



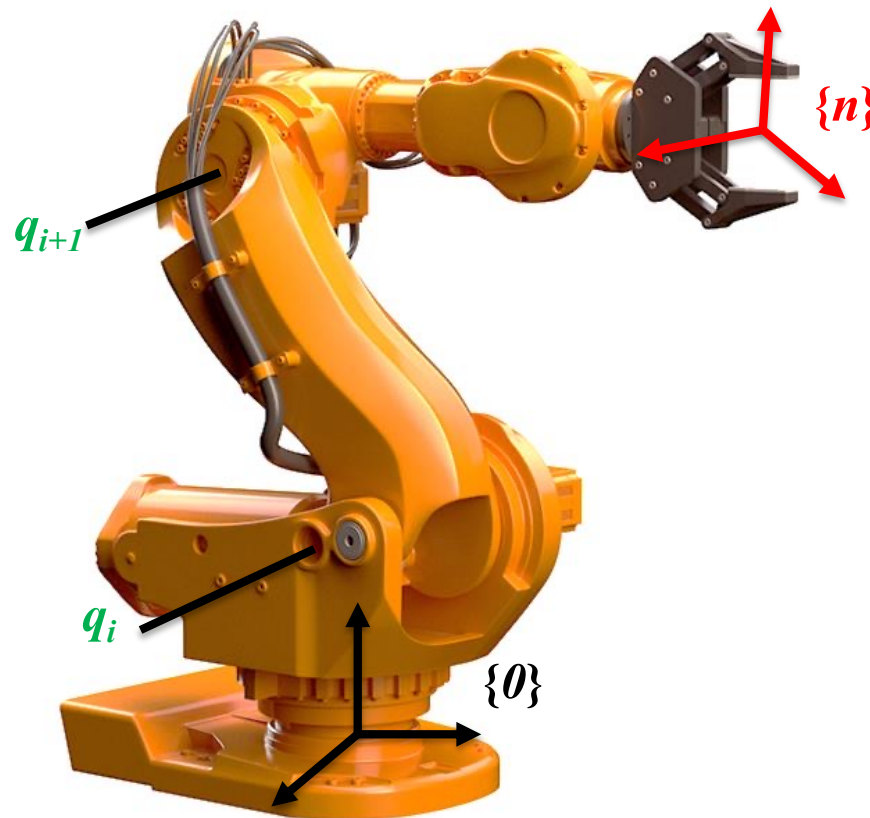
Kinematics

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Relation between **joints** (q_i) and the **pose** (position/orientation) of some point (e.g.: frame $\{n\}$)

- Primary Workspace (*reachable*): WS_1

Positions that can be reached with at least one orientation



Each point can be reached
(orientation “does not matter”)

- Out of WS_1 there is no solution to the problem
- For all $p \in WS_1$ (using a proper orientation), there is at least one solution

