# 2021 endterm solutions

Computer Vision II: Multiple view geometry (Technische Universität München)

Chair of Computer Vision and Artificial Intelligence
Department of Informatics
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# Computer Vision II: Multiple View Geometry

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| **Exam:** | IN2228 / Endterm | **Date:** | Monday 26th July, 2021 |
| **Examiner:** | Florian Bernard | **Time:** | 14:15 – 16:15 |

## Working instructions

- This exam consists of **18 pages** with a total of **9 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 120 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - This is an open book graded exercise: You are allowed to use any source that does not involve communication. In particular, you are allowed to use all lecture material (including solutions that we provide), textbooks, research papers, Matlab or existing webpages, etc. **Any type of communication, including actively seeking for help from others (e.g. in forums or chatrooms, etc.), is not allowed**.
- There is an additional 10 min submission buffer.
- This graded exercise uses the TUMexam platform which offers a student manual on their webpage[1]. Please also consult the information provided in Moodle.
- The boxes on the sides of the subproblems (those with numbers) are used for correction and ticking them is prohibited. A ticked box that is not part of a multiple choice question may result in zero points for the problem.
- In the multiple choice questions there is always **exactly one correct answer** (i.e., single choice). A correct answer (comprising of exactly one tick at the correct position) will give the indicated credits, whereas a wrong answer will result in 0 credits.

*Mark correct answers with a cross* ☒

*To undo a cross, completely fill out the answer option* ■

*To re-mark an option, use a human-readable marking* ×■

- **By submitting this online graded exercise to TUMexam you confirm that you did it completely alone without the help of anyone else.**

[1] https://www.tumexam.de/links.html

## Problem 1 MATLAB (19 credits)

Let `A` and `B` be two real matrices in Matlab, both of size `[p,q]`. In the questions below, you need to select the answer that holds **for all** choices of `A`, `B`, `p > 1`, and `q > 1` (unless exact values are given).

**Notes:**

- The symbol `1` is the integer one.
- The Matlab notation `A(s)` is equivalent to `Avec(s)` (for $s$ being an integer with $1 \leq s \leq pq$), where `Avec = A(:)` is the stack of columns of the matrix `A` (i.e. `Avec` is a column vector of size `[p*q,1]` that stacks all columns of the matrix `A`).

a) Let `C = kron(A,B)`. What is the value of `C(1,1)`? **(2 credit(s))**

- [ ] `A(q)*B(1)`
- [ ] `A(p)*B(1)`
- [ ] `A(1)*B(q)`
- [ ] `A(p)*B(q)`
- [ ] `A(q)*B(p)`
- [ ] None of the other options
- [x] `A(1)*B(1)` $C_{11} = A_{11}B_{11}$
- [ ] `A(1)*B(p)`

b) Let `C = kron(A,B)`. What is the value of `C(p,1)`? **(2 credit(s))**

- [ ] `A(1)*B(1)`
- [ ] None of the other options
- [x] `A(1)*B(p)` $C_{p1} = A_{11}B_{p1}$
- [ ] `A(q)*B(p)`
- [ ] `A(p)*B(1)`
- [ ] `A(p)*B(q)`
- [ ] `A(q)*B(1)`
- [ ] `A(1)*B(q)`

c) Let `C = kron(A,B)`. What is the value of `C(1,p)`? **(2 credit(s))**

- [ ] `A(p)*B(1)`
- [ ] `A(1)*B(1)`
- [ ] `A(p)*B(q)`
- [ ] `A(1)*B(p)`
- [ ] `A(1)*B(q)`
- [ ] `A(q)*B(1)`
- [x] None of the other options $C_{1p}$ cannot be computed from any of the options (can be verified e.g. in Matlab)
- [ ] `A(q)*B(p)`

— The problem continues on the next page —

d) Let `C = kron(B,A)` . What is the value of `C(p,1)` ? **(2 credit(s))**

- ☐ `A(p)*B(q)`
- ☐ `A(q)*B(p)`
- ☒ `A(p)*B(1)` $C_{p1} = A_{p1}B_{11}$
- ☐ `A(1)*B(q)`
- ☐ `A(q)*B(1)`
- ☐ `A(1)*B(p)`
- ☐ None of the other options
- ☐ `A(1)*B(1)`

e) Assume $p = q = 3$ and consider `[U,S,V] = svd(A)` . Which of the following matrices is an element of $SO(3)$ (up to numerical precision)? **(2 credit(s))**

- ☐ `U*diag([-det(U*V'),1,1])*V'` has a negative determinant
- ☐ None of the other options
- ☐ `U*V` can have a negative determinant
- ☒ `U*diag([1,1,det(U*V')])*V'` $U$ and $V$ are orthogonal, so that $UV^\top$ is also orthogonal. The term in the middle ensures that the result has the determinant +1 while remaining orthogonal
- ☐ `V'*U` can have a negative determinant
- ☐ `U*V'` can have a negative determinant
- ☐ `V*diag(diag(U))` may not be orthogonal
- ☐ `V*U` can have a negative determinant

f) Assume $p = q = 4$ and consider `[U,S,V] = svd(A)` . Which of the following matrices is an element of $SE(3)$ (up to numerical precision)? **(2 credit(s))**

