作业 8

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8.1. 设 $f(x) = e^{0.1x^2}$, 用基函数的 Lagrange 形式的三次 Hermite 插值多项式 H_3 来计算 $f(1.25)$ 的近似值. f 及 f' 的值如下

x_i	1	1.5
$f(x_i)$	1.105170918	1.252322716
$f'(x_i)$	0.2210341836	0.3756968148

解.

$$
\begin{aligned}
H_3(x) &= \sum_{j=0}^{1}[f(x_j)\alpha_j(x) + f'_j(x)\beta_j(x)] \\
&= f(x_0)\alpha_0(x) + f'(x_0)\beta_0(x) + f(x_1)\alpha_1(x) + f'(x_1)\beta_1(x)
\end{aligned}
$$

其中

$$
\begin{aligned}
\alpha_0(x) &= \left(1 - 2(x - x_0)\sum_{i=1}^{1}\frac{1}{x_0 - x_i}\right) l_0^2(x) = (4x-3)l_0^2(x) \\
\beta_0(x) &= (x - x_0)l_0^2(x) = (x-1)l_0^2(x) \\
\alpha_1(x) &= \left(1 - 2(x - x_1)\sum_{i=0, i\neq 1}^{1}\frac{1}{x_1 - x_i}\right) l_1^2(x) = (7-4x)l_1^2(x) \\
\beta_1(x) &= (x - x_1)l_1^2(x) = (x-1.5)l_1^2(x)
\end{aligned}
$$

基函数

$$
\begin{aligned}
l_0(x) &= \frac{x - x_1}{x_0 - x_1} = \frac{x - 1.5}{1 - 1.5} = -2(x - 1.5) \\
l_1(x) &= \frac{x - x_0}{x_1 - x_0} = \frac{x - 1}{1.5 - 1} = 2(x - 1)
\end{aligned}
$$

得到

$$
\begin{aligned}
H_3(x) = {} & 1.105170918(4x-3)(-2(x-1.5))^2 + 0.2210341836(x-1)(-2(x-1.5))^2 \\
& + 1.252322716(7-4x)(2(x-1))^2 + 0.3756968148(x-1.5)(2(x-1))^2 \\
\approx {} & 0.0324952256x^3 + 0.0328055352x^2 + 0.0579374364x + 0.9819327208
\end{aligned}
$$

则

$$
f(1.25) \approx H_3(1.25) \approx 1.16908040255
$$

8.2. 设 $f(x)=\sin(e^x-2)$, 其数据如下

x	$f(x)$	$f'(x)$
0.8	0.22363362	2.1691753
1.0	0.65809197	2.0466965

利用 Newton 形式的三次 Hermite 插值多项式 H_3 来计算 $f(0.9)$ 的近似值, 并计算其实际误差及相应的误差估计.

解. 均差

$$
\begin{aligned}
f[x_0] &= f(0.8) = 0.22363362 \\
f[x_0,x_0] &= f'(0.8) = 2.16917530 \\
f[x_0,x_0,x_1] &= \frac{f[x_0,x_1]-f[x_0,x_0]}{x_1-x_0} = \frac{\frac{f(1.0)-f(0.8)}{1.0-0.8}-2.1691753}{1.0-0.8} = 0.01558225 \\
f[x_0,x_0,x_1,x_1] &= \frac{f[x_0,x_1,x_1]-f[x_0,x_0,x_1]}{x_1-x_0} = \frac{\frac{f'(1.0)-f[x_0,x_1]}{1.0-0.8}+5.05541865}{1.0-0.8} = -3.21779250
\end{aligned}
$$

则 Newton 形式的三次 Hermite 插值多项式为

$$
\begin{aligned}
H_3(x) &= f(x_0)+f[x_0,x_0](x-x_0)+f[x_0,x_0,x_1](x-x_0)^2+f[x_0,x_0,x_1,x_1](x-x_0)^2(x-x_1) \\
&= 0.22363362+2.16917530(x-0.8)+0.01558225(x-0.8)^2-3.21779250(x-0.8)^2 \\
&\approx -3.21779250x^3+8.38184275x^2-5.06361150x+0.55765322
\end{aligned}
$$

则

$$f(0.9)\approx H_3(0.9)\approx 0.44392476$$

实际误差

$$f(0.9)-H_3(0.9)\approx -0.0003$$

误差估计

$$R_3(0.9)=f[x_0,x_0,x_1,x_1,0.9](x-0.8)^2(0.9-1.0)^2=-0.0003323$$

8.3. 求次数不超过 4 次的多项式 P, 使其满足

$$P(1)=P(3)=0, \quad P(2)=1, \quad P'(1)=0, \quad P''(1)=8;$$

并写出其 Newton 形式的余项.

