Analysis I Übungsblatt 4

Aufgabe 1

$f(x) = x^3 - 7x^2 + 10x = x(x-5)(x-2) \Rightarrow$ Die Nullstellen

des $f(x)$ auf X-Achse sind $x_1 = 0 \quad x_2 = 2 \quad x_3 = 5$

$$\Rightarrow \text{Flächen} = \int_0^2 x^3 - 7x^2 + 10x \, dx + \int_2^5 x^3 - 7x^2 + 10x \, dx$$

$$= \left| \frac{1}{4}x^4 - \frac{7}{3}x^3 + 5x^2 \Big|_0^2 \right| + \left| \frac{1}{4}x^4 - \frac{7}{3}x^3 + 5x^2 \Big|_2^5 \right|$$

$$= \left| \frac{16}{3} - 0 \right| + \left| -\frac{125}{12} - \frac{16}{3} \right|$$

$$= \frac{16}{3} + \frac{189}{12} = \frac{253}{12}$$

Aufgabe 2

a) i): Z: $x_{n,k} = \frac{k}{n} \qquad x_{n,k} - x_{n,k-1} = \frac{k}{n} - \frac{k-1}{n} = \frac{1}{n}$

$$\Rightarrow \lim_{n\to\infty} \max_{1<k<n} \left|\frac{1}{n}\right| = 0$$

Es sei $\forall \varepsilon > 0, \exists N \in \mathbb{N} = N(\varepsilon), \forall n > N(\varepsilon)$, dass

$$\left|\frac{1}{n} - 0\right| < \varepsilon \Rightarrow \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| < \left|\frac{1}{N(\varepsilon)}\right| = \varepsilon_0. \quad \blacksquare$$

ii): $f(\xi_{n,k}) = f(x_{n,k}) = f\left(\frac{k}{n}\right) = \left(\frac{k}{n}\right)^2$

$$x_{n,k} - x_{n,k-1} = \frac{k}{n} - \frac{k-1}{n} = \frac{1}{n}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2$$

$$\sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 = \int_0^1 x^2 \, dx = \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{3}$$

$$\lim_{n\to\infty} \frac{n}{n} \cdot \frac{1}{3} = \cancel{\frac{1}{3} \cdot \lim_{n\to\infty} \left(\frac{n}{n}\right) = 0}$$

$$\frac{1}{3} \cdot \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} = \frac{1}{3} \cdot 1 = \frac{1}{3}$$

b) i): ξ: $x_{n,k} = (\sqrt[n]{2})^k \quad x_{n,k} - x_{n,k-1} = 2^{\frac{k}{n}} - 2^{\frac{k-1}{n}} = 2^{\frac{k}{n}}(1 - 2^{-\frac{1}{n}})$

$$\lim_{n\to\infty} \max_{1\le k\le n} \left|2^{\frac{k}{n}}\left(1 - \frac{1}{\sqrt[n]{2}}\right)\right| = 0$$

Es sei $\forall \varepsilon_1 > 0 \ \exists N \in \mathbb{N}(\varepsilon_1) \ \forall n_1 > N(\varepsilon_1)$, dass

$$\left|\left(1 - \frac{1}{\sqrt[n]{2}}\right) - 0\right| < \varepsilon_1 \Rightarrow \left|\left(1 - \frac{1}{\sqrt[n]{2}}\right)\right| < \left|\left(1 - \frac{1}{\sqrt[N(\varepsilon_1)]{2}}\right)\right| = \varepsilon_1$$

$$\Rightarrow \lim_{n\to\infty} \max_{1\le k\le n} \left|\left(1 - \frac{1}{\sqrt[n]{2}}\right)\right| = 0$$

