Analysis 2 Übungsblatt $\cancel{6}$

Aufgabe 1

a) $\int_0^\infty e^{-ax}\,dx = \lim_{b\to\infty}\int_0^b e^{-ax}\,dx = \lim -\frac{e^{-ax}}{a}\Big|_0^b$

$= \lim_{b\to\infty}\left(-\frac{e^{-ab}}{a}-\left(-\frac{1}{a}\right)\right)$

$= \lim_{b\to\infty}\frac{1-e^{-ab}}{a} = \frac{1}{a}$

Es sei $\forall \varepsilon>0, \exists N\in \mathbb{N}(\varepsilon)\ \forall m > N(\varepsilon)$, dass

$\left|\frac{1-e^{-ab}}{a}-\frac{1}{a}\right| = \left|\frac{-e^{-ab}}{a}\right| < \varepsilon \Rightarrow \int_0^\infty e^{-ax}\,dx = \frac{1}{a}$

b) $\int_0^\infty xe^{-ax^2}\,dx = \lim_{b\to\infty}\int_0^b e^{-ax}\,dx = \lim -\frac{1}{2a}e^{-ax^2}\Big|_0^b$

$= \lim_{b\to\infty}\left(-\frac{e^{-ab^2}}{2a}-\left(-\frac{1}{2a}\right)\right)$

$= \lim_{b\to\infty}\frac{1-e^{-ab^2}}{2a} = \frac{1}{2a}$

Es sei $\forall \varepsilon>0$, $\exists N\in \mathbb{N}(\varepsilon)\ \forall m > N(\varepsilon)$, dass

$\left|\frac{1-e^{-ab^2}}{2a}-\frac{1}{2a}\right| = \left|\frac{-e^{-ab}}{2a}\right| < \varepsilon \Rightarrow \int_0^\infty xe^{-ax^2}\,dx = \frac{1}{2a}$

c) $\int_{-\infty}^\infty xe^{-ax^2}\,dx = \int_0^\infty xe^{-ax^2}\,dx + \int_{-\infty}^0 xe^{-ax^2}\,dx$

$= \lim_{b\to\infty} -\frac{1}{2a}e^{-ax^2}\Big|_0^b + \lim_{b\to-\infty} -\frac{1}{2a}e^{-ax^2}\Big|_b^0$

$= \lim_{b\to\infty}\left(-\frac{e^{-ab^2}}{2a}-\left(\frac{-1}{2a}\right)\right) + \lim_{b\to\infty}\left(-\frac{1}{2a}-\left(-\frac{e^{-ab^2}}{2a}\right)\right)$

$= \frac{1}{2a}-\frac{1}{2a} = 0.$

Es sei $\forall \varepsilon>0$, $\exists N\in \mathbb{N}(\varepsilon)\ \forall m > N(\varepsilon)$ dass

$\left|\frac{1-e^{-ab^2}}{2a}-\frac{1}{2a}\right| = \left|\frac{-e^{-ab}}{2a}\right| < \varepsilon \quad \left|\frac{e^{-ab^2}-1}{2a}-\frac{-1}{2a}\right| = \left|\frac{e^{-ab^2}}{2a}\right| < \varepsilon$

$\Rightarrow \int_{-\infty}^\infty xe^{-ax^2}\,dx = 0.$

Aufgabe 2

a) $\int_1^\infty |2x+5|^r \, dx = \lim_{a\to\infty} \int_1^a |2x+5|^r = \frac{1}{2(r+1)} |2x+5|^{r+1} \Big|_1^a$

Fall 1. $r+1=0 \Rightarrow r=-1 \Rightarrow \lim_{a\to\infty} \frac{1}{2(r+1)} |2x+5|^0 \Big|_1^a$ $\frac{1}{0}$

Fall 2. $r+1=1. \Rightarrow r=0 \Rightarrow \lim_{a\to\infty} \frac{1}{2} |2x+5| \Big|_1^a = \lim_{a\to\infty} (|2a+5|-7)$

$= \lim_{a\to\infty} 2a-2 \Rightarrow$ divergent.

Fall 3. $r+1>0 \Rightarrow r>-1 \Rightarrow \lim_{a\to\infty} \frac{1}{2(r+1)} |2x+5|^{r+1} \Big|_1^a$

$= \lim_{a\to\infty} \frac{1}{2(r+1)} |2a+5|^{r+1} - \frac{7^{r+1}}{2(r+1)} \Rightarrow$ divergent

Fall 4. $r+1<0 \quad r<-1 \Rightarrow \lim_{a\to\infty} \frac{1}{(2r+2)} |2x+5|^{r+1} \Big|_1^a$

$= \lim_{a\to\infty} \frac{1}{2(r+1)} |2a+5|^{r+1} - \frac{7^{r+1}}{2(r+1)}$

$= \lim_{a\to\infty} \frac{|2a+5|^{r+1} - 7^{r+1}}{2(r+1)}$

b) $\int_{-3}^1 |2x+5|^r \, dx = \cancel{\lim} \frac{1}{2(r+1)} |2x+5|^{r+1} \Big|_{-3}^1$

$= \frac{1}{2(r+1)} 7^{(r+1)} - \frac{1}{2(r+1)} 1^{(r+1)} = \frac{7^{(r+1)}-1}{2(r+1)} \quad r \neq -1$