解. 设 Hermite 插值节点为 $x_0=1,\ x_1=1,\ x_2=1,\ x_3=2,\ x_4=3$, 定义 $f(x)=P(x)$，则

$$f(1)=0, \quad f'(1)=0, \quad f''(1)=8, \quad f(2)=1, \quad f(3)=0.$$

均差

$$f[x_0]=f(1)=0,$$

$$f[x_0,x_1]=f'(1)=0 \quad (x_0=x_1=1),$$

$$f[x_0,x_1,x_2]=\frac{f''(1)}{2!}=\frac{8}{2}=4 \quad (x_0=x_1=x_2=1).$$
$$f[x_2,x_3]=\frac{f(2)-f(1)}{2-1}=\frac{1-0}{1}=1,$$
$$f[x_1,x_2]=f'(1)=0,$$
$$f[x_1,x_2,x_3]=\frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}=\frac{1-0}{2-1}=1,$$
$$f[x_0,x_1,x_2,x_3]=\frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}=\frac{1-4}{2-1}=-3.$$
$$f[x_2,x_3]=1, \quad f[x_3,x_4]=\frac{f(3)-f(2)}{3-2}=\frac{0-1}{1}=-1,$$
$$f[x_2,x_3,x_4]=\frac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}=\frac{-1-1}{3-1}=-1,$$
$$f[x_1,x_2,x_3,x_4]=\frac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}=\frac{-1-1}{3-1}=-1,$$
$$f[x_0,x_1,x_2,x_3,x_4]=\frac{f[x_1,x_2,x_3,x_4]-f[x_0,x_1,x_2,x_3]}{x_4-x_0}=\frac{-1-(-3)}{3-1}=1.$$

因此 Newton 形式为

$$\begin{aligned}
P(x)&=f[x_0]+f[x_0,x_1](x-x_0)+f[x_0,x_1,x_2](x-x_0)(x-x_1)\\
&\quad+f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)\\
&\quad+f[x_0,x_1,x_2,x_3,x_4](x-x_0)(x-x_1)(x-x_2)(x-x_3)\\
&=0+0(x-1)+4(x-1)^2-3(x-1)^3+(x-1)^3(x-2). \quad =x^4-8x^3+22x^2-24x+9.
\end{aligned}$$

余项为

$$R_4(x)=f[1,1,1,2,3,x]\,(x-1)^3(x-2)(x-3).$$

8.4. 设 $f(x)=\dfrac{1}{a-x}$, 证明

(1) $f[x_0,x_1,\cdots,x_n]=\displaystyle\prod_{j=0}^{n}\left(\frac{1}{a-x_j}\right)$.

证明. 由数学归纳法

当 $n=0$ 时, $f[x_0]=f(x_0)=\dfrac{1}{a-x_0}$, 命题成立.

假设当 $n=k$ 时, 命题成立, 即 $f[x_0,x_1,\cdots,x_k]=\displaystyle\prod_{j=0}^{k}\left(\frac{1}{a-x_j}\right)$.

则当 $n=k+1$ 时,

$$
\begin{aligned}
f[x_0,x_1,\cdots,x_{k+1}] &= \frac{f[x_1,x_2,\cdots,x_{k+1}]-f[x_0,x_1,\cdots,x_k]}{x_{k+1}-x_0}\\
&= \frac{\prod_{j=1}^{k+1}\left(\frac{1}{a-x_j}\right)-\prod_{j=0}^{k}\left(\frac{1}{a-x_j}\right)}{x_{k+1}-x_0}\\
&= \frac{\frac{(a-x_0)-(a-x_{k+1})}{(a-x_0)(a-x_{k+1})}\prod_{j=1}^{k}\left(\frac{1}{a-x_j}\right)}{x_{k+1}-x_0}\\
&= \frac{1}{(a-x_0)(a-x_{k+1})}\prod_{j=1}^{k}\left(\frac{1}{a-x_j}\right)\\
&= \prod_{j=0}^{k+1}\left(\frac{1}{a-x_j}\right)
\end{aligned}
$$

故命题成立. □

(2) $\frac{1}{a-x}=\frac{1}{a-x_0}+\frac{1}{(a-x_0)(a-x_1)}(x-x_0)+\cdots+\frac{1}{(a-x_0)\cdots(a-x_n)}(x-x_0)\cdots$

