Exam ss20 retake solution

Computer Vision II: Multiple view geometry (Technische Universität München)

Computer Vision Group
Department of Informatics
Technical University of Munich

Esolution

Place student sticker here

Note:

- During the attendance check a sticker containing a unique code will be put on this exam.
- This code contains a unique number that associates this exam with your registration number.
- This number is printed both next to the code and to the signature field in the attendance check list.

Computer Vision II: Multiple View Geometry

Exam:	IN2228 / Retake	**Date:**	Friday 2nd October, 2020
Examiner:	Florian Bernard	**Time:**	14:15 – 16:15

Working instructions

- This exam consists of **20 pages** with a total of **11 problems**.
  Please make sure now that you received a complete copy of the exam.
- The total amount of achievable credits in this exam is 120 credits.
- Detaching pages from the exam is prohibited.
- Allowed resources:
  - This is an open book graded exercise: you are allowed to use any source that does not involve communication. In particular, you are allowed to use all lecture material (including solutions that we provide), textbooks, research papers, Matlab or existing webpages, etc. **Any type of communication, including actively seeking for help from others (e.g. in forums or chatrooms, etc.), is not allowed**.
- This graded exercise uses the TUMexam platform which offers a student manual on their webpage[1].
- The boxes on the sides of the subproblems (those with numbers) are used for correction and ticking them is prohibited. A ticked box that is not part of a multiple choice question may result in zero points for the problem.
- In the multiple choice questions there is always exactly one correct answer. A correct answer (comprising of exactly one tick at the correct position) will give the indicated credits, whereas a wrong answer will result in 0 credits.

Mark correct answers with a cross ☒

To undo a cross, completely fill out the answer option ■

To re-mark an option, use a human-readable marking × ■

Left room from ________ to ________	/	Early submission at ________

[1] `https://www.tumexam.de/links.html`

Problem 1 Personal Information (1 credit)

0
1

Please enter your matriculation number with leading zero. **(1 credit(s))**

This field should contain your matriculation number.

Problem 2 MATLAB Operations (14 credits)

Let `A` and `B` be given, where `size(A) = [m,n]` and `size(B) = [n,p]` . Assume `m,n,p` are mutually different dimensions.

a) How can we compute `norm(A(:))*norm(A(:))` ? **(2 credit(s))**

- [ ] `sum(sum(A'*A))`
- [ ] `trace(A'.*A)`
- [ ] `sum(sum(A*A'))`
- [x] `trace(A'*A)`
- [ ] The expression will lead to an error
- [ ] `trace(A.*A)`
- [ ] `trace(A.*A')`

b) How was `C` obtained if `size(C) = [m*p,n*n]` ? **(2 credit(s))**

- [ ] `C = kron(B,A')`
- [ ] `C = kron(A',B)`
- [ ] `C = kron(B,A)'`
- [ ] `C = kron(A,B)`
- [ ] `C = kron(B,A)`
- [ ] `C = kron(A,B)'`
- [x] `C = kron(A',B)'`

c) What is the dimension of the output of `kron(A(:),B(:)')` ? **(2 credit(s))**

- [x] `[m*n, n*p]`
- [ ] `[m*m*n*p, 1]`
- [ ] `[m*n*n*p, 1]`
- [ ] `[m*n, m*p]`
- [ ] `[m*n*p*p, 1]`
- [ ] `[m*p, n*n]`
- [ ] `[m*n*p*p, 1]`
- [ ] The expression will lead to an error
- [ ] `[n*p, m*n]`

— The problem continues on the next page —

d) What is the dimension of the output of `kron(A(:),B(:))` ? **(2 credit(s))**

- ☐ `[n*p, m*n]`
- ☐ `[m*p, n*n]`
- ☐ `[m*n, m*p]`
- ☐ `[m*m*n*p, 1]`
- ☐ `[m*n, n*p]`
- ☐ `[m*n*p*p, 1]`
- ☐ `[m*n*p*p, 1]`
- ☐ The expression will lead to an error
- ☒ `[m*n*n*p, 1]`

e) How many scalar multiplication operations are necessary to compute `A(:)*B(:)'` for dense `A,B` ? **(2 credit(s))**

- ☐ `m*m*n*p`
- ☐ The expression will lead to an error
- ☐ `2*m*n*p`
- ☒ `m*n*n*p`
- ☐ `m*n*p*p`
- ☐ `m*n*p*p`
- ☐ `2*m*m*n*p`
- ☐ `m*m*p*p`

f) How many scalar multiplication operations are necessary to compute `A.*B` for dense `A,B` ? **(2 credit(s))**