- Secondary Workspace (*dexterous*): WS_2

Positions can be reached with any orientation

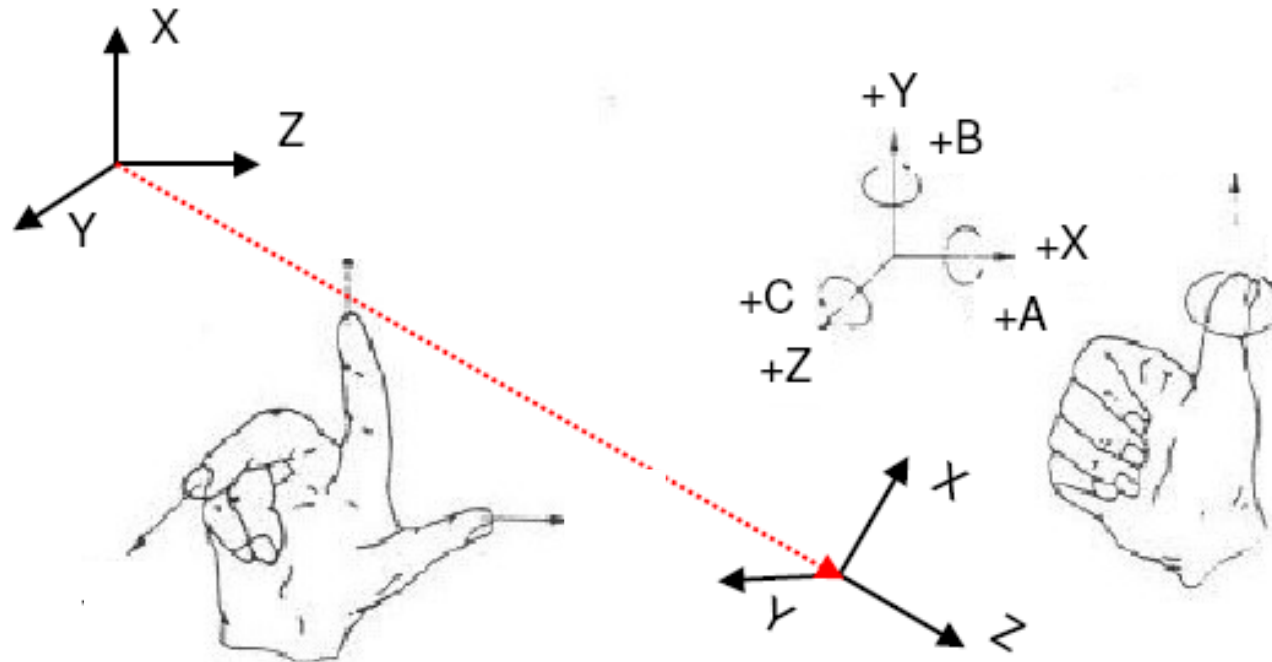


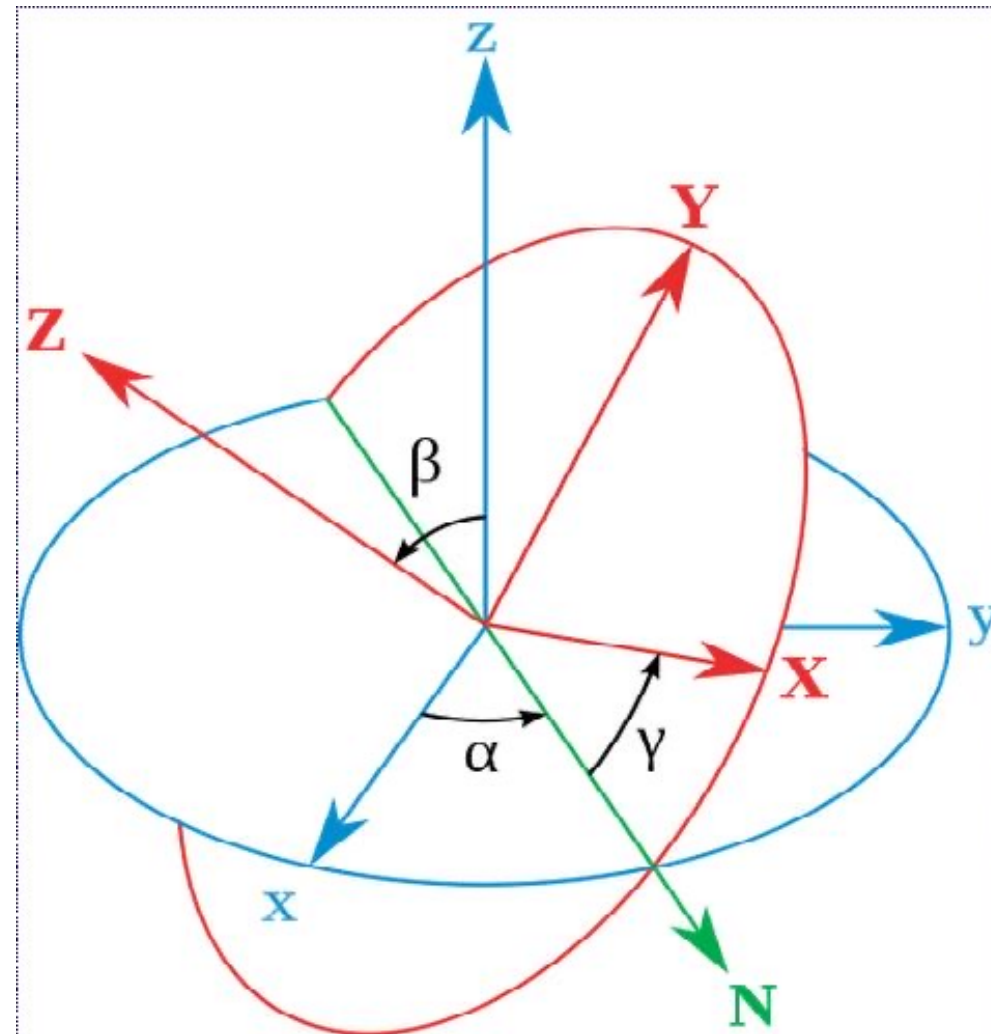
Reach every point with all
possible orientations

- For all $p \in WS_2$ there is (at least) one solution for every orientation

- Relation between WS_1 y WS_2 : $WS_2 \subseteq WS_1$

Degrees of Freedom N – number of independent motion parameters of a body in space





□ $q = (\text{position, orientation}) = (x, y, z, ???)$

□ Parametrization of orientations by matrix:

