作业 12

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12.1. 确定求积公式

$$\int_0^1 \sqrt{x} f(x)\,\mathrm{d}x \approx A_0 f(x_0) + A_1 f(x_1)$$

的节点 x_0, x_1 和系数 A_0, A_1 使该求积公式具有三次代数精度.

解. 设 2 次正交多项式为

$$\omega_2(x) = x^2 + ax + b$$

则有

$$\int_0^1 \sqrt{x}\omega_2(x)\,\mathrm{d}x = \frac{2}{7} + \frac{2}{5}a + \frac{2}{3}b = 0$$
$$\int_0^1 \sqrt{x}x\omega_2(x)\,\mathrm{d}x = \frac{2}{9} + \frac{2}{7}a + \frac{2}{5}b = 0$$

解得 $a = -\frac{10}{9}, b = \frac{5}{21}$，故

$$\omega_2(x) = x^2 - \frac{10}{9}x + \frac{5}{21}$$

因此节点为 $x_0 = \frac{35 - 2\sqrt{70}}{63}, x_1 = \frac{35 + 2\sqrt{70}}{63}$.

$$\int_0^1 \sqrt{x}\,dx = \frac{2}{3} = A_0 + A_1, \qquad \int_0^1 x\sqrt{x}\,dx = \frac{2}{5} = A_0x_0 + A_1x_1.$$

解得

$$A_0 = \frac{1}{3} - \frac{\sqrt{70}}{150}, \qquad A_1 = \frac{1}{3} + \frac{\sqrt{70}}{150}.$$

12.2. 用 Gauss-Chebyshev 求积公式证明

$$\int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x = \frac{\pi}{2}$$

解.

$$x_k = \cos\frac{2k+1}{4}\pi, \quad A_k = \frac{\pi}{2}, \quad k = 0, 1$$

则有

$$x_0 = \frac{\sqrt{2}}{2}, \quad x_1 = -\frac{\sqrt{2}}{2}, \quad A_0 = A_1 = \frac{\pi}{2}.$$

因此

$$\int_{-1}^1 \frac{x^2}{\sqrt{1-x^2}}\,\mathrm{d}x \approx A_0 f(x_0) + A_1 f(x_1) = \frac{\pi}{2}\cdot\frac{1}{2} + \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{2}.$$

12.3. 用奇点分离方法计算

$$\int_0^1 x^{-\frac{1}{4}} \sin x \mathrm{d}x$$

提示：令 $G(x) = \begin{cases} x^{-\frac{1}{4}}(\sin x - P(x)), & 0 < x \leqslant 1 \\ 0, & x = 0. \end{cases}$ ，对 $\int_0^1 G(x)dx$ 用 $n=2$ 的复合 Simpson 求积公式，$P(x) = x - \frac{x^3}{6}$

解. 由 Simpson 公式

$$\begin{aligned} S_n(f) &= \frac{h}{6}\sum_{k=1}^{n}\left[f(x_{k-1}) + 4f(x_{k-1/2}) + f(x_k)\right] \\ &= \frac{h}{6}\left[f(a) + 4\sum_{k=1}^{n} f(x_{k-1/2}) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right] \end{aligned}$$

其中 $h = \frac{b-a}{n} = \frac{1}{2}$，$x_0 = 0, x_1 = 0.5, x_2 = 1$，则

$$\begin{aligned} S_2(G) &= \frac{1/2}{6}\left[G(0) + 4G(0.25) + 2G(0.5) + 4G(0.75) + G(1)\right] \\ &= \frac{1}{12}\left[0 + 4\cdot 0.00001149 + 2\cdot 0.0003079 + 4\cdot 0.002997 + 0.0081377\right] \\ &\approx 0.0017 \end{aligned}$$

$$\int_0^1 P(x)x^{-\frac{1}{4}}\mathrm{d}x = \int_0^1 \left(x^{\frac{3}{4}} - \frac{x^{\frac{11}{4}}}{6}\right)\mathrm{d}x = \frac{4}{7} - \frac{2}{45} \approx 0.5270.$$

则有

$$\int_0^1 x^{-\frac{1}{4}}\sin x\mathrm{d}x = \int_0^1 G(x)\mathrm{d}x + \int_0^1 P(x)x^{-\frac{1}{4}}\mathrm{d}x \approx 0.0017 + 0.5270 = 0.5287.$$

12.4. 试确定数值微分公式的误差项

（1） $f'(x_0) \approx \frac{1}{4h}[f(x_0+3h) - f(x_0-h)]$

解. 由 Taylor 展开式

$$f(x_0+3h) = f(x_0) + 3hf'(x_0) + \frac{9h^2}{2}f''(\xi_1)$$
$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(\xi_2)$$

故误差项为

$$f'(x_0) - \frac{1}{4h}[f(x_0+3h) - f(x_0-h)] = -\frac{9h}{8}f''(\xi_1) + \frac{h}{8}f''(\xi_2)$$

由于

$$-h\max f'' \leq -\frac{9h}{8}f''(\xi_1) + \frac{h}{8}f''(\xi_2) \leq -h\min f''$$

故误差项为 $-hf''(\xi)$.

（2） $f'(x_0) \approx \dfrac{1}{2h}[4f(x_0+h) - 3f(x_0) - f(x_0+2h)]$

解. 由三阶 Taylor 展开式

$$
\begin{aligned}
f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{6}f'''(\xi_1) \\
f(x_0+2h) &= f(x_0) + 2hf'(x_0) + 2h^2 f''(x_0) + \frac{4h^3}{3}f'''(\xi_2)
\end{aligned}
$$

故误差项为

$$
f'(x_0) - \frac{1}{2h}[4f(x_0+h) - 3f(x_0) - f(x_0+2h)] = \frac{h^2}{3}f'''(\xi)
$$