- ☒ The expression will lead to an error
- ☐ `m*m*p*p`
- ☐ `2*m*m*n*p`
- ☐ `m*n*n*p`
- ☐ `2*m*n*p`
- ☐ `m*n*p*p`
- ☐ `m*n*p*p`
- ☐ `m*m*n*p`

g) Let `P = kron(eye(n),eye(m))` . What does `kron(A(:),A(:))'*P(:)` compute? **(2 credit(s))**

- ☐ `sum(sum(A*A'))`
- ☒ `trace(A*A')`
- ☐ The expression will lead to an error
- ☐ `sum(sum(A'*A))`
- ☐ `trace(A'.*A)`
- ☐ `trace(A.*A')`

Problem 3 Matrix Algebra in MATLAB (12 credits)

a) Write down a symmetric matrix `Q` of size `[2,2]` such that `Q'*Q = eye(2)` . **(2 credit(s))**

0 1 2

Q=[1 0; 0 1] or Q=[-1 0; 0 1] or Q=[1 0; 0 -1] or Q=[-1 0; 0 -1]

b) Write down a symmetric orthogonal matrix `Q` of size `[2,2]` with negative determinant. **(2 credit(s))**

0 1 2

Q=[-1 0; 0 1] or Q=[1 0; 0 -1]

c) Let `[U,S,V] = svd(D)` . Which of the following holds for *any* choice of `D` with `size(D) = [n,n]` (up to numerical precision)? **(2 credit(s))**

- ☐ `U*V'*det(U*V')` is an element of $SO(n)$
- ☐ `expm(U*V')` is an element of $SO(n)$
- ☐ `U*V'` is an element of $SE(n)$
- ☐ `U*V` is an element of $SO(n)$
- ☐ `U*det(U)` and `V` are both elements of $SO(n)$
- ☒ None

d) Let `A` with `size(A) = [n,n]` and `b` with `size(b) = [n,1]` be given. Which of the following statements is true in general (up to numerical precision)? **(2 credit(s))**

- ☒ `x = pinv(A)*b` is the only solution to `A*x = b` if `det(A) ~= 0`
- ☐ `pinv(A)*b` is equal to `A\b`
- ☐ `x = pinv(A)*b` is the only solution to `A*x = b` if `det(A) == 0`
- ☐ None
- ☐ `x = A\b` gives a solution to `A*x = b`

— The problem continues on the next page —

e) Let `[U,S,V] = svd(E)` . Which of the following holds for *any* choice of `E` with `size(E) = [3,3]` (up to numerical precision)? **(2 credit(s))**

- [ ] `expm(U*V')` is an element of $SO(3)$
- [ ] `U*V'` is an element of $SE(3)$
- [ ] `U*V` is an element of $SO(3)$
- [ ] `U*det(U)` and `V` are both elements of $SO(3)$
- [ ] None
- [x] `U*V'*det(U*V')` is an element of $SO(3)$

f) Let `[x,y] = eig(rand(3))` . Which of the following holds for any random seed? **(2 credit(s))**

- [x] None
- [ ] `x, y` are real matrices
- [ ] `x, y` are symmetric matrices
- [ ] `x, y` are orthogonal matrices
- [ ] `x, y` have orthogonal columns

Problem 4 Definitions (15 credits)

a) Let $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ with $c = 0$. Consider the set that contains all matrices of the form $\begin{pmatrix} A & b \\ c^\top & 1 \end{pmatrix}$. We denote $\circ$ as the matrix multiplication. Is $(G, \circ)$ a group? If so, give the name of this group. If no, why not? **(2 credit(s))**

0 1 2

It's not a group.Indeed the elements of G are not invertible if their top left matrix A is not invertible.

b) Let $M = [c_1, \dots, c_n] \in \mathbb{R}^{m\times n}$. State the name of the following quantity: $\dim(\mathrm{span}(\{c_1, \dots, c_n\}))$. **(2 credit(s))**

0 1 2

The rank of the matrix M.

c) Let $r : \mathbb{R}^n \to \mathbb{R}^m$ and $J_r : \mathbb{R}^n \to \mathbb{R}^{m\times n}$ be its Jacobian. We consider the following optimization problem

