statistics have a well-behaved limit distribution.[10] Unlike in the correctly specified case, however, this distribution is nonstandard. Nevertheless, critical values and simultaneous confidence regions can be obtained as follows:

COROLLARY 2: *For* $\kappa(\alpha)$ *denoting the* α *quantile of* $\mathcal{K}$ *and* $\hat{\kappa}(\alpha)$ *any consistent estimate of it* (*e.g., the estimate defined below*), $\lim_{n\to\infty} P\{(R(\tau)'\beta(\tau) - r(\tau)) \in \widehat{I}_n(\tau)$, *for all* $\tau \in \mathcal{T}\} = \alpha$, *where* $\widehat{I}_n(\tau) = [u(\tau) : |V(\tau)^{-1/2}\sqrt{n}(R(\tau)' \times \hat{\beta}(\tau) - r(\tau) - u(\tau))| \leq \hat{\kappa}(\alpha)]$. *If* $R(\tau)'\beta(\tau) - r(\tau)$ *is scalar, the simultaneous confidence interval is* $\widehat{I}_n(\tau) = [R(\tau)'\hat{\beta}(\tau) - r(\tau) \pm \hat{\kappa}(\alpha) \cdot \sqrt{V(\tau)}/\sqrt{n}\,]$. *This result is not affected by replacing* $V(\tau)$ *with an estimate that is consistent uniformly in* $\tau \in \mathcal{T}$.

A consistent estimate of the critical value, $\hat{\kappa}(\alpha)$, can be obtained by subsampling. Let $j = 1, \ldots, B$ index B randomly chosen subsamples of $((Y_i, X_i), i \leq n)$ of size b, where $b \to \infty$, $b/n \to 0$, and $B \to \infty$ as $n \to \infty$. Compute

[10]In practice, by stochastic equicontinuity of the QR process, we can replace any continuum of quantile indices $\mathcal{T}$ by a finite grid $\mathcal{T}_{K_n}$, where the distance between adjacent grid points goes to zero as $n \to \infty$.

the test statistic for each subsample as $K_j = \sup_{\tau\in\mathcal{T}} |\hat{V}(\tau)^{-1/2}\sqrt{b}R(\tau)'(\widehat{\beta}_j(\tau) - \widehat{\beta}(\tau))|$, where $\widehat{\beta}_j(\tau)$ is the QR estimate using the jth subsample. Then set $\widehat{\kappa}(\alpha)$ to be the α quantile of $\{K_1, \dots, K_B\}$. Chernozhukov and Fernández-Val (2005) discuss subsampling for QR inference in greater detail.

Finally, the inference procedure above requires estimators of $\Sigma(\tau, \tau')$ and $J(\tau)$ that are uniformly consistent in $(\tau, \tau') \in \mathcal{T} \times \mathcal{T}$. These are given by

$$\hat{\Sigma}(\tau, \tau') = n^{-1}\sum_{i=1}^{n}\bigl(\tau - \mathbb{1}\{Y_i \le X_i'\hat{\beta}(\tau)\}\bigr)\bigl(\tau' - \mathbb{1}\{Y_i \le X_i'\hat{\beta}(\tau')\}\bigr) \cdot X_iX_i',$$

$$\hat{J}(\tau) = (2nh_n)^{-1}\sum_{i=1}^{n}\mathbb{1}\bigl\{|Y_i - X_i'\hat{\beta}(\tau)| \le h_n\bigr\} \cdot X_iX_i',$$

where $\hat{\Sigma}(\tau, \tau')$ differs from its usual counterpart $\hat{\Sigma}_0(\tau, \tau') = [\min(\tau, \tau') - \tau\tau'] \cdot n^{-1}\sum_{i=1}^{n} X_iX_i'$ used in the correctly specified case and where $\hat{J}(\tau)$ is Powell's (1986) estimator of the Jacobian, with h_n such that $h_n \to 0$ and $h_n^2 n \to \infty$. Koenker (1994) suggested $h_n = C \cdot n^{-1/3}$ and provided specific choices of C. The Appendix shows that these estimates are consistent uniformly in (τ, τ') under the additional condition that $E\|X\|^4$ is finite.

4. APPLICATION TO U.S. WAGE DATA

In this section we study the approximation properties of QR in widely used U.S. Census micro data sets.[11] The main purpose of this section is to show that linear QR indeed provides a useful approximation to the conditional distribution of wages, accurately capturing changes in the wage distribution from 1980 to 2000. We also report new substantive empirical findings that arise from the juxtaposition of data from the 2000 census with earlier years. The inference methods derived in the previous section facilitate this presentation. In our analysis, Y is the real log weekly wage for U.S.-born men aged 40–49, calculated as the log of reported annual income from work divided by weeks worked in the previous year, and the regressor X consists of a years-of-schooling variable and other basic controls.[12]

[11] The data were drawn from the 1% self-weighted 1980 and 1990 samples, and the 1% weighted 2000 sample, all from the Integrated Public Use Microdata Series website (Ruggles et al. (2004)). The sample consists of U.S.-born black and white men aged 40–49 with five or more years of education, positive annual earnings, and positive hours worked in the year preceding the census. Individuals with imputed values for age, education, earnings or weeks worked were also excluded from the sample. The resulting sample sizes are 65,023, 86,785, and 97,397 for 1980, 1990, and 2000.