- ☒ None of the other options An element of $SE(3)$ must be a matrix of $4 \times 4$, where the last row is $[0, 0, 0, 1]$. None of the given matrices fulfills this.
- ☐ `V*U`
- ☐ `V*diag(diag(U))`
- ☐ `U*diag([1,1,1,det(U*V')])*V'`
- ☐ `U*V`
- ☐ `V'*U`
- ☐ `U*V'`
- ☐ `U*diag([-det(U*V'),1,1,1])*V'`

— The problem continues on the next page —

g) Assume $p = q = 3$. Which of the following matrices is an element of $so(3)$ (up to numerical precision)? **(2 credit(s))**

- [ ] `A*A'`
- [ ] `A.*A`
- [ ] `A./A`
- [ ] `A\A`
- [ ] None of the other options
- [x] `A-A'` The matrix $A - A^\top$ is skew-symmetric, and hence in $so(3)$
- [ ] `svd(A)`
- [ ] `A+A'`

h) Assume $p = 4, q = 3$ and consider `[U,S,V] = svd(A)` . Which of the following matrices is an element of $SE(3)$ (up to numerical precision)? **(2 credit(s))**

- [ ] `S*S'`
- [ ] `U*S`
- [ ] `S*V`
- [ ] `S*V'`
- [ ] `U'*S`
- [ ] `V*V`
- [x] `U*U'` $UU^\top$ is the identity matrix of size $4 \times 4$, and belongs to $SE(3)$
- [ ] None of the other options

i) Assume `p=q` , `B = A'*A` , and that `u` is a vector of size `[q,1]` . Which of the following computes `norm(A*u)^2` (up to numerical precision)? **(3 credit(s))**

- [ ] `sum(kron(A,u)'*B(:))`
- [ ] `kron(u,u)'*A(:)`
- [ ] `sum(kron(A(:),B)*u)`
- [ ] `sum(sum(kron(A,u)*B))`
- [ ] `sum(sum(kron(u,u')'*A))`
- [x] `kron(u,u)'*B(:)` $\|Au\|^2 = u^\top A^\top Au = u^\top Bu = (u^\top \otimes u^\top)B^s$
- [ ] None of the other options
- [ ] `sum(sum(kron(u,u')'*B))`

## Problem 2 Linear Algebra (23 credits)

**Note:** You can use Matlab syntax whenever appropriate (e.g. `A'` , or write matrices as `[1 2 3; 4 5 6]` ).

a) Are there matrices that are symmetric and skew-symmetric at the same time? If your answer is 'yes', give an example, if your answer is 'no', explain why. **(2 credit(s))**

0 1 2

Yes. Zero matrices (of any dimension) are symmetric and skew-symmetric.

b) Let $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$, so that $X^s = [a, b, c, d]^\top \in \mathbb{R}^4$. **Derive and write down** the matrix $B$ such that the constraint $BX^s = 0$ is equivalent to the matrix $X$ being skew-symmetric. **(3 credit(s))**

0 1 2 3

Skew-symmetry means that $X^\top = -X \Leftrightarrow X^\top + X = 0$. Hence, $a = 0$, $b + c = 0$, $d = 0$.
Thus, for $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ we can express these constraints in matrix form, where we have $BX^s = [a, b + c, d]^\top = 0 \Leftrightarrow X^\top = -X$.
(Additional explanations: it is not sufficient to specify a matrix $B$ which satisfies $BX^S = 0$ if $X$ is skew-symmetric. In addition, $BX^S \neq 0$ must also hold if $X$ is not skew-symmetric. Our $B$ must satisfy $\ker(B) = \text{range}([0, -1, 1, 0]^T)$, where $[0, -1, 1, 0]^T$ is a basis for (vectorised) $2 \times 2$ skew-symmetric matrices.)

c) For $C = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$, write down a matrix $Q$ such that $\text{range}(Q) = \ker(C)$. **(3 credit(s))**

0 1 2 3

$Q = [1, 0, 0, -1]^T$ is one possible solution (We observe that $CQ = 0$. Moreover, $C$ has four columns and three independent rows, so its rank is 3 and its nullspace is of dimension one. Hence, a matrix $Q$ with single column is sufficient to represent the nullspace of $C$. Note that writing "$\ker(C) = [1, 0, 0, -1]^T$" is incorrect and does not give full credits (as $\ker(C) = \{\lambda[1, 0, 0, -1]^T : \lambda \in \mathbb{R}\}$).)

d) The Euclidean projection $P$ of the matrix $A \in \mathbb{R}^{n\times n}$ onto the set of skew-symmetric matrices $\mathcal{S}_n = \{X \in \mathbb{R}^{n\times n} \mid X^\top = -X\}$ can be obtained by solving the **constrained** least-squares problem $P = \text{argmin}_{X\in\mathcal{S}_n} \|A - X\|_f^2$ (i.e. we seek a minimiser of $\|A - X\|_f^2$ subject to the constraint $X \in \mathcal{S}_n$).

0 1 2 3 4

For $A^s \in \mathbb{R}^{n^2}$ being the stack of the columns of $A$, explain how we can utilise an **unconstrained** linear least-squares problem of the form

$$... = \underset{...}{\text{argmin}} \|A^s - ...\|_2^2$$

in order to express the Euclidean projection that obtains the vector $P^s$ (the stack of the matrix $P$). **(4 credit(s))**

**Hints:** Complete the three missing parts (specify the missing parts indicated by the dots in the equation above), and then specify $P^s$. First solving subproblem (e) may be helpful.

We first solve $a^* = \text{argmin}_a \|A^s - Za\|_2^2$, where $Z$ has the same meaning as in subproblem e). With that, $P^s = Za^*$.