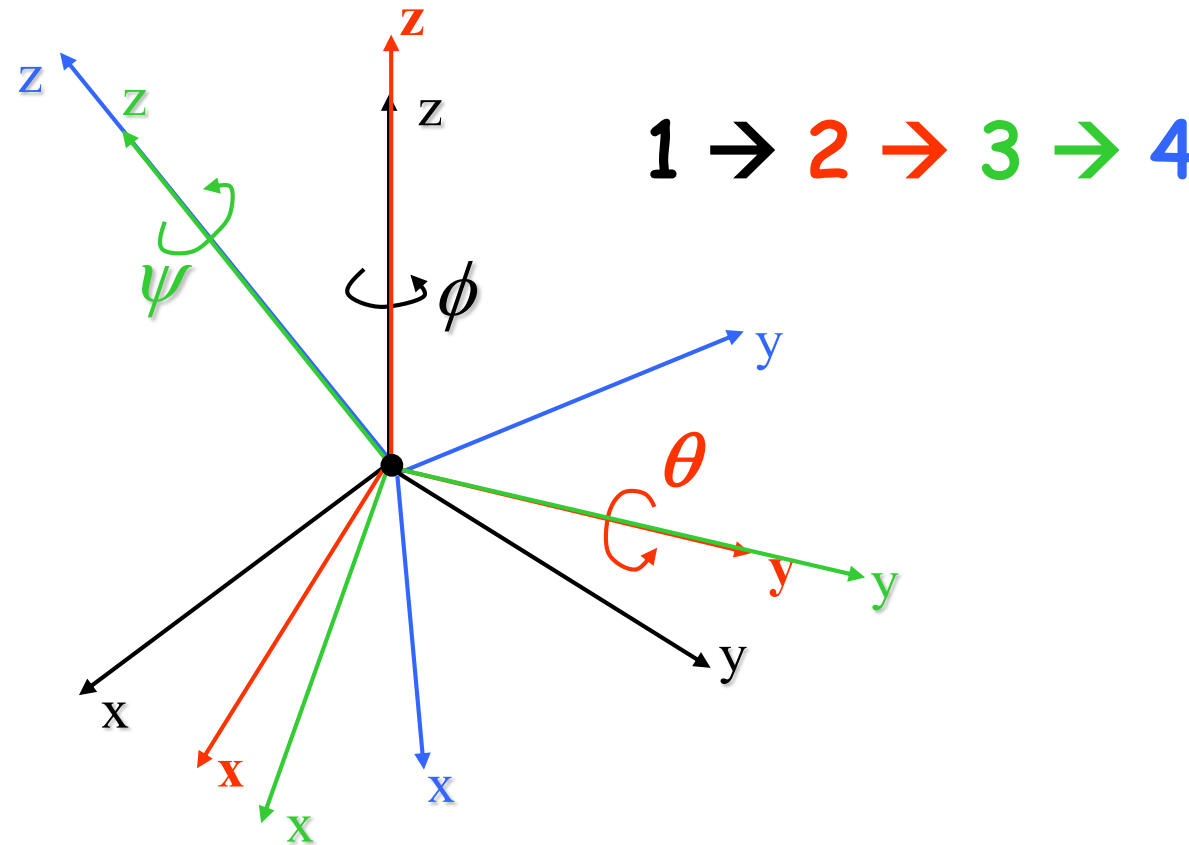
$q = (r_{11}, r_{12}, \dots, r_{33}, r_{33})$ where $r_{11}, r_{12}, \dots, r_{33}$ are the elements of rotation matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

with

- $r_{1i}^2 + r_{2i}^2 + r_{3i}^2 = 1$ for all i ,
- $r_{1i} r_{1j} + r_{2i} r_{2j} + r_{3i} r_{3j} = 0$ for all $i \neq j$,
- $\det(R) = +1$

- Parametrization of orientations by Euler angles: $\phi/\theta/\psi$



$${}^A_B R_{XYZ}(\gamma, \beta, \alpha) = R_Z(\alpha) R_Y(\beta) R_X(\gamma)$$

$$= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix},$$

$${}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}$$

$$\beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}),$$

$$\alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta),$$

$$\gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta),$$

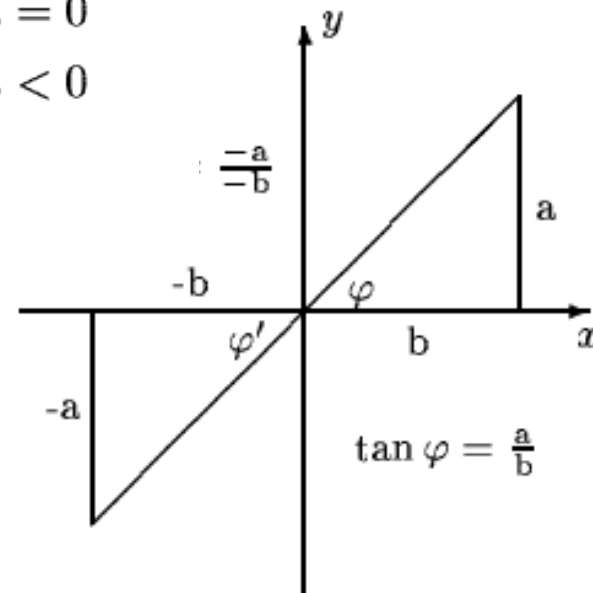
$$\beta = 90.0^\circ,$$

$$\alpha = 0.0.$$

$$\gamma = \text{Atan2}(r_{12}, r_{22}),$$

Proper Euler angles		
$X_1 Z_2 X_3 =$	$\begin{bmatrix} c_2 & -c_3 s_2 & s_2 s_3 \\ c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 \\ s_1 s_2 & c_1 s_3 & c_2 c_3 s_1 + c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	
$X_1 Y_2 X_3 =$	$\begin{bmatrix} c_2 & s_2 s_3 & c_3 s_2 \\ s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 \\ -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	
$Y_1 X_2 Y_3 =$	$\begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 & c_1 s_3 + c_2 c_3 s_1 \\ s_2 s_3 & c_2 & -c_3 s_2 \\ -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 & c_1 c_2 c_3 - s_1 s_3 \end{bmatrix}$	
$Y_1 Z_2 Y_3 =$	$\begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 & c_3 s_1 + c_1 c_2 s_3 \\ c_3 s_2 & c_2 & s_2 s_3 \\ -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 & c_1 c_3 - c_2 s_1 s_3 \end{bmatrix}$	
$Z_1 Y_2 Z_3 =$	$\begin{bmatrix} c_1 c_2 c_3 - s_1 s_3 & -c_3 s_1 - c_1 c_2 s_3 & c_1 s_2 \\ c_1 s_3 + c_2 c_3 s_1 & c_1 c_3 - c_2 s_1 s_3 & s_1 s_2 \\ -c_3 s_2 & s_2 s_3 & c_2 \end{bmatrix}$	
$Z_1 X_2 Z_3 =$	$\begin{bmatrix} c_1 c_3 - c_2 s_1 s_3 & -c_1 s_3 - c_2 c_3 s_1 & s_1 s_2 \\ c_3 s_1 + c_1 c_2 s_3 & c_1 c_2 c_3 - s_1 s_3 & -c_1 s_2 \\ s_2 s_3 & c_3 s_2 & c_2 \end{bmatrix}$	

