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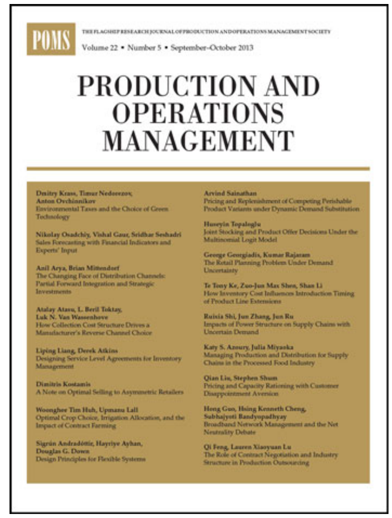

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Recent Developments in Dynamic Pricing Research: Multiple Products, Competition, and Limited Demand Information

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Abstract

Dynamic pricing enables a firm to increase revenue by better matching supply with demand, responding to shifting demand patterns, and achieving customer segmentation. In the last twenty years, numerous success stories of dynamic pricing applications have motivated a rapidly growing research interest in a variety of dynamic pricing problems in the academic literature. A large class of problems that arise in various revenue management applications involve selling a given amount of inventory over a finite time horizon without inventory replenishment. In this paper, we identify most recent trends in dynamic pricing research involving such problems. We review existing research on three new classes of problems that have attracted a rapidly growing interest in the last several years, namely, problems with multiple products, problems with competition, and problems with limited demand information. We also identify a number of possible directions for future research.

Keywords: Survey, Dynamic Pricing, Multiple Products, Competition, Limited Demand Information.

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1 Introduction

Dynamic pricing is one of the most fundamental and commonly used revenue management tools. It enables firms to increase revenue by better matching supply with demand, responding to shifting demand patterns, and achieving customer segmentation. Since its early success in the airline industry, dynamic pricing has now gained popularity in many other industries where revenue management plays a central role, including the hotel industry, the car rental industry, the cruise line industry, the entertainment industry, and the retail industry.

Most dynamic pricing problems in revenue management share the following three main characteristics. First, there is a given and finite selling season (or time horizon). Products in revenue management applications are typically time-sensitive with a fixed selling season. For example, most airlines start to sell seats several months before the departure time, and most fashion apparel has a selling season that may only last for six to eight weeks. Second, there is a given and finite amount of inventory of a product available at the beginning of the selling season to be sold over time, and no inventory replenishment is possible during the selling season. For example, airlines typically commit a particular type of aircraft to a particular flight, and hence the number of seats on each flight is fixed and no new seats can be added to a given flight. Similarly, due to a long supply lead time, retailers often make a one-time order long before the beginning of the selling season, and once the selling season starts, there is no opportunity to replenish the inventory if the demand turns out to be higher than expected. Third, pricing is dynamic in nature and the selling season consists of multiple periods such that a different price may be set for a different period rather than having a single price for the entire season (which is called fixed or static pricing). It is possible that in some extreme cases, the optimal price in each period happens to be identical and hence a fixed pricing policy is optimal.

Over the last two decades, numerous success stories of dynamic pricing applications have motivated a rapidly growing research interest in a variety of dynamic pricing models in the academic literature. In this paper, we identify major trends in the recent dynamic pricing literature for revenue management applications and give a state-of-the-art review on three emerging classes of dynamic pricing problems, most of which share the three characteristics discussed above (i.e., finite time horizon, finite initial inventory without replenishment, and dynamic pricing over time).

Several other classes of dynamic pricing problems, which do not share all these characteristics, have also been studied in the literature, but since they are not closely related to revenue management applications, most of such problems will not be reviewed in this paper. They include, among others, (i) problems where inventory replenishment occurs during the selling season, such as joint pricing-inventory and joint pricing-production problems (see survey papers, Chan et al. 2004 and Yano and Gilbert 2004, and more recent papers, e.g., Geunes et al. 2006, Ahn et al. 2007, Huh and Janakiraman 2008, and Chen and Yang 2010); (ii) problems with an infinite selling season, such as those that involve pricing and replenishing time-insensitive products over a long period of time (e.g., Chen and Simchi-Levi 2004, Chao and Zhou 2006), and those that involve selling to repeated customers with reference price effect (e.g., Popescu and Wu 2007, Huh et al. 2010); and (iii) problems with an infinite initial inventory which are studied extensively in the economics and marketing literatures. These classes of problems are in general quite different from most of the problems we review. Customer shopping behavior for time-insensitive products typically involved in the infinite selling season case is very different from that for time-sensitive products involved in most of our problems. The optimal price path for problems with an infinite selling season or/and an infinite initial inventory is very different from that for problems with a finite selling season and a finite initial inventory.

We may include in our survey a handful of studies on problems with an infinite horizon or/and infinite initial inventory because these problems are closely related to some problems that are within the scope of our survey.

1.1 Major Trends

Academic research on dynamic pricing problems in revenue management started about twenty years ago, but has experienced an explosive growth in the last seven to eight years. Like in most other quantitative research areas, the evolution of academic research in this area is driven partly by practical issues and partly by the complexity of the resulting mathematical models and the availability of technical tools for solving these models. Consequently, research attention collectively has moved over time from simple models with specialized or sometimes unrealistic assumptions to more sophisticated models with richer features and more general assumptions. We find that year 2006 represents a clear transition point in the evolution of the literature. Almost all the papers published before 2006 make the following four assumptions:

(i) There is a single product.

(ii) There is a single firm selling products in a monopoly market with *no competition*.

(iii) The customers are *myopic* such that they always make a "buy-or-leave" decision at the time of arrival according to the following myopic rule: if their valuation is higher than the selling price, they buy the product immediately; and otherwise they leave the store (in this case, the demand is lost for the firm).

(iv) There is complete demand information such that the demand is either deterministic or stochastic but with a known probability distribution.

In practice, most firms sell a variety of products at the same time, e.g., different models of digital cameras, different colors of T-shirts, etc. If a firm makes pricing decisions for individual products separately without considering demand correlation among the products (e.g., substitutability or complementarity), then the solution would be suboptimal. It is well understood that the optimal price paths for products considered jointly (e.g., Dong et al. 2009) are very different from the optimal price paths for products considered separately (e.g., Gallego and van Ryzin 1994).

The assumption of a monopoly market may not reflect the reality of many practical settings. The reality is that many companies compete on product prices. In fact, price competition is a classical problem and has been studied extensively in the economics literature (e.g., Vives 1999). However, most economics literature is focused on static (i.e., single-period) pricing. Dynamic pricing with competition has received little attention until very recently. To solve a dynamic pricing problem with competition, demand modeling and price optimization require game theoretic approaches, which are generally more complex than the typical demand modeling and optimization approaches used for problems with no competition.

Under the assumption of myopic customer shopping behavior, the demand can be represented as a function of the current price only. Often times, this is not the case in a real market where customers can observe and predict the price dynamics with a certain level of accuracy based on their own shopping experience or with the aid of advanced information technologies such as online deal forums or prediction websites. In such a situation, some customers may behave strategically in the sense that they may delay their purchases in anticipation for a price drop in the future. Such customers are referred to as *strategic* (a.k.a. rational or forward-looking) customers. With the presence of strategic customers, the demand can no longer be simply represented as a function of the current price only. A commonly-adopted approach is to use a game theoretic framework where the firm acts as a Stackelberg leader and determines the pricing policy first, and then the customers act as followers and choose when to buy to maximize their expected utility by taking into account the firm's decision as well as other customers' decisions. When determining a price schedule over time, ignoring such strategic customer behavior may result in suboptimal solutions and incur a substantial revenue loss (e.g., Zhang and Cooper 2008, Dasu and Tong 2010).

Finally, in practical settings involving time-sensitive products with a short selling season, there is often little or no sales history available, which makes it difficult, if not impossible, to precisely characterize the probability distribution of the demand as a function of price. In cases involving multiple products or multiple competing firms, it is usually extremely difficult to characterize demand correlation among different products or firms. Therefore, often times there is limited (or incomplete) demand information in practice.

Since 2006, there has been a rapidly growing interest in dynamic pricing problems with multiple products, dynamic pricing problems with competition, dynamic pricing problems with strategic customers, and dynamic pricing problems with limited demand information. This has resulted in a sizable stream of literature in each of these problem areas.

There are several existing reviews on dynamic pricing and related problems (Elmaghraby and Keskinocak 2003, Bitran and Caldentey 2003, Chan et al. 2004, Shen and Su 2007, Aviv et al. 2009). Elmaghraby and Keskinocak (2003) and Chan et al. (2004) review both dynamic pricing problems with and without inventory replenishment. However, for problems without inventory replenishment, they mainly focus on the class of single-product problems with myopic customers and no competition. Bitran and Caldentey (2003) provide a limited review on the same class of problems. Shen and Su (2007) provide a review of the literature on customer behavior in the areas of revenue management and auction including dynamic pricing problems with strategic customers. They cover most of the papers published or written in 2007 or earlier on dynamic pricing problems with strategic customers. Aviv et al. (2009) present a detailed review of several studies that explore strategies for counteracting the adverse impact of strategic customer behavior. These existing surveys provide excellent reviews of the problems or issues that they focus on up to the time when they were written. Research on these classes of problems has continued at an even faster pace since the publication of these surveys. Dozens of new papers have appeared in the last several years on single-product problems without competition. Representative recent papers include Levin et al. (2008), Celik et al. (2009), and Xu and Hopp (2009) on such problems with myopic customers, and Osadchiy and Vulcano (2010), Su (2010), Mersereau and Zhang (2012) on such problems with strategic customers. Given the most recent developments discussed above, it is not surprising that these surveys collectively have covered very few papers on problems with multiple products, problems with competition, and problems with limited demand information.

In this paper we provide a review on the three emerging classes of dynamic pricing problems in revenue management that have not been covered in existing surveys, namely, problems with multiple products, problems with competition, and problems with limited demand information. Our goal is to provide a structured and detailed review of existing models, their solutions and related managerial insights. In addition, we identify several topics for future research. Clearly, our survey complements the existing surveys.

The remainder of this paper is organized as follows. In Section 1.2, we define key model components shared by all problem classes. We then provide a review of the literature on each emerging problem class in Sections 2, 3, and 4, respectively. Finally we conclude this survey in Section 5 by identifying a number of possible directions for future research.

1.2 Terminology

Key model components of a dynamic pricing problem include the structure of the time horizon, the nature of the allowable prices, the underlying demand model, and the pricing scheme used. Below we define the common characteristics and terms associated with each of the model components except for the demand model. Demand modeling depends heavily on other problem characteristics such as whether customers are myopic or strategic, whether there are multiple products, and whether there is competition. Hence, the demand models used in the literature differ significantly for different classes of problems. Therefore, we discuss how demand is modeled later in the following sections when specific problem classes are reviewed.

In terms of the structure of the time horizon, all the problems involve one of the following three types:

(i) *Continuous time horizon* where price changes can occur at any point in time. A continuous horizon is suitable for situations where prices can be easily adjusted over time at little or no cost (e.g., online stores). In many papers, for the purpose of finding an efficient algorithm, a continuous time horizon is discretized into many sufficiently short time intervals such that there is at most one customer arrival within each time interval. In this case, we may call the time horizon *discretized continuous*.

(ii) *Discrete time horizon* which consists of a finite number (two or more) of time periods such that price changes can only occur periodically at the beginning of a time period. A discrete horizon is appropriate for settings where price changes may incur a significant cost and prices are usually adjusted according to a fixed schedule (e.g., most physical stores).

(iii) *Customer based time horizon* where price changes can only occur at the time points of customer arrivals, as opposed to the continuous (discrete) time horizon case where price changes can be made at any time point (at the beginning of a time period) independent of customer arrivals.

In terms of the allowable prices that can be used, all the existing problems involve the following two cases:

(i) *Continuous allowable prices*, i.e., any price can be used (in some cases, there might be a lower and an upper bound for the price). Continuous allowable prices can be used in an ideal situation where firms have complete flexibility in setting the prices.

(ii) *Discrete allowable prices*, i.e., the price can only be chosen from a given finite set of price points. This reflects more closely real-world practices. There are certain price points at which consumers become much more willing to buy, and hence retailers that follow a sound pricing strategy often use a small set of popular price points for a product (e.g., Anderson and Simester 2003, Allen 2011). For example, a store selling a particular camera model may set allowable prices of the camera to be {\$299, \$349, \$399, \$449, \$499}.

