Dynamic Pricing & Revenue Management: Learning under Misspecified Model

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April 21, 2022

I. Preliminaries: Nash & Colluding

Bertrand Model

Consider a duopoly market where two sellers sell exactly the same product, each seller sets its price to maximize its revenue. The demand function is given by:

$$d_i(p_i, p_{-i}) = G_i - \alpha_i \cdot p_i + \beta_i \cdot p_{-i}, i \in \{1, 2\}$$

Assumption:

$$G_i > 0, \alpha_i > 0, \beta_i > 0$$

I. Preliminaries: Nash & Colluding

The demand function is given by:

$$d_1 = G_1 - \alpha_1 \cdot p_1 + \beta_1 \cdot p_2$$
$$d_2 = G_2 - \alpha_2 \cdot p_2 + \beta_2 \cdot p_1$$

Assumptions:

- the marginal cost is 0;
- the sellers are the same, the products are homogeneous;
- the sellers do not face any inventory or capacity constraints;
- this is a complete information static game.

I. Preliminaries: Nash & Colluding

Nash Equilibrium

In a (pure strategy) NE each seller chooses a price that is best response to its competitor's price. For each seller:

$$p_i^N = \arg\max_{p_i} \pi(p_i, p_{-i}) = \arg\max_{p_i} p_i \cdot d_i(p_i, p_{-i})$$

$$p_i^N = \frac{G_{-i}\beta_i - 2G_i\alpha_{-i}}{4\alpha_i\alpha_{-i} - \beta_i\beta_{-i}}$$

More specifically:

$$p_1^N = \frac{G_2\beta_1 - 2G_1\alpha_2}{4\alpha_1\alpha_2 - \beta_1\beta_2}, p_2^N = \frac{G_1\beta_2 - 2G_2\alpha_1}{4\alpha_1\alpha_2 - \beta_1\beta_2}$$

I. Preliminaries: Nash & Colluding

Colluding Solution

If the two sellers cooperated with each other, then they choose their prices to maximize the total revenue.

$$\arg\max_{p_i, p_{-i}} \sum_{i=1}^{2} p_i \cdot d_i(p_i, p_{-i})$$

$$p_i^C = \frac{G_{-i}(\beta_i+\beta_{-i})-2G_i\alpha_{-i}}{4\alpha_i\alpha_{-i}-(\beta_i+\beta_{-i})^2}$$

More specifically:

$$p_1^C = \frac{G_2(\beta_1+\beta_2)-2G_1\alpha_2}{4\alpha_1\alpha_2-(\beta_1+\beta_2)^2}, p_2^C = \frac{G_1(\beta_1+\beta_2)-2G_2\alpha_1}{4\alpha_1\alpha_2-(\beta_1+\beta_2)^2}$$

Mention: first-order condition is required:

$$4\alpha_1\alpha_2 - (\beta_1 + \beta_2)^2 > 0$$

I. Preliminaries: Nash & Colluding

Comparison of the two solutions:

$$\sum \pi_i(p_i^C, p_{-i}^C) \geq \sum \pi_i(p_i^N, p_{-i}^N)$$

Prisoner's Dilemma An example(symmetric parameter):

π Seller2 / Seller 1	p_2^N	p_2^C
p_1^N	(4,4)	(7,2)
p_1^C	(2,7)	(6,6)

II. Learning under Misspecified Model

Sequence of the Bertrand model

Suppose that over a sequence of time periods the sellers face repeated instances of the pricing game described above. The time is indexed by k. In period $k+1$, each seller$i$ chooses p_i^k, and the system will return d_i^{k+1} based on:

$$d_i^{k+1} = d_i(p_i, p_{-i}) + \epsilon_i^k$$
$$d_i(\cdot) = G_i - \alpha_i \cdot p_i + \beta_i \cdot p_{-i}, i \in \{1, 2\}$$

Assumption:

- ϵ_i^k is a martingale difference noise satisfy:
  - $\mathbb{E}[\epsilon_i^{k+1} | \mathcal{F}^k] = 0$
  - $\mathbb{E}[(\epsilon_i^{k+1})^2 | \mathcal{F}^k] < M$
- the underlying model is "static": meaning it doesn't modify as time goes.

II. Learning under Misspecified Model

Misspecified Leaning Model
However, we assume that the sellers assume the demand function is given by:

$$\tilde{d}_i(\cdot) = d_i(p_i)$$

rather than the "real" one: $d_i(\cdot) = d_i(p_i, p_{-i}) + \epsilon_i$

Because the seller don't consider its competitor's price's effect, so it's a misspecified model.

Assumption:

- both sellers don't know the structure of the underlying model;
- both sellers use misspecified model to predict its demand;
- both sellers don't know competitor's historical demand,revenue and parameters estimated.