0 1 2

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|r(x)\|^2 .$$

State the name of the algorithm whose update is given by:

$$x^{(k+1)} = \operatorname*{argmin}_{x\in\mathbb{R}^n} \frac{1}{2}(x - x^{(k)})^\top J_r^\top J_r(x - x^{(k)}) + (x - x^{(k)})^\top J_r^\top r ,$$

where r and J_r are evaluated at $x^{(k)}$. **(2 credit(s))**

Gauss-Newton.

d) Let $E = \{A \in \mathbb{R}^{n\times n} \mid A^\top = A\}$. What is the dimension of this space? **(2 credit(s))**

0 1 2

The dimension is : $\dfrac{n(n+1)}{2}$.

e) Let $A \in \mathbb{R}^{m\times n}$, $b \in range(A)$ and let $S = \{x \in \mathbb{R}^n \mid Ax = b\}$. Write down (without justification) the solution of

0 1 2

$$x^* = \operatorname*{argmin}_{x\in S} \|x\|^2 .$$

(2 credit(s))

$x^* = A^\dagger b$, where $A^\dagger \in \mathbb{R}^{n\times m}$ is the pseudo inverse of A.

f) Let $L : \mathbb{R}^n \to \mathbb{R}^n$ such that $\forall x \in \mathbb{R}^n$, $L(x) = Ax$ where $A \in \mathbb{R}^{n\times n}$. Let $\{e_i\}_i$ be the canonical basis of $\mathbb{R}^n$ and $\{e_i'\}_i$ be another basis of $\mathbb{R}^n$. We define $B = P^{-1}AP$ where $P = [e_1', \dots, e_n']$. Describe briefly how B is related to L. **(2 credit(s))**

0 1 2

Either of the following answers is acceptable and will give full credits: 1) B represents the linear transformation L in the basis $\{e_i'\}_i$.2) B transform the coordinates of x according to $\{e_i'\}_i$ into the coordinates of $L(x)$ according to $\{e_i'\}_i$.

g) Tick the wrong statement. **(1 credit(s))**

- [x] $\mathrm{SL}(n) \subset \mathrm{O}(n)$
- [ ] $\mathrm{E}(n) \subset \mathrm{A}(n)$
- [ ] $\mathrm{SO}(n) \subset \mathrm{O}(n)$
- [ ] $\mathrm{SE}(n) \subset \mathrm{GL}(n+1)$

— The problem continues on the next page —

h) Let $R = (r_{ij}) \in SO(3)$, which represents a rotation by an angle $\theta \in (0, \pi)$ around an axis spanned by a unitary vector $\vec{u}$. Which of the following corresponds to the correct expressions of *theta* and $\vec{u}$? **(2 credit(s))**

- [ ] $\theta = cos\left(\frac{\text{trace}(R) - 1}{2}\right)$ and $\vec{u} = \frac{1}{\sin(\theta)}(r_{32} - r_{23}, r_{13} - r_{31}, r_{21} - r_{12})^\top$
- [ ] $\theta = cos^{-1}\left(\frac{\text{trace}(R) - 1}{2}\right)$ and $\vec{u} = \frac{1}{\sin(\theta)}(r_{32} - r_{23}, r_{21} - r_{12}, r_{13} - r_{31})^\top$
- [ ] $\theta = cos^{-1}\left(\frac{\text{trace}(R)}{2}\right)$ and $\vec{u} = \frac{1}{2\sin(\theta)}(r_{32} - r_{23}, r_{21} - r_{12}, r_{13} - r_{31})^\top$
- [ ] $\theta = cos^{-1}\left(\frac{\text{trace}(R) + 1}{2}\right)$ and $\vec{u} = \frac{1}{2\sin(\theta)}(r_{32} + r_{23}, r_{13} + r_{31}, r_{21} + r_{12})^\top$
- [x] $\theta = cos^{-1}\left(\frac{\text{trace}(R) - 1}{2}\right)$ and $\vec{u} = \frac{1}{2\sin(\theta)}(r_{32} - r_{23}, r_{13} - r_{31}, r_{21} - r_{12})^\top$

Problem 5 Linear Algebra I (8 credits)

a) Let $USV^\top$ be the singular value decomposition of $A \in \mathbb{R}^{n\times n}$, and let $XLX^\top$ denote the eigenvalue decomposition of $AA^\top$. How can you compute X and L from U, S, V? Write down the solution as well as a **derivation**. **(4 credit(s))**