In terms of the pricing scheme used, there are two schemes studied in the literature:

(i) *Preannounced pricing* under which the firm determines and announces all the prices for the entire selling season at the very beginning without considering possible future sales.

(ii) *Contingent pricing* under which the firm dynamically adjusts and announces the price over time based on the realized sales history and the remaining inventory.

In a market setting where all customers are myopic and there is no competition, a contingent pricing policy is clearly better than a preannounced pricing policy as the latter policy lacks the flexibility in reacting to sales realizations. However, it is not immediately clear which policy is better in a market setting where customers behave strategically (e.g., Aviv and Pazgal 2008, Levin et al. 2010) or there is competition (see Section 3.1). Under some circumstances, the firms may be better off using a preannounced pricing policy.

2 Models with Multiple Products

Gallego and van Ryzin (1997) is the first paper that investigates a dynamic pricing problem with multiple products in the context of revenue management. Since this early work, little research on similar problems that involve multiple products has appeared in the literature until very recently. However, the last several years have witnessed a rapid growth of research in this area. The literature on multi-product dynamic pricing can be divided into two streams according to the selling mechanism involved. Under the first selling mechanism, all available products are displayed in the beginning, and once a customer makes her purchasing decision, the firm does not take any follow-up action such that the customer's decision is final. Most models reviewed in this paper involve this selling mechanism. We call such pricing models *models without follow-up*. Under the second selling mechanism, the firm uses a more active selling tactic which can be generally described as follows. After a customer makes her initial purchasing decision, the firm may use the information revealed (e.g., the customer's valuation) to influence the customer's final purchasing decision by offering a different product or additional products, which were not displayed initially, for the customer to purchase. The customer may maintain her initial purchasing decision or end up with buying a product different from the one initially chosen, or buying multiple products including the one chosen initially and additional products offered by the firm. Such active selling techniques include *upgrading*, *upselling*, and *cross-selling* that are defined in Section 2.2. We call such pricing models *models with follow-up*. In the following, we review the existing models without follow-up actions and the existing models with follow-up actions in Sections 2.1 and 2.2, respectively.

2.1 Without Follow-up

We first (in Section 2.1.1) give an overview of the existing multi-product dynamic pricing models without follow-up based on some main model characteristics. We then (in Section 2.1.2) review the main structural results and managerial insights derived in the existing papers.

2.1.1 Overview of Existing Models

Table 1 categorizes the existing multi-product dynamic pricing problems without follow-up according to some main model characteristics including the relationship among the products, the demand model, the time horizon, and the allowable prices. Below we provide an overview of some of these model characteristics.

Consider a firm selling N products simultaneously. In general, the demand of one product not only depends on its own price, but may also depend on the prices of other products. Customers compare the prices, availability and other product attributes (e.g., quality, style, feature) of all products and may end up purchasing one or multiple of the given products. In practice, the products are viewed as *substitutable* (or *complementary*) if increasing the price of one product can increase (or decrease) the demand of other products. In both cases, optimizing the price for each individual product independently may result in a suboptimal solution. In order to maximize the total revenue, one needs to jointly optimize the prices of all the products by taking into account the inventory positions and demand interdependency of all the products.

Three types of relationship among multiple products are considered in the literature: (i) Substitutable, as defined above. A majority of the existing literature considers problems with substitutable products and explicitly models demand substitutability among the products. (ii) General, i.e., neither substitutability nor complementarity relationship is assumed for the products. Several papers (Gallego and van Ryzin 1997, Maglaras and Meissner 2006, Bulut et al. 2009, Koenig and Meissner 2010) use fairly general demand functions to model a general relationship among the products. (iii) Independent, i.e., demand of one product is independent of the prices of the other products. Several papers (Erdelyi and Topaloglu 2011, Caro and Gallien 2012, Wang and Ye 2013) assume that the products in their model are independent.

Substitutable products can be either *horizontally differentiated* if customers' valuations of product attributes are idiosyncratic, e.g., some customers prefer a blue shirt while others may prefer a white shirt if the prices and other product attributes are the same, or *vertically differentiated* if customers' valuations of the product attributes are uniform, e.g., customers always prefer a deluxe room over a standard room if the prices are the same. Among the papers that consider substitutable products, only a few of them (Akcay et al. 2010, Bitran et al. 2006, Parlakturk 2012) consider vertically differentiated products, whereas all the other papers (including one model in Akcay et al. 2010) consider horizontally differentiated products.

With only one exception, most papers assume that customers are myopic and hence the underlying demand can be simply described as a function of the current prices. Parlakturk (2012) is the only paper that considers strategic customers such that the demand can no longer be simply represented by an exogenously given function. In this case, a game theoretic framework is required to formulate the pricing and purchasing decisions between the firm and the customers. Both the firm's pricing policy and the customer demand emerge as the equilibrium of the game.

In general, there are three types of demand models, deterministic models, stochastic models, and models with limited information. With a deterministic model, given the prices of all the products, the demand for each product is known exactly. In contrast, with a stochastic model, given the prices of all the products, the demand for each product is unknown, but follows a known distribution. In the case with limited demand information, even the underlying demand distribution is unknown. A handful of studies on multi-product problems (Lim et al. 2008, Chen and Chen 2014, Besbes and Zeevi 2012, Gallego and Talebian 2012, Keskin and Zeevi 2014) consider the case with limited demand information. These papers are reviewed in Section 4 with all other papers that study problems with limited demand information. All other studies on multi-product problems consider stochastic demand models. Interestingly, most of these studies assume that customer arrivals follow a Poisson process. We refer to these models as *Poisson models* and divide them into the following two groups.

General Poisson (GP) models: the vector of demand rates (or intensities) $\bar{\lambda} = (\lambda_1, \ldots, \lambda_N)$ is a general function of time t and the price vector $\bar{p} = \{p_1, \ldots, p_N\}$ at time t, i.e., $\bar{\lambda} = \Lambda(\bar{p}, t)$. GP models do not explicitly capture consumer purchasing behavior.

Poisson with consumer choice (PCC) models: the total customer arrival rate λ_t at time t for all the products together is given exogenously, and the prices of all the products jointly determine the purchasing probability P_i for a particular product i, for $i = 1, \ldots, N$, where $\sum_{n=0}^{N} P_i = 1$ and P_0 denotes the non-purchasing probability. The purchasing probability P_i may be modeled differently in different papers. We discuss different ways of modeling consumer choice when we review individual papers in Section 2.1.2.

2.1.2 Review of Existing Results and Insights

In this section, we review the existing literature according to the product relationship and the demand model involved. In the single-product case, Gallego and van Ryzin (1994) derive the following monotonicity properties: the optimal price p is non-increasing in the remaining inventory x and non-decreasing in the remaining time t until the end of the horizon (which is also called time-to-go). In the multi-product setting, many problems possess one or more of the following four properties. Much of our review below will center around these structural results.

Property PQ: Higher quality product should be priced higher;
Property PT: The optimal price p_i for product i is non-decreasing in time-to-go t;
Property PI: The optimal price p_i for product i is non-increasing in its own remaining inventory x_i;
Property PJ: The optimal price p_i for product i is non-increasing in the remaining inventory of another product x_j, where $j \neq i$.

General Products. GP demand models are used by Gallego and van Ryzin (1997), Maglaras and Meissner (2006), and Koenig and Meissner (2010) to model a general relationship among the products. Gallego and van Ryzin (1997) study a problem involving a given initial inventory of M resources that can be used to produce N products. They show that the deterministic counterpart provides an upper bound on the revenue of the original stochastic problem. They propose two heuristics according to the deterministic solution and prove that both heuristics are asymptotically optimal when the volume of the expected sales goes to infinity. In the case when $M = 1$, Maglaras and Meissner (2006) show that the multi-product pricing problem and the capacity control problem studied by Lee and Hersh (1993) can be reduced to a common formulation, and thus can be treated as a single-product pricing problem where the firm controls the aggregate capacity consumption rate. Similar to Gallego and van Ryzin, Maglaras and Meissner also propose heuristics based on deterministic solution and prove that the heuristics are asymptotically optimal. Their work is further extended by Koenig and Meissner (2010) who compare a dynamic pricing policy and a list-price capacity control policy (where the price for each product is given and fixed over the time horizon) via numerical experiments. They show that when the initial inventory is sufficient relative to the demand, there is little disadvantage of using list-price capacity control compared to dynamic pricing; this disadvantage increases as the initial inventory decreases.

Similarly, a general relationship of the products is assumed in the problem studied by Bulut et al. (2009). There are two products that can be sold separately or together as a bundle. The authors use a PCC demand model with reservation prices to capture customer preferences among the two individual products and the bundled product. They assume that a customer will choose the product (an individual one or the bundle) that brings the maximum surplus which is the difference between her reservation price and the actual price. They show via numerical examples that offering the bundle mid-season is more effective in increasing revenue than updating the prices of individual products, particularly when the product reservation prices are negatively correlated.

Substitutable & Horizontally Differentiated Products. A number of papers use PCC demand models for substitutable products, which allows them to explicitly model consumer choice decisions and derive structural properties of the optimal pricing policy. The *random utility model* is most commonly used to characterize consumer choice among multiple substitutable products. The model assumes that a consumer's utility U_i for a particular product i consists of an observable component u_i and an unobservable component ϵ_i, i.e., $U_i = u_i + \epsilon_i$. Consumers choose the product with the highest utility among all available products, and hence the purchasing probability for product i is $P_i = Prob(U_i = \max_{\{k=1,\dots,N\}} U_k)$. Typically the observable component u_i is a function of the quality index q_i and the price p_i.

When products are horizontally differentiated, two special cases of the random utility model, multinomial logit (MNL) model and nested logit (NL) model, are most commonly used. The MNL applies to the situation where all the products are equally dissimilar and the *independent-from-irrelevant-alternatives* (IIA) property holds. The NL model is more appropriate when product dissimilarity is asymmetric and products belong to different groups. Several papers (Akcay et al. 2010, Dong et al. 2009, Suh and Aydin 2011, Li and Graves 2012) use the MNL model to characterize consumer choice and formulate their problem as a stochastic dynamic program. Akcay et al. (2010) show that the key driver of price differentiation in a horizontal assortment is individual product inventory availability. They show that the optimal price of a product is equal to the marginal value of inventory of this product plus a *uniform price*. The uniform price is also termed as "profit margin/markup" in some other papers. Thus, the optimal prices should be chosen such that the markup is equal across all products (*equal markup property*). Consequently, products with an inventory surplus (the available inventory of a product is more than its maximum potential demand) has zero marginal value of inventory and should be priced the same (the uniform price) regardless of the quality difference. Whereas products with an inventory shortage (the available inventory is less than the maximum potential demand) should be charged with a premium (marginal value of inventory) over the uniform price. They demonstrate via numerical examples that some monotonicity properties including PQ, PT, and PJ may not hold. These findings are consistent with the findings by Dong et al. (2009) and Suh and Aydin (2011). Dong et al. demonstrate that the complex price behavior is due to the interplay between inventory scarcity and quality difference among the products. As a result, some monotonicity properties that hold for the single-product case do not hold in the multi-product setting. In addition, Dong et al. compare the full-scale dynamic pricing with several restricted pricing strategies and demonstrate that the full-scale dynamic pricing offers great value when inventory is scarce. Suh and Aydin (2011) focus on the problem with two products only. Despite the fact that Properties PT and PJ do not hold, they prove that Property PI does hold. In addition, they show that the marginal value of inventory of a product is increasing in the remaining time and decreasing in the inventory level of either product. Li and Graves (2012) use a similar model to study a dynamic pricing problem during inter-generational product transition in which a new product is replacing an old one. Unlike most other papers that assume that the quality index of a product is fixed, they assume time-varying quality index and characterize consumer preference between the old product and the new product over time. In addition to some common structural results as discussed above, they also identify some interesting pricing behavior which arises specifically under their setting with two inter-generational products during the transition period. In particular, they show that inter-generational transition causes the firm to decrease and then increase the prices of both products due to the higher risk that customers might switch to other outside options.

