作业 4

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4.1. 下列向量序列 $\{x^{(k)}\}$ 是否有极限? 若有, 写出其极限向量.

(1) $\boldsymbol{x}^{(k)} = \left(\mathrm{e}^{-k}\cos k, k\sin\frac{1}{k}, 3+\frac{1}{k^2}\right)^{\mathrm{T}}$

解. 该向量序列有极限.

$\lim\limits_{k\to\infty} \mathrm{e}^{-k}\cos k = 0,$

$\lim\limits_{k\to\infty} k\sin\frac{1}{k} = \lim\limits_{k\to\infty} \frac{\sin\frac{1}{k}}{\frac{1}{k}} = 1,$

$\lim\limits_{k\to\infty} 3+\frac{1}{k^2} = 3.$

所以 $\lim\limits_{k\to\infty} \boldsymbol{x}^{(k)} = (0,1,3)^{\mathrm{T}}.$

(2) $\boldsymbol{x}^{(k)} = \left(k\mathrm{e}^{-k^2}, \frac{\cos k}{k}, \sqrt{k^2+k}-k\right)^{\mathrm{T}}$

解. 该向量序列有极限.

$\lim\limits_{k\to\infty} k\mathrm{e}^{-k^2} = 0,$

$\lim\limits_{k\to\infty} \frac{\cos k}{k} = 0,$

$\lim\limits_{k\to\infty} \sqrt{k^2+k}-k = \lim\limits_{k\to\infty} \frac{(\sqrt{k^2+k}-k)(\sqrt{k^2+k}+k)}{\sqrt{k^2+k}+k} = \lim\limits_{k\to\infty} \frac{k}{\sqrt{k^2+k}+k} =$
$\lim\limits_{k\to\infty} \frac{1}{\sqrt{1+\frac{1}{k}}+1} = \frac{1}{2}.$

所以 $\lim\limits_{k\to\infty} \boldsymbol{x}^{(k)} = \left(0,0,\frac{1}{2}\right)^{\mathrm{T}}.$

4.2. 分析方程组

$$\begin{bmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

J 法和 GS 法的收敛性.

解.

$$a_{11} = 1 > 0$$

$$\begin{vmatrix} 1 & a \\ a & 1 \end{vmatrix} = 1 - a^2 > 0 \Rightarrow |a| < 1$$

$$\begin{vmatrix} 1 & a & 0 \\ a & 1 & a \\ 0 & a & 1 \end{vmatrix} = 1 - 2a^2 > 0 \Rightarrow |a| < \frac{\sqrt{2}}{2}$$

综上所述，当 $|a| < \dfrac{\sqrt{2}}{2}$ 时收敛.

4.3. 设 A 对称正定, 其特征值 $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. 证明迭代法

$$x^{(k+1)} = x^{(k)} + \omega(b - Ax^{(k)}), \quad k = 0, 1, \cdots$$

当 $\omega \in \left(0, \dfrac{2}{\lambda_1}\right)$ 时收敛，并讨论 ω 取什么值时收敛最快？

解. 迭代矩阵 $B = I - \omega A$，其特征值为 $\mu_i = 1 - \omega\lambda_i$，$i = 1, 2, \cdots, n$.
收敛时，由充要条件 $\rho(B) = \max\{|1 - \omega\lambda_1|, |1 - \omega\lambda_n|\} < 1$.
得到

$$\begin{cases} |1 - \omega\lambda_1| < 1 \\ |1 - \omega\lambda_n| < 1 \end{cases} \Rightarrow 0 < \omega < \frac{2}{\lambda_1}.$$

所以，当 $\omega \in \left(0, \dfrac{2}{\lambda_1}\right)$ 时收敛.
由渐进收敛率 $R(B) = -\ln\max\{|1 - \omega\lambda_1|, |1 - \omega\lambda_n|\}$

$$|1 - w\lambda_1| > |1 - w\lambda_n|$$

$$①1 - w\lambda_n \leq 0 \Rightarrow \frac{1}{\lambda_n} \leq w < \frac{1}{\lambda_1}$$

$$②\begin{cases} w\lambda_1 - 1 \geqslant 1 - w\lambda_n \Rightarrow \dfrac{2}{\lambda_1 + \lambda_n} \leqslant w \leqslant \dfrac{2}{\lambda_1} \\ w\lambda_1 - 1 \geqslant 0 \end{cases}$$

$$|1 - w\lambda_n| > |1 - w\lambda_1|$$

$$①1 - w\lambda_1 \leq 0 \Rightarrow 0 < w \leq \frac{1}{\lambda_1}$$

$$②\begin{cases} 1 - w\lambda_n \geq 0 \\ 1 - w\lambda_1 \leq 0 \end{cases} \Rightarrow \frac{1}{\lambda_1} \leq w \leq \frac{1}{\lambda_n}$$

4.4.