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e) Assume $B$ is a matrix such that $BX^s = 0$ is equivalent to the matrix $X$ being skew-symmetric. Moreover, let $Z$ be a matrix such that range($Z$) = ker($B$), and let $\alpha$ be a real (column) vector that has a size such that $Z\alpha$ is well-defined. For which of the following vectors $Y^s$ is the unstacked matrix $Y$ skew-symmetric? **(3 credit(s))**

- [ ] $Y^s = M^s - Z\alpha$, where $M^s$ is the stack of the columns of any symmetric matrix $M$ of appropriate dimensions
- [ ] $Y^s = M^s + Z\alpha$, where $M^s$ is the stack of the columns of any symmetric matrix $M$ of appropriate dimensions
- [ ] $Y^s = I^s + Z\alpha$, where $I^s$ is the stack of the columns of the identity matrix
- [x] $Y^s = Z\alpha$ Since $BX^s = 0$ means that $X$ is skew-symmetric, and range($Z$) = ker($B$), any $Y$ obtained by unstacking $Y^s = Z\alpha$ is skew-symmetric
- [ ] None of the other options
- [ ] $Y^s = I^s - Z\alpha$, where $I^s$ is the stack of the columns of the identity matrix

f) What is the rank of the matrix $C = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$? **(2 credit(s))**

- [ ] None of the other options
- [x] 3 Three independent cols.
- [ ] 2
- [ ] 1
- [ ] 4

g) What is the dimension of ker($C$) for $C = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$? **(2 credit(s))**

- [ ] 2
- [x] 1 $4 - 3 = 1$ (MVG1, p. 16)
- [ ] 3
- [ ] 4
- [ ] None of the other options

h) Let $U\Sigma V^\top$ be obtained via singular value decomposition of the symmetric full-rank matrix $D$. Which of the following is true? **(2 credit(s))**

- [x] If $U = V$, the matrix $D$ is positive semi-definite If $U = V$, the SVD reads $U\Sigma V^\top = U\Sigma U^\top$, which can be understood as the eigenvalue decomposition of $D$ ($\Sigma$ is the diagonal matrix of eigenvalues, and $U$ the matrix of eigenvectors). Moreover, the diagonal matrix $\Sigma$ contains only non-negative values (per definition of SVD), so that all eigenvalues are non-negative, and hence $D$ is p.s.d. (MVG1, p. 19)
- [ ] If $D$ is the identity matrix, $U$ and $V$ can only be identity matrices With $U = -I, \Sigma = I, V = -I$ we have $D = U\Sigma V^\top = I$ as counterexample
- [ ] The matrix $\Sigma$ contains the eigenvalues of $D$ on its diagonal
- [ ] If $\Sigma$ is the identity matrix, $U$ must be equal to $V$
- [ ] If $\Sigma$ is positive semi-definite, the matrix $D$ is also positive semi-definite
- [ ] None of the other options

i) Which of the following statements is incorrect? **(2 credit(s))**

- [ ] The singular value decomposition can be used to compute the inverse of an invertible matrix correct: $A^{-1} = (U\Sigma V^\top)^{-1} = V\Sigma^{-1}U^\top$, where $\Sigma^{-1}$ is easily inverted by taking the reciprocal of each diagonal element of $\Sigma$
- [ ] The eigenvalue decomposition can be used to compute the singular value decomposition correct: MVG1, p. 25
- [ ] The singular value decomposition can be understood as generalisation of the eigenvalue decomposition to non-square matrices correct: MVG1, p. 23
- [x] The singular value decomposition and the eigenvalue decomposition must be equivalent for symmetric matrices Consider the matrix diag($-1, 1$). Its eigenvalues are $-1$ and 1; its singular values are 1 and 1
- [ ] The singular value decomposition can be used to compute the pseudo inverse of a matrix correct: MVG1, p. 28

## Problem 3 Image Formation (12 credits)

A 3D point is given in the world coordinate system as $P = [-2, -3, -4]^\top$. The point is observed by a camera which is canonically oriented (i.e., with respect to the world coordinate system, the camera origin is at $[0, 0, 0]$, the camera's rotation is given by $R = \mathbb{I}_{3\times3}$, and the camera's optical axis points along positive direction of z-axis). The intrinsic parameter matrix $K$ of the camera is given by

$$K = \begin{bmatrix} 100 & 0 & 320 \\ 0 & 100 & 240 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) Where is the point $P$ observed on the image captured by this camera? **(3 credit(s))**

- ☐ $[u, v] = [370, 315]$
- ☐ $[u, v] = [640, 480]$
- ☐ $[u, v] = [160, 120]$
- ☒ The given point is not observed in the image

The given point $P$ lies behind the camera origin $[0, 0, 0]$ and the camera looks away from the point, towards positive z-axis.

A 3D point is given in the world coordinate system as $P = [-3, -2, 5]^\top$. The point is observed by a camera which is canonically oriented (i.e., with respect to the world coordinate system, the camera origin is at $[0, 0, 0]$, the camera's rotation is given by $R = \mathbb{I}_{3\times3}$, and the camera's optical axis points along positive direction of z-axis). The intrinsic parameter matrix $K$ is given by

$$K = \begin{bmatrix} 250 & 0 & 640 \\ 0 & 500 & 360 \\ 0 & 0 & 1 \end{bmatrix}.$$

b) Where is the point $P$ observed on the image captured by this camera? **(3 credit(s))**

- ☐ The given point is not observed in the image
- ☐ $[u, v] = [70, 40]$
- ☐ $[u, v] = [880, 820]$
- ☒ $[u, v] = [490, 160]$

Solving $\lambda[u, v, 1]^\top = K\Pi_0[-3, -2, 5, 1]^\top$ would yield the result.