A special case of the MNL model with disjoint consideration sets (MNLD) is adopted by Zhang and Lu (2013) to study a dynamic pricing problem in network revenue management. In the MNLD model, customers belong to several different segments, and customers in each segment consider a distinct subset of products only. Zhang and Lu use a dynamic programming decomposition approach to solve their problem. They show that the decomposition approach provides a tighter upper bound than the deterministic approximation. In addition, they demonstrate via numerical results that dynamic pricing can provide a significant revenue lift compared to static pricing or choice-based availability control.

Although the MNL model is widely used in the pricing literature to model consumer choice behavior, the model has its limitations and does not apply to certain situations particularly when the products are correlated. Motivated by this, Li and Huh (2011) and Gallego and Wang (2012) use the NL model to examine multi-product pricing problems. Li and Huh (2011) assume that the price-sensitivity may vary across nests; however, products in the same nest do have the same price-sensitivity. They show that in this case, the profit function is concave in market share. In contrast to the MNL model with identical price-sensitivity parameters in which the equal markup property holds, the property no longer holds in a more general setting (the MNL with different price-sensitivity parameters or the NL model), and the optimal markup depends on the price sensitivity of products and the dissimilarity indices of product groups. Gallego and Wang (2012) study a more general problem where the price-sensitivity parameters are different both across and within nests, and the nest coefficients are not limited to be in the unit interval. They show that adjusted markup, which is defined as the price minus cost minus the reciprocal of the product's price sensitivity parameter, is constant for all products in the same nest. In addition, the adjusted nest-level markup is also constant across nests.

In contrast to all the papers discussed above, Zhang and Cooper (2009) do not use a specific choice model (e.g., MNL or NL). In addition, in their problem the allowable prices belong to a given discrete set instead of a continuous interval. This makes their DP formulation computational intractable due to its multi-dimensional state and action spaces. Therefore, rather than deriving structural results, they focus on analyzing bounds and developing various heuristics. Their numerical experiments demonstrate that requiring all products to be priced the same may result in a significant revenue loss. On the other hand, requiring price changes to be at pre-specified time points has a small impact on the revenue.

Although as Zhang and Cooper show, requiring all products to be priced the same may hurt the firm's revenue, Liu and Milner (2006) discuss various practical situations in which despite differences in products, differential pricing among products may make the firm appear unfair or opportunistic, and thus require the firm to apply a common price to all the products. Unlike in PCC models where the prices of all products jointly determine the purchase probability of individual products, they assume that the total customer arrival rate is determined by the common price while the purchasing probability for each individual product is given and fixed over the planning horizon regardless of the price used. They prove that Property PT holds. However, they demonstrate via a numerical example that Properties PI and PJ may not hold (note that in this case, properties PI and PJ are identical). They develop optimal policies for two special cases of the problem and several heuristics for the general problem.

Substitutable & Vertically Differentiated Products. When products are vertically differentiated, Akcay et al. (2010) propose a specific random utility model to capture consumer preference. In this model, $u_i = \theta q_i - p_i$ and $\epsilon_i = 0$, where θ captures customers' sensitivity to q_i and is uniformly distributed between 0 and 1. Unlike in the case of horizontal differentiation where the price of a product is driven by its individual inventory level, they demonstrate that the optimal price of a product i is determined by the aggregated inventory of higher quality products $\sum_{k=1,\ldots,i} x_k$. Note that by convention we assume that $q_1 > q_2 > \ldots > q_N$. They also show that properties PQ, PT, PI and PJ all hold in this case. Finally, they show that the optimal price of a product i is equal to the optimal price of its adjacent lower quality product plus a markup, which is determined by the aggregate inventory $\sum_{k=1,\ldots,i} x_k$ only. Bitran et al. (2006) propose a Walrasian Choice (WAL) model in which two parameters are used to characterize a customer's choice behavior, the non-purchase utility U_0 and her budget w. These two parameters are unknown to the firm and follow a given distribution. They also assume that the utility for each product U_i, for $i = 1, \ldots, N$, is known and can be ranked as follows, $U_1 > U_2 > \ldots > U_N$. Given the prices for all the products, a customer will choose product i if product i has the highest utility among those with a price below her budget, $U_i = \max_k\{U_k | p_k \leq w\}$. Unlike a random utility model where a customer simply chooses a product with the highest utility, a WAL model incorporates customers' budget constraint. Within their framework, they demonstrate Property PQ. Due to the tractability issue of the original stochastic problem, they focus on two special cases of the problem.

Parlakturk (2012) is the only paper that considers strategic customers. He investigates how intertemporal substitution interacts with variety substitution and thus affects the firm's benefit. Specifically, he considers a two-period model where a single firm sells two vertically differentiated products to consumers who differ in their valuation of product quality. It is shown that the firm's loss due to strategic customer waiting can be reduced when more product variants are offered. Therefore, the firm may find it beneficial to sell a product that would be otherwise unprofitable to sell when customers are myopic. On the other hand, adding a lower-quality variant may also hurt the firm's profit when customer impatience and the firm's costs are moderate. Furthermore, he shows that ignoring strategic customer behavior results in not only suboptimal pricing but also suboptimal product selection.

Independent Products. Unlike all the papers we have discussed so far, there are several papers that assume demand independency across the products, i.e., the demand for a product is determined by its own price only. Erdelyi and Topaloglu (2011) study a dynamic pricing problem in a network revenue management setting, where the arrival rate for an itinerary depends on its own price only. Similar to Zhang and Lu (2013), they use dynamic programming decomposition approaches to decompose the problem by flight legs. They show that compared to the deterministic solution approaches proposed by Gallego and van Ryzin (1997), their approaches achieve tighter upper bounds and and higher total expected revenues. Wang and Ye (2013) investigate a special pricing phenomenon called *hidden-city ticking* commonly observed in the airline industry where the price for an itinerary connecting at an intermediate city is less expensive than the price for a ticket from the origin to the intermediate city only. They build mathematical models to explore the cause of this phenomenon and the resulting managerial implications. They demonstrate that hidden-city opportunities may arise when the price elasticities of demand for different O-D pairs differ significantly. They further show that when passengers take advantage of the hidden-city opportunities, (i) it hurts the airlines' revenue substantially even when the airlines react in an optimal way, and (ii) the overall consumer surplus may fall if the airlines react optimally. While most papers develop highly stylized models and focus on deriving structural properties of the optimal pricing policies, Caro and Gallien (2012) is the only paper that describes a large-scale application of a multi-product markdown optimization implemented by a real firm, the Spanish fashion apparel retailer Zara. Their model incorporates a number of practical constraints that are frequently encountered in retail settings. With some approximation, the problem is formulated as an integer program. The model is solved periodically with updated demand estimates and the solution is implemented on a rolling horizon basis. They show that by implementing the markdown optimization system at Zara, the clearance sales revenue was increased by about 6%.

2.2 With Follow-up

With the aid of advanced information technologies, it is now much easier and cheaper for both traditional and online firms to collect real-time customer purchasing information and use the information to upsell or cross-sell other products to the customer immediately after a customer decides to buy a product. Once a customer decides to purchase product A, *upselling* is the practice of inducing the customer to purchase a higher quality product B instead at a generally higher price. The special case where the firm offers product B to the customer at the price of product A is called *upgrading.* The practice of *cross-selling* is to induce the customer to purchase one or more additional products, possibly at a discount.

Dynamic pricing problems with cross-selling are considered by Netessine et al. (2006) and Aydin and Ziya (2008). They both consider the setting where only one additional product is offered to a customer who has bought a product. Netessine et al. (2006) consider the case where there are a general number of multiple products and assume that the price for each individual product is fixed. Their main decisions include (i) packaging, i.e., which product to cross sell with each purchased product; and (ii) pricing, i.e., what price to charge for the cross-sold product. They consider two models, an emergency replenishment (ER) model where the firm has the capability to procure out-of-stock product at an extra cost, and a lost-sales (LS) model where product inventory cannot be replenished. For the ER model with a static packaging scheme, they show that the multi-dimensional dynamic program for the problem can be decomposed into multiple one-dimensional DPs (*decomposition property*). They also show that the optimal package price exhibits Properties PT and PI. The decomposition property, however, does not hold for the LS model. A number of heuristics are developed. Aydin and Ziya (2008) consider the case with two products, one regular and one promotional, and an extension with multiple regular products and one promotional product. The promotional product is the only one to be cross sold with a regular product. Each regular product has a fixed price and unlimited inventory. However, the promotional product has a fixed initial inventory. The problem is to dynamically determine the price of the promotional product and the discount level of this product when cross sold with a regular product. They show that the optimal announced price and cross-selling price of the promotional product satisfy Properties PT and PI. In addition, they show that the use of purchasing information is highly beneficial when the firm needs to use a static price for the promotional product but can adjust the discount level dynamically. The cross-selling models studied in both papers have limitations. Both involve cross selling only one additional product, and both assume that regular products have fixed prices. In practice, however, firms often cross sell multiple products and may dynamically adjust the prices for both the regular and promotional products. This can be an interesting future research avenue.

Gallego and Stefanescu (2012) study problems involving upgrading and upselling with a general number of multiple products. They first consider the case where the prices for all the products are given, demands are independent, and the only decision is product upgrading. They show that under mild conditions imposing a fairness constraint (upgrade priority should be given to customers who purchase higher quality products) does not incur a loss in optimality. They then identify conditions under which the optimal revenues are the same under both *limited cascading* (upgrading to the next higher quality product) and *full cascading* (upgrading to any higher quality product). When different products have different commissions, they show that resellers of capacity may fulfill demand with higher quality products and this practice can improve the reseller's revenue. They also consider the case where demand is characterized by the MNL model and the prices are decision variables. They show that if the commission margins are homogeneous and the optimal prices are used, upgrading or upselling cannot improve revenue; otherwise, upgrading or upselling can be beneficial.

A different dynamic pricing problem with follow-up is considered by Kuo and Huang (2012). There are two generations of a product where only one generation is displayed while the other is initially hidden and is offered only if a customer decides *not* to purchase the one being displayed. Customers are either price-takers or bargainers. The authors study two models, posted-pricing-first (negotiation-first) model in which the displayed product is sold under the posted pricing (negotiation) mechanism, whereas the hidden product is sold under the negotiation (posted-pricing) mechanism. This problem is different from upselling and cross-selling problems where all the products are displayed and upselling and cross-selling to a customer takes place only when the customer decides to buy a product. In addition, in contrast to upselling and cross-selling where the initial purchasing information is used in making the pricing and discount decisions in the second stage, in Kuo and Huang's problem, the distributions of customer reservation prices for the two products are assumed to be independent and therefore the purchasing decision in the first stage cannot be used in the second stage. They show via numerical examples that the posted price of a product may not satisfy Property PT.

3 Models with Competition

Price competition has been an important area of research in economics for a long time. The earliest work dates back to Bertrand (1883) who studies a duopoly model in which two firms, each with an unlimited capacity, sell identical products in a single period. This represents the simplest case where firms compete purely on price. In this case, customers simply buy from the firm that charges a lower price and hence the lower-price firm attracts all the demand. Bertrand argues that as a result of competition, each firm charges a price equal to the marginal cost and earns zero profit. This finding, derived from a model with overly simplified assumptions, is obviously not consistent with most real-world situations. Later studies in the economics literature relax some of the assumptions by, for instance, adding capacity constraints (Edgeworth 1897), considering differentiated products (Hotelling 1929, Chamberlin 1933), and incorporating search costs (Diamond 1971). An extensive treatment of the subject can be found in Vives (1999).

Price competition issues that arise from revenue management applications have received little attention until very recently. In contrast to most problems studied in the economics literature which are focused on single-period static pricing with an unlimited capacity (or initial inventory), price competition problems in the revenue management context typically involve a finite initial inventory and multiple time periods over which firms may have to dynamically adjust their pricing strategies in response to their competitors' actions. To the best of our knowledge, Dudey (1992) is the first paper that studies such a problem. In contrast to

Edgeworth's static model where a pure strategy equilibrium does not exist in general, Dudey proves the existence of a pure strategy equilibrium for any specification of the firms' capacities. Many recent models are built upon his work and most of them appeared in the last five years.