$$\text{ATAN2}(a, b) = \begin{cases} \arctan\left(\frac{a}{b}\right) & \text{falls } b > 0 \\ \frac{\pi}{2} & \text{falls } b = 0, a > 0 \\ \text{undefiniert} & \text{falls } b = 0, a = 0 \\ -\frac{\pi}{2} & \text{falls } b = 0, a < 0 \\ \arctan\left(\frac{a}{b}\right) + \pi & \text{falls } b < 0 \end{cases}$$



Let's start from a geometric view point. Imagine a coordinate with a vector \vec{X} where \vec{k} is the unit vector representing the axis of rotation. Let the vector \vec{x} be the result of rotating \vec{X} by an angle θ about \vec{k} . You can imagine a circle created by \vec{X} and \vec{x} with the axis of rotation going through its center (see Figure 1).

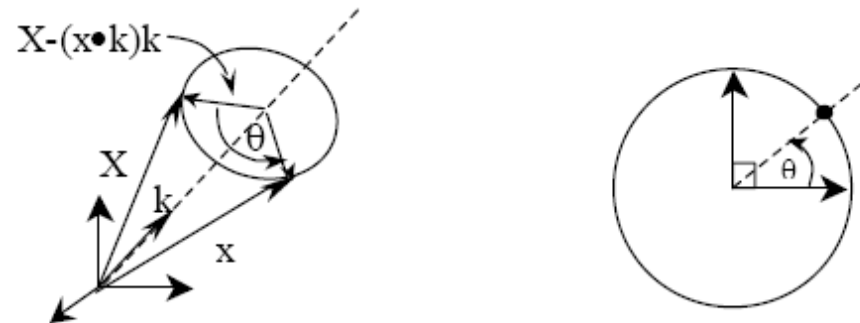


Figure 1: Axis and angle of rotation

$$\text{Hence } \vec{x} = (\vec{X} \cdot \vec{k})\vec{k} + (\vec{X} - (\vec{X} \cdot \vec{k})\vec{k}) \cos \theta + (\vec{k} \times \vec{X}) \sin \theta \quad (1)$$

(Also a good exercise to prove that $\vec{k} \times \vec{X}$ is perpendicular to $\vec{X} - (\vec{X} \cdot \vec{k})\vec{k}$)

Let define a skew symmetric matrix K such that $K = J(\vec{k})$. This means

$$K = \begin{bmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{bmatrix} \text{ and we know that } K\vec{v} = \vec{k} \times \vec{v}$$

Now we can write \vec{x} as

$$\begin{aligned}
 \vec{x} &= \vec{X} - \vec{X} + (\vec{X} \bullet \vec{k})\vec{k} + (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) \cos \theta + (K\vec{X}) \sin \theta \\
 &= \vec{X} - (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) + (\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) \cos \theta + (K\vec{X}) \sin \theta \\
 &= \vec{X} - (1 - \cos \theta)(\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) + (K\vec{X}) \sin \theta
 \end{aligned} \tag{2}$$

There exists an identity that $a \times (a \times b) = (a \bullet a)b - (a \bullet b)a$. You can also try to prove this for exercise as well. Now we can rewrite $(\vec{X} - (\vec{X} \bullet \vec{k})\vec{k})$ using this identity as

$$(\vec{X} - (\vec{X} \bullet \vec{k})\vec{k}) = (\vec{k} \bullet \vec{k})\vec{X} - (\vec{X} \bullet \vec{k})\vec{k} = \vec{k} \times (\vec{X} \times \vec{k}) = -\vec{k} \times (\vec{k} \times \vec{X}) \tag{3}$$

Note here that $(\vec{k} \bullet \vec{k})$ is just 1, so this doesn't change anything. Then rewrite the result using the property of the skew symmetric matrix K , we get

$$-\vec{k} \times (\vec{k} \times \vec{X}) = -\vec{k} \times K\vec{X} = -(K(K\vec{X})) = -K^2 \vec{X} \quad (4)$$

Substitute (4) in (2), we get

$$\begin{aligned} \vec{x} &= \vec{X} - (1 - \cos \theta)(K^2 \vec{X}) + (K\vec{X}) \sin \theta \\ &= (I + (1 - \cos \theta)K^2 + \sin \theta K)\vec{X} \end{aligned} \quad (5)$$

Since $\vec{x} = R\vec{X}$, therefore, the rotation matrix is described by

$$R = (I + (1 - \cos \theta)K^2 + \sin \theta K) \quad (6)$$

Rodrigues formula.

