

# October Math Gems

## PROBLEM OF THE WEEK 24

### §1 Problems

**Problem 1.1.** Solve the Diophantine equation  $x^5 - y^5 = 1993$

**Problem 1.2.** If  $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$  then  $\tan^2 x = ?$

**Problem 1.3.**  $\sin^{100}(x) + \cos^{100}(x) = 1$  Solve for the value of  $x$

**Problem 1.4.**  $a + b = 6\sqrt{ab}$ . Find  $\frac{a}{b}$ .

**Problem 1.5.**  $\log_{4x} x + \log_{\frac{x}{2}} x = 2$ . Solve for  $x$ .

**Problem 1.6.** Solve the exponential equation  $2^x - 3^x = \sqrt{6^x - 9^x}$

**Problem 1.7.** The following product can be expanded into a power series with coefficients  $a_k$ :

$$\prod_{n=1}^{100} \left[ \frac{x^3}{4^n} + \left( \frac{\pi}{2^n} \right)^2 \right] = \sum_{k=0}^{\infty} a_k x^k$$

Find the coefficients  $a_k$  in front of the individual  $x^k$  terms for all  $k \in N$

**Problem 1.8.** Determine all  $x$  that solve the equation  $x^{2x} + 27^2 = 54x^x$

**Problem 1.9.** Find the value of this infinite sum:  $\sum_{n=0}^{\infty} \frac{n^{2n} + 2^n}{2^{3n}}$

**Problem 1.10.** If  $a = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots$  Find the value of  $a$

**Problem 1.11.** Find  $f(x)$  for  $f\left(\frac{x-3}{x+1}\right) + f\left(\frac{x+3}{1-x}\right) = x$

**Problem 1.12.** Solve the equation  $x^4 - 8x + 63 = 0$

**Problem 1.13.** Let  $x$  and  $y$  be real numbers satisfying

$$(x^2 + x - 1)(x^2 - x + 1) = 2(y^3 - 2\sqrt{5} - 1) \text{ and } (y^2 + y - 1)(y^2 - y + 1) = 2(x^3 + 2\sqrt{5} - 1)$$

Find  $8x^2 + 4y^3$

**Problem 1.14.** Find positive integer  $n$  so that  $\frac{80-6\sqrt{n}}{n}$  is the reciprocal of  $\frac{80+6\sqrt{n}}{n}$