$$\begin{cases} 10x_1 - x_2 = 9, \\ -x_1 + 10x_2 - 2x_3 = 7, \\ -2x_2 + 10x_3 = 6. \end{cases}$$

(1) 写出 SOR 法计算公式;

解. 设 $A = \begin{bmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{bmatrix}$，$b = \begin{bmatrix} 9 \\ 7 \\ 6 \end{bmatrix}$，则 $D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$，$L = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$，
$U = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

则 SOR 迭代格式为

$$x^{(k+1)} = (D - \omega L)^{-1} \left[\omega b + (1-\omega) D x^{(k)} + \omega U x^{(k)}\right]$$

(2) 求最优松弛因子及 $\omega = \omega_b$ 时 SOR 法的渐近收敛率;

解. $B_J = D^{-1}(L+U) = \begin{bmatrix} 0 & -0.1 & 0 \\ -0.1 & 0 & -0.2 \\ 0 & -0.2 & 0 \end{bmatrix}$
计算特征值，得 $\lambda_1 = \dfrac{\sqrt{5}}{10}, \lambda_2 = \dfrac{-\sqrt{5}}{10}, \lambda_3 = 0$
则 $\rho(B_J) = \dfrac{\sqrt{5}}{10}$
最优松弛因子 $\omega_b = \dfrac{2}{1+\sqrt{1-\rho(B_J)^2}} = \dfrac{2}{1+\sqrt{1-\left(\frac{\sqrt{5}}{10}\right)^2}} = 40 - 4\sqrt{95}$
迭代矩阵 $B = (D-\omega L)^{-1}\left[(1-\omega)D + \omega U\right] = \begin{bmatrix} -0.0128 & -0.1013 & 0 \\ 0.0013 & -0.0026 & -0.2026 \\ -0.0003 & 0.0005 & 0.0282 \end{bmatrix}$
谱半径 $\rho(B) = 0.0128$
渐进收敛率 $R(B) = -\ln \rho(B) = 4.358$

(3) 取 $x^{(0)} = (0,0,0)^{\mathrm{T}}$, 用 $\omega = \omega_b$ 的 SOR 法求 $x^{(1)}, x^{(2)}, x^{(3)}$

解. 由（1）中的公式得

$$x^{(1)} = \begin{bmatrix} 0.911540 \\ 0.801299 \\ 0.770008 \end{bmatrix}^T, x^{(2)} = \begin{bmatrix} 0.981009 \\ 0.954036 \\ 0.791074 \end{bmatrix}^T x^{(3)} = \begin{bmatrix} 0.995588 \\ 0.957822 \\ 0.791571 \end{bmatrix}^T$$

4.5. 取初始向量为零向量，用共轭梯度法解方程组

$$\begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -5 \end{bmatrix}.$$

解. 选取 $\begin{matrix} x^{(0)} \in R^n, \\ p^{(0)} = r^{(0)} = b - Ax^{(0)}, \end{matrix}$
对 $k = 0, 1, \cdots$
$\alpha_k = \dfrac{\left(r^{(k)}, r^{(k)}\right)}{\left(Ap^{(k)}, p^{(k)}\right)},$
$x^{(k+1)} = x^{(k)} + \alpha_k p^{(k)},$
$r^{(k+1)} = r^{(k)} - \alpha_k A p^{(k)},$
$\beta_k = \dfrac{\left(r^{(k+1)}, r^{(k+1)}\right)}{\left(r^{(k)}, r^{(k)}\right)},$
$p^{(k+1)} = r^{(k+1)} + \beta_k r^{(k)}.$

第一次迭代：

$$\alpha_0 = \frac{r^{(0)T} r^{(0)}}{p^{(0)T} A p^{(0)}} = \frac{59}{376} \approx 0.1569.$$

$$x^{(1)} = x^{(0)} + \alpha_0 p^{(0)} = \begin{bmatrix} 0.4707 \\ 0.7846 \\ -0.7846 \end{bmatrix}, \quad r^{(1)} = r^{(0)} - \alpha_0 A p^{(0)} = \begin{bmatrix} -1.2367 \\ -0.3351 \\ -1.0771 \end{bmatrix}.$$

第二次迭代：

$$\alpha_1 = \frac{r^{(1)T} r^{(1)}}{p^{(1)T} A p^{(1)}} \approx 0.2311.$$

$$x^{(2)} = x^{(1)} + \alpha_1 p^{(1)} = \begin{bmatrix} 0.2179 \\ 0.7620 \\ -1.0884 \end{bmatrix}, \quad r^{(2)} = \begin{bmatrix} -0.1575 \\ 0.2100 \\ 0.1155 \end{bmatrix}.$$

第三次迭代：

$$\alpha_2 \approx 1.1490.$$

$$x^{(3)} = x^{(2)} + \alpha_2 p^{(2)} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad r^{(3)} = 0.$$