Now we can use this formula to find back \vec{k} and θ . Knowing that $R^T(\vec{k}, \theta) = R(\vec{k}, -\theta)$ applying Rodrigues formula for both sides, we will get

$$\begin{aligned} R - R^T &= 2 \sin \theta K \\ K &= \frac{R - R^T}{2 \sin \theta} \end{aligned} \quad (7)$$

Hence, $\vec{k} = \frac{1}{2 \sin \theta} \text{vect}(K)$ and θ can be determined by solving $2 \sin \theta = \|\text{vect}(R - R^T)\|$

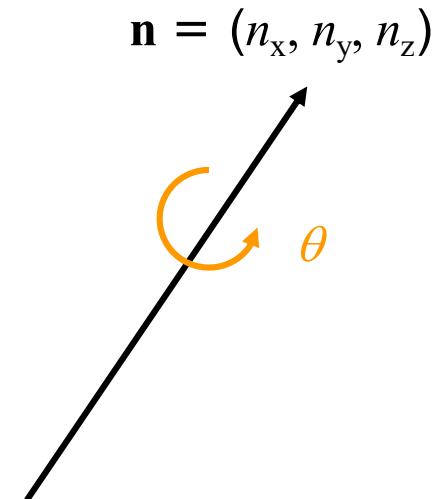
Note: Problems arise when θ is small since the axis of rotation is ill-defined and that (\vec{k}, θ) and $(-\vec{k}, -\theta)$ result in the same orientation.

TUM Example: rigid obot in 3-D workspace

- Parametrization of orientations by **unit**

quaternion: $u = (u_1, u_2, u_3, u_4)$ with $u_1^2 + u_2^2 + u_3^2 + u_4^2 = 1$.

- Note $(u_1, u_2, u_3, u_4) = (\cos \theta/2, n_x \sin \theta/2, n_y \sin \theta/2, n_z \sin \theta/2)$ with $n_x^2 + n_y^2 + n_z^2 = 1$.



- Compare with representation of orientation in 2-D:
 $(u_1, u_2) = (\cos \theta, \sin \theta)$



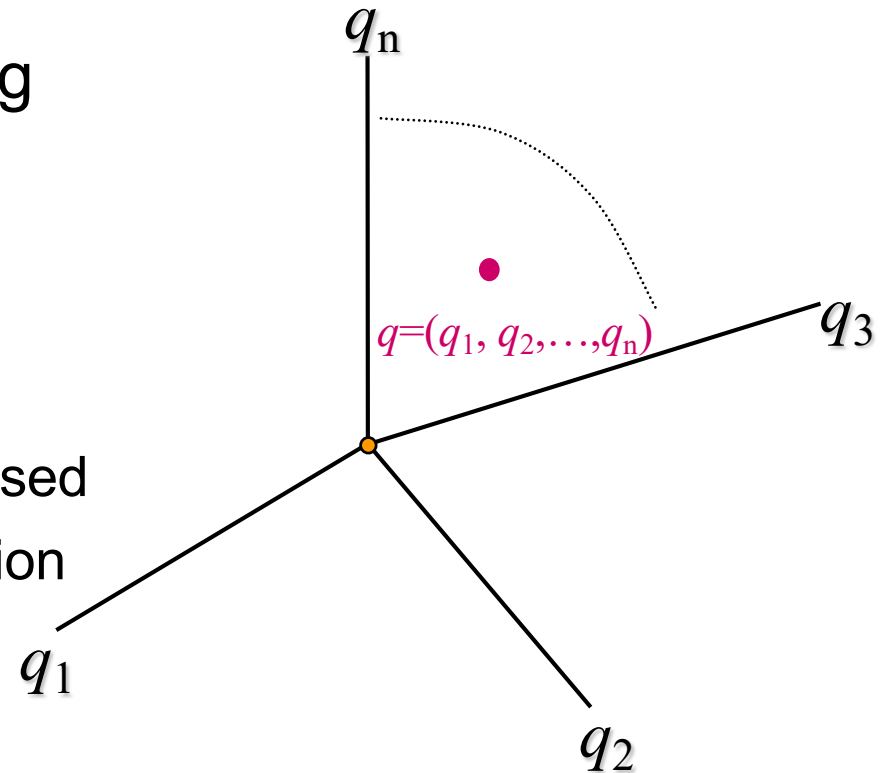
Example: rigid robot in 3-D workspace

- Advantage of unit quaternion representation
 - Compact
 - No singularity
 - Naturally reflect the topology of the space of orientations

- Number of dofs = 6
- Topology: $\mathbb{R}^3 \times \text{SO}(3)$

- The **configuration** of a moving object is a specification of the position of **every** point on the object.