[12] Annual income is expressed in 1989 dollars using the Personal Consumption Expenditures Price Index. The schooling variable for 1980 corresponds to the highest grade of school com-

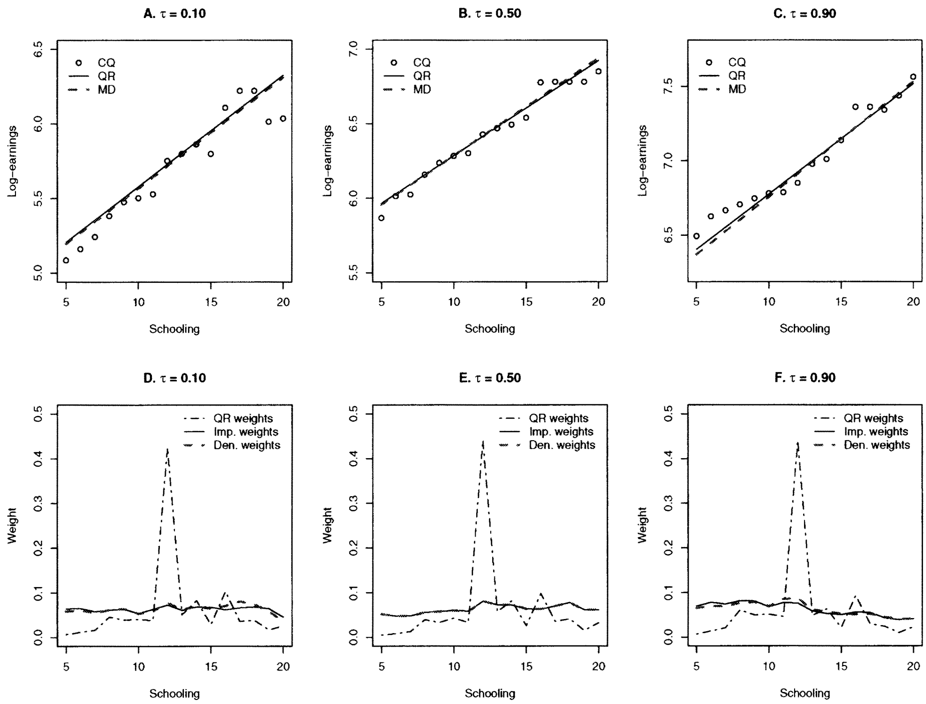

FIGURE 1.—Conditional quantile function and weighting schemes in the 1980 Census (for U.S.-born black and white men aged 40–49). Panels A–C plot the conditional quantile function, the linear quantile regression fit, and Chamberlain's minimum distance fit for log earnings given years of schooling. Panels D–F plot the QR weighting function (histogram × importance weights), the importance weights, and the density weights.

The nature of the QR approximation property is illustrated in Figure 1. Panels A–C plot a nonparametric estimate of the conditional quantile function $Q_\tau(Y|X)$, along with the linear QR fit for the 0.10, 0.50, and 0.90 quantiles, where X includes only the schooling variable. Here we take advantage of the discreteness of schooling and the large census sample to compare QR fits to the nonlinear CQFs computed at each point in the support of X. We focus on the 1980 data for this figure because the 1980 Census has a true highest grade completed variable, whereas for more recent years this must be imputed. It should be noted, however, that the approximation results for the 1990 and 2000 censuses are similar.

Our theorems establish that QR implicitly provides a weighted minimum distance approximation to the true nonlinear CQF. It is therefore useful to

pleted. The categorical schooling variables in the 1990 and 2000 Census were converted to years of schooling using essentially the same coding scheme as in Angrist and Krueger (1999). See Angrist, Chernozhukov, and Fernández-Val (2006) for details.

compare the QR fit to an explicit minimum distance (MD) fit similar to that discussed by Chamberlain (1994).[13] The MD estimator for QR is the sample analog of the vector $\tilde{\beta}(\tau)$ that solves

$$\tilde{\beta}(\tau) = \arg\min_{\beta \in \mathbb{R}^d} E\big[\big(Q_\tau(Y|X) - X'\beta\big)^2\big] = \arg\min_{\beta \in \mathbb{R}^d} E[\Delta_\tau^2(X, \beta)].$$

In other words, $\tilde{\beta}(\tau)$ is the slope of the linear regression of $Q_\tau(Y|X)$ on X, weighted only by the probability mass function of X, $\pi(x)$. In contrast to QR, this MD estimator relies on the ability to estimate $Q_\tau(Y|X)$ in a nonparametric first step, which, as noted by Chamberlain (1994), may be feasible only when X is low dimensional, the sample size is large, and sufficient smoothness of $Q_\tau(Y|X)$ is assumed.