$(x-x_{n-1})+\frac{1}{(a-x_0)\cdots(a-x_n)(a-x)}(x-x_0)\cdots(x-x_n)$

证明. 由 Newton 插值多项式的定义可知

$$
\begin{aligned}
P_n(x) &= f[x_0]+f[x_0,x_1](x-x_0)+\cdots+f[x_0,x_1,\cdots,x_n](x-x_0)\cdots(x-x_{n-1})\\
&= \frac{1}{a-x_0}+\frac{1}{(a-x_0)(a-x_1)}(x-x_0)+\cdots+\frac{1}{(a-x_0)\cdots(a-x_n)}(x-x_0)\cdots(x-x_{n-1})
\end{aligned}
$$

由插值余项公式可知

$$
R_n(x)=f[x_0,x_1,\cdots,x_n,x](x-x_0)(x-x_1)\cdots(x-x_n)=\frac{1}{(a-x_0)\cdots(a-x_n)(a-x)}(x-x_0)\cdots(x-x_n)
$$

故

$$
\begin{aligned}
f(x) &= P_n(x)+R_n(x)\\
&= \frac{1}{a-x_0}+\frac{1}{(a-x_0)(a-x_1)}(x-x_0)+\cdots+\frac{1}{(a-x_0)\cdots(a-x_n)}(x-x_0)\cdots(x-x_{n-1})\\
&+\frac{1}{(a-x_0)\cdots(a-x_n)(a-x)}(x-x_0)\cdots(x-x_n)
\end{aligned}
$$

□

8.5. 给定数据表

x	-3	-2	1	4
$f(x)$	2	0	3	1

试求三次样条插值函数 S, 使其分别满足边界条件

(1) $S'(x_0)=-1, S'(x_3)=1.$

解.

$$
\begin{aligned}
S_0(x) &= a_0+b_0(x+3)+c_0(x+3)^2+d_0(x+3)^3\\
S_1(x) &= a_1+b_1(x+2)+c_1(x+2)^2+d_1(x+2)^3\\
S_2(x) &= a_2+b_2(x-1)+c_2(x-1)^2+d_2(x-1)^3
\end{aligned}
$$

由插值条件得

$$
\begin{aligned}
S_0(-3) &= a_0 = 2 \\
S_0(-2) &= a_0 + b_0 + c_0 + d_0 = 0 \\
S_1(-2) &= a_1 = 0 \\
S_1(1) &= a_1 + 3b_1 + 9c_1 + 27d_1 = 3 \\
S_2(1) &= a_2 = 3 \\
S_2(4) &= a_2 + 3b_2 + 9c_2 + 27d_2 = 1
\end{aligned}
$$

由连续性条件得

$$
\begin{aligned}
S_0'(-2) &= b_0 + 2c_0 + 3d_0 = S_1'(-2) = b_1 \\
S_1'(1) &= b_1 + 6c_1 + 27d_1 = S_2'(1) = b_2
\end{aligned}
$$

由二阶连续性条件得

$$
\begin{aligned}
S_0''(-2) &= 2c_0 + 6d_0 = S_1''(-2) = 2c_1 \\
S_1''(1) &= 2c_1 + 18d_1 = S_2''(1) = 2c_2
\end{aligned}
$$

$$S_2'(4) = b_2 + 6c_2 + 27d_2 = 1.$$

由边界条件得

$$
\begin{aligned}
S_0'(-3) &= b_0 = -1 \\
S_2'(4) &= b_2 + 6c_2 + 27d_2 = 1
\end{aligned}
$$

解得

$$
\begin{aligned}
S_0(x) &= 2 - (x+3) - \frac{76}{31}(x+3)^2 + \frac{45}{31}(x+3)^3 \\
S_1(x) &= -\frac{48}{31}(x+2) + \frac{59}{31}(x+2)^2 - \frac{98}{279}(x+2)^3 \\
S_2(x) &= 3 + \frac{12}{31}(x-1) - \frac{39}{31}(x-1)^2 + \frac{253}{837}(x-1)^3.
\end{aligned}
$$

(2) $S''(x_0) = 0, S''(x_3) = 0.$

解. 同理可得

$$
\begin{aligned}
S_0(x) &= 2 - \frac{215}{87}(x+3) + \frac{41}{87}(x+3)^3, & -3 \le x \le -2 \\
S_1(x) &= -\frac{92}{87}(x+2) + \frac{41}{29}(x+2)^2 - \frac{190}{783}(x+2)^3, & -2 \le x \le 1 \\
S_2(x) &= 3 + \frac{76}{87}(x-1) - \frac{67}{87}(x-1)^2 + \frac{67}{783}(x-1)^3, & 1 \le x \le 4.
\end{aligned}
$$