**— The problem continues on the next page —**

For the next two subproblems, consider the following setting.

The 3D points $X_1 = [2, 3, 5]^\top$, and $X_2 = [1, -1, 1]^\top$ in the world coordinate system, appear in the image at pixel coordinates $x_1 = [372, 462]^\top$ and $x_2 = [480, 30]^\top$, respectively. The image is captured by a camera which is canonically oriented (i.e., with respect to the world coordinate system, the camera origin is at $[0, 0, 0]$, the camera's rotation is given by $R = \mathbb{I}_{3\times3}$, and the camera's optical axis points along positive direction of z-axis). Assuming a perspective pinhole camera with no skew, the intrinsic matrix is given by

$$K = \begin{bmatrix} f_x & 0 & o_x \\ 0 & f_y & o_y \\ 0 & 0 & 1 \end{bmatrix}.$$

What are the values of the intrinsic camera parameters?

c) What is the value of $f_x$? **(3 credit(s))**

- [x] $f_x = 180$

  We write down the projection equation for the two points and solve for the parameters of $K$.
  $\lambda_1[372, 462, 1]^\top = K\Pi_0[2, 3, 5, 1]^\top$,
  $\lambda_2[480, 30, 1]^\top = K\Pi_0[1, -1, 1, 1]^\top$.
  Simplifying the above equation system gives us the following equations.
  $\lambda_1 = 5$, $2f_x + 5o_x = 372 \times 5$, $3f_y + 5o_y = 462 \times 5$,
  $\lambda_2 = 1$, $f_x + o_x = 480$, and $-f_y + o_y = 30$.
  Solving the above equations gives the us the vales of $f_x$, and $f_y$.
- [ ] $f_x = 150$
- [ ] $f_x = 100$
- [ ] $f_x = 200$

d) What is the value of $f_y$? **(3 credit(s))**

- [ ] $f_y = 300$
- [ ] $f_y = 200$
- [ ] $f_y = 150$
- [x] $f_y = 270$

## Problem 4 Essential and Fundamental Matrix (9 credits)

a) Let $E$ be a valid, non-zero essential matrix and F be a valid, non-zero fundamental matrix. Let their singular value decompositions be given by

$$E = U_E \text{diag}(\lambda_1, \lambda_2, \lambda_3) V_E^\top \quad \text{and} \quad F = U_F \text{diag}(\mu_1, \mu_2, \mu_3) V_F^\top$$

where the singular values are ordered from largest to smallest, i.e. $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$ and $\mu_1 \geq \mu_2 \geq \mu_3 \geq 0$. Which of the following properties does **not** hold true in general?

*Remark:* The operator $\widehat{\cdot}$ maps a vector $u \in \mathbb{R}^3$ to the corresponding skew-symmetric matrix $\widehat{u} \in \mathbb{R}^{3\times3}$. **(2 credit(s))**

- ☐ $\text{rank}(F) = 2$ $F = K^{-\top} E K^{-1}$. $K$ has full rank, therefore $F$ has same rank as $E$, i.e. 2
- ☐ $\mu_3 = 0$ $F$ has rank 2, i.e. the smallest singular value is 0
- ☐ $EE^\top = -\widehat{a}\widehat{a}$ for some $a \in \mathbb{R}^3$ $EE^\top = \widehat{T}R(\widehat{T}R)^\top = \widehat{T}RR^\top(-\widehat{T}) = -\widehat{T}\widehat{T}$
- ☐ $\lambda_1 = \lambda_2$ True according to the definition of the essential matrix
- ☐ $\lambda_3 = 0$ $E$ has rank 2, i.e. the smallest singular value is 0
- ☒ $\mu_1 = \mu_2$ While the first two singular values are equal for the essential matrix, this is not true for the fundamental matrix

b) Consider the matrix

☐ 0
☐ 1
☐ 2
☐ 3

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

State a **full rank** matrix $A \in \mathbb{R}^{3\times3}$ that would yield the matrix $E$ when projected onto the normalised essential space. Justify your choice. **(3 credit(s))**

Any matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

with $a, b \geq c > 0$ is fine.✓ In this case $A = \mathbb{I} \cdot \text{diag}(a, b, c) \cdot \mathbb{I}$ forms an SVD of $A$, where $\mathbb{I}$ means the identity matrix. To project onto the normalised essential space we need to replace the matrix $\Sigma$ that contains the singular values by $\text{diag}(1, 1, 0)$, which yields the matrix $E$.✓ ✓

— The problem continues on the next page —

c) Assume that we have a camera with motion constrained to the $YZ$-plane, i.e. the camera center can only be moved to locations of the form $O = [0, t_1, t_2]$ for $t_1, t_2 \in \mathbb{R}$, and the rotation is constrained to a rotation around the $X$-axis (both in world coordinates). For $a, b, c, d, e, f \in \mathbb{R}$, which form does the essential matrix $E$ take for any two poses of this constrained camera? **(4 credit(s))**