Dynamic pricing problems with competition can be generally classified into two types, i.e., problems involving *identical products* and problems involving *differentiated products*. These two types of problems require different demand models. In the first type of problems, firms sell identical products such that customers simply buy from the firm that charges the lowest price. In this case, price is the only determinant of the demand such that each firm will have an all-or-nothing type of demand subject to its inventory availability. This is also referred to as *perfect competition* in the literature. In the second type of problems, firms sell substitutable but differentiated products. This includes two cases, a case where the products differ in quality and other attributes, and a case where the products are identical but customers have heterogeneous preferences among the firms due to various reasons, e.g., loyalty. In both cases, a customer's purchasing behavior and hence the demand for a particular firm's product is influenced not only by the prices of the products but also by the product attributes or/and customer preferences. Consequently, a customer may buy from a firm that charges a higher price. This is referred to as *imperfect competition*. This type of problems typically require a model to specify how demand is split among the firms given the prices charged by all the firms. Such problems can be viewed as multi-product dynamic pricing problems in a non-cooperative setting.

In addition to the nature of products, other important characteristics involved in a dynamic pricing problem with competition include the type of customers (myopic or strategic), the pricing policy (contingent or preannounced), and the nature of market (duopoly or oligopoly). Table 2 summarizes the existing literature according to these problem characteristics.

In what follows, we provide an overview of the main findings and managerial insights reported in the literature in Section 3.1, review models with identical products in Section 3.2.1, and models with differentiated products in Section 3.2.2, respectively.

3.1 Main Findings and Managerial Insights

In this section, we summarize the main findings and managerial insights reported in the existing literature.

Validity of market response hypothesis. It has been conjectured in the revenue management literature (Talluri and van Ryzin 2004, Phillips 2005) that monopoly models may incorporate the effects of competition because the parameter estimates of the monopoly models are based on data collected in the presence of competition. Cooper et al. (2013) call this conjecture *market response hypothesis* and study its validity. They show that this hypothesis holds only in some special cases, and does not hold in general. However, they also show that in some cases, the firms can be better off if they use monopoly models than they would be if they use models that explicitly incorporate competition.

Impact of competition on pricing and initial inventory decisions. Generally, compared to the monopolistic case, competition drives firms to price lower and thus hurts firms' profits (Xu and Hopp 2006, Mantin et al. 2011, Mookherjee and Friesz 2008). Anderson and Schneider (2007), on the other hand, show that when search costs are present, prices are higher but profits are lower in a duopoly market than in a monopoly market. Lin and Sibdari (2009) demonstrate that Property PT which holds for the monopolistic case (as described in Section 2.1.2) does not hold in general in a competitive market. Both Dudey (1992) and Martinez-de-Albeniz and Talluri (2011) show that competition may lead to a price below the marginal cost of the product. When the initial inventory is a decision variable, it is shown that competition drives each firm to overstock (Xu and Hopp 2006).

Contingent versus preannounced pricing. In a monopoly market, when customers are myopic, a contingent pricing policy always outperforms a preannounced pricing policy simply because a contingent pricing policy allows a firm to respond to demand realizations and gives the firm more pricing flexibility. However, this may not be the case in a competitive market. Based on a two-period problem, Dasci and Karakul (2009) show that in a duopoly market, (i) in most cases, a contingent pricing policy does not perform better than a preannounced pricing policy, and (ii) a contingent pricing policy may outperform a preannounced pricing policy only when the total initial inventory of the two firms is very low relative to the total demand in the market and customers' reservation price in the second period is higher than in the first period. Xu and Hopp (2006) show that whether a contingent or a preannounced pricing policy is more profitable depends on the level of competition. They show that when the number of firms is small (i.e., the level of competition is not high), a contingent pricing policy tends to perform better; otherwise, a preannounced pricing policy tends to dominate. Liu and Zhang (2013) examine a similar problem in a duopoly market with strategic customers. One firm offers a higher-quality product while the other firm offers a lower-quality product. They show that in general both firms benefit when either firm commits to static pricing (a special case of preannounced pricing). In addition, when the firm that offers higher quality product commits to static pricing, it benefits both firms more.

Impact of strategic customer behavior. As in the monopolistic case, it is found that in a competitive market strategic customer behavior reduces firms' revenue; in addition, when customers are strategic, ignoring customers' strategic behavior may incur a significant revenue loss and the firms that provide lower-quality products in general suffer more (Levin et al. 2009, Liu and Zhang 2013). Levin et al. (2009) also show that the impact of strategic customer behavior increases with the increased level of competition.

Level of competition. As one would expect, Xu and Hopp (2006) show that the level of competition increases with the number of firms in the market and competition ultimately drives firms' profits to zero when the number of firms goes to infinity. Martinez-de-Albeniz and Talluri (2011) prove that when the total initial inventory of the competing firms is given, the lowest total profit is achieved when all the firms have an equal amount of initial inventory because in this case the level of competition is the highest.

3.2 Review of Existing Literature

As discussed above, situations involving identical products can be very different from situations involving differentiated products. They may require fundamentally different demand models and pricing strategies.

Below we review existing papers on problems with identical products in Section 3.2.1 and existing papers on problems with differentiated products in Section 3.2.2, respectively.

3.2.1 Identical Products

When products sold by different firms are identical, price is the predominant factor that determines the demand. In the absence of search costs, a customer will simply compare the prices and buy from the firm that offers the lowest price. However, the presence of search costs may limit a customer's ability to find the lowest price. A firm's optimal pricing strategy depends on the total inventory, individual firms' inventories, market size and customers' valuation of the product.

Dudey (1992) develops a simple duopoly model with limited capacities in which the total number of customers is known and all customers have a common known valuation of the product. The model can be viewed as a dynamic version of the Edgeworth model. Unlike the Edgeworth model for which a pure strategy equilibrium does not exist in general, Dudey demonstrates that his model has an equilibrium in pure strategies for any specification of the firms' capacities. In addition, in contrast to the result shown in Bertrand's model that both firms set a price equal to the marginal cost and earn zero profit, he shows that if at least one firm cannot supply the entire market, both firms earn positive profits. An interesting case arises when at least one firm does not have adequate capacity to supply the entire market while the total capacity of the two firms is more than the market. In this case, he shows that the firm with a lower capacity has some competitive advantage and will sell its product at a price equal to the customers' valuation before the other firm starts to make a sale. Interestingly, he finds that when the two firms have an equal capacity, the equilibrium price can be below the marginal cost.

In contrast to Dudey's model where there is only a single customer segment (i.e., all customers have a common known valuation), Dasci and Karakul (2009) consider the case where there are two customer segments (with different valuations) who arrive in two periods, respectively. They compare two dynamic pricing schemes, a contingent pricing scheme and a special type of preannounced pricing scheme which they call *fixed-ratio pricing* in which the first-period prices are determined by the two firms simultaneously but the second-period price of each firm is a given ratio of its first-period price. They find that in most cases, the fixed-ratio pricing policy outperforms the contingent pricing policy. In addition, they show that in equilibrium, one firm assumes the role of a low-cost high-volume alternative (i.e., charges a lower price and makes more sales) while the other assumes the role of a high-cost low-volume alternative (i.e., charges a higher price and makes less sales).

The above two papers assume that both the total number of customers and customers' valuations of the product are known. Several papers incorporate uncertainty into customers' valuations of the product such that a customer's valuation is no longer a known value; instead, it follows a common known distribution. Anderson and Schneider (2007) investigate the impact of customers' search costs on the optimal pricing policy and the resulting firms' profits. When search costs are present, a customer may not always buy from the firm that charges the lowest price. This assumption more closely resembles the real world but can substantially change the dynamics of the game between the firms. They show that when search costs are present, prices are higher and profits are lower in a duopoly market than in a monopoly market with an equal total capacity. This implies that firms have an incentive to merge. Mantin et al. (2011) study a closely related problem. However, they do not model search costs explicitly. Instead, they assume that a consumer can only visit a single store (or firm) in each period. Customers are assumed to follow the first-order Markov visiting pattern. Specifically, in any period, if the quoted price from a store is above her valuation, she will return to the same store in the following period with a certain probability P or switch to the other store with probability $1 - P$. They show that unless P is 1, or P is very close to 1 and the time horizon is short, the prices decrease exponentially over time (as opposed to the monopolistic case where the price decreases linearly over time). They further demonstrate that the results remain valid even when customers behave strategically. Christou et al. (2007) consider a problem where two firms first choose capacities, and then compete on prices over three periods with one customer arrival in each period. They do not consider search costs. Unlike Anderson and Schneider (2007) and Mantin et al. (2011) who assume that pricing decisions are made at the beginning of each period without knowing customer's valuation, they assume that firms observe the customer's valuation in the current period before making the pricing decisions. However, the valuations of customers in the future periods are unknown. They show that in this case, a firm may forego a sale in the current period if the expected marginal value of inventory is higher in future periods. The result found in Dudey (1992), that the firm with lower capacity sells out its capacity first before the other firms starts to sell, does not always hold when customers' valuations are unknown. In addition, they show that in a duopoly market, firms do not invest in excess capacity. However, firms may over-invest in capacity if they expect to merge in the future.

Demand uncertainty is modeled in a different way in Martinez-de-Albeniz and Talluri (2011). They assume that the total number of customers is unknown while customers' valuation is known. They show that in a duopoly market the firm with a lower capacity always starts to sell its products first at a price which is equal to the reservation value (i.e., minimum price a firm is willing to sell the product) of its competitor. The firm with a higher capacity has no choice but to wait until the other firm sells out its entire capacity, and then acts as a monopoly for the remaining time horizon. When the two firms have an equal capacity, the equilibrium price can be negative as each firm tries to be the one with a lower capacity to gain competitive advantage for the remaining horizon. These findings are similar to the ones identified by Dudey (1992). In addition, they observe a similar price pattern as identified by Gallego and van Ryzin (1994) in a non-competitive setting, i.e., price decreases over time and jumps up each time when a unit is sold. Unlike Martinez-de-Albeniz and Talluri (2011) who use a general stochastic counting process to model customer arrivals, Xu and Hopp (2006) use geometric Brownian motion to model customer arrivals, which allows them to explicitly capture demand correlation across time. They also assume that the demand function is isoelastic and demand is split among the firms that charge the lowest price according to some random process. They find that firms are willing to behave cooperatively in a decentralized pricing system (a non-cooperative setting). However, competition drives each firm to overstock and lower price, and hence hurts firms' profits. Competition will ultimately drive the profits to zero as the level of competition increases. They also demonstrate that compared to preannounced pricing, contingent pricing is beneficial only if the level of competition is not high.

Besides Mantin et al. (2011), Anton et al. (2013) is another paper that incorporates strategic customer behavior. In their model, there are two incumbent sellers who compete in two periods. In addition, many potential sellers may choose to enter the market as well. Unlike in the models considered in most of the above papers where each customer demands one unit, their model involves a single customer with a demand of 3 units, where the customer not only chooses the time to buy, but also the quantity to buy in each period. They show that a pure strategy does not exist. Compared to static (i.e. single-period) competition, dynamic competition is less intense and makes the sellers better off while hurting the customer. The customer has several options to mitigate such negative impact, e.g., commit to making no purchase in the second period, commit to myopic behavior (period-by-period optimization), or integrate with one of the sellers.

3.2.2 Differentiated Products

Currie et al. (2008) consider a relatively simple duopoly model with deterministic demand. They assume that customer arrival rate is given and the market share for each firm at any given point in time is a general function of the prices charged by both firms. Under some assumptions on the form of this function, they prove the existence and uniqueness of the optimal solution. The solution is static in nature (open-loop solution) as both firms determine and announce the price path for the entire horizon in the very beginning. Gallego and Hu (2014) study a problem with an oligopolistic market and time-varying demand functions involving both substitutable and complementary products. They first formulate the deterministics case of the problem as a differential game in continuous time and show that the equilibrium strategy has a simple structure such that the equilibrium prices at any time can be calculated from a one-shot price competition game under the current-time demand structure, taking into account a set of time-invariant shadow prices that measure the aggregate capacity externalities. They also show re-solving the open-loop Nash equilibrium continuously result in a feedback Nash equilibrium (also referred to as closed-loop Nash equilibrium). They then use the solution from this deterministic game to construct heuristic policies for the stochastic counterpart and demonstrate that the proposed policies are asymptotic equilibria. The model proposed by Lin and Sibdari (2009) can be viewed as a discretized continuous-time horizon counterpart of the model studied in Gallego and Hu (2014). In addition, Lin and Sibdari use a specific model (MNL model) to describe customers' choice behavior. For the complete information setting where each firm knows the real-time inventory information of all other firms, they prove the existence of a pure-strategy closed-loop Nash equilibrium. They show that Property PT may not hold in a competitive setting. For the incomplete information setting where each firm only knows the initial inventory levels of other firms, they develop a heuristic policy and demonstrate via numerical experiments that the heuristic policy works very well under different scenarios.