Figure 1 plots this MD fit with a dashed line. The QR and MD regression lines are close, as predicted by our approximation theorems, but they are not identical because the additional weighting by $w_\tau(X, \beta)$ in the QR fit accentuates the quality of the fit at values of X where Y is more densely distributed near true quantiles. To further investigate the QR weighting function, panels D–F in Figure 1 plot the overall QR weights $w_\tau(X, \beta(\tau)) \cdot \pi(X)$ against the regressor X. The panels also show estimates of the importance weights from Theorem 1, $w_\tau(X, \beta(\tau))$, and their density approximations, $f_Y(Q_\tau(Y|X)|X)/2$.[14] The importance weights and the actual density weights are fairly close. The importance weights are stable across X and tend to accentuate the middle of the distribution a bit more than other parts. The overall weighting function ends up placing the highest weight on 12 years of schooling, implying that the linear QR fit should be the best in the middle of the design.

Also of interest is the ability of QR to track changes over time in quantile-based measures of conditional inequality. The column labeled CQ in panel A of Table I shows nonparametric estimates of the average 90–10 quantile spread conditional on schooling, potential experience, and race. This spread increased

[13]See Ferguson (1958) and Rothenberg (1973) for general discussions of MD. Buchinsky (1994) and Bassett, Tam, and Knight (2002) present other applications of MD to quantile problems.

[14]The importance weights defined in Theorem 1 are estimated at $\beta = \hat{\beta}(\tau)$ as

$$\widehat{w}_\tau(X, \widehat{\beta}(\tau)) = (1/(U+1)) \times \sum_{u=0}^{U} \big[(1 - u/U) \cdot \widehat{f}_Y\big((u/U) \cdot X'\widehat{\beta}(\tau) + (1 - u/U) \cdot \widehat{Q}_\tau(Y|X)|X\big)\big], \tag{17}$$

where U is set to 100; $\hat{f}_Y(y|X)$ is a kernel density estimate of $f_Y(y|X)$, which employs a Gaussian kernel and Silverman's rule for bandwidth; $\widehat{Q}_\tau(Y|X)$ is a nonparametric estimate of $Q_\tau(Y|X)$ for each cell of the covariates X; and $X'\widehat{\beta}(\tau)$ is the QR estimate. Approximate weights are calculated similarly. (See Angrist, Chernozhukov, and Fernández-Val (2006) for further details.) Weights based on Theorem 2 are similar and therefore are not shown.

TABLE I

COMPARISON OF CQF AND QR-BASED INTERQUANTILE SPREADS

		Interquantile Spread					
		90–10		90–50		50–10	
Census	Obs.	CQ	QR	CQ	QR	CQ	QR
		A. Overall					
1980	65,023	1.20	1.19	0.51	0.52	0.68	0.67
1990	86,785	1.35	1.35	0.60	0.61	0.75	0.74
2000	97,397	1.43	1.45	0.67	0.70	0.76	0.75
		B. High school graduates					
1980	25,020	1.09	1.17	0.44	0.50	0.65	0.67
1990	22,837	1.26	1.31	0.52	0.55	0.74	0.76
2000	25,963	1.29	1.32	0.59	0.60	0.70	0.72
		C. College graduates					
1980	7,158	1.26	1.19	0.61	0.54	0.65	0.64
1990	15,517	1.44	1.38	0.70	0.66	0.74	0.72
2000	19,388	1.55	1.57	0.75	0.80	0.80	0.78

Notes: The sample consist of U.S.-born black and white men aged 40–49. The table shows average measures calculated using the distribution of the covariates in each year. The covariates are schooling, race, and a quadratic function of experience. Sampling weights were used for the 2000 Census.

from 1.2 to about 1.35 from 1980 to 1990, and then to about 1.43 from 1990 to 2000. Quantile regression estimates match this almost perfectly, not surprisingly because an implication of our theorems is that QR should fit (weighted) average quantiles exactly. The fit is not as good, however, when averages are calculated for specific schooling groups, as reported in panels B and C of the table. These results highlight the fact that QR is only an approximation. Table I also documents two important substantive findings, apparent in both the CQ and QR estimates. First, the table shows conditional inequality increasing in both the upper and lower halves of the wage distribution from 1980 to 1990, but in the top half only from 1990 to 2000. Second, the increase in conditional inequality since 1990 has been much larger for college graduates than for high school graduates.

Figure 2 provides a useful complement to, and a partial explanation for, the patterns and changes in Table I. In particular, Panel A of the figure shows estimates of the schooling coefficient quantile process, along with robust simultaneous 95% confidence intervals. These estimates are from quantile regressions of log earnings on schooling, race, and a quadratic function of experience, using data from the 1980, 1990, and 2000 censuses.[15] The robust simultaneous

[15]The simultaneous bands were obtained by subsampling using 500 repetitions with subsample size $b = 5n^{2/5}$ and a grid of quantiles $\mathcal{T}_n = \{0.10, 0.11, \dots, 0.90\}$.