0 1 2 3 4

We have that $A = USV^\top$, so $AA^\top = USV^\top(USV^\top)^\top = USV^\top VS^\top U^\top = USS^\top U^\top$,
thus we need to identify X and L as follows:
$X = U$
$L = SS^T$(since S is diagonal, $L = SS$ is also correct)

b) Show that if $Y \in \mathbb{R}^{3\times 3}$ is a skew-symmetric matrix, then $z^\top Yz = 0$ for every $z \in \mathbb{R}^3$. **(4 credit(s))**

0 1 2 3 4

Y is skew-symmetric, so there exists $w \in \mathbb{R}^3$, s.t. $\hat{w} = Y$. Moreover, $\hat{w}z = w \times z$. Hence, $z^\top Yz = z^\top \hat{w}z = z^\top(w \times z) = 0$, since z and $w \times z$ are orthogonal.
Alternative solution (valid for $Y \in \mathbb{R}^{n\times n}$): We can split $z^\top Yz = \frac{1}{2}z^\top Yz + \frac{1}{2}z^\top Yz$, where each summand is a scalar.Transposing the second (scalar) term does not change it, hence $= \frac{1}{2}z^\top Yz + \frac{1}{2}(z^\top Yz)^\top = \frac{1}{2}z^\top Yz + \frac{1}{2}z^\top Y^\top z$. Plugging in the definition of skew-symmetry $Y^\top = -Y$ gives $= \frac{1}{2}z^\top Yz - \frac{1}{2}z^\top Yz = 0$

Problem 6 Linear Algebra II (10 credits)

The Euclidean projection R^* of the matrix $A \in \mathbb{R}^{n\times n}$ onto the set $SO(n) = \{Q \in \mathbb{R}^{n\times n} \mid Q^\top Q = \mathbb{I}_{n\times n}, \det(Q) = 1\}$ is defined as

$$R^* = \underset{R\in SO(n)}{\operatorname{argmin}} \|A - R\|_f\,,$$

i.e. we seek a minimiser R^* of $\|A-R\|_f$ subject to $R \in SO(n)$. It can be computed as $R^* = U\operatorname{diag}(1, \dots, 1, \det(UV^\top))V^\top$ for $U\Sigma V^\top$ being the singular value decomposition of A with $\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$ where $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$.

0 1 2

a) What is the difference between the sets $O(n)$ and $SO(n)$? **(2 credit(s))**

Elements of $SO(n)$ have determinant 1, while elements of $O(n)$ have determinant ± 1

0 1

b) How many different values can the inner term $\operatorname{diag}(1, \dots, 1, \det(UV^\top))$ take? **(1 credit(s))**

2

0 1 2

c) Explain your answer in b) and list the cases. **(2 credit(s))**

Since U and V are orthogonal, $\det(UV^\top) \pm 1$, so there are two cases.

0 1 2

d) Explain the effect of the different cases identified in c) on $U\operatorname{diag}(1, \dots, 1, \det(UV^\top))V^\top$. **(2 credit(s))**

If $\det(UV^\top) = 1$, the matrix $UV^\top$ is already in $SO(n)$(the inner term becomes $\mathbb{I}_{n\times n}$)
If $\det(UV^\top) = -1$, the matrix $UV^\top$ is not in $SO(n)$ (it has negative determinant), so $U\operatorname{diag}(1, \dots, 1, \det(UV^\top))V^\top \in SO(n)$.

0 1 2 3

e) Explain why $S = U\operatorname{diag}(\det(UV^\top), 1, \dots, 1)V^\top$ does not necessarily compute a Euclidean projection onto $SO(n)$ for general A? **(3 credit(s))**

Formal solution: $\|A - S\|_f = \|A - U\operatorname{diag}(\det(UV^\top), 1, \dots, 1)V^\top\|_f = \|U^\top AV - \operatorname{diag}(\det(UV^\top), 1, \dots, 1)\|_f$ (invariance of Frobenius norm under orthogonal transformations). Since, $U^\top AV = \Sigma$, we obtain $\|A - S\|_f = \|\Sigma - \operatorname{diag}(\det(UV^\top), 1, \dots, 1)\|$. The norm may increase when flipping the sign at the position of the largest SV, compared to the smallest SV.
High-level solution: We need to minimise the norm $\|A - R\|_f$ over $R \in SO(n)$. When flipping the direction of a column of U (or V), we need to ensure that it is the column corresponding to the smallest SV. Flipping the sign for the column corresponding to the largest SV may increase the norm.