☐ $E = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & 0 \end{bmatrix}$

☐ $E = \begin{bmatrix} 0 & a & b \\ c & d & 0 \\ e & 0 & f \end{bmatrix}$

☐ $E = \begin{bmatrix} 0 & a & 0 \\ b & 0 & c \\ 0 & d & 0 \end{bmatrix}$

☐ $E = \begin{bmatrix} a & 0 & b \\ 0 & c & d \\ e & f & 0 \end{bmatrix}$

☐ $E$ does not have a special structure, all entries can be non-zero

☒ $E = \begin{bmatrix} 0 & a & b \\ c & 0 & 0 \\ d & 0 & 0 \end{bmatrix}$

A rotation around the $X$-axis does not change a point that is located on the $X$-axis. Therefore, the rotation matrix has the structure

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix},$$

where $*$ means a potentially non-zero entry. (Any point $p = [a, 0, 0]^\top$ for $a \in \mathbb{R}$ is not affected by the rotation). The translation vector looks like

$$T = -RO = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \begin{bmatrix} 0 \\ * \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ * \\ * \end{bmatrix}.$$

From that we see that the essential matrix has the structure

$$E = \widehat{T}R = \begin{bmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix} = \begin{bmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$$

## Problem 5 Corresponding Points (10 credits)

In the following we consider two cameras. The transformation between them consists only of a translation $T = [2, 2, -1]^\top$, i.e. a 3D point $X_1$ in the coordinates of camera 1 reads $X_2 = X_1 + T$ in the coordinates of camera 2. The intrinsics matrices of the first and the second camera are given as

$$K_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad K_2 = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

a) Consider the point $x_1' = [30, 20]^\top$ given in pixel coordinates. Which of the following pixels $x_2'$ can be a correspondence of $x_1$? **(3 credit(s))**

- ☐ $x_2' = [20, 16]^\top$
- ☐ $x_2' = [35, 12]^\top$
- ☐ $x_2' = [18, 15]^\top$
- ☒ None of the other options
- ☐ $x_2' = [15, 5]^\top$

We need to check the epipolar constraint for uncalibrated cameras for this.

$$(x_2')^\top K_2^{-\top} \widehat{T} K_1^{-1} x_1' = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \\ 1 \end{bmatrix} = (x_2')^\top \begin{bmatrix} 3/2 \\ -3 \\ -12 \end{bmatrix} = 0$$

(We now use a homogeneous version of $x_2'$). None of the points fulfills this constraint.

b) Describe how you obtained the result in a). You do not need to give numeric values. Formulas and/or a verbal explanation of your approach are sufficient. **(3 credit(s))**

☐ 0 ☐ 1 ☐ 2 ☐ 3

To check if the point pairs match, we need to check the epipolar constraint for uncalibrated cameras, which none of the point pairs fulfill. ✓ ✓ ✓ (The epipolar constraint reads $(x_2')^\top F x_1' = (x_2')^\top K_2^{-\top} \widehat{T} K_1^{-1} x_1' = 0$ in this case, since $R = I$)

c) Consider the 3D point $X_1 \in \mathbb{R}^3$. Let $x_1 = [4, 3, 1]^\top$ be its **normalised** coordinates in image 1, i.e. $X_1 = \lambda_1 x_1$ where $\lambda_1$ is the depth of $X_1$ in frame 1. Corresponding to $x_1$, which of the following is the equation of the epipolar line in image 2 given in **pixel coordinates**? **(4 credit(s))**

- ☐ $y = \frac{1}{2}x - \frac{5}{3}$
- ☐ $y = \frac{5}{3}x - \frac{3}{2}$
- ☐ $y = \frac{4}{3}x - \frac{1}{3}$
- ☐ None of the other options
- ☒ $y = \frac{5}{6}x - \frac{4}{3}$

We can compute the coimage of the epipolar line in pixel coordinates as

$$l' \sim K_2^{-\top} \widehat{T} x_1 = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/4 \\ -3/2 \\ -2 \end{bmatrix}$$

Note that the point $x_1$ in image 1 is given in normalised coordinates while we are interested in the epipolar line in pixel coordinates. Therefore we need to consider $K_2$ but not $K_1$. From this we can compute the parameters of the epipolar line $y = mx + b$ as

$$m = -\frac{l_1'}{l_2'} = \frac{5}{6} \quad \text{and} \quad b = -\frac{l_3'}{l_2'} = -\frac{4}{3}$$

## Problem 6 Radial Distortion (7 credits)

a) Which of the following statements about radial distortion is true? **(2 credit(s))**

- ☐ A radial distortion model applied after the pinhole projection can model cameras with a field of view of more than 180°.
- ☐ Radial distortion is caused by image sensor noise.
- ☐ Rectangles in an image appear helical due to radial distortion.
- ☒ Radial distortion is caused by imperfect lenses. (MVG3, p. 17 and p. 18)

b) If the camera exhibits radial distortion, one can obtain a rectified virtual image through undistortion. Assuming perfect undistortion, the rectified virtual image perfectly follows the pinhole camera model. Which of the following statements about a straight line oriented in any direction in 3D is true? **(2 credit(s))**

- ☒ It will either appear as a straight line or a point in the rectified virtual image.
  In the rectified image, a straight line in 3D appears as a straight line if it is not aligned with the ray passing through the camera origin and pixel location in image plane. When it is aligned, it is a point.
- ☐ It will appear as a straight line in the rectified virtual image.
- ☐ The angle between the line and another line in 3D is preserved in the rectified virtual image.
- ☐ It will appear as a point in the rectified virtual image.