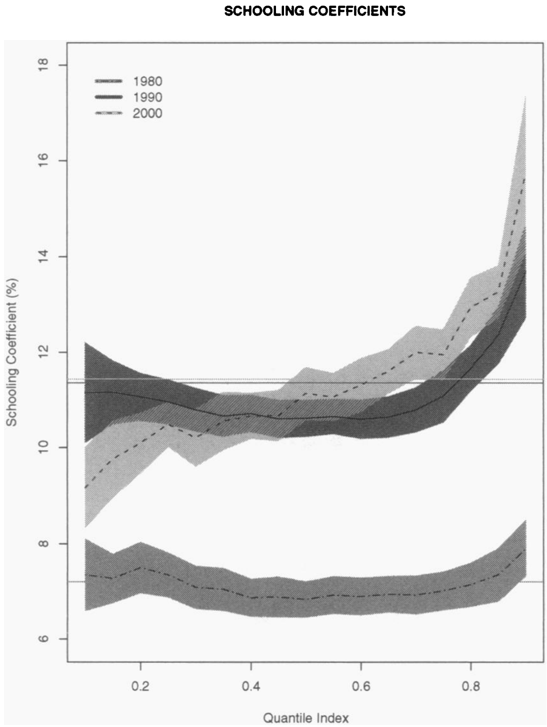

FIGURE 2A.—Schooling coefficients in the 1980, 1990, and 2000 Censuses (for U.S.-born black and white men aged 40–49). The figure shows the quantile process for the QR of log earnings on years of schooling, race, and a quadratic function of experience; robust simultaneous 95% confidence bands are given by the shaded regions. The horizontal lines indicate OLS estimates of the schooling coefficients.

confidence intervals allow us to asses the significance of changes in schooling coefficients across quantiles and across years. The horizontal lines in the figure indicate the corresponding OLS estimates.

The figure suggests the returns to schooling were low and essentially constant across quantiles in 1980, a finding similar to Buchinsky's (1994) using Current Population Surveys for this period. On the other hand, the returns increased sharply and became much more heterogeneous in 1990 and especially in 2000, a result we also confirmed in Current Population Survey data. Because the simultaneous confidence bands do not contain a horizontal line, we reject the hypothesis of constant returns to schooling for 1990 and 2000. The fact that there are quantile segments where the simultaneous bands do not overlap indicates statistically significant differences across years at those segments. For instance, the 1990 band does not overlap with the 1980 band, suggesting a marked and statistically significant change in the relationship between school-

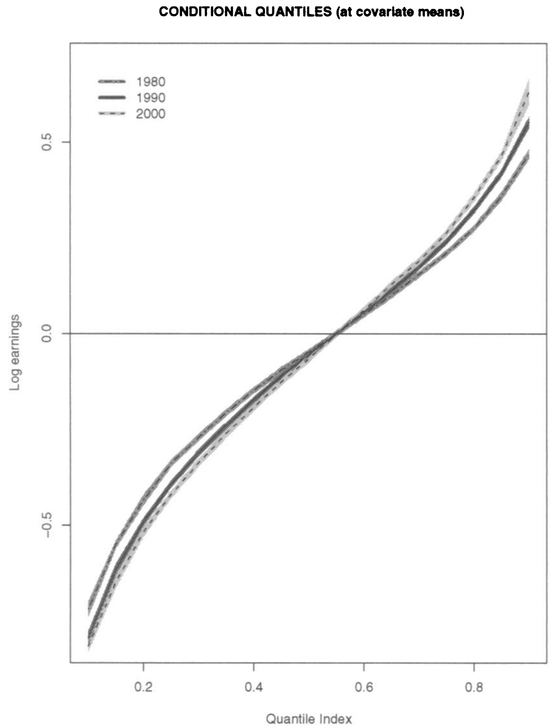

FIGURE 2B.—Conditional quantiles of log earnings in the 1980, 1990, and 2000 Censuses (for U.S.-born black and white men aged 40–49). The figure shows simultaneous 95% confidence bands for the QR approximation to the conditional quantile function given schooling, race, and a quadratic function of experience. Covariates are evaluated at sample mean values for each year, and distributions are centered at median earnings for each year (i.e., for each quantile τ and year, $E[X]'(\widehat{\beta}(\tau) - \widehat{\beta}(0.5))$ is plotted).

ing and the conditional wage distribution in this period.[16] The apparent twist in the schooling coefficient process explains why inequality increased for college graduates from 1990 to 2000. In the 2000 census, higher education was associated with increased wage dispersion to a much greater extent than in earlier years.

Another view of the stylized facts laid out in Table I is given in Figure 2B. This figure plots changes in the approximate conditional quantiles, based on a QR fit, with covariates evaluated at their mean values for each year. The figure also shows simultaneous 95% confidence bands. This figure provides a

[16]Due to the independence of samples across Census years, the test that looks for overlapping in two 95% confidence bands has a significance level of about 10%, namely $1 - 0.95^2$. Alternately, an α-level test can be based on a simultaneous α-level confidence band for the difference in quantile coefficients across years, again constructed using Theorem 3.

visual representation of the finding that between 1990 and 2000 conditional wage inequality increased more in the upper half of the wage distribution than in the lower half, whereas between 1980 and 1990 the increase in inequality occurred in both tails. Changes in schooling coefficients across quantiles and years, sharper above the median than below, clearly contributed to the fact that recent (conditional) inequality growth has been mostly confined to the upper half of the wage distribution.