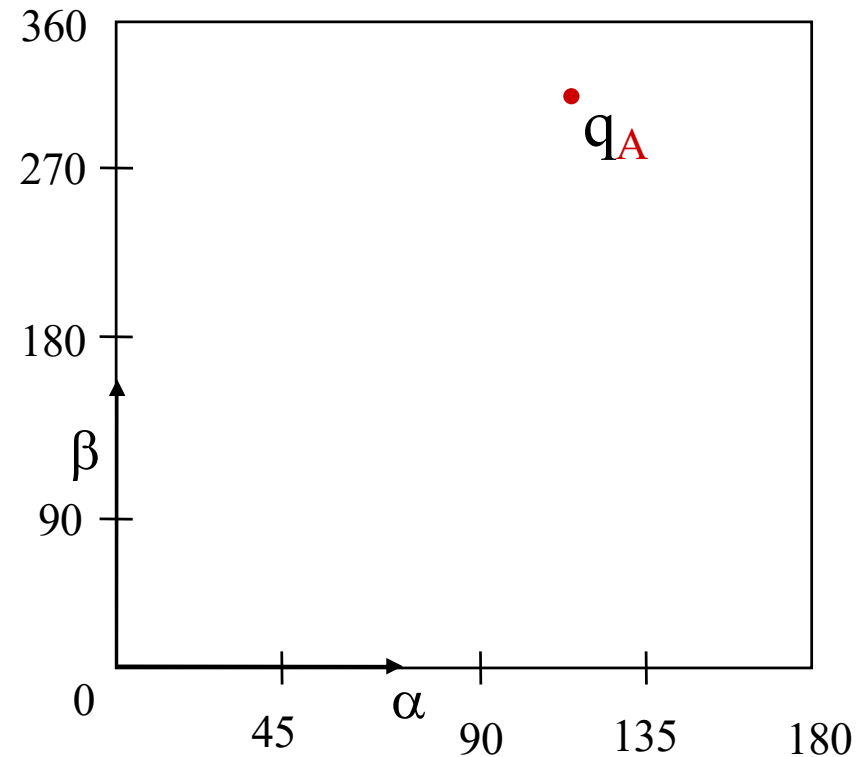
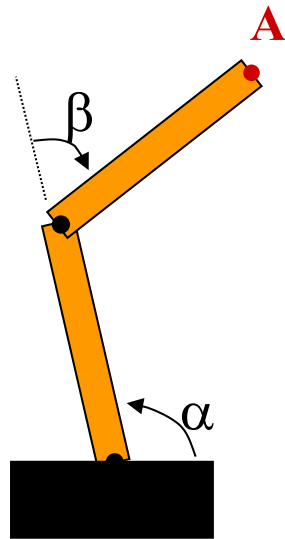
- Usually a configuration is expressed as a vector of position & orientation parameters: $q = (q_1, q_2, \dots, q_n)$



- The **configuration space** C is the set of all possible configurations.

- A configuration is a point in C .

Where can we put $\bullet q_B$?

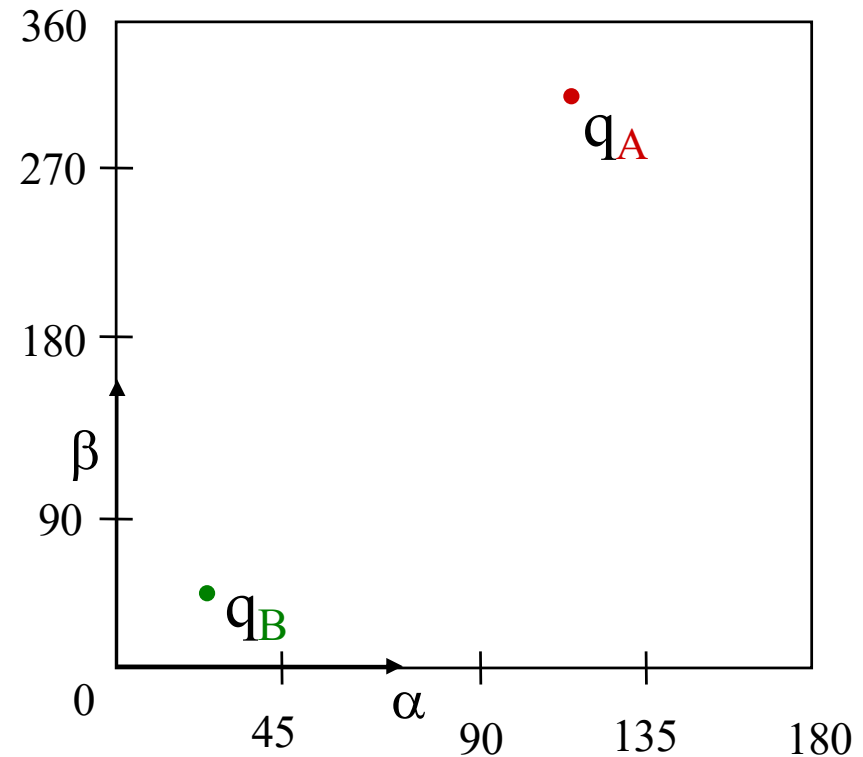
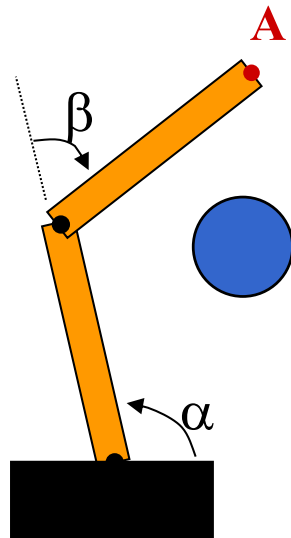


An obstacle in the robot's workspace

Torus

(wraps horizontally and vertically)

Where do we put  ?