Problem 7 Image Formation (12 credits)

A 3D point is given in the camera coordinate frame by $P = (-1, 1, 8)^\top$. The intrinsic parameter matrix K is given by

$$\begin{pmatrix} 640 & 0 & 320 \\ 0 & 480 & 240 \\ 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

Calculate the pixel-position of the projected point in the image and tick the correct answer.

a) What is the value of u? **(2 credit(s))**

- ☐ u = 310
- ☒ u = 240
- ☐ u = 150
- ☐ u = 550

b) What is the value of v? **(2 credit(s))**

- ☐ v = 130
- ☒ v = 300
- ☐ v = 490
- ☐ v = 910

Given is the transform

$$g_{CamToWorld} = \begin{pmatrix} 0 & 0 & 1 & 4 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{SE}(3)$$

that transforms a point given in camera coordinates X to the point given in world coordinates $X_0 = g_{CamToWorld} X$. Consider the point $P_0 = (8, -1, 1)^\top$ given in world coordinates. Considering the transform $g_{CamToWorld}$ and the intrinsic matrix from Equation 1, calculate the pixel-position of the projected point in the image.

c) Explain how you calculate the pixel-position of the projected point in the image. Do not give numeric values, only the derivation counts. **(2 credit(s))**

☐ 0 ☐ 1 ☐ 2

Transform to camera coordinates: $P_{cam} = g_{WorldToCam} P_0$, with $g_{WorldToCam} = (g_{CamToWorld})^{-1}$.
Given $(x, y, z) \equiv P_{cam}$, calculate pixel-position: $u = f_x \frac{x}{z} + o_x$, $v = f_y \frac{y}{z} + o_y$.

d) What is the value of u? **(3 credit(s))**

- ☒ u = 800
- ☐ u = 120
- ☐ u = 230
- ☐ u = 210

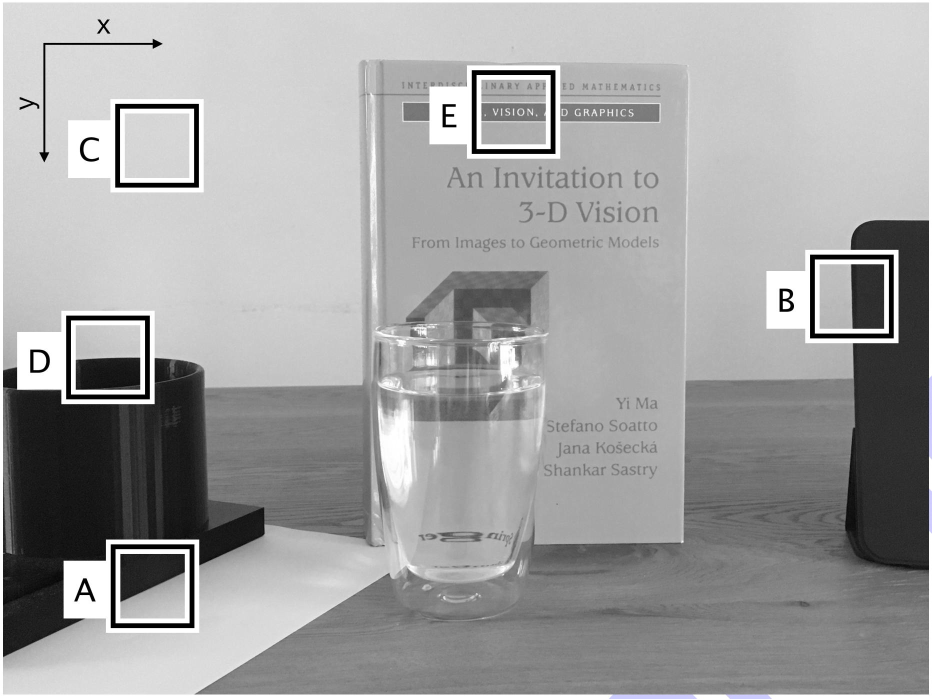
— The problem continues on the next page —

e) What is the value of v? **(3 credit(s))**

- [x] v = 480
- [ ] v = 400
- [ ] v = 160
- [ ] v = 830

Problem 8 Lucas-Kanade Algorithm: The Structure Tensor (14 credits)

The black and white boxes in the image below are annotations used within this exercise and were not present in the image used to calculate the structure tensors.