Finally, it is worth noting that the simultaneous bands differ from the corresponding pointwise bands (the latter are not plotted). Moreover, the simultaneous bands allow multiple comparisons across quantiles without compromising confidence levels. Even more importantly in our context, accounting for misspecification substantially affects the width of simultaneous confidence intervals in this application. Uniform bands calculated assuming correct specification can be constructed using the critical values for the Kolmogorov statistic $\mathcal{K}$ reported in Andrews (1993). In this case, the resulting bands for the schooling coefficient quantile process are 26%, 23%, and 32% narrower than the robust bands plotted in Figure 2A for 1980, 1990, and 2000.[17]

5. SUMMARY AND CONCLUSIONS

We have shown how linear quantile regression provides a weighted least squares approximation to an unknown and potentially nonlinear conditional quantile function, much as OLS provides a least squares approximation to a nonlinear conditional expectation function. The QR approximation property leads to a partial quantile regression relationship and an omitted variables bias formula analogous to those for OLS. Although misspecification of the CQF functional form does not affect the usefulness of QR, it does have implications for inference. We therefore also present a misspecification-robust distribution theory for the QR process. This provides a foundation for simultaneous confidence intervals and a basis for global tests of hypotheses about distributions.

An analysis of wage data from the U.S. census data illustrates the sense in which QR fits the CQF. The empirical example also shows that QR accurately captures changes in the wage distribution from 1980 to 2000. An important substantive finding is the sharp twist in schooling coefficients across quantiles in the 2000 census. We use simultaneous confidence bands that are robust to misspecification to show that this pattern is highly significant. A related finding is that most inequality growth after 1990 has been in the upper part of the wage distribution.

[17]The simultaneous bands take the form of $\hat{\beta}(\tau) \pm \hat{\kappa}(\alpha) \cdot$ robust std.error($\hat{\beta}(\tau)$). Using the procedure described in Corollary 2, we obtain estimates for $\hat{\kappa}(0.05)$ of 3.78, 3.70, and 3.99 for 1980, 1990, and 2000. The simultaneous bands that impose correct specification take the form $\hat{\beta}(\tau) \pm \kappa_0(\alpha) \cdot$ std.error($\hat{\beta}(\tau)$), where $\kappa_0(\alpha)$ is the α quantile of the supremum of (the absolute value of) a standardized tied-down Bessel process of order 1. For example, $\kappa_0(0.05) = (9.31)^{1/2} = 3.05$ from Table I in Andrews (1993).

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APPENDIX: PROOFS OF THEOREMS AND COROLLARIES

This appendix provides proofs of the theorems and corollaries not proved in the main text. In particular, it contains proofs for Theorem 3 and the corollaries of this theorem, along with the proofs of uniform consistency for the estimators of the components of the covariance kernel.

A.1. *Proofs of Theorem 3 and Its Corollaries*

The proof has two steps.[18] The first step establishes uniform consistency of the sample QR process. The second step establishes asymptotic Gaussianity of the sample QR process.[19] For $W = (Y, X)$, let $\mathbb{E}_n[f(W)]$ denote $n^{-1}\sum_{i=1}^n f(W_i)$ and let $\mathbb{G}_n[f(W)]$ denote $n^{-1/2}\sum_{i=1}^n (f(W_i) - E[f(W_i)])$. If $\widehat{f}$ is an estimated function, $\mathbb{G}_n[\widehat{f}(W)]$ denotes $n^{-1/2}\sum_{i=1}^n (f(W_i) - E[f(W_i)])_{f=\widehat{f}}$. For a matrix A, mineig$[A]$ denotes the minimum eigenvalue of A.

A.1.1. *Uniform consistency of* $\hat{\beta}(\cdot)$

For each τ in $\mathcal{T}$, $\hat{\beta}(\tau)$ minimizes $Q_n(\tau, \beta) := \mathbb{E}_n[\rho_\tau(Y - X'\beta) - \rho_\tau(Y - X'\beta(\tau))]$. Define $Q_\infty(\tau, \beta) := E[\rho_\tau(Y - X'\beta) - \rho_\tau(Y - X'\beta(\tau))]$. It is easy to show that $E\|X\| < \infty$ implies that $E|\rho_\tau(Y - X'\beta) - \rho_\tau(Y - X' \times \beta(\tau))| < \infty$. Therefore, $Q_\infty(\tau, \beta)$ is finite and, by the stated assumptions, it is uniquely minimized at $\beta(\tau)$ for each τ in $\mathcal{T}$.

We first show the uniform convergence, namely for any compact set $\mathcal{B}$, $Q_n(\tau, \beta) = Q_\infty(\tau, \beta) + o_{p^*}(1)$ uniformly in $(\tau, \beta) \in \mathcal{T} \times \mathcal{B}$. This statement holds pointwise by the Khinchine law of large numbers. The empirical process $(\tau, \beta) \mapsto Q_n(\tau, \beta)$ is stochastically equicontinuous because $|Q_n(\tau', \beta') - Q_n(\tau'', \beta'')| \le C_{1n} \cdot |\tau' - \tau''| + C_{2n} \cdot \|\beta' - \beta''\|$, where $C_{1n} = 2 \cdot \mathbb{E}_n\|X\| \cdot$

[18]Basic concepts used in the proof, including weak convergence in the space of bounded functions, stochastic equicontinuity, and Donsker and Vapnik–Červonenkis (VC) classes, are defined as in Van der Vaart and Wellner (1996).

[19]This step does not rely on Pollard's (1991) convexity argument, because this argument does not apply to the process case.

$\sup_{\beta\in\mathcal{B}}\|\beta\| = O_p(1)$ and $C_{2n} = 2\cdot\mathbb{E}_n\|X\| = O_p(1)$. Hence, the convergence also holds uniformly.