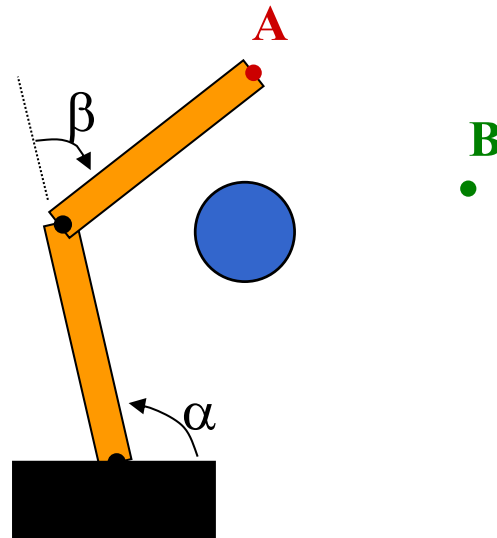


An obstacle in the robot's workspace

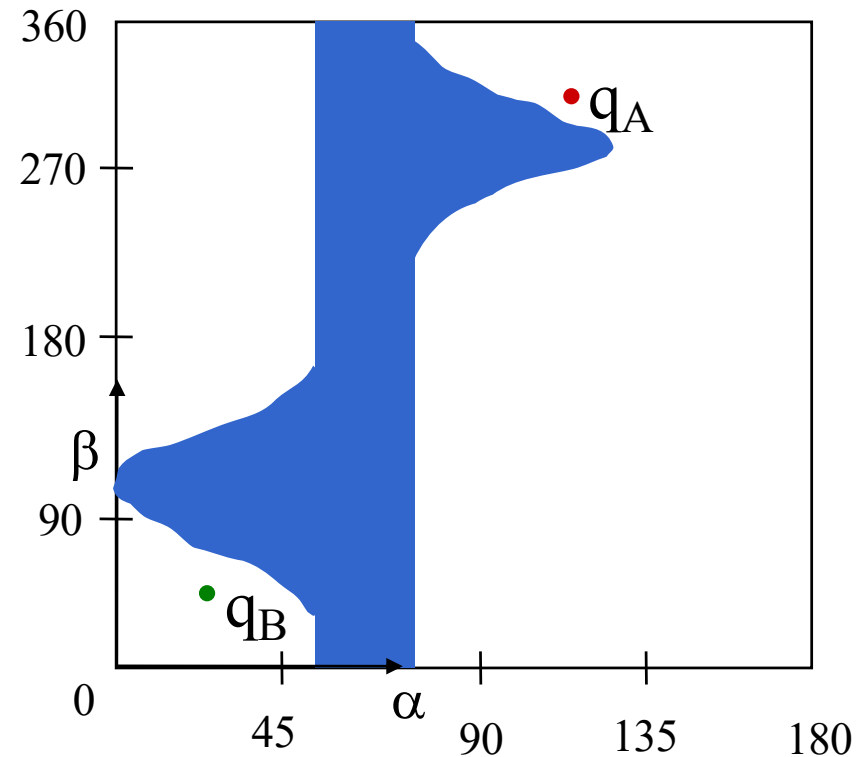
Torus

(wraps horizontally and vertically)

How do we get from **A** to **B**?

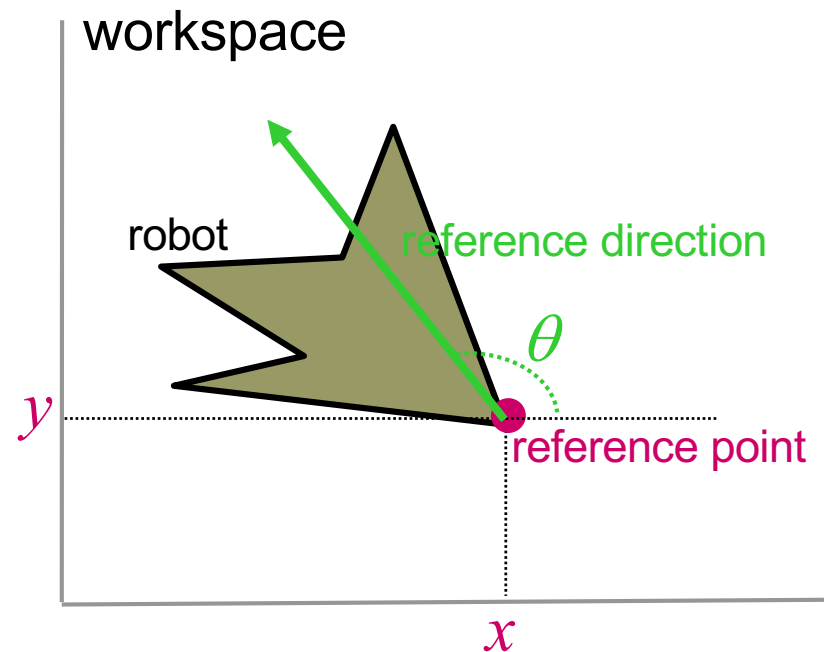


An obstacle in the robot's workspace



The C-space representation
of this obstacle...

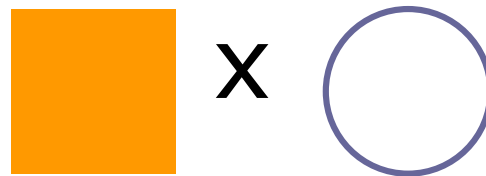
- The **dimension of a configuration space** is the **minimum** number of parameters needed to specify the configuration of the object completely.
- It is also called the **number of degrees of freedom** (dofs) of a moving object.



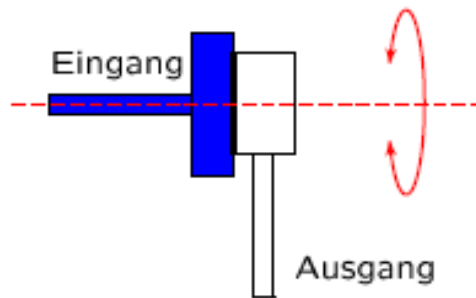
- 3-parameter specification: $q = (x, y, \theta)$ with $\theta \in [0, 2\pi)$.
 - 3-D configuration space