II. Learning under Misspecified Model

Learning Process

A. Each seller uses a model of demand as a function of its own decisions. The model is incorrect in the sense that it does not explicitly incorporate the effect of competitors' decisions on demand or revenue.
eg.:

$$\tilde{d}_i = b_i - s_i \cdot p_i$$

B. Each seller uses data comprised only of its own past decisions and its own past demands to estimate the parameters of its model.
e.g. At period k, the seller i uses data $(p_i^0, d_i^1), (p_i^1, d_i^2), \cdot, (p_i^{k-1}, d_i^k)$ and uses OLS to estimate $\tilde{b_i^k}, \tilde{s_i^k}$.

Learning Process

C. With the parameter estimates in hand, each seller then treats its model and the associated parameter estimates as if they were correct and optimizes the objective of the model to make a decision.
eg. At period k+1 the seller i choose $p_i = \frac{\tilde{b}_i^k}{2\tilde{s}_i^k}$
D. As new data are obtained, each seller updates its parameter estimates with the hope of getting better estimates and making better decisions.
eg. At period k+1 the seller i obtain new data p_i^k, d_i^{k+1} and use them to re-optimize its parameters.

II. Learning under Misspecified Model

What is our purpose of designing this system

- Using the misspecified model without considering competitor's decision's influence, is this strategy effective for the enterprise?
- For two sellers using the same misspecified model, what situation does the market evolve to?
- Will the price converge? To NE or Colluding solution? Or any solution else? If so, why?

II. Learning under Misspecified Model

Review of the Former Research

- From 1975 to 1983, Alan Kirman designed the idea of misspecified model and proposes the model in the former example. But in his settings the sellers have symmetric underlying demand functions with $\epsilon_i = 0$. Using simulation results, he conjectured a "pseudo equilibrium": given arbitrary initial prices, the prices would converge.
- From 1992 to 1995, Kirman overturned his conjecture, pointing that the price is not converging, but moving slowly. In addition, he proved that the convergence can happen if the initial prices were chosen properly.

II. Learning under Misspecified Model

Review of the Former Research

- From 2010 to 2015, Omar Besbes focus on the misspecified model in the monopoly market and find that just using OLS has surprisingly high efficiency.
- From 2001 to now, more and more literature are studying the misspecified model in the duopoly system using different parameter estimating methods, some of them are proven to converge to optimal solution or Nash Equilibrium.

III. Example1: UCB algorithm

Model Assumption

- Both sellers have symmetric underlying demand models;
- Both sellers can only select prices from a discrete set $P = \{p_1, p_2, \ldots, p_l\}$;
- Both firms use index algorithm to decide which price to choose;
- $\epsilon_i^k \sim U[-\frac{1}{\delta}, \frac{1}{\delta}]$, δhere is called the signal to noise ratio(SNR).

III. Example1: UCB algorithm

UCB algorithm

The idea of this algorithm is to maintain an index for each arm. Intuition: For each arm p_l, at each time k we track the empirical average of the profit from that arm(Exploitation) and the number of times that arm has been pulled(Exploration).

$$I_l^k = \bar{\pi}_l^k + \sqrt{\frac{2\ln k}{n_l^k}}$$

Under this algorithm, an arm would be selected if:

- the arm's empirical average past profits is high(exploitation);
- or the number being selected is low, we select it to be more confident about its reward(exploration).

III. Example1: UCB algorithm

Simulation Process
We set $d_i = 0.48 - 0.9p_i + 0.6p_j$.

$$p_i^N = 0.4, p_i^C = 0.8$$

$$P = \{0.10, 0.11, 0.12, ;0.99, 1.00\}$$

In each trial, we consider the algorithm run 2 million times and analyse the last 1000 periods. **Conclusions:**

When the SNR is high(low), independent bandits result in colluding solution(Nash Equilibrium).

III. Example1: UCB algorithm

Simulation Results

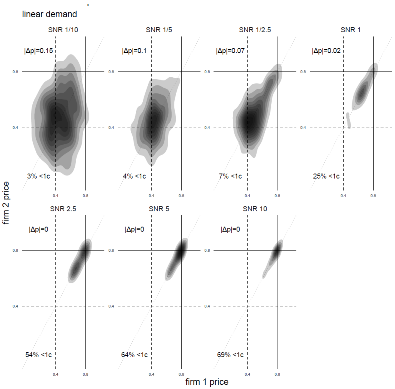

Figure 1: Simulation Results

III. Example1: UCB algorithm

Theoretical Analysis(Not proven)
First Step
We consider the underlying demand model is deterministic($\epsilon = 0$),and we let P $= \{p_i^N, p_i^C\}$, which construct a prisoner's dilemma. In this system, p_i^N is the dominant strategy but if both sellers choose p_i^C, they would have more profits.
Using UCB algorithm, there are two possibilities at the first two rounds:
1. Matched Prices,i.e.(p^C, p^C) in one round, (p^N, p^N) in another;
2. Mismatched Prices, i.e..(p^C, p^N) in one round, (p^N, p^C)in another.