Next, we show uniform consistency. Consider a collection of closed balls $B_M(\beta(\tau))$ of radius M and center $\beta(\tau)$, and let $\beta_M(\tau) = \beta(\tau) + \delta_M(\tau)\cdot v(\tau)$, where $v(\tau)$ is a direction vector with unity norm $\|v(\tau)\| = 1$ and $\delta_M(\tau)$ is a positive scalar such that $\delta_M(\tau) \geq M$. Then uniformly in $\tau \in \mathcal{T}$,

$$\begin{aligned}
&(M/\delta_M(\tau))\cdot\big(Q_n(\tau,\beta_M(\tau)) - Q_n(\tau,\beta(\tau))\big)\\
&\overset{(a)}{\geq} Q_n(\tau,\beta_M^*(\tau)) - Q_n(\tau,\beta(\tau))\\
&\overset{(b)}{\geq} Q_\infty(\tau,\beta_M^*(\tau)) - Q_\infty(\tau,\beta(\tau)) + o_{p^*}(1)\\
&\overset{(c)}{>} \epsilon_M + o_{p^*}(1),
\end{aligned}$$

for some $\epsilon_M > 0$, where (a) follows by convexity in β for $\beta_M^*(\tau)$ the point of the boundary of $B_M(\beta(\tau))$ on the line connecting $\beta_M(\tau)$ and $\beta(\tau)$; (b) follows by the uniform convergence established above; and (c) follows because $\beta(\tau)$ is the unique minimizer of $Q_\infty(\beta,\tau)$ uniformly in $\tau\in\mathcal{T}$, by convexity and assumption (iii). Hence for any $M > 0$, the minimizer $\hat\beta(\tau)$ must be in the radius-M ball centered at $\beta(\tau)$ uniformly for all $\tau\in\mathcal{T}$, with probability approaching 1.

A.1.2. *Asymptotic Gaussianity of* $\sqrt{n}(\hat\beta(\cdot) - \beta(\cdot))$

First, by the computational properties of $\hat\beta(\tau)$, for all $\tau\in\mathcal{T}$ (cf. Theorem 3.3 in Koenker and Bassett (1978)) we have that $\|\mathbb{E}_n[\varphi_\tau(Y - X'\hat\beta(\tau))X]\| \leq \text{const}\cdot\sup_{i\leq n}\|X_i\|/n$, where $\varphi_\tau(u) = \tau - \mathbb{1}\{u\leq 0\}$. Note that $E\|X_i\|^{2+\varepsilon} < \infty$ implies $\sup_{i\leq n}\|X_i\| = o_{p^*}(n^{1/2})$, because $P(\sup_{i\leq n}\|X_i\| > n^{1/2}) \leq nP(\|X_i\| > n^{1/2}) \leq nE\|X_i\|^{2+\varepsilon}/n^{(2+\varepsilon)/2} = o(1)$. Hence uniformly in $\tau\in\mathcal{T}$,

$$\sqrt{n}\mathbb{E}_n\big[\varphi_\tau(Y - X'\hat\beta(\tau))X\big] = o_p(1). \tag{A.1}$$

Second, $(\tau,\beta)\mapsto\mathbb{G}_n[\varphi_\tau(Y - X'\beta)X]$ is stochastically equicontinuous over $\mathcal{B}\times\mathcal{T}$, where $\mathcal{B}$ is any compact set, with respect to the $L_2(P)$ pseudometric

$$\begin{aligned}
&\rho\big((\tau',\beta'),(\tau'',\beta'')\big)^2\\
&:= \max_{j\in 1,\dots,d} E\big[\big(\varphi_{\tau'}(Y - X'\beta')X_j - \varphi_{\tau''}(Y - X'\beta'')X_j\big)^2\big]
\end{aligned}$$

for $j\in 1,\dots,d$ indexing the components of X. Note that the functional class $\{\varphi_\tau(Y - X'\beta)X, \tau\in\mathcal{T}, \beta\in\mathcal{B}\}$ is formed as $(\mathcal{T} - \mathcal{F})X$, where $\mathcal{F} = \{\mathbb{1}\{Y\leq X'\beta\}, \beta\in\mathcal{B}\}$ is a VC subgraph class and hence a bounded Donsker class. Hence $\mathcal{T} - \mathcal{F}$ is also bounded Donsker and $(\mathcal{T} - \mathcal{F})X$ is, therefore, Donsker with a square-integrable envelope $2\cdot\max_{j\in 1,\dots,d}|X|_j$ by Theorem 2.10.6 in

Van der Vaart and Wellner (1996). Stochastic equicontinuity then is part of being Donsker.