c) Consider the radial distortion model

$$x_d = g(r)\frac{x}{\|x\|}, \tag{6.1}$$

which maps coordinates $x \in \mathbb{R}^2$ in the normalized image plane to distorted coordinates $x_d \in \mathbb{R}^2$. The factor $g(r)$ depends on the radius $r = \|x\|$ and is given as

$$g(r) = s \log(1 + \frac{r}{k}), \tag{6.2}$$

where, $s \in \mathbb{R}^+$, and $k \in \mathbb{R}^+$ are model parameters. Choose the function $f$ such that the inverse distortion can be written as

$$x = f(r_d)\frac{x_d}{\|x_d\|}, \tag{6.3}$$

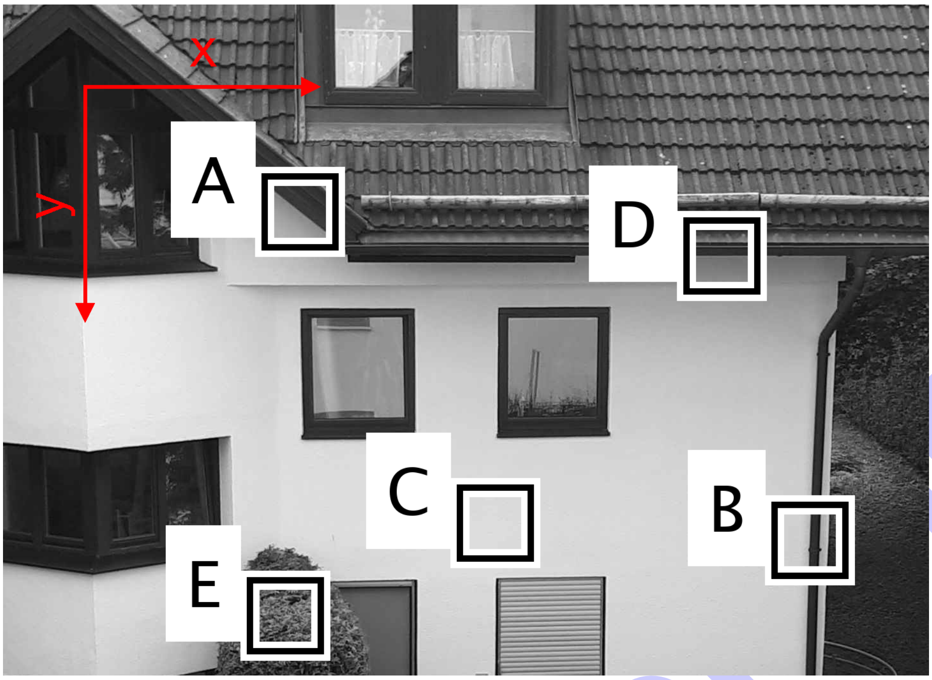
where, $r_d = \|x_d\|$. **(3 credit(s))**

- ☐ $f(r_d) = \frac{\exp(\frac{r_d}{s}) - 1}{k}$
- ☐ $f(r_d) = \frac{1 - \exp(\frac{r_d}{s})}{k}$
- ☐ $f(r_d) = \frac{\exp(\frac{r_d}{s}) - 1}{k}$
- ☒ $f(r_d) = k(\exp(\frac{r_d}{s}) - 1)$
  Taking the norm of $x_d = g(r)\frac{x}{\|x\|}$, we get $r_d = g(r)$. Similarly, taking the norm of $x = f(r_d)\frac{x_d}{\|x_d\|}$, we get $r = f(r_d)$. (Since $g(r)$, and $f(r_d)$ are always positive.)
  Now,

$$\begin{aligned}
& r_d = g(r) = s\log(1 + \frac{r}{k}) \\
\Longrightarrow\ & \frac{r_d}{s} = \log(1 + \frac{r}{k}) \\
\Longrightarrow\ & \exp(\frac{r_d}{s}) - 1 = \frac{r}{k} \\
\Longrightarrow\ & r = k(\exp(\frac{r_d}{s}) - 1) \\
\Longrightarrow\ & f(r_d) = k(\exp(\frac{r_d}{s}) - 1).
\end{aligned}$$

## Problem 7 Lucas-Kanade Algorithm: The Structure Tensor (14 credits)

The black and white boxes in the image below are annotations used within this exercise and were not present in the image used to calculate the structure tensors.




In Equation 7.1 you are given the eigenvalues $\lambda_i^{(1)}, \dots, \lambda_i^{(5)}$ and eigenvectors $v_i^{(1)}, \dots, v_i^{(5)}$ for five structure tensors $M^{(1)}, \dots, M^{(5)}$. For each tensor you have to choose the corresponding image patch A – E. The neighborhood used to compute the structure tensor $W(x)$ is set to be the same as the inside of the corresponding annotation box. A uniform weight is used throughout the window. There is a one-to-one correspondence between tensors and patches. (To obtain readable numbers, the values below are scaled and rounded.)

