

# October Math Gems

## PROBLEM OF THE WEEK 35

### §1 Problems

**Problem 1.1.** The real root of the equation  $8x^3 - 3x^2 - 3x - 1 = 0$  can be written in the form  $\frac{\sqrt[3]{a} + \sqrt[3]{b+1}}{c}$ , where  $a$ ,  $b$ , and  $c$  are positive integers. Find  $a + b + c$ .

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**Problem 1.2.** Let  $m$  be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4.$$

There are positive integers  $a, b, c$  such that  $m = a + \sqrt{b + \sqrt{c}}$ . Find  $a + b + c$ .

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**Problem 1.3.** In a Martian civilization, all logarithms whose bases are not specified are assumed to be base  $b$ , for some fixed  $b \geq 2$ . A Martian student writes down

$$3 \log(\sqrt{x} \log x) = 56$$

$$\log_{\log(x)}(x) = 54$$

and finds that this system of equations has a single real number solution  $x > 1$ . Find  $b$ .

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**Problem 1.4.** Let  $r$ ,  $s$ , and  $t$  be the three roots of the equation

$$8x^3 + 1001x + 2008 = 0.$$

Find  $(r + s)^3 + (s + t)^3 + (t + r)^3$ .

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**Problem 1.5.** Real numbers  $x$  and  $y$  satisfy the equation  $x^2 + y^2 = 10x - 6y - 34$ . What is  $x + y$ ?

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**Problem 1.6.** What is the value of

$$\frac{2^{2014} + 2^{2012}}{2^{2014} - 2^{2012}}?$$

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**Problem 1.7.** For certain real numbers  $a$ ,  $b$ , and  $c$ , the polynomial

$$g(x) = x^3 + ax^2 + x + 10$$

has three distinct roots, and each root of  $g(x)$  is also a root of the polynomial

$$f(x) = x^4 + x^3 + bx^2 + 100x + c.$$

What is  $f(1)$ ?

**Problem 1.8.** Positive integers  $a$  and  $b$  satisfy the condition

$$\log_2(\log_{2^a}(\log_{2^b}(2^{1000}))) = 0.$$

Find the sum of all possible values of  $a + b$ .

**Problem 1.9.** A hexagon that is inscribed in a circle has side lengths 22, 22, 20, 22, 22, and 20 in that order. The radius of the circle can be written as  $p + \sqrt{q}$ , where  $p$  and  $q$  are positive integers. Find  $p + q$ .

**Problem 1.10.** What is the sum of all possible values of  $k$  for which the polynomials  $x^2 - 3x + 2$  and  $x^2 - 5x + k$  have a root in common?

**Problem 1.11.** If  $y + 4 = (x - 2)^2$ ,  $x + 4 = (y - 2)^2$ , and  $x \neq y$ , what is the value of  $x^2 + y^2$ ?

**Problem 1.12.** Points  $(\sqrt{\pi}, a)$  and  $(\sqrt{\pi}, b)$  are distinct points on the graph of  $y^2 + x^4 = 2x^2y + 1$ . What is  $|a - b|$ ?

**Problem 1.13.** Suppose that  $a$ ,  $b$ , and  $c$  are positive real numbers such that  $a^{\log_3 7} = 27$ ,  $b^{\log_7 11} = 49$ , and  $c^{\log_{11} 25} = \sqrt{11}$ . Find

$$a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}.$$

**Problem 1.14.** How many positive integers  $n$  satisfy

$$\frac{n + 1000}{70} = \lfloor \sqrt{n} \rfloor?$$

(Recall that  $\lfloor x \rfloor$  is the greatest integer not exceeding  $x$ .)

**Problem 1.15.** In rectangle  $ABCD$ ,  $AB = 6$ ,  $AD = 30$ , and  $G$  is the midpoint of  $\overline{AD}$ . Segment  $AB$  is extended 2 units beyond  $B$  to point  $E$ , and  $F$  is the intersection of  $\overline{ED}$  and  $\overline{BC}$ . What is the area of  $BFDG$ ?

**Problem 1.16.** Find  $ax^5 + by^5$  if the real numbers  $a$ ,  $b$ ,  $x$ , and  $y$  satisfy the equations

$$\begin{aligned} ax + by &= 3, \\ ax^2 + by^2 &= 7, \\ ax^3 + by^3 &= 16, \\ ax^4 + by^4 &= 42. \end{aligned}$$

**Problem 1.17.** Given that  $\sin a + \sin b = \frac{1}{10}$  and  $\cos a + \cos b = \frac{1}{9}$ , find  $\lfloor \tan^2(a + b) \rfloor$

**Problem 1.18.** In a triangle  $ABC$ ,  $D$  is midpoint of  $BC$ . If  $\angle ADB = 45^\circ$  and  $\angle ACD = 30^\circ$ , determine  $\angle BAD$ .

**Problem 1.19.** Find  $a + 2b + 3c$  If

$$a + \frac{3}{b} = 3$$

$$b + \frac{2}{c} = 2$$

$$c + \frac{1}{a} = 1$$

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**Problem 1.20.** Find the minimum value of

$$f(x) = \frac{x^2 + x + 1}{x^2 + 2x + 1}$$

for all  $x$  in the domain of  $f(x)$

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