Report on the outcomes of a Short-Term Scientific Mission[1]

Action number: CA20111

Grantee name: Thomas Jan Mikhail

Details of the STSM

Title: The $(\infty,1)$-category of ∞-groupoids in the spirit of Lawvere

Start and end date: 08.05.2025 to 21.05.2025

Description of the work carried out during the STSM

Description of the activities carried out during the STSM. Any deviations from the initial working plan shall also be described in this section.

(max. 500 words)

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During the visit Christian Sattler and I met on a regular basis and had multiple discussions per week. I participated in the local seminar group and gave a talk which served as an introduction to Finster and Mimram´s type theory CaTT describing (∞,∞)-categories, as well as an overview of my work related to this type theory.

Two themes prevailed in the discussions with Christian Sattler, both of which relate to the foundations of globular approaches to higher categories.

On the syntactic side:

In contrast to (∞,n)-categories, there is more than one reasonable notion of (∞,∞)-categories. The definition which homotopy theorists are interested in comes equipped with an underlying ∞-groupoid, and the inherited notion of equivalences is that based on the pre-existing notion of equivalence provided by the underlying ∞-groupoids. In the second definition, obtained by localization, the equivalences are precisely the coinductive equivalences. This seems to be the notion of (∞,∞)-categories the type theory CaTT relates to, since all coherences are naturally equipped with the structure of a coinductive equivalence.

[1] This report is submitted by the grantee to the Action MC for approval and for claiming payment of the awarded grant. The Grant Awarding Coordinator coordinates the evaluation of this report on behalf of the Action MC and instructs the GH for payment of the Grant.

This motivates the introduction of an intrinsic notion of equivalences in the type theory CaTT. The ability to access equivalences internally is not only convenient and desirable, but would also bring the CaTT closer to the notion of (∞,∞)-categories in the sense relevant to homotopy theorists. We discussed ideas for implementations of such an “identity type”. One approach that emerged during the discussions is the addition of a rule which essentially implements a closure property of equivalences.

Now, modulo size issues, the collection of (∞,∞)-categories not only forms an $(\infty,1)$-category, but in fact an entire (∞,∞)-category. Another question that arose early on in our discussions was, whether CaTT can be used to describe functors of (∞,∞)-categories. In our discussions, we discovered rules by which CaTT is extended in such a way that it describes two (∞,∞)-categories and a functor between them. As a matter of fact, we can generalize these rules so as to encode also natural transformations and all higher morphisms (aka transfors). With this at hand, the discussions turned towards the development of a type theory of the (∞,∞)-category of (∞,∞)-categories with an emphasis on the implementation of a function type encoding (∞,∞)-functors.

On the semantic side:

The models of CaTT are known to be precisely the (∞,∞)-categories in the sense of Grothendieck and Maltsiniotis. A major and very difficult open problem in this area is the homotopy hypothesis, i.e. proving that the ∞-groupoids à la Grothendieck and Maltsiniotis are equivalent to spaces. Thanks to the work of Henry Simon, this question has been reduced to the construction of cylinder objects. A considerable amount of our time was also invested in understanding the current state of affairs and developing strategies for solving this problem.

Description of the STSM main achievements and planned follow-up activities

Description and assessment of whether the STSM achieved its planned goals and expected outcomes, including specific contribution to Action objective and deliverables, or publications resulting from the STSM. Agreed plans for future follow-up collaborations shall also be described in this section.

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All the questions addressed and mentioned above, although also of independent interest, are related to the search for a foundational language for ∞-groupoids in the spirit of Lawvere, as we envision this to be a globular theory. A theory of ∞-groupoids also requires an understanding of functors and all higher transfors between them. The ideas developed and discussed during the visit in the setting of (∞,∞)-categories can all be restricted to (∞,n)-categories for any finite number n. In the special case for $n=0$, the theory restricts to ∞-groupoids, and such a theory could serve as a candidate theory for the $(\infty,1)$-category of ∞-groupoids in the style of Lawvere’s ETCS.

A proof of the homotopy hypothesis on the other hand would first of all solidify and justify the globular models. Now, as mentioned above, solving the homotopy hypothesis has been reduced to the constructions of cylinder objects. Cylinder objects are closely related to cone objects and the study thereof go hand in hand. The latter is an essential ingredient in the theory of limits, and the construction of cones has played an important role in previous work of mine. As a result, a better understanding of cylinder objects, and thereby cones, would very likely also help develop the theory of limits in the globular setting, another component of the theory of ∞-groupoids in the spirit of Lawvere.

In light of the above, we consider the goals of the given STSM to have been met with success. The visit has also been successful in terms of getting to know the other members of the Logic and Types group at Chalmers, with whom I had the chance to share and exchange ideas, and some interactions of which might lead to future collaborations. Christian Sattler and I plan to continue working on these ideas and we are considering another visit in August.