The following two papers study joint pricing and inventory allocation problems frequently encountered in the airline industry. Zhao and Atkins (2011) develop a parsimonious model where two airlines compete over two periods for two fare classes. Given the price of the discount market (in period 1), the two airlines simultaneously determine the prices and the inventory protection levels for the full-fare market (in period 2). They use an additive function to model stochastic demand with the expected demand assumed to be a specific type of linear function of the prices charged by both firms. They prove the existence of a pure-strategy Nash equilibrium. They show that (i) compared to a monopoly case, competition will never lead to a higher price or a lower protection level for the full-fare market; and (ii) compared to the case where neither firm adopts Revenue Management (serving both the full-fare and discount markets rather than serving the full-fare market only), both airlines choosing Revenue Management result in a win-win situation. Mookherjee and Friesz (2008) extend Zhao and Atkins' model to a more general setting with more firms, more time periods, and more fare classes. In addition, their model incorporates overbooking. Unlike Zhao and Atkins, they focus on a multiplicative demand form. They propose a variational inequality formulation to characterize the market equilibrium and prove the existence of a pure-strategy open-loop Nash equilibrium. It is found that competition leads to lower equilibrium prices and higher overbooking limits.

Several papers study problems with the presence of strategic customers. Levin et al. (2009) use a general linear random utility model to capture consumer choice behavior among multiple products including the option of delaying purchase to future periods due to strategic behavior. In their model, customers belong to multiple segments and each segment has a different product valuation and a different degree of strategic behavior. They demonstrate that the impact of strategic customer behavior increases with increased competition among the firms. In addition, firms that ignore strategic customer behavior can incur a significant revenue loss and those that provide better quality products are generally less affected. This is consistent with the findings by Liu and Zhang (2013) who specifically investigate the asymmetric impact of strategic behavior on firms' profits when two firms offer vertically differentiated products. Liu and Zhang assume customers are inter-temporal utility maximizers. Their demand model is quite similar to the one proposed by Akcay et al. (2010) (reviewed in Section 2) for vertically differentiated products in a non-competitive setting. In addition to the findings identified by Levin et al., they also show that (i) when either firm commits to static pricing, in general it benefits both firms; and (ii) when the higher-quality firm commits to static pricing, it benefits both firms more compared to the case where the lower-quality firm commits to static pricing. Jerath et al. (2010) is another paper that considers strategic customer behavior. However, their focus is on comparing the benefits of last-minute sales by direct channels versus opaque channels. They develop a model in which two firms with equal capacity offer horizontally differentiated products and compete in two periods (a regular period and a clearance period). All customers have an identical valuation of the product but heterogeneous preferences between the two firms. They show that (i) when demand is deterministic, opaque selling is slightly better than last-minute selling via direct channel; and (ii) when demand is stochastic, opaque selling tends to dominate last-minute selling when consumer valuation is low and/or the probability of having high demand is high. They also investigate a case where the two firms have different capacities and demonstrate that in this case, the imperfect masking of the opaque product reduces the efficacy of the opaque channel and the firm with a larger capacity is at greater disadvantage.

In all of the above reviewed papers, demand is either deterministic or stochastic with known distribution. Perakis and Sood (2006), Bertsimas and Perakis (2006), Kachani et al. (2007), and Cooper et al. (2013) study dynamic pricing problems with competition when there is limited demand information. Their work is reviewed in Section 4 together with all other work involving limited demand information.

4 Models with Limited Demand Information

All the papers we reviewed in Sections 2 and 3 assume that the demand as a function of price is either deterministic or stochastic with a known probability distribution. In reality, demand information is often limited with one of the following two cases:

Nonparametric Setting: The demand function form is unknown to the decision maker who only knows that the demand function belongs to a broad functional class that satisfies some basic regularity conditions, or that there is a known uncertainty set where all possible demand realizations belong.

Parametric Setting: The demand function form is known to the decision maker, but some parameters that specify the demand function are unknown.

There is a rapidly growing interest in the operations management literature in dynamic pricing problems with limited demand information in both parametric and nonparametric settings. There are two types of modeling approaches studied in the literature for solving such problems. One type of approach incorporates active demand learning (*exploration*), which is to gain information about the unknown demand function through price experimentation, into price optimization (*exploitation*), which is to maximize the expected revenue under the estimated demand function. In demand learning models, pricing has two goals: 1) to maximize revenue, and 2) to obtain information about demand function in order to increase future revenues. These goals are usually in conflict, requiring a tradeoff between the two. A key part of a solution approach is thus to balance the effort spent on exploration and the effort spent on exploitation.

The other type of approach does not engage active demand learning and are essentially *robust optimization* modeling approaches which often optimize a worst-case performance measure such as maximizing minimum revenue, minimizing maximum regret, or maximizing minimum competitive ratio, among others, over all possible demand realizations from an uncertainty set.

For parametric problems, most existing research uses demand learning approaches. For nonparametric problems, a small number of existing studies use demand learning approaches, whereas the majority use robust optimization approaches.

In the following subsections, we review existing research that uses robust optimization approaches and existing research that uses demand learning approaches, respectively.

4.1 Robust Optimization Approaches

Robust optimization approaches are developed by Lim and Shanthikumar (2007) and Cohen et al. (2012) for problems with a single product, and by Perakis and Sood (2006), Lim et al. (2008) and Chen and Chen (2014) for problems with multiple products. The problem considered by Perakis and Sood is parametric, whereas the problems considered in the other papers are all nonparametric.

Single Product. Lim and Shanthikumar (2007) use the notion of relative entropy to model uncertainty (a.k.a. *ambiguity*) in the demand rate. Instead of assuming that the demand rate is a known function of price, they characterize the error in the nominal demand rate model through a constraint on relative entropy. They formulate the problem as a two-player zero-sum stochastic differential game in which the firm determines the pricing policy while the "nature" chooses a probability measure within the constraint set to minimize the firm's revenue. They characterize the solution through a version of the Isaacs' equation. A closed-form solution is obtained for the special case with an exponential nominal demand rate. Cohen et al. (2012), on the other hand, propose a sampling-based model that uses only a limited number of demand data. This allows them to generate solutions that are less conservative compared to the ones generated by traditional robust optimization models that typically consider the entire uncertainty space. They provide a theoretical performance guarantee on their sampling-based solution according to the number of samples used.

Multiple Products. Lim et al. (2008) extend their early work (Lim and Shanthikumar, 2007) on a single-product problem to the multi-product setting. Similarly, they apply the notion of relative entropy to model demand uncertainty. They formulate the problem as a worst-case stochastic intensity control problem in which different products can have different levels of ambiguity. They show that the multiple-ambiguity robust dynamic pricing problem is equivalent to a risk-sensitive pricing problem. They further show that if the demand rate for each product depends on its own price only, their multi-product problem can be decomposed into several single-product problems by introducing a cost to account for the consumption of common resources by each product. Chen and Chen (2014) consider a problem with two substitutable products subject to a number of business rules. They use three types of bounds to model uncertain demand for any given prices: (i) lower and upper bounds for the demand of each individual product in each period; (ii) lower and upper bounds for the total demand of the two products in each period; and (iii) lower and upper bounds for the total demand of the two products from period 1 to each period t. These bounds define the uncertainty space and enable them to model inter-product and intertemporal demand substitutions. They propose a robust optimization framework for maximizing the worst-case revenue and develop a fully polynomial time approximation scheme.

The problems reviewed above all involve a single firm in a monopolistic setting. Perakis and Sood (2006) consider an oligopolistic setting where multiple firms sell differentiated products. They assume that demand for each firm in a given period is a function of the prices of all the firms in that period. The exact values of the demand function parameters, however, are unknown and belong to a known uncertainty set. They propose a robust optimization approach that maximizes the revenue for each firm under the most adverse instances of parameters within their uncertainty set. They prove the existence of equilibrium policies and develop an iterative learning algorithm for finding them. Their numerical results show that typically prices are higher in periods where the demand sensitivity is lower.

4.2 Demand Learning Approaches

The value of demand learning in pricing is well documented. Many existing studies (e.g., Lobo and Boyd 2003, Carvalho and Puterman 2005, Bertsimas and Perakis 2006) provide empirical evidence that, in a variety of settings, pricing policies that perform some sort of active exploration outperform greedy policies that focus on short-term revenue maximization, indicating that there is some intrinsic value to price experimentation.

However, incorporating demand learning into a pricing problem makes the problem more difficult to solve. This can be illustrated by looking at the simple case with an unlimited inventory. Without demand learning, under certain conditions, a myopic policy which focuses on the current time period only is optimal, and hence one can just use a static pricing policy. However, when there is learning involved, a static policy would no longer be optimal because the price used in an earlier period affects information collection which in turn affects future pricing decisions.

Also, compared to the case without demand learning, it can be expected that the optimal price path under demand learning may vary more over time because of price experimentation. This may have an impact on customer shopping behavior, which in turn may change the demand function in future periods. We are not aware of any existing research that incorporates such an issue into a demand learning model. Except for a small number of studies that consider time-varying demand patterns, all other existing papers assume that the demand function is time-invariant (i.e., stationary).

Certain types of pricing problems with demand learning, especially parametric problems with an infinite inventory and an infinite time horizon, have been extensively studied in the economics literature. Most such work uses Bayesian updating schemes to update the probability distributions of unknown parameters associated with the demand function. Without inventory and horizon length limit, the model is much simpler such that one can focus on demand learning issues without the inventory and time constraints. Furthermore, with or without inventory limit, the structure for learning is significantly different (den Boer and Zwart 2011), and the only connection between periods occurs through the belief process associated with Bayesian updating, as opposed to operational issues in general (Araman and Caldentey 2009). One of the most fundamental questions addressed in the economics literature is whether a seller who follows an optimal policy will eventually obtain complete information about the underlying demand function. Rothschild (1974) considers the case where the seller can choose prices from a finite set and shows that a seller who follows an optimal policy may never learn what the demand model actually is, an outcome which is referred to as *incomplete learning* (McLennan 1984).

Given the focus of our survey, we review the work in the operations management literature only. Since most of the existing dynamic pricing studies in the OM literature that involve demand learning are closely related, we review every such study that we are aware of, including the work on problems with an infinite inventory or/and an infinite horizon.

4.2.1 Parametric Problems

The parametric problems studied in the literature are highly stylized. Most of these problems involve a monopolistic seller selling a single product to myopic customers and assume that the demand function is time-invariant with a small number of parameters unknown to the seller. The unknown parameters or their distributions are updated over time through learning using the sales and price history. A typical approach is to formulate the joint learning and earning decisions into a dynamic programming formulation. In most cases, such a DP is intractable due to the high dimensionality or lack of a proper structure. A majority of the existing research focuses on analyzing performance bounds of some heuristic approaches.

A majority of the existing studies on parametric problems use what we call a *willingness-to-pay demand model* which is also widely used in dynamic pricing models without demand learning and can be described as follows. Homogeneous customers arrive following a random process with a certain arrival rate. Their willingness-to-pay (WtP) at any time point follows an i.i.d. distribution. Some studies (Lin 2006, Araman and Caldentey 2009, Farias and van Roy 2010) assume that the arrival rate is unknown whereas the WtP distribution is known. Some other studies (Carvalho and Puterman 2005, Cope 2007, Broder and Rusmevichientong 2012) assume that the arrival rate is known but some parameters of the WtP distribution are unknown. There are also studies (Sen and Zhang 2009) that assume that both the arrival rate and some parameters of the WtP distribution are unknown. Linear demand functions with unknown parameters are also quite commonly used (Lobo and Boyd 2003, Bertsimas and Perakis 2006, Kachani et al. 2007, Cooper et al. 2013, Keskin and Zeevi 2013, 2014). There are also several more complex demand models used to deal with time-varying demand function (Aviv and Pazgal 2005, Besbes and Zeevi 2011, Keskin and Zeevi 2013, Chen and Farias 2013), strategic customer behavior (Levina et al. 2009), or multiple products (Gallego and Talebian 2012). These demand models will be described when we review the corresponding papers below.