In Equation 2 you are given the eigenvalues $\lambda_i^{(1)}, \dots, \lambda_i^{(5)}$ and eigenvectors $v_i^{(1)}, \dots, v_i^{(5)}$ for five structure tensors $M^{(1)}, \dots, M^{(5)}$. For each tensor you have to choose the corresponding image patch A – E. The neighborhood used to compute the structure tensor $W(x)$ is set to be the same as the inside of the corresponding annotation box. A uniform weight is used throughout the window. There is a one-to-one correspondence between tensors and patches. (To obtain readable numbers, the values below are scaled and rounded.)

$$
\begin{array}{llll}
M^{(1)}: & \lambda_1^{(1)} = 1.38\,,\ \lambda_2^{(1)} = 0.86 & v_1^{(1)} = (-0.08,\ \ 1.00)^\top, & v_2^{(1)} = (-1.00, -0.08)^\top \\
M^{(2)}: & \lambda_1^{(2)} = 0.86\,,\ \lambda_2^{(2)} = 0.05 & v_1^{(2)} = (\ \ 0.04,\ \ 1.00)^\top, & v_2^{(2)} = (-1.00,\ \ 0.04)^\top \\
M^{(3)}: & \lambda_1^{(3)} = 0.01\,,\ \lambda_2^{(3)} = 0.01 & v_1^{(3)} = (-1.00,\ \ 0.06)^\top, & v_2^{(3)} = (-0.06, -1.00)^\top \\
M^{(4)}: & \lambda_1^{(4)} = 0.40\,,\ \lambda_2^{(4)} = 0.06 & v_1^{(4)} = (\ \ 0.44,\ \ 0.90)^\top, & v_2^{(4)} = (-0.90,\ \ 0.44)^\top \\
M^{(5)}: & \lambda_1^{(5)} = 0.55\,,\ \lambda_2^{(5)} = 0.04 & v_1^{(5)} = (-1.00,\ \ 0.03)^\top, & v_2^{(5)} = (-0.03, -1.00)^\top
\end{array}
\tag{2}
$$

a) Choose the correct image patch for the structure tensor $M^{(1)}$ and explain your choice. **(4 credit(s))**

0 1 2 3 4

The correct region is E. The two eigenvalues are both large. Therefore, the patch must have sufficient structure in x- and y-direction, which is not the case for an edge or the white wall.

b) Choose the correct image patch for the structure tensor $M^{(2)}$ and explain your choice. **(4 credit(s))**

0 1 2 3 4

The correct region is D. The larger eigenvalue is large and the smaller eigenvalue is close to zero. Therefore the patch must contain an edge. The eigenvector corresponding to the large eigenvalue points approximately along the y-axis and thus the edge must be horizontal.

— The problem continues on the next page —

c) Choose the correct image patch for the structure tensor $M^{(3)}$. **(2 credit(s))**

- [ ] A
- [ ] E
- [ ] B
- [ ] D
- [x] C

d) Choose the correct image patch for the structure tensor $M^{(4)}$. **(2 credit(s))**

- [ ] C
- [ ] E
- [ ] B
- [ ] D
- [x] A

e) Choose the correct image patch for the structure tensor $M^{(5)}$. **(2 credit(s))**

- [ ] D
- [x] B
- [ ] C
- [ ] E
- [ ] A

Problem 9 **3D Geometry I (8 credits)**

a) We assume that there are three cameras whose rotations and intrinsics are all the identity matrix $\mathbb{I}_{3\times3}$. Let $(F_{ij})_{1\leqslant i<j\leqslant 3}$ be the fundamental matrix between images i and j. We consider the pixels in homogeneous coordinates $x_1 = (10, 6, 1)^\top$, $x_2 = (5, 15, 1)^\top$ and $x_3 = (8, 9, 1)^\top$ of image 1,2 and 3, respectively. Moreover, we assume that

$$x_j^\top F_{ij} x_i = 0 \quad \forall (i, j) \in \{1, 2, 3\}^2 \text{ with } i < j\,.$$

A statement is suggested: "There exists a unique 3D point whose projections in the different images correspond to x_1, x_2 and x_3". What can be said about this statement? **(1 credit(s))**

- [ ] It's impossible to know whether this statement is true or not.
- [x] The statement is true.
- [ ] The statement is wrong.

b) Explain your choice in the previous question. **(3 credit(s))**

☐ 0 ☐ 1 ☐ 2 ☐ 3

The vectors x_1, x_2 and x_3 are not coplanar, hence the pairwise epipolar constraints ensure the existence and uniqueness of the preimage, i.e. those pixels are projections of a unique 3D point.

