

作业 7

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7.1. 作 $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & -4 & 5 \end{bmatrix}$ 的 QR 分解

解. 用构建 Householder 矩阵的方法: 设 A 的列向量为 $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, 则

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \|\mathbf{a}_1\|_2 = \sqrt{9} = 3$$

取 $\mathbf{u}_1 = \mathbf{a}_1 - 3\mathbf{e}_1 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}$, 则

$$P_1 = I - \frac{2}{\mathbf{u}_1^T \mathbf{u}_1} \mathbf{u}_1 \mathbf{u}_1^T = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

计算 $P_1 A$:

$$P_1 A = \begin{bmatrix} 3 & -3 & 3 \\ 0 & 3 & -3 \\ 0 & 0 & 3 \end{bmatrix}$$

此时已是上三角矩阵, 故 $R = P_1 A$, $Q = P_1^T = P_1$.

7.2. 用正交相似变换化下列矩阵为上 Hessenberg 矩阵:

$$(1) \begin{bmatrix} 2 & -1 & 3 \\ 2 & 0 & 1 \\ -2 & 1 & 4 \end{bmatrix};$$

解. 构造 Householder 矩阵: 设 A 的列向量为 $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, 则

$$\mathbf{a}_1^{(1)} = \begin{bmatrix} a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad \|\mathbf{a}_1^{(1)}\|_2 = 2\sqrt{2}$$

取 $\mathbf{u}_1 = \mathbf{a}_1^{(1)} - 2\sqrt{2}\mathbf{e}_1 = \begin{bmatrix} 2 - 2\sqrt{2} \\ -2 \end{bmatrix}$, 则

$$\bar{P}_1 = I - \frac{2}{\mathbf{u}_1^T \mathbf{u}_1} \mathbf{u}_1 \mathbf{u}_1^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{2}{\sqrt{2}} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$\text{则 } P_1 = \begin{bmatrix} 1 & 0 \\ 0 & \bar{P}_1 \end{bmatrix}$$

$$A^{(2)} = P_1 A P_1 = \begin{bmatrix} 2 & -2\sqrt{2} & -\sqrt{2} \\ -2\sqrt{2} & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

$$(2) \begin{bmatrix} 4 & 1 & -2 & 2 \\ 1 & 2 & 0 & 1 \\ -2 & 0 & 3 & -2 \\ 2 & 1 & -2 & 1 \end{bmatrix}.$$

解. 构造 Householder 矩阵: 设 A 的列向量为 $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$, 则

$$\mathbf{a}_1^{(1)} = \begin{bmatrix} a_{21} \\ a_{31} \\ a_{41} \end{bmatrix}, = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \|\mathbf{a}_1\|_2 = 3$$

$$\text{取 } \mathbf{u}_1 = \mathbf{a}_1^{(1)} - 3\mathbf{e}_1 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}, \text{ 则}$$

$$\bar{P}_1 = I - \frac{2}{\mathbf{u}_1^T \mathbf{u}_1} \mathbf{u}_1 \mathbf{u}_1^T = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$\text{则 } P_1 = \begin{bmatrix} 1 & 0 \\ 0 & \bar{P}_1 \end{bmatrix}$$

$$A^{(2)} = P_1 A P_1 = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & \frac{38}{9} & -\frac{4}{9} & -\frac{5}{9} \\ 0 & -\frac{9}{5} & -\frac{1}{9} & -\frac{9}{17} \\ 0 & -\frac{9}{9} & -\frac{8}{9} & \frac{17}{9} \end{bmatrix}$$

$$\mathbf{a}_2^{(2)} = \begin{bmatrix} a_{32}^{(2)} \\ a_{42}^{(2)} \end{bmatrix} = \begin{bmatrix} -\frac{4}{9} \\ \frac{9}{5} \\ -\frac{9}{9} \end{bmatrix}, \quad \|\mathbf{a}_2^{(2)}\|_2 = \frac{\sqrt{41}}{9}$$

$$\text{取 } \mathbf{u}_2 = \mathbf{a}_2^{(2)} - \frac{13}{9}\mathbf{e}_1 = \begin{bmatrix} -\frac{4-\sqrt{41}}{9} \\ \frac{9}{5} \\ -\frac{9}{9} \end{bmatrix}, \text{ 则}$$

$$\bar{P}_2 = I_2 - \frac{2}{\mathbf{u}_2^T \mathbf{u}_2} \mathbf{u}_2 \mathbf{u}_2^T = \frac{1}{\sqrt{41}} \begin{bmatrix} -4 & -5 \\ -5 & 4 \end{bmatrix}.$$

$$\text{则 } P_2 = \begin{bmatrix} I_2 & 0 \\ 0 & \bar{P}_2 \end{bmatrix}$$

$$A^{(3)} = P_2 A^{(2)} P_2 = \begin{bmatrix} 4 & 3 & 0 & 0 \\ 3 & \frac{38}{9} & \frac{\sqrt{41}}{89} & 0 \\ 0 & \frac{\sqrt{41}}{9} & \frac{369}{48} & -\frac{41}{63} \\ 0 & 0 & -\frac{41}{41} & \frac{41}{41} \end{bmatrix}.$$

$A^{(3)}$ 即为所求的上 Hessenberg 矩阵.