Third, by stochastic equicontinuity of $(\tau, \beta) \mapsto \mathbb{G}_n[\varphi_\tau(Y - X'\beta)X]$ we have that

$$\text{(A.2)} \qquad \mathbb{G}_n\big[\varphi_\tau(Y - X'\hat{\beta}(\tau))X\big] = \mathbb{G}_n\big[\varphi_\tau(Y - X'\beta(\tau))X\big] + o_{p^*}(1) \quad \text{in } \ell^\infty(\mathcal{T}),$$

which follows from $\sup_{\tau\in\mathcal{T}} \|\hat{\beta}(\tau) - \beta(\tau)\| = o_{p^*}(1)$ and resulting convergence with respect to the pseudometric $\sup_{\tau\in\mathcal{T}} \rho[(\tau, \hat{\beta}(\tau)), (\tau, \beta(\tau))]^2 = o_p(1)$. The latter is immediate from $\sup_{\tau\in\mathcal{T}} \rho[(\tau, b(\tau)), (\tau, \beta(\tau))]^2 \le C_3 \cdot \sup_{\tau\in\mathcal{T}} \|b(\tau) - \beta(\tau)\|^{\varepsilon/(2+\varepsilon)}$, where $C_3 = (\bar{f} \cdot (E\|X\|^2)^{1/2})^{\varepsilon/(2(2+\varepsilon))} \cdot (E\|X\|^{2+\varepsilon})^{2/(2+\varepsilon)} < \infty$ and $\bar{f}$ is the a.s. upper bound on $f_Y(Y|X)$. (This follows by the Hölder's inequality and Taylor expansion.)

Furthermore, the following expansion is valid uniformly in $\tau \in \mathcal{T}$:

$$\text{(A.3)} \qquad E[\varphi_\tau(Y - X'\beta)X]|_{\beta=\hat{\beta}(\tau)} = [J(\tau) + o_p(1)]\big(\hat{\beta}(\tau) - \beta(\tau)\big).$$

Indeed, by Taylor expansion, $E[\varphi_\tau(Y - X'\beta)X]|_{\beta=\hat{\beta}(\tau)} = E[f_Y(X'b(\tau)| X)XX']|_{b(\tau)=\beta^*(\tau)}(\hat{\beta}(\tau) - \beta(\tau))$, where $\beta^*(\tau)$ is on the line connecting $\hat{\beta}(\tau)$ and $\beta(\tau)$ for each τ, and can be different for each row of the Jacobian matrix. Then (A.3) follows by the uniform consistency of $\hat{\beta}(\tau)$, and the assumed uniform continuity and boundedness of the mapping $y \mapsto f_Y(y|x)$, uniformly in x over the support of X.

Fourth, we have that

$$\text{(A.4)} \qquad o_p(1) = [J(\cdot) + o_p(1)]\big(\hat{\beta}(\cdot) - \beta(\cdot)\big) + \mathbb{G}_n\big[\varphi_\cdot(Y - X'\beta(\cdot))X\big],$$

because the left-hand side of (A.1) is equal to the left-hand side of $n^{1/2}$ (A.3) plus the left-hand side of (A.2). Therefore, using that $\operatorname{mineig}[J(\tau)] \ge \lambda > 0$ uniformly in $\tau \in \mathcal{T}$,

$$\text{(A.5)} \qquad \sup_{\tau\in\mathcal{T}} \big\|\mathbb{G}_n\big[\varphi_\tau(Y - X'\beta(\tau))X\big] + o_p(1)\big\| \ge (\sqrt{\lambda} + o_p(1)) \cdot \sup_{\tau\in\mathcal{T}} \sqrt{n}\|\hat{\beta}(\tau) - \beta(\tau)\|.$$

Fifth, the mapping $\tau \mapsto \beta(\tau)$ is continuous by the implicit function theorem and stated assumptions. In fact, because $\beta(\tau)$ solves $E[(\tau - \mathbb{1}\{Y \le X'\beta\})X] = 0$, $d\beta(\tau)/d\tau = J(\tau)^{-1}E[X]$. Hence $\tau \mapsto \mathbb{G}_n[\varphi_\tau(Y - X'\beta(\tau))X]$ is stochastically equicontinuous over $\mathcal{T}$ for the pseudometric given by $\rho(\tau', \tau'') := \rho((\tau', \beta(\tau')), (\tau'', \beta(\tau'')))$. Stochastic equicontinuity of $\tau \mapsto \mathbb{G}_n[\varphi_\tau(Y - X'\beta(\tau))X]$ and a multivariate central limit theorem imply that

$$\text{(A.6)} \qquad \mathbb{G}_n\big[\varphi_\cdot(Y - X'\beta(\cdot))X\big] \Rightarrow z(\cdot) \quad \text{in } \ell^\infty(\mathcal{T}),$$

where $z(\cdot)$ is a Gaussian process with covariance function $\Sigma(\cdot,\cdot)$ specified in the statement of Theorem 3. Therefore, the left-hand side of (A.5) is $O_p(n^{-1/2})$, implying $\sup_{\tau\in\mathcal{T}}\|\sqrt{n}(\hat\beta(\tau)-\beta(\tau))\| = O_{p^*}(1)$.

