清华大学机械工程系

数值分析 A

2025 年秋季学期

作业 14

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14.1. 试推导 Hamming 公式

$$y_{n+3} = \frac{1}{8}(9y_{n+2} - y_n) + \frac{3}{8}h[f(x_{n+3}, y_{n+3}) + 2f(x_{n+2}, y_{n+2}) - f(x_{n+1}, y_{n+1})]$$

的局部截断误差主项.

解. 由 $T_{n+k} = y(x_{n+k}) + \sum_{j=0}^{k-1} \alpha_j y(x_{n+j}) - h\sum_{j=0}^{k} \beta_j f(x_{n+j}, y(x_{n+j}))$ 得

$$\begin{aligned}
T_{n+3} &= y(x_{n+3}) - \frac{1}{8}(9y(x_{n+2}) - y(x_n)) - \frac{3}{8}h[f(x_{n+3}, y(x_{n+3})) + 2f(x_{n+2}, y(x_{n+2})) - f(x_{n+1}, y(x_{n+1}))]. \\
&= y(x_{n+3}) - \frac{1}{8}(9y(x_{n+2}) - y(x_n)) - \frac{3}{8}h[y'(x_{n+3}) + 2y'(x_{n+2}) - f'(x_{n+1})].
\end{aligned}$$

泰勒展开得

$$\begin{aligned}
y(x_{n+3}) &= y(x_n) + 3hy' + \frac{9}{2}h^2y'' + \frac{9}{2}h^3y''' + \frac{27}{8}h^4y^{(4)} + O(h^5) \\
y(x_{n+2}) &= y(x_n) + 2hy' + 2h^2y'' + \frac{4}{3}h^3y''' + \frac{2}{3}h^4y^{(4)} + O(h^5) \\
y'(x_{n+3}) &= y' + 3hy'' + \frac{9}{2}h^2y''' + \frac{9}{2}h^3y^{(4)} + O(h^4) \\
y'(x_{n+2}) &= y' + 2hy'' + 2h^2y''' + \frac{4}{3}h^3y^{(4)} + O(h^4) \\
y'(x_{n+1}) &= y' + hy'' + \frac{1}{2}h^2y''' + \frac{1}{6}h^3y^{(4)} + O(h^4)
\end{aligned}$$

$$\begin{aligned}
T_{n+3} = y(x_n)&\left[1 - \frac{1}{8}(9-1)\right] \\
&+ hy'(x_n)\left[3 - \frac{1}{8}(18) - \frac{3}{8}(1+2-1)\right] \\
&+ h^2y''(x_n)\left[\frac{9}{2} - \frac{1}{8}(18) - \frac{3}{8}(3+4-1)\right] \\
&+ h^3y'''(x_n)\left[\frac{9}{2} - \frac{1}{8}(12) - \frac{3}{8}\left(\frac{9}{2} + 4 - \frac{1}{2}\right)\right] \\
&+ h^4y^{(4)}(x_n)\left[\frac{27}{8} - \frac{1}{8}(6) - \frac{3}{8}\left(\frac{9}{2} + \frac{8}{3} - \frac{1}{6}\right)\right] + O(h^5) \\
= O(h^5)&
\end{aligned}$$

14.2. 试用数值积分方法直接推导二步方法

$$y_{n+2} - y_{n+1} = \frac{h}{12}[5f(x_{n+2}, y_{n+2}) + 8f(x_{n+1}, y_{n+1}) - f(x_n, y_n)].$$

解. 由数值积分方法

$$y(x_{n+k}) = y(x_{n+k-l}) + \int_{x_{n+k-l}}^{x_{n+k}} f(x, y(x))\mathrm{d}x.$$

取 $k=2, l=1$，则

$$y(x_{n+2}) = y(x_{n+1}) + \int_{x_{n+1}}^{x_{n+2}} f(x, y(x))\mathrm{d}x.$$

$f(x, y(x))$ 在 x_{n+1}, x_{n+2} 构建二次插值多项式

$$\begin{aligned}
L_2(x) = &\sum_{i=n}^{n+2} f(x_i, y(x_i)) \prod_{\substack{j=n \\ j\neq i}}^{n+2} \frac{x-x_j}{x_i-x_j} \\
=&f(x_n, y(x_n))\frac{(x-x_{n+1})(x-x_{n+2})}{(x_n-x_{n+1})(x_n-x_{n+2})} \\
&+ f(x_{n+1}, y(x_{n+1}))\frac{(x-x_n)(x-x_{n+2})}{(x_{n+1}-x_n)(x_{n+1}-x_{n+2})} \\
&+ f(x_{n+2}, y(x_{n+2}))\frac{(x-x_n)(x-x_{n+1})}{(x_{n+2}-x_n)(x_{n+2}-x_{n+1})} \\
=&y'(x_n)
\end{aligned}$$

积分得

$$\begin{aligned}
\int_{x_{n+1}}^{x_{n+2}} L_2(x)\mathrm{d}x =&h\left[\frac{1}{2}f(x_n, y(x_n))\int_1^2 (t-1)(t-2)\mathrm{d}x - f(x_{n+1}, y(x_{n+1}))\int_1^2 t(t-2)\mathrm{d}x\right. \\
&\left.+\frac{1}{2}f(x_{n+2}, y(x_{n+2}))\int_1^2 t(t-1)\mathrm{d}x\right] \\
=&\frac{h}{12}[5f(x_{n+2}, y(x_{n+2})) + 8f(x_{n+1}, y(x_{n+1})) - f(x_n, y(x_n))].
\end{aligned}$$