III. Example1: UCB algorithm

Case1: Matched Prices

After the first two rounds, $\pi_{C,C} > \pi_{N,N}$, so in the 3rd round both sellers would choose p^C.

Because two sellers have symmetric model, so they do their first switch to p^C at time k only when:

$$\pi_{C,C} + \sqrt{\frac{2 \ln k}{k-2}} \leq \pi_{N,N} + \sqrt{2 \ln k}.$$

This condition can be reached as k is large enough, but once they switched to p^N, the next period they would go back to p^C. The time difference between choosing p^N is proven to go to infinity as k go to infinity, so the frequency of the p^N goes to zero. So the market would converge to Colluding Solution.

III. Example1: UCB algorithm

Case2: Mismatched Prices
After the first two rounds, $\pi_{N,C} > \pi_{C,N}$, so in the 3rd round both sellers would choose p^N.
It is not difficult to see that since round 3, every stage they would pick up the same price. Then they would go to the colluding solution.
There is another conclusion that when the market converge to colluding solution, then the correlation of two sellers' price would be larger.

Second Step: Intuition

The upper analysis explains half of our conclusion;
If the SNR is small, meaning the variance is large, then the two sellers may won't act the same. Then the exploitation of the index would be disturbed, under such condition sellers would realize p^N is a dominant strategy at a greater possibility, then the market would be more likely to converge to the Nash Equilibrium.

IV. Example1: OLS

Condition 1: Known Slope

Assumptions:

- the price to be selected is continuous;
- the sellers have asymmetric demand functions;
- the underlying demand function is $d_i(p_i, p_{-i}) = G_i - \alpha_i \cdot p_i + \beta_i \cdot p_{-i}, i \in \{1, 2\}$;
- the perceived demand function is $\tilde{d}_i = b_i - s_i \cdot p_i$;
- both sellers knows that $\tilde{s}_i = \alpha_i$;
- both sellers use OLS to optimize its parameters at each stage.

IV. Example1: OLS

At period k+1 each seller would get its would get its $\hat{b}_i^k$ and use $p_i^k = \frac{\hat{b}_i^k}{2\alpha_i}$

Demand Consistency

An estimator $\hat{b}_i^k$ has the property of demand consistency if:
Whenever $(p_1^k, p_2^k) \rightarrow (p_1^\infty, p_2^\infty)\ as\ k \rightarrow \infty, \tilde{d}_i^k(p_i^k) \rightarrow d_i(p_i^\infty, p_{-i}^\infty)$

Then we can get $\tilde{b}_i^\infty = \frac{1\alpha_i(2G_i\alpha_{-i})}{4\alpha_1\alpha_2 - \beta_1\beta_2}, p_i^\infty = \frac{G_{-i}\beta_i - 2G_i\alpha_{-i}}{4\alpha_i\alpha_{-i} - \beta_i\beta_{-i}}$

Proposition1: Under such system, $\lim_{k\rightarrow\infty}, \tilde{b}_i^k = \tilde{b}_i^\infty$.

IV. Example1: OLS

Condition 2: Known Intercept

Assumptions:

- both sellers knows that $\tilde{b}_i = G_i$

Using nearly the same methods in condition 1, the author proves that this would converge to the colluding solution

Condition 3: Both Unkown

Assumptions:

- both sellers don't know the real parameters;
- $\epsilon_i = 0$
- $p_i^0 \neq p_i^1$

IV. Example1: OLS

Introduced $r_i^k = \frac{Cov(\{p_i^l, p_{-i}^l\}_{l=0}^{k-1})}{Var(\{p_i^l\}_{l=0}^{k-1})}$

$R := \{(r_1, r_2) \in \mathbb{R}^2 : r_i < -\alpha_i/\beta_i$ and either $r_{-i} = r_i = 0$ or $0 < r_i r_{-i} \leq 1\}$;

$P := \{(p_1(r_1, r_2), p_2(r_1, r_2)) : (r_1, r_2) \in R\}$;

Where $p_i(r_1, r_2) = (-2G_i\alpha_{-i} + G_{-i}\beta_i - G_i\beta_{-i}r_{-i})\cdot$
$(4\alpha_{-i}\alpha_i + 2\alpha_{-i}\beta_i r_i + 2\alpha_i\beta_{-i}r_{-i} - \beta{-}i\beta_i(1 - r_{-i}r_i))^{-1}$

Proposition: For any $(p_1^*, p_2^*) \in P$, there exists initial points $(p_1^k, p_2^k), k = 0, 1, 2,$ such that the process would become stationary since period 3.

Note the the NE and colluding solution is whithin the set P.

V. Our Works

- We consider an asymmetric model;
- We assume that both sellers don't know any parameter;
- $\epsilon_i \neq 0$;
- We want to study how the system would change as SNR changes.