Es sei $\forall \varepsilon_2 > 0$, $\exists N \in \mathbb{N}(\varepsilon_2) \ \forall n_2 > N(\varepsilon_2)$, dass

$$\left|2^{\frac{k}{n}} - 1\right| < \varepsilon_2 \Rightarrow \left|2^{\frac{k}{n}} - 2^0\right| < \left|\left(2^{\frac{k}{N(\varepsilon_2)}} - 2^0\right)\right| = \varepsilon_2$$

$$\Rightarrow \lim_{n\to\infty} \max_{1\le k\le n} \left|2^{\frac{k}{n}}\right| = 1$$

$$\Rightarrow \lim_{n\to\infty} \max_{1\le k\le n} \left|2^{\frac{k}{n}}\left(1 - \frac{1}{\sqrt[n]{2}}\right)\right| = 0. \quad \blacksquare$$

ii): $f(\xi_{n,k}) = \cancel{f[(\sqrt[n]{2})^k]^{-2}} = f[(\sqrt[n]{2})^k] = \frac{1}{(\sqrt[n]{2})^{2k}}$

$x_{n,k} - x_{n,k-1} = 2^{\frac{k}{n}}(1 - 2^{-\frac{1}{n}})$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f[(\sqrt[n]{2})^k] \cdot 2^{\frac{k}{n}}(1 - 2^{-\frac{1}{n}}) \qquad \sum_{k=1}^{n} f(x) = \int_1^2 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^2 = \frac{1}{2}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x)\,(x_{n,k} - x_{n,k-1}) = \lim_{n\to\infty} \frac{1}{2} \cdot 2^{\frac{k}{n}}(1 - 2^{-\frac{1}{n}})$$

$$\cancel{= \frac{1}{2} \cdot \lim_{n\to\infty} 2^{\frac{k}{n}}(1 - 2^{-\frac{1}{n}}) = 0}$$

$$= \frac{1}{2} \cdot \lim_{n\to\infty} \sum_{k=1}^{n} 2^{\frac{k}{n}}(1 - 2^{-\frac{1}{n}})$$

$$= \frac{1}{2} \sum_{k=1}^{n} 2^{\frac{k}{n}}$$

$$= \frac{1}{2} \int_0^1 2^x = \frac{1}{2} \cdot \frac{2^x}{\ln 2}\Big|_0^1$$

$$= \frac{1}{2 \cdot \ln 2}$$

c) i): Z: $x_{n,k} = 1 + \frac{k}{n}$ $\quad x_{n,k} - x_{n,k-1} = 1 + \frac{k}{n} - (1 + \frac{k-1}{n}) = \frac{1}{n}$

$$\lim_{n\to\infty} \max_{1\le k\le n} |x_{n,k} - x_{n,k-1}| = \lim_{n\to\infty} \max_{1\le k\le n} |\tfrac{1}{n}| = 0$$

Es sei $\forall \varepsilon < 0$, $\exists N \in \mathbb{N}(\varepsilon)$ $\forall m > N(\varepsilon)$, dass

$$|(\tfrac{1}{n}) - 0| < \varepsilon \Rightarrow |\tfrac{1}{n} - 0| = |\tfrac{1}{n}| < |\tfrac{1}{N(\varepsilon)}| < \varepsilon.$$

$$\Rightarrow \lim_{n\to\infty} \max_{1\le k\le n} |x_{n,k} - x_{n,k-1}| = 0 \quad \blacksquare$$

ii) $f(\xi_{n,k}) = f(\sqrt{x_{n,k-1}\, x_{n,k}}) = f(\sqrt{(1+\frac{k}{n})(1+\frac{k-1}{n})})$

$$x_{n,k} - x_{n,k-1} = \tfrac{1}{n}$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x)(x_{n,k} - x_{n,k-1}) = \sum_{k=1}^{n} f(x) = \int_{\sqrt{2}}^{2} \frac{1}{x^2}\, dx = -\frac{1}{x}\Big|_{\sqrt{2}}^{2}$$

$$= \frac{\sqrt{2}-1}{2}$$

$$= \frac{\sqrt{2}-1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} (\tfrac{1}{n}) = \frac{\sqrt{2}-1}{2} \cdot 1 = \frac{\sqrt{2}-1}{2}$$