Assume you would like to determine the 3D rigid body motion (with unknown scale) between two images taken of the same static scene. If you know the intrinsic camera calibration, what is the minimal number of 2D to 2D point-correspondences needed ...

c) ... for a planar scene? **(2 credit(s))**

- [ ] 5
- [ ] 6
- [ ] 7
- [ ] 8
- [x] 4

d) ... in general? **(2 credit(s))**

- [ ] 6
- [x] 5
- [ ] 4
- [ ] 8
- [ ] 7

Problem 10 3D Geometry II (14 credits)

We consider in the following two cameras for which the rigid body motion is only an unknown rotation $R \in SO(3)$. This means that given a 3D point whose coordinates in the two cameras are X_1 and X_2, we have that $X_2 = RX_1$. We assume that both cameras have the same known intrinsic matrix $K \in \mathbb{R}^{3\times3}$. The aim of this problem is to suggest different ways to estimate the rotation R, as well as the minimum number of 2D to 2D point-correspondences needed for this task.

0 1 2

a) Can the 8-point algorithm be used for this task? If yes, explain how and at which step the zero translation constraint has to be included. If no, why not? **(2 credit(s))**

The 8-point algorithm cannot be usedsince it makes use of the epipolar constraint, which now become trivial as there is no translation.

b) Let $(p_1, p_2) \in (\mathbb{R}^3)^2$ be a pair of corresponding pixels (in homogeneous coordinates). Which of the following pairs of vectors $(w, v) \in (\mathbb{R}^3)^2$ fulfill $w = Rv$? (Hint: the derivation resembles the one for the epipolar constraint) **(3 credit(s))**

- ☐ $w = \frac{K^{-1}p_2}{\|p_2\|}$, $v = \frac{Kp_1}{\|p_1\|}$
- ☐ $w = \frac{Kp_2}{\|Kp_2\|}$, $v = \frac{Kp_1}{\|Kp_1\|}$
- ☐ $w = K^{-1}p_2$, $v = K^{-1}p_1$
- ☐ $w = Kp_2$, $v = Kp_1$
- ☒ $w = \frac{K^{-1}p_2}{\|K^{-1}p_2\|}$, $v = \frac{K^{-1}p_1}{\|K^{-1}p_1\|}$
- ☐ $w = \frac{Kp_2}{\|p_2\|}$, $v = \frac{Kp_1}{\|p_1\|}$

0 1 2 3

c) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. What is the minimum required number N of linearly independent vectors $\{v_i\}_{i=1..N}$ with known image under f, i.e. $\{f(v_i)\}_{i=1..N}$, in order to compute $f(v)$ for any $v \in \mathbb{R}^n$? Justify your answer. **(3 credit(s))**

We require that any $v \in \mathbb{R}^n$ can be expressed as a linear combination of the $\{v_i\}$, so that $span(v_1, \dots, v_N) = \mathbb{R}^n$, which in turn means that the minimum number is $N = n$.

0 1 2

d) Let $(v_1, v_2) \in (\mathbb{R}^3)^2$ and $(w_1, w_2) = (Rv_1, Rv_2)$. Write (without justification) the expression of $R(v_1 \times v_2)$ in terms of w_1 and w_2, where $\times$ denotes the cross product. We deduce from this relation that if the image under R of v_1 and v_2 are known, then the image (under R) of $v_1 \times v_2$ is also known. **(2 credit(s))**

$R(v_1 \times v_2) = w_1 \times w_2$

— The problem continues on the next page —

e) What is the minimal number of 2D to 2D correspondences needed in order to estimate the rotation matrix R? (Hint: results from the previous parts of this problem may be useful) **(2 credit(s))**

- ☐ 3
- ☐ 4
- ☐ 5
- ☒ 2
- ☐ 7
- ☐ 8
- ☐ 1
- ☐ 6

f) Let $\{(w_i, v_i)\}_i$ be a family of tuples in $\mathbb{R}^3$ such that $\forall i, w_i = Rv_i$. Let W be the matrix whose columns are the vectors $\{w_i\}_i$ and V the matrix whose columns are the vectors $\{v_i\}_i$. We assume that V and W are invertible. What is the expression of the rotation matrix R? **(2 credit(s))**