令 $y_n = y(x_n)$，则得

$$y_{n+2} - y_{n+1} = \frac{h}{12}[5f(x_{n+2}, y_{n+2}) + 8f(x_{n+1}, y_{n+1}) - f(x_n, y_n)].$$

14.3. 证明线性二步法

$$y_{n+2} + (b-1)y_{n+1} - by_n = \frac{1}{4}h[(b+3)f(x_{n+2}, y_{n+2}) + (3b+1)f(x_n, y_n)]$$

当 $b \neq -1$ 时是二阶的, 当 $b=-1$ 时是三阶的

解. 由 $T_{n+k} = y(x_{n+k}) + \sum_{j=0}^{k-1}\alpha_j y(x_{n+j}) - h\sum_{j=0}^{k}\beta_j f(x_{n+j}, y(x_{n+j}))$ 得

$$\begin{aligned}
T_{n+2} =& y(x_{n+2}) + (b-1)y(x_{n+1}) - by(x_n) - \frac{1}{4}h[(b+3)f(x_{n+2}, y(x_{n+2})) + (3b+1)f(x_n, y(x_n))]. \\
=& y(x_{n+2}) + (b-1)y(x_{n+1}) - by(x_n) - \frac{1}{4}h[(b+3)y'(x_{n+2}) + (3b+1)y'(x_n)].
\end{aligned}$$

泰勒展开得

$$\begin{aligned}
y(x_{n+2}) &= y(x_n) + 2hy' + 2h^2y'' + \frac{4}{3}h^3y''' + \frac{2}{3}h^4y^{(4)} + O(h^5)\\
y(x_{n+1}) &= y(x_n) + hy' + \frac{1}{2}h^2y'' + \frac{1}{6}h^3y''' + \frac{1}{24}h^4y^{(4)} + O(h^5)\\
y'(x_{n+2}) &= y' + 2hy'' + 2h^2y''' + \frac{4}{3}h^3y^{(4)} + O(h^4)\\
y'(x_n) &= y'
\end{aligned}$$

代入得

$$\begin{aligned}
T_{n+2} = &\left[y(x_n) + 2hy' + 2h^2y'' + \frac{4}{3}h^3y''' + \frac{2}{3}h^4y^{(4)}\right]\\
&+ (b-1)\left[y(x_n) + hy' + \frac{1}{2}h^2y'' + \frac{1}{6}h^3y''' + \frac{1}{24}h^4y^{(4)}\right] - by(x_n)\\
&- \frac{h}{4}(b+3)\left[y' + 2hy'' + 2h^2y''' + \frac{4}{3}h^3y^{(4)}\right] - \frac{h}{4}(3b+1)y' + O(h^5)\\
= &-\frac{1+b}{3}h^3y'''(x_n) - \frac{7b+9}{24}h^4y^{(4)}(x_n) + O(h^5)
\end{aligned}$$

当 $b \neq -1$ 时，$T_{n+2} = O(h^3)$，为二阶方法；当 $b = -1$ 时，$T_{n+2} = O(h^4)$，为三阶方法。

14.4. 讨论线性多步法

$$y_{n+3} + \frac{1}{4}y_{n+2} - \frac{1}{2}y_{n+1} - \frac{3}{4}y_n = \frac{1}{8}h[19f(x_{n+2}, y_{n+2}) + 5f(x_n, y_n)]$$

的收敛性。

解. 第一多项式与第二多项式分别为

$$\rho(\lambda) = \lambda^3 + \frac{1}{4}\lambda^2 - \frac{1}{2}\lambda - \frac{3}{4}, \quad \rho'(\lambda) = 3\lambda^2 + \frac{1}{2}\lambda - \frac{1}{2}, \quad \sigma(\lambda) = \frac{19}{8}\lambda^2 + \frac{5}{8}.$$

$$\rho(1) = 1 + \frac{1}{4} - \frac{1}{2} - \frac{3}{4} = 0, \quad \rho'(1) = 3 + \frac{1}{2} - \frac{1}{2} = 3, \quad \sigma(1) = \frac{19}{8} + \frac{5}{8} = 3.$$

由 $\rho(1) = 0$ 且 $\rho'(1) = \sigma(1)$ 可知该方法是相容的
计算第一多项式的零点

$$\lambda^3 + \frac{1}{4}\lambda^2 - \frac{1}{2}\lambda - \frac{3}{4} = 0.$$

解得 $\lambda_1 = 1, \lambda_2 = \dfrac{-5 - i\sqrt{23}}{8}, \lambda_3 = \dfrac{-5 + i\sqrt{23}}{8}$
其中有单重零点 $\lambda = 1$ 位于单位圆上，单重零点 $\lambda_2 = \dfrac{-5 \pm i\sqrt{23}}{8}$ 在单位圆内，因为该方法收敛.

14.5. 试证明隐式 Euler 方法是 A-稳定的.

解. 由隐式 Euler 方法

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}).$$

将隐式 Euler 方法用于试验方程得到

$$y_{n+1} = y_n + h\lambda y_{n+1} \Rightarrow (1 - h\lambda)y_{n+1} = y_n \Rightarrow y_{n+1} = \frac{y_n}{1 - h\lambda}.$$

$$\left|\frac{1}{1-h\lambda}\right| < 1 \Rightarrow |1-h\lambda| > 1 \Rightarrow (1-a)^2 + b^2 > 1.$$

包含整个负半平面，因此隐式 Euler 方法是 A-稳定的.