7.3. 设 $A = \begin{bmatrix} 2 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$, 试计算一步 QR 迭代. 用基本 QR 算法.

解. 对 A_1 做 QR 分解

取第一列

$$a_1 = \begin{bmatrix} 2 \\ \varepsilon \end{bmatrix}, \quad \|a_1\| = \sqrt{4 + \varepsilon^2} =: s.$$

定义

$$q_1 = \frac{1}{s} \begin{bmatrix} 2 \\ \varepsilon \end{bmatrix}.$$

取与 q_1 正交的单位向量

$$q_2 = \frac{1}{s} \begin{bmatrix} -\varepsilon \\ 2 \end{bmatrix},$$

$$Q_1 = Q = \frac{1}{s} \begin{bmatrix} 2 & -\varepsilon \\ \varepsilon & 2 \end{bmatrix}.$$

由 $R_1 = Q_1^T A_1$ 得

$$Q^T = \frac{1}{s} \begin{bmatrix} 2 & \varepsilon \\ -\varepsilon & 2 \end{bmatrix},$$

因此

$$R_1 = R = Q^T A = \frac{1}{s} \begin{bmatrix} 2 & \varepsilon \\ -\varepsilon & 2 \end{bmatrix} \begin{bmatrix} 2 & \varepsilon \\ \varepsilon & 1 \end{bmatrix} = \begin{bmatrix} s & \frac{3\varepsilon}{s} \\ 0 & \frac{2 - \varepsilon^2}{s} \end{bmatrix},$$

其中 $s = \sqrt{4 + \varepsilon^2}$.

$$A_2 = RQ = \begin{bmatrix} s & \frac{3\varepsilon}{s} \\ 0 & \frac{2 - \varepsilon^2}{s} \end{bmatrix} \cdot \frac{1}{s} \begin{bmatrix} 2 & -\varepsilon \\ \varepsilon & 2 \end{bmatrix} = \begin{bmatrix} \frac{5\varepsilon^2 + 8}{\varepsilon^2 + 4} & \frac{\varepsilon(2 - \varepsilon^2)}{\varepsilon^2 + 4} \\ \frac{\varepsilon(2 - \varepsilon^2)}{\varepsilon^2 + 4} & \frac{2(2 - \varepsilon^2)}{\varepsilon^2 + 4} \end{bmatrix}.$$

7.4. 2. 设 $f(x) = 3xe^x - 2e^x$, 取 $x_0 = 1.0, x_1 = 1.05, x_2 = 1.07$, 构造二次 Lagrange 插值多项式 L_2 , 并计算 $f(1.03)$ 的近似值. 给出实际计算误差及估计误差界.

解. 插值节点为

$$\begin{aligned} f(x_0) &= 3 \cdot 1 \cdot e^1 - 2e^1 \approx 2.71828, & f(x_1) &= 3 \cdot 1.05 \cdot e^{1.05} - 2e^{1.05} = 3.28630, \\ f(x_2) &= 3 \cdot 1.07 \cdot e^{1.07} - 2e^{1.07} = 3.52761. \end{aligned}$$

则二次 Lagrange 插值多项式为

$$\begin{aligned} L_2(x) &= 2.71828 \cdot \frac{(x-1.05)(x-1.07)}{(1.0-1.05)(1.0-1.07)} + 3.28630 \cdot \frac{(x-1.0)(x-1.07)}{(1.05-1.0)(1.05-1.07)} \\ &\quad + 3.52761 \cdot \frac{(x-1.0)(x-1.05)}{(1.07-1.0)(1.07-1.05)} \\ &\Rightarrow L_2(1.03) \approx 3.0525. \end{aligned}$$

计算误差

$$R_2(1.03) = f(1.03) - 3.0525 \approx 0.000114$$

估计误差界:

$$\begin{aligned} h &= \max\{|1.05-1.0|, |1.07-1.05|\} = 0.05 \\ \|f - L_n\|_\infty &\leq \frac{h^{n+1}}{4(n+1)} \|f^{(n+1)}\|_\infty = \frac{0.05^3}{12} \max_{x \in [1, 1.07]} |f^{(3)}(x)| \approx 0.0007 \end{aligned}$$

7.5. 给定数据

x	1	1.5	0	2
$f(x)$	3	3.25	3	5/3

试构造出 f 的均差表和三次 Newton 插值多项式, 并写出均差型余项.

解.

x	1	1.5	0	2
$f[x_i]$	3	3.25	3	5/3
$f[x_i, x_{i+1}]$	0.5	-0.5	$-\frac{2}{3}$	
$f[x_i, x_{i+1}, x_{i+2}]$	1	$-\frac{2}{3}$		
$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$-\frac{5}{3}$			

则三次 Newton 插值多项式为

$$N_3(x) = 3 + 0.5(x-1) + 1(x-1)(x-1.5) - \frac{5}{3}(x-1)(x-1.5)x$$

均差型余项为

$$R_3(x) = f[x_0, x_1, x_2, x_3] \cdot (x-1)(x-1.5)x(x-2)$$