- ☐ $R = V^{-1}W$
- ☐ $R = VWV^{-1}$
- ☐ $R = W^{-1}VW$
- ☐ $R = V^{-1}WV$
- ☐ $R = WV$
- ☒ $R = WV^{-1}$

Problem 11 Epipolar Lines (12 credits)

In the following we consider two cameras for which there is only a translation $T = (-1, 3, 3)^\top$. This means that given a 3D point whose coordinates in the two cameras are X_1 and X_2, we have that $X_2 = X_1 + T$. We denote the intrinsics of the first and the second camera as $K_1 \in \mathbb{R}^3$ and $K_2 \in \mathbb{R}^3$, respectively.

a) Assume that the intrinsics are given by

$$K_1 = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad K_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We consider a 3D point whose coordinates in the first frame are $X_1 \in \mathbb{R}^3$. Let $x_1 = (2, 2, 1)^\top$ be its **normalized** coordinates, i.e. $zx_1 = X_1$ where z is the depth of X_1 according to the first frame. Important: Note the difference between normalized coordinates and pixel coordinates. Let $(x, y) \in \mathbb{R}^2$ be pixel coordinates on the epipolar line in the second image associated to x_1. Which of the following relations holds? **(1 credit(s))**

- [x] $y = \frac{15}{14}x + \frac{40}{7}$
- [ ] None
- [ ] $y = -\frac{24}{11}x + \frac{70}{11}$
- [ ] $y = -\frac{30}{11}x + \frac{20}{11}$
- [ ] $y = \frac{9}{8}x + \frac{17}{4}$
- [ ] $y = -\frac{9}{10}x + \frac{17}{2}$

0 1 2 3

b) Explain your choice in the previous question (without calculation). **(3 credit(s))**

For any pixel p_1 of the first image (in homogeneous coordinates), the equation of the associated epipolar line in the second image is given by : $ax + by + c = 0$ where $(a, b, c)^\top = Fp_1$, and F is the fundamental matrix, whose expression is $F = K_2^{-\top} \hat{T} K_1^{-1}$.
Since x_1 corresponds to the normalized coordinates (and not the pixel) of X_1, we have that $(a, b, c)^\top = FK_1x_1 = K_2^{-\top}\hat{T}x_1$, which lead to the result after few calculation steps.

0 1 2

c) We consider a 3D point whose coordinates according to both frames are $(X_1, X_2) \in (\mathbb{R}^3)^2$. By $x_1 \in \mathbb{R}^3$ and $x_2 \in \mathbb{R}^3$ we denote its projections in image planes 1 and 2 respectively (in homogeneous coordinates), and $(z_1, z_2) \in \mathbb{R}^2$ the corresponding depth values according to each frame. Justify the following relation: **(2 credit(s))**

$$z_2 K_2^{-1} x_2 = z_1 K_1^{-1} x_1 + T$$

x_1 and x_2 are the projections of the same 3D point whose coordinates in both frames are given by (X_1, X_2). Thus we have that:
$z_1 K_1^{-1} x_1 = X_1$
$z_2 K_2^{-1} x_2 = X_2$ (because the rigid body motion is restricted to a translation).
Since $X_2 = X_1 + T$, we deduce that : $z_2 K_2^{-1} x_2 = X_1 + T = z_1 K_1^{-1} x_1 + T$

— The problem continues on the next page —

d) We assume that the inverses of the intrinsic matrices are:

$$K_1^{-1} = \begin{pmatrix} \frac{1}{10} & 0 & -1 \\ 0 & \frac{1}{10} & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad K_2^{-1} = \begin{pmatrix} \frac{1}{10} & 0 & -5 \\ 0 & \frac{1}{5} & -20 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $x_1 = (20, 30, 1)^\top$ be a pixel of the first image in homogeneous coordinates. The equation of the associated epipolar line in the second image is $y = 105$. By using question c), which of the following pixels in the second image can be a correspondence of x_1, where the corresponding 3D point has positive depth in both camera frames? **(3 credit(s))**

- ☐ $x_2 = (40, 105, 1)^\top$
- ☐ $x_2 = (20, 90, 1)^\top$
- ☐ $x_2 = (20, 100, 1)^\top$
- ☐ $x_2 = (30, 100, 1)^\top$
- ☐ $x_2 = (30, 90, 1)^\top$
- ☒ None