

作业 10

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10.1. 设 $f(x) = \sin \frac{\pi}{2}x, x \in [-1, 1]$, 试在 $[-1, 1]$ 上用 Legendre 多项式作 f 的三次最佳平方逼近多项式.

解. 用 Legendre 多项式作三次最佳平方逼近多项式为

$$s_3(x) = \frac{1}{2}a_0P_0(x) + \frac{3}{2}a_1P_1(x) + \frac{5}{2}a_2P_2(x) + \frac{7}{2}a_3P_3(x),$$

其中 $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$. 系数 $a_i (i = 0, 1, 2, 3)$:

$$a_i = \int_{-1}^1 f(x)P_i(x)dx.$$

计算得:

$$a_0 = \int_{-1}^1 \sin \frac{\pi}{2}x dx = 0,$$

$$a_1 = \int_{-1}^1 x \sin \frac{\pi}{2}x dx = \frac{8}{\pi^2},$$

$$a_2 = \int_{-1}^1 \frac{1}{2}(3x^2 - 1) \sin \frac{\pi}{2}x dx = 0,$$

$$a_3 = \int_{-1}^1 \frac{1}{2}(5x^3 - 3x) \sin \frac{\pi}{2}x dx = \frac{48(\pi^2 - 10)}{\pi^4}.$$

因此,

$$s_3(x) = \frac{12}{\pi^2}x + \frac{84(\pi^2 - 10)}{\pi^4}(5x^3 - 3x)$$

10.2. $f(x) = \ln(1+x)$ 的 Padé 逼近 $R_{3,2}(x)$.

解. $f(x) = \ln(1+x)$ 的麦克劳林级数展开为

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$$

设 $R_{3,2}(x) = \frac{p_0 + p_1x + p_2x^2 + p_3x^3}{1 + q_1x + q_2x^2}$, 则有方程组

$$\begin{cases} -p_0 = 0, \\ 1 - p_1 = 0, \\ q_1 - \frac{1}{2} - p_2 = 0, \\ q_2 - \frac{1}{2}q_1 + \frac{1}{3} - p_3 = 0, \\ -\frac{1}{2}q_2 + \frac{1}{3}q_1 - \frac{1}{4} = 0, \\ \frac{1}{3}q_2 + \frac{1}{5} = 0. \end{cases}$$

解得:

$$p_0 = 0, p_1 = 1, p_2 = -\frac{13}{20}, p_3 = -\frac{23}{120}, q_1 = -\frac{3}{20}, q_2 = -\frac{3}{5}.$$

因此,

$$R_{3,2}(x) = \frac{x - \frac{13}{20}x^2 - \frac{23}{120}}{1 - \frac{3}{20}x - \frac{3}{5}x^2}.$$

10.3. 已知数据

x_i	-3	-1	1	3
y_i	15	5	1	5

试用 $y = ax^2 + bx + c$ 来拟合上述数据.

解. 法方程为

$$\begin{bmatrix} 4 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 164 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 26 \\ -34 \\ 186 \end{bmatrix}.$$

解得: $a = \frac{7}{8}, b = -\frac{17}{10}, c = \frac{17}{8}$. 因此拟合多项式为

$$y = \frac{7}{8}x^2 - \frac{17}{10}x + \frac{17}{8}.$$

10.4. 给定数据

x_i	0.5	1.0	1.5	2.0
y_i	0.4000	0.3333	0.2857	0.2500

试用线性化方法以 $y = \frac{a}{x+b}$ 的形式拟合上述数据.

解. 对方程线性化, 得到

$$\frac{1}{y} = \frac{b}{a} + \frac{1}{a}x.$$

设 $\frac{1}{y} = Y, \frac{1}{a} = A, \frac{b}{a} = B$, 则有 $Y = Ax + B$. 用最小二乘法拟合得到法方程:

$$\begin{bmatrix} 4 & 5 \\ 5 & 7.5 \end{bmatrix} \begin{bmatrix} B \\ A \end{bmatrix} = \begin{bmatrix} 11.5 \\ 15.25 \end{bmatrix}.$$

得 $Y = 0.7x + 2$ 即 $y = \frac{1}{0.7x + 2}$.

10.5. 给定数据

x_i	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
y_i	-1	0	1	2	1

试用 $y = a \sin \pi x + b \cos \pi x$ 形式来拟合上述数据.

解. 由最小二乘法拟合得到法矩阵:

$$\begin{bmatrix} \sum_{i=0}^4 \sin^2 \pi x_i & \sum_{i=0}^4 \sin \pi x_i \cos \pi x_i \\ \sum_{i=0}^4 \sin \pi x_i \cos \pi x_i & \sum_{i=0}^4 \cos^2 \pi x_i \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

则法方程为:

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^4 y_i \sin \pi x_i \\ \sum_{i=0}^4 y_i \cos \pi x_i \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

解得 $y = \sin \pi x + \frac{1}{3} \cos \pi x$.