$$
\begin{aligned}
M^{(1)}:&\quad \lambda_1^{(1)} = 5.54\,,\ \lambda_2^{(1)} = 3.10 &\quad v_1^{(1)} = (\ \ 0.14,\ \ 0.99)^\top,\ v_2^{(1)} = (-0.99,\ \ 0.14)^\top\\
M^{(2)}:&\quad \lambda_1^{(2)} = 2.07\,,\ \lambda_2^{(2)} = 0.07 &\quad v_1^{(2)} = (-0.00,\ \ 1.00)^\top,\ v_2^{(2)} = (-1.00,-0.00)^\top\\
M^{(3)}:&\quad \lambda_1^{(3)} = 0.01\,,\ \lambda_2^{(3)} = 0.00 &\quad v_1^{(3)} = (\ \ 0.03,\ \ 1.00)^\top,\ v_2^{(3)} = (-1.00,\ \ 0.03)^\top\\
M^{(4)}:&\quad \lambda_1^{(4)} = 0.69\,,\ \lambda_2^{(4)} = 0.04 &\quad v_1^{(4)} = (-0.63,\ \ 0.78)^\top,\ v_2^{(4)} = (-0.78,-0.63)^\top\\
M^{(5)}:&\quad \lambda_1^{(5)} = 4.94\,,\ \lambda_2^{(5)} = 0.25 &\quad v_1^{(5)} = (-1.00,-0.05)^\top,\ v_2^{(5)} = (\ \ 0.05,-1.00)^\top
\end{aligned}
\tag{7.1}
$$

a) Choose the correct image patch for the structure tensor $M^{(1)}$ and explain your choice. **(4 credit(s))**

0 1 2 3 4

The correct region is E. The two eigenvalues are both large. Therefore, the patch must have sufficient structure in x- and y-direction, which is not the case for an edge or the white wall.

b) Choose the correct image patch for the structure tensor $M^{(2)}$ and explain your choice. **(4 credit(s))**

0 1 2 3 4

The correct region is D. The larger eigenvalue is large and the smaller eigenvalue is close to zero. Therefore the patch must contain an edge. The eigenvector corresponding to the large eigenvalue points approximately along the y-axis and thus the edge must be horizontal.

**— The problem continues on the next page —**

c) Choose the correct image patch for the structure tensor $M^{(3)}$. **(2 credit(s))**

- ☐ A
- ☐ E
- ☐ B
- ☐ D
- ☒ C Since both eigenvalues are small, the patch must be in an area with little structure, therefore the white wall in patch C is correct.

d) Choose the correct image patch for the structure tensor $M^{(4)}$. **(2 credit(s))**

- ☐ E
- ☐ C
- ☐ D
- ☒ A $\lambda_1^{(4)} \gg \lambda_2^{(4)}$, therefore the patch contains an edge. The eigenvector corresponding to the larger eigenvalue is oriented diagonally, therefore the edge must be tilted. This leads to patch A.
- ☐ B

e) Choose the correct image patch for the structure tensor $M^{(5)}$. **(2 credit(s))**

- ☐ D
- ☐ A
- ☒ B $\lambda_1^{(4)} \gg \lambda_2^{(4)}$, therefore the patch contains an edge. The eigenvector corresponding to the larger eigenvalue is oriented horizontally, therefore the edge must be vertical. This leads to patch B.
- ☐ C
- ☐ E

## Problem 8 Bundle Adjustment (9 credits)

a) Which of the following statements is wrong? **(2 credit(s))**

- ☐ Bundle adjustment requires initialisation
- ☐ We can define bundle adjustment for different noise models
- ☐ Bundle adjustment can estimate structure and motion simultaneously
- ☐ We can use a linear algorithm to initialise bundle adjustment
- ☐ Bundle adjustment is a non-convex problem
- ☒ The bundle adjustment problem can be solved in closed-form bundle adjustment is a difficult non-convex problem for which no closed-form solution is known

Consider the bundle adjustment energy for two views

$$E(R, T, X_1, \dots X_N) = \sum_{j=1}^{N} \|\tilde{x}_1^j - \pi(X_j)\|^2 + \|\tilde{x}_2^j - \pi(R, T, X_j)\|^2. \tag{8.1}$$

We denote by $\tilde{x}_i^j \in \mathbb{R}^2$ the observed position of point $j$ in image $i$. Note that these points are not in homogeneous coordinates. For this exercise we assume that we are working in normalised image coordinates.

b) Does equation 8.1 define a direct approach? Give an explanation for your answer. **(2 credit(s))**

0 1 2

The equation does not describe a direct approach, but an indirect approach. ✓ A direct approach would compare image intensities, while equation 8.1 aims at minimizing a geometric error (the reprojection error). ✓

c) What does the term $\pi(R, T, X_j)$ represent? Describe all the steps involved in evaluating it. **(3 credit(s))**

0 1 2 3

The term $\pi(R, T, X_j)$ first transforms the point $X_j$ to the coordinate system of camera 2, and then projects it to the image plane. ✓ To evaluate it, we can consider the equation

$$\lambda x = RX_j + T.$$

The result of the term is the first two components of the right hand side divided by the third component. ✓ ✓ (Note that we do not have the intrinsics matrix $K$ here, since we are working in normalised coordinates).

d) Why is $\pi(X_j)$ in the first term in the summation of equation 8.1 not depending on the pose parameters? **(2 credit(s))**

0 1 2

We choose the coordinate system of camera 1 as our world coordinate system (i.e. the system in which we represent the $X_j$). Therefore we do not have to perform any coordinate transform before projection to the first image. ✓ ✓

*Remark:* The world coordinate system is the coordinate system in which we represent the 3D points $X_j$. This coordinate system does not need to coincide with the coordinate system of the first camera! This is an active choice that we need to make, that results in the formulation of the problem as given in equation 8.1.