Bayesian updating is by far the most commonly used approach for updating unknown parameters over time in demand learning. It starts with the seller's initial knowledge (or a prior probability distribution) of each unknown parameter, optimizes pricing decisions with respect to this prior, and then updates the seller's belief (which is then called posterior) based on the actual demand realizations and the prices used by a Bayesian updating scheme. This process is repeated over time. A major shortcoming of this approach is that it requires prior knowledge about the unknown parameters. In reality, prior knowledge is either vague or non-existent, and hence it can be very difficult to specify a unique prior distribution. Different people may suggest different priors which may lead to different solutions. Hence it runs the risk of model mis-specification. Among the papers we review below, Bayesian updating is used by Lobo and Boyd (2003), Aviv and Pazgal 2005, Lin (2006), Cope (2007), Araman and Caldentey (2009), Levina et al. (2009), Sen and Zhang (2009), Farias and van Roy (2010), Harrison et al. (2012), and Gallego and Talebian (2012).

Maximum likelihood estimation is also quite commonly used (Carvalho and Puterman 2005, Broder and Rusmevichientong 2012, den Boer and Zwart 2011, 2014). Other approaches used include linear least squares estimation (Bertsimas and Perakis 2006, Kachani et al. 2007, Cooper et al. 2013, Keskin and Zeevi 2013, 2014, Besbes and Zeevi 2014), and simple empirical estimation which is quite different from all other approaches and is described in detail below when we review the corresponding papers (Besbes and Zeevi 2009, 2011, Wang et al. 2014, Chen and Farias 2013). Unlike the Bayesian approach, these approaches do not require prior knowledge of an unknown parameter.

Below we review each of the existing studies involving parametric problems with demand learning. Our review is organized according to the parameter updating approaches used.

Bayesian Updating. We first review existing studies that use Bayesian updating for problems with an unlimited inventory. Lobo and Boyd (2003) consider a linear demand function: $d_t(p_t) = a - bp_t + e_t$, where a and b are unknown and assumed to have priors with normal distributions, and e_t is a normal distribution with known parameters. The authors consider: (1) a static policy which uses the prior distributions of the unknown parameters and does not update these distributions, (2) myopic policy which updates the distributions and focuses on one period only but does not incorporate the impact of learning on future periods, (3) myopic policy with *dithering* which is a modification of the myopic policy by adding a random perturbation to the price. The motivation behind this is that price variation will "excite" the learning process and increase the amount of information learned. They show through simple numerical examples that the myopic policy with dithering increases revenue as compared to the static or myopic policy. Cope (2007) considers a willingness-to-pay demand model where the WtP distribution is unknown. The seller only knows a prior of this distribution which follows the Dirichlet distribution. Some approximation strategies are studied. Harrison et al. (2012) consider a problem where the seller only knows that the demand function is one of the two given functions (the two-hypothesis model). The firm has a given prior probability that the demand function is one of the two given functions. They study a myopic Bayesian policy (MBP) under which in each period, the problem is to maximize the next period's expected revenue given the posterior belief updated using Bayes' rule. They show that MBP may get stuck with an uninformative price and fail to learn the demand function completely (i.e., incomplete learning). A constrained variant of this policy which excludes a small neighborhood of the uninformative price is also considered. This policy is proved to have at most a constant gap from the clairvoyant who knows the underlying demand function.

Next we review existing studies using Bayesian updating for problems with a finite inventory. Lin (2006) uses a willingness-to-pay demand model and assumes that customer's WtP distribution is known while the customer arrival rate is unknown. Gamma distribution is used to characterize the firm's knowledge about the customer arrival rate, and this distribution is updated over time according to the real-time sales data. He proposes a *variable-rate policy* which updates the price in real time based on the latest information about the customer arrival rate. Numerical experiments demonstrate that this policy achieves a near-optimal solution and is also quite robust even when the initial knowledge about the customer arrival rate deviates significantly from its true value. Araman and Caldentey (2009) and Farias and van Roy (2010) consider similar problems with a willingness-to-pay demand model. In both problems, the arrival rate is unknown to the firm. Araman and Caldentey (2009) assume that the rate is one of the two given values, representing low or high market size, respectively, whereas Farias and Van Roy (2010) assume that the rate follows a finite mixture of Gamma distributions, where a kth order mixture of Gamma distributions is parameterized by three k-dimensional vectors. These parameters are updated over time based on the available sales information. Since the underlying dynamic programming formulations are intractable, they both study approximation and heuristic approaches and analyze their performance. Sen and Zhang (2009) consider a more complex problem with a willingness-to-pay demand model where both the arrival rate and the WtP distribution are unknown. The seller only knows that the WtP distribution is one of the several given functions with certain prior probabilities and that the arrival rate follows a gamma distribution with unknown parameters. Numerical results show that compared to the pricing model without demand learning, their model is particularly beneficial when the initial estimate of demand rate is inaccurate, the actual demand mismatches supply, and the demand is price-sensitive.

All the problems studied in the above reviewed papers are rather simplistic, involving a single product with myopic customers and stationary demand. Only a handful of existing studies consider demand learning problems with time-varying demand, strategic customer behavior, or multiple products. Aviv and Pazgal (2005) consider a problem with a finite inventory where the demand environment may fluctuate over time and hence the demand function is not stationary. They develop a stylized partially observed Markov decision process (POMDP) framework under which the market environment is characterized by a finite number of core states, where each state represents a particular demand environment. The exact state is unobservable to the firm. State may switch from one to another from period to period according to a given transition probability matrix. The sales data in the previous periods is used to update the seller's knowledge about the core state in the next period. An approximation technique and a heuristic pricing policy are proposed. Levina et al. (2009) consider a problem involving selling a finite inventory over a finite horizon to strategic customers where the demand is a function of time, the remaining inventory level and the price, specified by some unknown parameters. An adaptive simulation based optimization algorithm is proposed to solve the problem. Computational results demonstrate that the algorithm is robust to deviation from actual market if learning is incorporated. It is also shown that ignoring strategic customer behavior may result in inferior solutions even if demand learning is incorporated. Gallego and Talebian (2012) study a problem with multiple products that share a core value v which is unknown to the seller. Customers choose which product to buy following a known multinomial logit choice model which is a function of customer utilities. The customer utility for each product i at time t given price p_t is $u_t^i(p_t) = k^i v - p_t^i$ where the multipliers k^i are all known. However, the customer arrival rate is unknown. Bayesian updating is used to estimate the arrival rate, whereas maximum likelihood estimation is used to update the core value. They use a rolling horizon based heuristic approach to solve the problem. Their computational results show that (i) early demand observations play a very important role in learning, (ii) when inventory is limited, dynamic pricing even without learning is quite efficient, (iii) joint dynamic pricing with learning remains superior.

Maximum Likelihood Estimation. Carvalho and Puterman (2005) consider a problem with an unlimited inventory and a willingness-to-pay demand model. The arrival rate is known and the WtP distribution is a logistic function of price p, $\exp(\alpha+\beta p)/[1+\exp(\alpha+\beta p)]$, where α and β are unknown parameters which are updated using maximum likelihood estimation based on the sales information up to the current period. They compare three pricing policies: (i) a myopic policy which is only concerned with maximizing revenue in each period; (ii) a myopic policy with random exploration under which in each period t with some probability η_t the firm chooses a price which maximizes the current period's revenue, and with probability $1-\eta_t$ the firm chooses a random price, where η_t decreases with t; (iii) a one-step look ahead policy which explicitly accounts for the trade-off between revenue maximization and learning. Their computational results show that the myopic policy performs poorly by getting stuck at a price level away from the optimum, and the one-step look ahead policy performs superior to the other two policies. Broder and Rusmevichientong (2012) consider a problem where a seller prices a product to a sequence of T customers who arrive sequentially with their WtP following an i.i.d. distribution. The seller knows the form of the WtP distribution, but does not know the values of the parameters involved. The objective is to minimize the total regret of the revenue. For the general case where the WtP distribution can have any form that satisfies some basic conditions, the authors construct a forced-exploration policy based on maximum likelihood estimation which achieves the optimal $O(\sqrt{T})$ order of regret. They also consider a special case of a well-separated demand family, for which a myopic maximum likelihood policy achieves the optimal $O(\log T)$ order of regret.

den Boer and Zwart (2011, 2014) consider two closely related problems and investigate a fundamental question in demand learning: using the certainty equivalent pricing, will the estimates of unknown parameters converge to the true values? In certainty equivalent pricing (a.k.a. myopic pricing, passive learning), an optimal price is found by maximizing the revenue of the current period based on the estimated parameters, and no price experimentation is considered and learning (or parameter updating) is merely a passive action. den Boer and Zwart (2014) consider a problem with an infinite initial inventory and a fairly general demand function where the mean demand $E[D(p)]$ and variance of demand $Var[D(p)]$ are known functions of the price p with unknown parameters estimated by maximum quasi-likelihood estimation. They show that using certainty equivalent pricing, there is a positive probability that the seller may never learn the optimal price. This is mainly because there is not enough price dispersion, leading to insufficient information being collected for parameter estimation. They then propose a pricing policy called *controlled variance pricing* that sets a lower bound on the sample variance of the chosen prices in each period. Using this policy, they show that the estimated parameters converge to the true values and hence the prices converge to the true optimal price. The resulting regret is in the order of $O(T^{1/2+\delta})$, where δ is an arbitrarily small number. den Boer and Zwart (2011) consider a similar problem but with a finite inventory and multiple consecutive selling seasons, each with a finite horizon. At the beginning of each selling season, there are C units of inventory and all unsold units in the end of a selling season perish. The demand function in each selling season is the same and Bernouli distributed with the mean being a fairly general function with unknown parameters estimated using maximum likelihood estimation based on all historical demand realizations (including the data from previous seasons). They show that certainty equivalent pricing can guarantee that the estimated parameters converge to their true values. They call this *endogenous learning* property. The intuition is that with a limited inventory, the optimal price changes with the remaining inventory and the length of the remaining selling season and hence an optimal certainty equivalent pricing policy naturally induces price dispersion. They also show that under this algorithm the regret (defined as the revenue loss due to not using the optimal prices) after T selling seasons is $O(\log(T)^2)$.

Linear Least Squares. Several papers study problems in a monopolistic setting. Bertsimas and Perakis (2006) consider a firm selling a finite inventory of a single product over a finite number of time periods with a linear demand in each period t, $d_t = a + bp_t + \epsilon_t$, where ϵ_t is a random noise following normal distribution $N(0, \sigma^2)$, and a, b and σ are unknown and need to be learned over time. They give a 5-dimensional DP approximation by assuming that σ for future periods is the same as that in the current period t. In addition, they consider several heuristics including a 1-dimensional DP where the values of a, b and σ in future periods are all assumed to be the same as in the current period t, and a myopic policy which considers one period at a time by ignoring future periods. Their computational experiment shows that (i) all the methods succeed in estimating accurately the demand parameters over time, and (ii) the 5-dimensional DP outperforms the 1-dimensional DP which outperforms the myopic policy. Keskin and Zeevi (2014) consider problems of selling an unlimited inventory of a single product or multiple products across T time periods in a monopolistic setting with a similar linear demand function. They show that for both the single-product and multi-product problems, the smallest achievable revenue loss (due to not using the optimal prices) of any pricing policy is in the order of $O(\sqrt{T})$. They give sufficient conditions for a pricing policy to achieve asymptotic optimality, and show that semi-myopic policies based on iterative linear least squares updating scheme of the unknown parameters in the demand function can achieve asymptotic optimality. Keskin and Zeevi (2013) consider a more general single-product problem where the unknown parameters in the linear demand model may change over time. They use a quadratic variation metric to measure the amount of change over T periods and allow a finite budget (B) for such changes. They propose a weighted least squares estimation procedure that discounts old observations at a certain rate. They show that the lower bound on the revenue loss is in the order of $O(B^{1/3}T^{2/3})$. They construct a family of pricing policies and prove that the revenue performance of these policies asymptotically matches the lower bound. Furthermore, they find that the revenue loss is significantly smaller in the case of bursty changes (in the order of $O(T^{1/2}logT)$) compare to the case of smooth changes. Besbes and Zeevi (2014) investigate the problem of demand model misspecification. They assume that the seller uses a simple linear demand model while the actual demand model might be substantially different. They propose a family of pricing policies that operate in repeated stages where each stage involves price testing, demand model recalibration, and price optimization. Surprisingly, they show that the performance of their proposed pricing policies can be quite good under reasonable conditions. The price converges to the optimal price under the true demand model while the revenue loss is in the order of $O(logT)^2\sqrt{T}$, as compared to $O(\sqrt{T})$ in the case where the demand model is correctly specified, i.e., the actual demand is linear (Keskin and Zeevi, 2014).