Finally, the latter fact and (A.4)–(A.6) imply that in $\ell^\infty(\mathcal{T})$,

$$J(\cdot)\sqrt{n}\big(\hat\beta(\cdot)-\beta(\cdot)\big) = -\mathbb{G}_n\big[\varphi_\cdot(Y-X'\beta(\cdot))\big]+o_{p^*}(1)\Rightarrow z(\cdot). \tag{A.7}$$ *Q.E.D.*

A.1.3. *Proof of corollaries*

PROOF OF COROLLARY 1: This result follows by the continuous mapping theorem in $\ell^\infty(\mathcal{T})$. Absolute continuity of $\mathcal{K}$ follows from Theorem 11.1 in Davydov, Lifshits, and Smorodina (1998). *Q.E.D.*

PROOF OF COROLLARY 2: This result follows by absolute continuity of $\mathcal{K}$. The consistency of the subsampling estimator of $\hat\kappa(\alpha)$ follows from Theorem 2.2.1 and Corollary 2.4.1 in Politis, Romano, and Wolf (1999) for the case where $V(\tau)$ is known. When $V(\tau)$ is estimated consistently uniformly in $\tau\in\mathcal{T}$, the result follows by an argument similar to the proof of Theorem 2.5.1 in Politis, Romano, and Wolf (1999). *Q.E.D.*

A.1.4. *Uniform consistency of* $\widehat{\Sigma}(\cdot,\cdot)$ *and* $\widehat{J}(\cdot)$

Here it is shown that under the conditions of Theorem 3 and the additional assumption that $E\|X\|^4<\infty$, the estimates described in the main text are consistent uniformly in $(\tau,\tau')\in\mathcal{T}\times\mathcal{T}'$.[20]

First, recall that $\hat J(\tau) = [1/(2h_n)]\cdot\mathbb{E}_n[\mathbb{1}\{|Y_i - X_i'\hat\beta(\tau)|\le h_n\}\cdot X_iX_i']$. We will show that

$$\hat J(\tau) - J(\tau) = o_{p^*}(1) \quad \text{uniformly in } \tau\in\mathcal{T}. \tag{A.8}$$

Note that $2h_n\hat J(\tau) = \mathbb{E}_n[f_i(\hat\beta(\tau),h_n)]$, where $f_i(\beta,h) = \mathbb{1}\{|Y_i - X_i'\beta|\le h\}\cdot X_iX_i'$. For any compact set B and positive constant H, the functional class $\{f_i(\beta,h),\beta\in B, h\in(0,H]\}$ is a Donsker class with a square-integrable envelope by Theorem 2.10.6 in Van der Vaart and Wellner (1996), because this is a product of a VC class $\{\mathbb{1}\{|Y_i-X_i'\beta|\le h\},\beta\in B,h\in(0,H]\}$ and a square-integrable random matrix X_iX_i' (recall $E\|X_i\|^4<\infty$ by assumption). Therefore, $(\beta,h)\mapsto\mathbb{G}_n[f_i(\beta,h)]$ converges to a Gaussian process in $\ell^\infty(B\times(0,H])$, which implies that $\sup_{\beta\in B,0<h\le H}\|\mathbb{E}_n[f_i(\beta,h)] - E[f_i(\beta,h)]\| = O_{p^*}(n^{-1/2})$. Letting B be any compact set that covers $\bigcup_{\tau\in\mathcal{T}}\beta(\tau)$, this implies $\sup_{\tau\in\mathcal{T}}\|\mathbb{E}_n[f_i(\hat\beta(\tau),h_n)] - E[f_i(\beta,h_n)]|_{\beta=\hat\beta(\tau)}\| = O_{p^*}(n^{-1/2})$. Hence (A.8) follows by using $2h_n\hat J(\tau) = \mathbb{E}_n[f_i(\hat\beta(\tau),h_n)]$, $1/(2h_n)\cdot E[f_i(\beta,h_n)]|_{\beta=\hat\beta(\tau)} = J(\tau)+o_p(1)$, and the assumption $h_n^2 n\to\infty$.

[20]Note that the result for $\widehat{J}(\tau)$ is not covered by Powell (1986) because his proof applies only pointwise in τ, whereas we require a uniform result.

Second, we can write $\hat{\Sigma}(\tau, \tau') = \mathbb{E}_n[g_i(\hat{\beta}(\tau), \hat{\beta}(\tau'), \tau, \tau')X_iX_i']$, where $g_i(\beta', \beta'', \tau', \tau'') = (\tau - \mathbb{1}\{Y_i \leq X_i'\beta'\})(\tau' - \mathbb{1}\{Y_i \leq X_i'\beta''\}) \cdot X_iX_i'$. We will show that

$$\hat{\Sigma}(\tau, \tau') - \Sigma(\tau, \tau') = o_{p^*}(1) \quad \text{uniformly in } (\tau, \tau') \in \mathcal{T} \times \mathcal{T}. \tag{A.9}$$

It is easy to verify that $\{g_i(\beta', \beta'', \tau', \tau''), (\beta', \beta'', \tau', \tau'') \in B \times B \times \mathcal{T} \times \mathcal{T}\}$ is Donsker, and hence a Glivenko–Cantelli class, for any compact set B, e.g., using Theorem 2.10.6 in Van der Vaart and Wellner (1996). This implies that $\mathbb{E}_n[g_i(\beta', \beta'', \tau', \tau'')X_iX_i'] - E[g_i(\beta', \beta'', \tau', \tau'')X_iX_i'] = o_{p^*}(1)$ uniformly in $(\beta', \beta'', \tau', \tau'') \in (B \times B \times \mathcal{T} \times \mathcal{T})$. The latter and continuity of $E[g_i(\beta', \beta'', \tau', \tau'')X_iX_i']$ in $(\beta', \beta'', \tau', \tau'')$ imply (A.9).

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