**Problem 1.15.** If  $a, b, c$  are non negative integers and  $2^a + 2^b = c!$

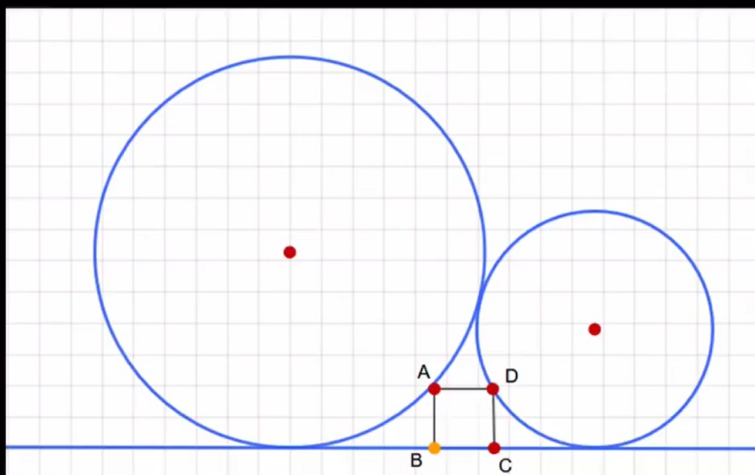
Solve for  $a$ ,  $b$  and  $c$  values

**Problem 1.16.** Solve for integer solutions of  $n$

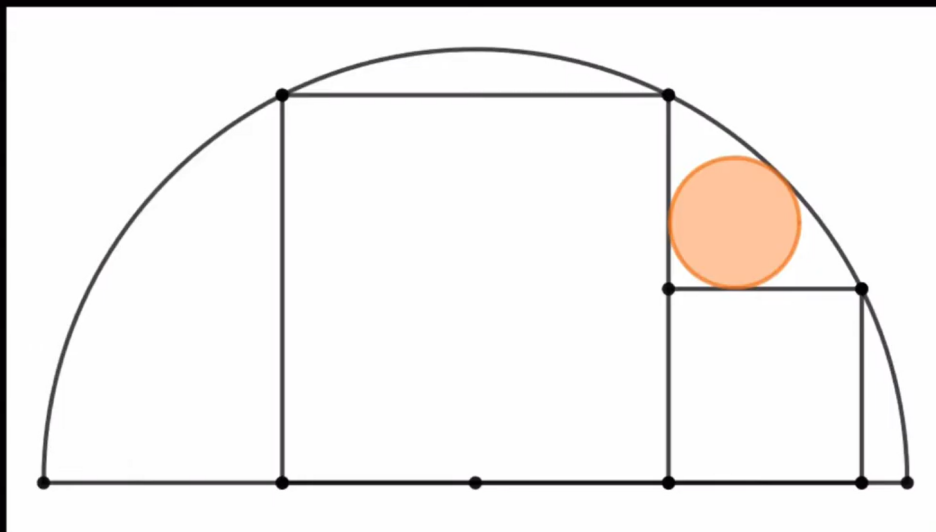
$$n! = n^3 - n$$

**Problem 1.17.** Find the value of  $x$

**A square with side length  $x$  is inscribed in the region between two circles that are tangent and a line that is tangent to both circles as shown below. The radius of the larger circle is 3 units and the radius of the smaller circle is 2 units. Find  $x$ .**

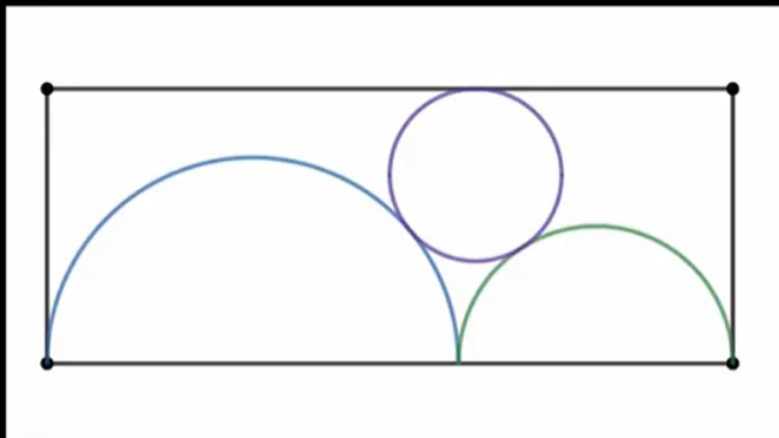


**Problem 1.18.** Find the radius of the shaded circle



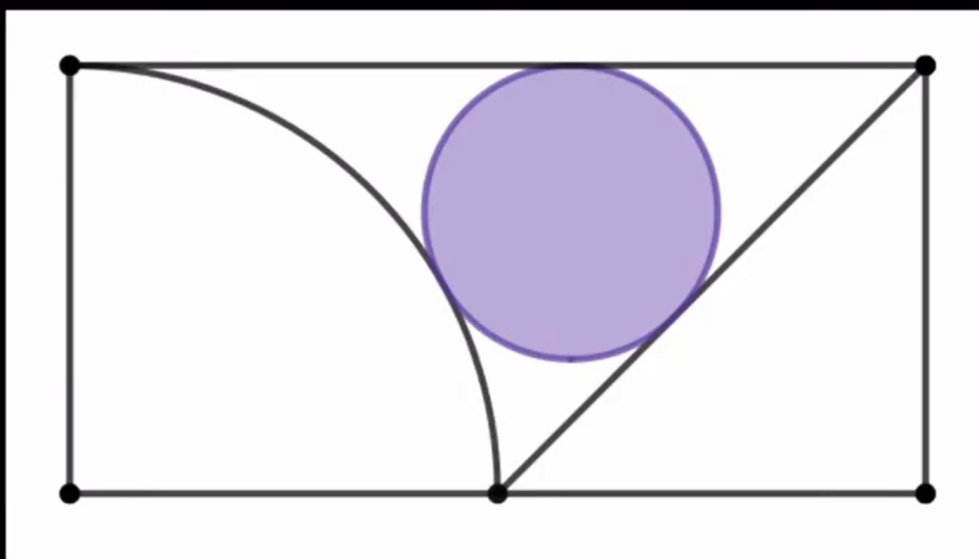
**Two squares and a circle are inscribed in a semicircle with radius 1 as shown above. Find the radius of the**

**Problem 1.19.** Find the radius of the circle



**Two semicircles with radii 3 and 2 and a circle with radius  $r$  are inscribed in a 4 by 10 rectangle as shown. Find  $r$ .**

**Problem 1.20.** Find the radius of the shaded circle



**A circle is inscribed in the region between a quarter circle and an isosceles right triangle in a 1 by 2 rectangle as shown above. Find the radius of the circle.**