There are also studies that consider demand learning problems with competition. Bertsimas and Perakis (2006) consider a problem involving multiple sellers competing in an oligopolistic setting and each seller's demand function is assumed to be linear in its own price and its competitors' prices plus a noise term. Each firm does not know the parameters of its own function and the parameters of its competitors' functions. Each seller knows its own realized demand, its own prices used, and its competitors' prices used in the past, but does not know its competitors' realized demand in the past. For a given seller, in addition to estimating unknown parameters of its demand function, it also needs to predict its competitors' prices over time. Both the case with a finite inventory and the case with an infinite inventory are considered. The case with an infinite inventory can be separated by time periods. For the finite inventory case, a myopic solution approach is proposed. Kachani et al. (2007) consider a similar problem and propose a solution approach consisting of three steps in each period. First, assuming that the demand parameters are given, each firm finds Nash equilibrium demand by solving a best-response problem. Next, each seller updates the demand parameters using the demand information obtained from the first step. Finally, each firm sets optimal prices of their product for the remaining periods based on the information from the earlier steps.

It has been conjectured in the literature that monopoly models may implicitly capture the effects of competition as the parameters are estimated based on data collected in the presence of competition. Cooper et al. (2013) refer to this as the *market response hypothesis* and investigate the validity of this hypothesis by studying a duopoly model involving selling an unlimited inventory of products over an infinite horizon with the assumption that each seller uses a monopoly demand function which is linear in its own price without incorporating the competitor's price. The parameters in the demand functions are unknown and estimated over time using realized demand. They find that this hypothesis is valid only in a limited way in certain settings. Specifically, they show that (i) for the case where the sellers know the slope in their demand function, the firms' prices obtained from the monopoly model converge to the Nash equilibrium of the duopoly model; (ii) for the case where the sellers know the intercept in their demand function, under certain conditions the firms' prices obtained from the monopoly model converge to the cooperative prices (which maximize the total expected revenue of both firms), but not the Nash equilibrium; (iii) for the general case where both the slope and intercept are unknown to the sellers, the firms' prices may converge under certain conditions and the limit prices may vary depending on the initial conditions.

Simple Empirical Estimation. The above reviewed demand learning approaches all involve fairly sophisticated statistical estimation procedures to update the unknown demand parameters based on the realized demand. In contrast, there are several studies that adopt much simpler estimation procedures and focus on learning the demand rate itself or some parameters that can be easily inferred based on the demand rate. In all these papers, the demand rate is estimated by simply counting the total sales over a period of time. Besbes and Zeevi (2009) consider a problem with a finite initial inventory and a finite planning horizon where the demand function has a known form but with some unknown parameters. The objective is to minimize the maximum regret, where regret is defined as the percentage loss of revenue relative to the full-information deterministic relaxation problem. They propose an algorithm which divides the time horizon into two phases, a price experimentation phase and a price optimization phase. They prove that the solution generated by the algorithm has a regret no more than $O(n^{-1/3})$ and is asymptotically optimal as the initial inventory and rate of demand go to infinity, where n is the market size (the initial inventory and the demand rate are proportional to n). They also show that no admissible pricing policy can achieve a regret less than $O(n^{-1/2})$ under the asymptotic regime they use. Wang et al. (2014) consider the same problem but propose a different algorithm which considers price experimentation and price optimization iteratively. Their algorithm achieves a better regret $O((\log n)^{4.5} n^{-1/2})$.

In contrast to the above two papers where the demand function is assumed to be the same over time, Besbes and Zeevi (2011) and Chen and Farias (2013) consider the case where the market condition may change over time. Besbes and Zeevi (2011) consider a problem where N customers arrive sequentially with i.i.d. WtP distribution. However, the distribution of WtP changes from one function F_a to another F_b starting from the τ-th customer. The firm knows F_a and F_b, but does not know the point of change (i.e., the value of τ). They consider the minimax regret objective. They show that any admissible policy must incur a revenue loss of at least in the order of $O(N^{1/2})$ relative to the best possible revenue achievable when the value of τ is known in advance. They propose an algorithm which regularly conducts price experimentation to detect the change in market response (WtP distribution) and show that this algorithm is near optimal. In contrast, Chen and Farias (2013) assume that customers' WtP distribution is known and fixed while customer arrival rate Λ_t at time t is unknown and can vary over time. They propose a simple pricing policy which involves frequent reoptimization and reestimation of the demand rate Λ_t and show that their policy has uniformly bounded relative performance losses compared to the optimal revenue an "idealized" seller (who has perfect knowledge of the market size process) can achieve. In contrast to the other three papers discussed in this section, their proposed policy does not involve active price experimentation.

4.2.2 Nonparametric Problems

Only a handful of existing studies deal with dynamic pricing problems with demand learning in a non-parametric setting. Besbes and Zeevi (2009) consider a problem with a finite initial inventory and a finite planning horizon. This problem is similar to the parametric problem considered in the same paper (reviewed earlier) except that the demand function form is unknown and can be any function satisfying some standard conditions. They propose an algorithm similar to the one for the parametric problem but requires more test prices. They show that the solution found by their algorithm is asymptotically optimal when the initial inventory and rate of demand go to infinity, and derive a lower ($O(n^{-1/2})$) and an upper ($O(n^{-1/4})$) bound of the performance of any admissible pricing policy. Wang et al. (2014) consider the same problem and propose an algorithm where price experimentation and exploitation are considered iteratively. Under the same asymptotic regime used by Besbes and Zeevi (2009), Wang et al. show that their algorithm achieves a better regret which is no more than $O((\log n)^{4.5} n^{-1/2})$.

Eren and Maglaras (2010) consider two problems with an infinite initial inventory and a finite horizon. In both problems, a willingness-to-pay demand model is assumed with the WtP distribution function F unknown. The objective is to maximize the worst-case competitive ratio over all possible F (defined as the ratio of the expected revenue of a pricing strategy without knowing F to the expected revenue that could have been earned if F is known). In the first problem, no learning is involved. In the second problem, the time horizon is divided into two periods where price experimentation is allowed in the first period, but not allowed in the second period. For both problems, they derive closed-form optimal pricing policies. Their computational results suggest that even price experimentation at a few points can increase the revenue performance significantly, and the relative benefit from continuing learning decreases after a point.

Besbes and Zeevi (2012) consider a multi-product problem with a finite inventory of each product and a finite horizon. The demand function is unknown except the fact that it satisfies some basic conditions. The objective is to maximize the worst-case competitive ratio. They consider two cases of the problem, one with a discrete allowable prices, and the other with continuous allowable prices. They propose an algorithm which involves an exploration phase and an exploitation phase. For the discrete allowable price case, they develop a simple linear programming based policy that determines the fraction of time each allowable price point should be applied. For the continuous allowable price case, they simply pick the "best" price to be applied for the entire exploitation phase. They prove that all the policies they develop are asymptotically optimal when the volume of sales grows large.

5 Future Research Directions

We have provided a review of the three new dynamic pricing problem areas in revenue management that have emerged since 2006, namely, problems with multiple products, problems with competition, and problems with limited demand information. These problems are motivated by the issues that arise in a number of practical settings that have not been studied prior to 2006. Although the existing literature collectively has addressed a large variety of practical issues and industrial practices over the last two decades, some important issues that are commonly encountered in practice and some emerging industrial practices have received very little or no attention in the academic literature. We believe that these issues and new practices, discussed below, are worth of investigation in future research.

Behavioral Issues. The vast majority of existing dynamic pricing models assume that customers are utility maximizers, fully rational and highly sophisticated in making purchasing decisions. Behavioral economics suggests that most people are not necessarily utility maximizers, are only partially rational, and may use simple rules to make decisions. Behavioral issues have received little attention in the dynamic pricing literature. We are only aware of a handful of papers in this area. Popescu and Wu (2007) study a reference price model to capture the fact that customers have memory and may make purchasing decisions by comparing the current price and a reference price formed by the price history. Nasiry and Popescu (2012) assume that customers are emotionally rational and take regret into consideration when maximizing their utility. Su (2009) assume that some customers are inertial who may delay their purchase even if an immediate purchase is in their best interest. There are many more behavioral issues that need to be incorporated into dynamic pricing models. We give three examples and discuss how the corresponding behavioral issues can be incorporated into pricing models:

(i) People often behave irrationally and buy more than they actually need just to obtain free accessories or opt for free shipping (Ariely 2010). When offered seemingly attractive deals such as "buy one get second one 50% off" or "free shipping for orders over $100", some customers may be tempted to buy a product or a bundle that may not have the maximum utility.

(ii) Experiments (Kivetz et al. 2004, Ozer and Zheng 2012) show that the presence of an inferior product

A amplifies the attractiveness of another product B if they are similarly priced. Similarly, the presence of a product A with a much higher price amplifies the attractiveness of a similar quality product B. In both cases, the presence of product A encourages some customers who would otherwise not buy either product to buy product B. Clearly these customers are not fully rational.
(iii) Researchers show that human decisions are subject to emotional concerns (Fehr and Fischbacher 2002). For example, people feel bad if their actions induce uneven allocation of welfare in the group. Since firm decisions are made by humans, it can be expected that firms are not always self-interested or motivated purely by revenue. Hence, firms' action may deviate from the equilibrium solution of the underlying game theoretical model commonly used in the existing literature with competition.

Examples (i) and (ii) above illustrate some behavioral issues that may exist in multi-product settings. The existing multi-product models all assume that customers buy a product or a bundle that maximizes their utility. It will be interesting to develop proper demand models to incorporate such behavioral issues and investigate how such shopping behavior impacts firms' pricing strategies and revenue. Example (iii) above illustrates social preference issues in price competition settings. A possible future research topic is to extend the existing dynamic pricing models with competition to incorporate such issues.

Big Data Implications. The rapid growth of E-commerce, the rise of social media, and the advances in information technology result in enormous amount of data being generated on a daily basis. These data, which are often called "Big Data" and were not available in the past, contain abundant information about consumers including their demographics, preferences, shopping behavioral patterns, and social interaction. One particular type of big data, which can be easily acquired now by retailers, is the clickstream data of online shoppers. A clickstream dataset tracks the activities of online users and records the virtual trail each user leaves behind while surfing the Web, including, the IP address, the type of browser used, each webpage viewed, a time stamp for each browsing activity, and other page specific variables (e.g., keywords) transmitted between the server and the user's computer. In the past decade or so, a significant amount of research has been done in the analysis of clickstream data in the marketing literature (Bucklin and Sismeiro 2009). There are models (e.g., Moe and Fader 2004, Montgomery et al. 2004) that can predict with a certain accuracy whether a customer is going to make a purchase based on the customer's browsing history.

The availability of personalized consumer information and the advances of technologies for storing and analyzing these data have made it feasible for retailers to segment consumers at the individual level and apply the so-called first degree price discrimination (i.e., offering different prices to different customers) and make personalized product offerings to consumers to maximize profitability (Brynjolfsson 2013, Ozimek 2013). Shiller (2013) shows through a simulation study that personalized pricing using individual customers' web browsing data can increase profit by 1.4%. In fact, personalized pricing (a.k.a. customized or tailored pricing) and other personalization issues have been studied in the economics and marketing literatures (e.g., Murthi and Sarkar 2003, Choudhary et al. 2005, Montgomery and Smith 2009). However, all the existing models on personalized pricing that have appeared in the economics and marketing literatures are static in nature and do not consider inventory limit. We believe that dynamic personalized pricing over multiple time periods and with inventory consideration is an exciting future research topic for the OM community. Aydin and Ziya (2009) is the only paper that we are aware of in the OM literature that considers such a dynamic personalized pricing problem. In their problem, it is assumed that the customer population can be divided into two segments and within each segment all the customers have the same willingness-to-pay (WtP) distribution. Each arriving customer provides a signal (e.g., demographic information) to the firm which enables the firm to update its belief about the segment the customer belongs to. In addition to the current on-hand inventory level and the remaining time, the firm also considers each individual customer's signal when determining the optimal price. The authors show some properties associated with the optimal prices.