## Problem 9 **Optimisation (17 credits)**

For all the questions in this problem, the following information holds.
Let $A \in \mathbb{R}^{n\times n}$ be a symmetric square matrix, $x \in \mathbb{R}^n$, and $\mathbb{I} \in \mathbb{R}^{n\times n}$ be the identity matrix. Further, the eigenvalues of $A$ are ordered as $0.9 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0.2$.

a) What is the value of $a$ in the following optimisation problem? **(3 credit(s))**

$$a^* = \min_{\|x\|_2=1} \langle x, Ax\rangle + 5\langle x, x\rangle. \tag{9.1}$$

- ☐ The given information is insufficient to determine the value of a.
- ☐ 0.7
- ☐ 3.2
- ☒ 5.2
  $a^* = \min_{\|x\|_2=1}\langle x, Ax\rangle + 5\langle x, x\rangle = \min_{\|x\|_2=1}\langle x, Ax\rangle + 5$, since $\|x\|_2 = 1$.
  We know $\min_{\|x\|_2=1}\langle x, Ax\rangle = \lambda_n$, the minimum eigenvalue of $A$. Therefore, $a^* = 5 + 0.2 \implies a^* = 5.2$.
- ☐ 2.2
- ☐ 0.2

0 1 2 3 4 5

b) Consider the following unconstrained version of the optimisation problem in Eq. (9.1).

$$\min_{x\in\mathbb{R}^n} \langle x, Ax\rangle + 5\langle x, x\rangle. \tag{9.2}$$

Can we always apply Newton's method to solve the problem (substantiate your answer)? Write down the update step for optimising the given problem using Newton's method with a stepsize $t$. **(5 credit(s))**

We can write the objective $\langle x, Ax\rangle + 5\langle x, x\rangle$ as $\langle x, (A + 5\mathbb{I})x\rangle$.
The gradient of this quadratic expression is $g = 2(A + 5\mathbb{I})x$, and the Hessian is $H = 2(A + 5\mathbb{I})$.

Given that the matrix $(A + 5\mathbb{I})$ is also symmetric and has non-zero eigenvalues, it is invertible. Hence, we can always apply Newton method to solve the given problem.
The update step for Newton method is $x_{k+1} = x_k - tH^{-1}g = x_k - t(2(A + 5\mathbb{I}))^{-1})2(A + 5\mathbb{I})x_k = (1 - t)x_k$.
$\implies x_{k+1} = (1 - t)x_k$.

0 1 2 3 4

c) What is the solution for the following optimisation problem (in terms of eigenvectors of $A$)? **(4 credit(s))**

$$x^* = \operatorname{argmin}_{\|x\|_2=1} \langle x, x\rangle - \langle x, Ax\rangle. \tag{9.3}$$

$\operatorname{argmin}_{\|x\|_2=1}\langle x, x\rangle - \langle x, Ax\rangle = \operatorname{argmin}_{\|x\|_2=1} 1 - \langle x, Ax\rangle = \operatorname{argmin}_{\|x\|_2=1} -\langle x, Ax\rangle$.
The eigenvector corresponding to the smallest eigenvalue of $-A$ is same as the eigenvector corresponding to the largest eigenvalue of $A$, and vice-versa. Hence, the minimiser is the eigenvector corresponding to the largest eigenvalue of $A$.

— The problem continues on the next page —

d) What is the value of the function $\langle x, x \rangle - \langle x, Ax \rangle$ at $x^*$ from the previous subproblem? **(2 credit(s))**

- [ ] 1
- [ ] 0
- [x] 0.1
  From the previous solution, $x^*$ is the eigenvector corresponding to the largest eigenvalue of $A$.
  Since $\|x^*\|_2 = 1$, $\langle x, x \rangle - \langle x, Ax \rangle = 1 - \langle x^*, Ax^* \rangle = 1 - \lambda_1 = 1 - 0.9 = 0.1$.
- [ ] 0.2

e) For a constant $b \in \mathbb{R}^n$, what is the Levenberg-Marquardt update step for the following optimisation problem? (Remember that $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix.) **(3 credit(s))**

$$\min_{x \in \mathbb{R}^n} \langle Ax + b, Ax + b \rangle. \tag{9.4}$$

- [x] $\Delta = -(A + A^{-1}\lambda \mathrm{diag}(A^2))^{-1}(Ax + b)$
  The jacobian of the given residual $r = Ax + b$ is $J_r = A$.
  The update step is $\Delta = -(A^\top A + \lambda \mathrm{diag}(A^\top A))^{-1} A^\top (Ax + b)$.
  Since $A^\top = A$, and $(XY)^{-1} = Y^{-1}X^{-1}$, $\Delta = -(A^{-1}(A^2 + \lambda \mathrm{diag}(A^2)))^{-1}(Ax + b)$
  $\implies \Delta = -(A^{-1}A^2 + A^{-1}\lambda \mathrm{diag}(A^2))^{-1}(Ax + b)$
  $\implies \Delta = -(A + A^{-1}\lambda \mathrm{diag}(A^2))^{-1}(Ax + b)$.
- [ ] $\Delta = -(A + A^{-1}\lambda \mathrm{diag}(A^2))^{-1}A(Ax + b)$
- [ ] $\Delta = -(A^\top A + \lambda \mathbb{I})^{-1} A^\top (Ax + b)$
- [ ] $\Delta = -2Ax - b$

**Please do not use the space below. It is merely used for technical reasons and answer written in this space will NOT BE CONSIDERED.**