Aydin and Ziya's model is quite stylized in that there are only two customer segments and each segment's WtP distribution is time-invariant and known in advance. In practice, however, due to the availability of big data and the tools to analyze such data, a much finer customer segmentation becomes possible now. Furthermore, abundant real-time product and price information available to customers at various product review and social media websites is likely to influence customer preference and WtP in a real-time fashion. Consequently, the WtP distribution of a specific customer segment in a specific time period t is likely a function of the sales volume up to time t and the price history. The more people within a customer's social cycle buy the product, the more likely this customer is going to buy the product (word-of-mouth effect). On the other hand, the prices other people paid for this product also have an influence on the customer's WtP. With available big data analytics tools, retailers now for the first time have the capability of updating customer segments and the associated WtP distributions dynamically in a real-time fashion. No one has considered dynamic pricing decisions in such a setting. It remains to be seen: (i) how the optimal prices for different customer segments look like in each period, and (ii) how the optimal price for a specific customer segment changes over time. In addition to setting personalized prices, online retailers can also make personalized product offering according to the available customer-specific information (e.g., demographics, customer taste, purchasing history). The cross-selling problems studied by Netessine et al. (2006) and Aydin and Ziya (2008) are two special examples where only the purchasing information for a product is used to determine additional products to be offered. It will be interesting to see how the multi-product models in Section 2 can be extended so that both product offering and pricing decisions are considered jointly over time in a more general setting.

Some related issues may also need to be studied. A direct consequence of personalized pricing is that different customers may pay different prices for the same product in the same time period. As customers learn through social media and online review websites the prices other customers have paid in the previous time periods, they may take a more active strategy rather than being a passive price-taker. For instance, some may bargain with the retailer for a better price. We are aware of only one study, Kuo et al. (2011), which has incorporated negotiation in a dynamic pricing model. In their model, there are two types of customers, i.e., price-takers who will purchase the product as long as his/her reservation price is above the posted price, and bargainers who always negotiate in order to get a discount. The firm and the bargainers have different bargaining power and agree on a final selling price that is between the posted price and a cut-off price under which the firm is not willing to sell. The firm dynamically determines the posted price and the cut-off price over time. They show that negotiation allows the firm to achieve price discrimination and thus obtain a higher revenue compared to the case without negotiation particularly when the inventory is high and the remaining time is short, and that the firm's benefit from negotiation increases with the fraction of bargainers in the customer population and the firm's bargaining power. The model by Kuo et al. (2011) is not about personalized pricing. It will be interesting to see how negotiation can be incorporated into personalized pricing models.

In a big data era where much more information becomes available in a much faster fashion, demand learning becomes much more important and valuable. It is also much easier for online retailers to test a large number of different prices (by simply charging different prices to different customers simultaneously) and get a lot of test data to quickly learn demand functions. However, knowing that retailers use clickstream data or test prices to learn their shopping behavior, customers may behave strategically by changing their shopping routine, e.g., use a different website or computer to shop. How such strategic behavior can be incorporated into a pricing model also requires some attention.

Collabetition. All the existing dynamic pricing models with competition reviewed in Section 3 assume that (i) each seller owns the product it sells, and (ii) each firm is an independent entity. Naturally, these assumptions lead to competition among the sellers. In many practical settings, the first assumption may not hold. For example, all major hotel chains sell their rooms on both their own websites and third-party online travel agencies such as Expedia, Orbitz, and Travelocity. Another example is PC manufacturers such as HP and Dell selling their PCs on both their own online stores and via retailers such as Walmart and Staples. In both examples, there are a product provider (e.g., hotel, PC manufacturer) which owns the product and a number of resellers (e.g., travel agencies, retail stores). On one hand, the product provider and resellers compete for customers because they are selling the same product. On the other hand, the sales generated by the resellers have a direct impact on the revenue of the product provider. Therefore, the product provider has an incentive to collaborate with the resellers. Most likely form of collaboration is signing a contract to specify inventory allocation, price constraints, and a revenue sharing mechanism. We view such a relationship between the product provider and the resellers as *collabetition* because it involves both *collab*oration and comp*etition*. For pricing problems with collabetition, different solution approaches (e.g., cooperative game theory) than the ones used for problems with competition only may need to be employed. Similarly, compared to the situation with competition only, optimal pricing strategies of the firms in a collabetition situation will certainly be different. Intuitively, one would expect the product provider to be more reluctant to engage in price competition because as the owner of the product, price competition has a double negative impact on its revenue. It will also be interesting to look into the impact on the firms' revenues due to strategic customer behavior, level of competition, or level of collaboration (specified by contract details). We note that the issues discussed here are somewhat related to the channel coordination and conflict issues studied in the supply chain literature. However, in the supply chain literature, the focus is on contract design issues (Tsay and Agrawal 2004, Cattani et al. 2004). In addition to contract design, a handful of papers in this area (Cattani et al. 2006, Zhao 2008, Ryan et al. 2012) also look into static pricing. To the best of our knowledge, no existing research has looked into dynamic pricing issues in a collabetition environment.

Price Insurance Schemes. In early 2011, the US Department of Transportation (DOT) announced a new regulation for the airline industry: all airlines must either allow passengers to cancel a ticket within 24 hours without penalty or hold a reservation for 24 hours for free. Clearly the new regulation provides customers with more flexibility and comfort. This seemingly harmful regulation to the airline industry may actually benefit them as customers may become more willing to purchase a ticket early. To increase profitability, some airlines have gone one step further. Continental Airlines (merged with United Airlines later) started in 2010 to offer the FareLock option to its customers which allows customers to hold a reservation for three to seven days, with a small fee (e.g., \$5 for a 3-day lock-in period, and \$9 for the 7-day option). Several other airlines (e.g., Lufthansa) also offer similar options to their customers for a fee. Such price insurance schemes are likely to achieve a win-win result. On one hand, they provide even more flexibility to customers than the DOT's new regulation. On the other hand, if such schemes are designed optimally, the airlines involved may make extra revenue by collecting the fees paid by the customers who buy such a price insurance and by encouraging more customers to purchase tickets early. Such schemes are called *options* in the finance literature. Option pricing problems have been studied extensively in the literature (e.g., Bates 2003, Broadie and Detemple 2004). In an option pricing problem, the decision maker has no control over the price of the underlying financial asset which is determined by the market. The only decision is to determine the price of the option. However, in a dynamic pricing problem with a price insurance scheme, the decision maker needs to determine both the price of the price insurance scheme and the price of the product over time. One may expect that if a company offers a price insurance, the company may adjust the price of the product less frequently and less significantly. It will be interesting to investigate how the product inventory, the expiration date and price of the insurance scheme, and the price of the product inter-play over time.

Business Rules and Constraints. In their survey paper, Elmaghraby and Keskinocak (2003) point out that "*Another disconnect between most of the academic literature and practice is the incorporation of business rules into pricing decisions.*" Unfortunately, after so many years, this is still the case with the latest literature. Except for a handful of papers (discussed below), the models studied in most existing papers are highly stylized, oversimplify practical situations, and do not closely reflect business norms and rules commonly encountered in practice. Using stylized models makes the underlying mathematical formulations tractable and enables one to obtain structural results and derive general managerial insights. Although the resulting insights and solutions might make sense from a theoretic point of view, they may not be optimal or even feasible from a practical point of view. For example, most papers allow an unlimited number of price changes and have no limit on the magnitude of each price change. However, it has been long understood (Hall and Hitch 1939) that frequent price changes can make a retailer appear unfair or dishonest, as customers try to interpret the retailer's motives behind a price change. Furthermore, frequent price changes and substantial price differences from one period to another may significantly change customers' purchasing behavior when customers are strategic. Consequently, it may change the underlying demand function which is assumed to be given exogenously in most papers. In such a case, an optimal solution based on the original demand function may no longer be optimal for the new demand function. In addition, it is also reasonable to limit the number of price changes in order to ease markdown implementation and save the associated implementation costs. Only a handful of existing papers incorporate business rules into dynamic pricing problems. Smith et al. (1998), Netessine (2006), Chen and Chen (2014) and Chen et al. (2014) consider a limited number of price changes, Chen and Chen (2014) and Chen et al. (2014) consider restrictions on the magnitude of each price change, and Caro and Gallien (2012) and Chen et al. (2014) consider the requirement of allocating at least a minimum inventory to each product category or each store. Clearly, consideration of business rules and constraints in a dynamic pricing model will make the model more practical and enable one to generate better solutions and more useful insights.

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Table 1: Overview of Existing Multi-Product Models without Follow-up

Relationship among products	Demand model	Specification of demand parameters	Time horizon & allowable prices	Paper(s)
General	GP	$\bar{\lambda} = \Lambda(\bar{p}, t)$	Continuous & Continuous	Gallego and van Ryzin, 1997
			Discretized continuous & Continuous	Maglaras and Meissner, 2006 Koenig and Meissner, 2010
	PCC	maximize consumer surplus $r_i - p_i$	Discrete & Continuous	Bulut et al., 2009
Substitutable & horizontally differentiated	PCC	MNL model, $u_i = q_i - p_i$	Discretized Continuous & Continuous	Akcay et al., 2010 Dong et al., 2009 Suh and Aydin, 2011
		MNL model, $u_i = q_i(t) - bp_i$, where quality index $q_i(t)$ is time-varying and b is price sensitivity	Discretized Continuous & Continuous	Li and Graves, 2012
		MNLD model, $u_{li} = q_{li} - p_i$, where u_{li} stands for a segment l customer's utility on product i and the product sets for different customer segments do not overlap	Discretized Continuous & Continuous	Zhang and Lu, 2013
		NL model, $u_i = q_{i\|g} - b_g p_{i\|g}$, where subscript $i\|g$ stands for product i in nest g; price sensitivity parameter b_g is identical for all products in the same nest g; nest coefficient $\gamma_g \in [0, 1]$	Discretized Continuous & Continuous	Li and Huh, 2011
		NL model, $u_i = q_{i\|g} - b_{i\|g} p_{i\|g}$, where price sensitivity $b_{i\|g}$ are product specific	Discretized Continuous & Continuous	Gallego and Wang, 2012
		no specific choice model is required as long as $P_i \geq 0$ and $\sum_{i=1,\ldots,N} P_i = 1$	Discretized Continuous & Discrete	Zhang and Cooper, 2009
	Other	$\lambda = \Lambda(p)$, where p is the common price for all products and and the purchasing probability for each product P_i is given with $\sum_{i=1,\ldots,N} P_i = 1$	Continuous & Continuous	Liu and Milner, 2006
Substitutable & vertically differentiated	PCC	$u_i = \theta q_i - p_i$ and θ is uniformly distributed between 0 and 1	Discretized Continuous & Continuous	Akcay et al., 2010
		WAL model	Continuous & Continuous	Bitran et al., 2006
	Other	equilibrium of the pricing-purchasing game	Discrete & Continuous	Parlakturk, 2012
Independent	Other	Demand is independent across products and $\lambda_i = \Lambda(p_i)$	Discretized Continuous & Discrete	Erdelyi and Topaloglu, 2011
			Discretized Continuous & Continuous	Wang and Ye, 2013
			Discrete & Discrete	Caro and Gallien, 2012

Table 2: Overview of Existing Models with Competition

Nature of products	Type of customers	Pricing policy	Nature of market	Papers
Identical	Myopic	Contingent	Duopoly	Dudey, 1992
				Dasci and Karakul, 2009
				Anderson and Schneider, 2007
				Mantin et al., 2011
				Christou et al., 2007
				Martinez-de-Albeniz and Talluri, 2011
			Oligopoly	Martinez-de-Albeniz and Talluri, 2011
				Xu and Hopp, 2006
		Preannounced	Duopoly	Dasci and Karakul, 2009
	Strategic	Contingent	Duopoly	Mantin et al., 2011
				Anton et al., 2013
Differentiated	Myopic	Contingent	Duopoly	Currie et al., 2008
			Oligopoly	Gallego and Hu, 2014
				Lin and Sibdari, 2009
		Preannounced	Duopoly	Zhao and Atkins, 2011
			Oligopoly	Gallego and Hu, 2014
				Mookherjee and Friesz, 2008
	Strategic	Contingent	Duopoly	Liu and Zhang, 2013
				Jerath et al., 2010
			Oligopoly	Levin et al., 2009