TUM Example: rigid robot in 2-D workspace

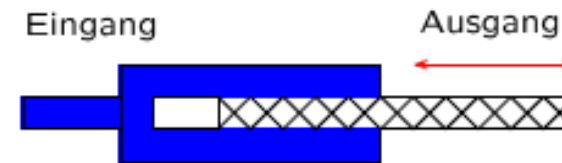
- 4-parameter specification: $q = (x, y, u, v)$ with $u^2 + v^2 = 1$. Note $u = \cos \theta$ and $v = \sin \theta$.
- dim of configuration space = ???
 - Does the dimension of the configuration space (number of dofs) depend on the parametrization?
- Topology: a 2-D cylinder $C = \mathbb{R}^2 \times S^1$



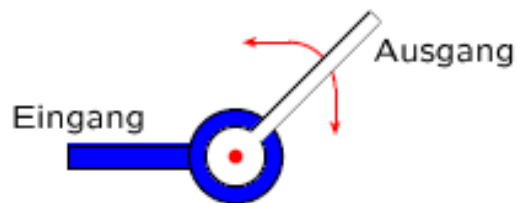
- Does the topology depend on the parametrization?



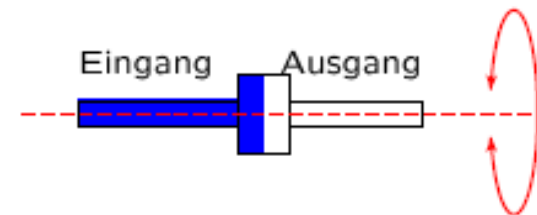
revolving joint



linear joint

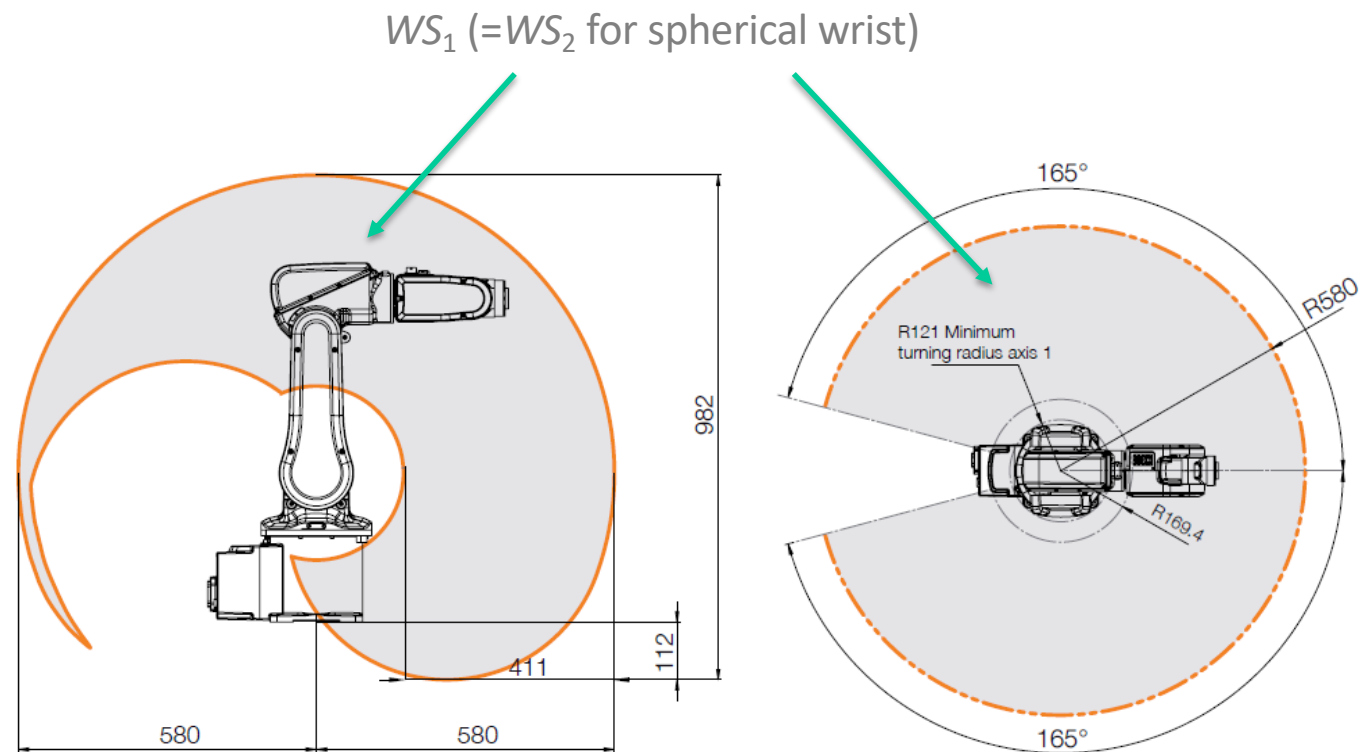


rotational joint



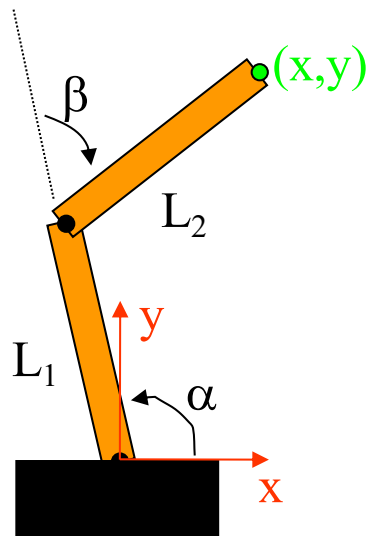
twisting joint

- Example: ABB's IRB 120 robot



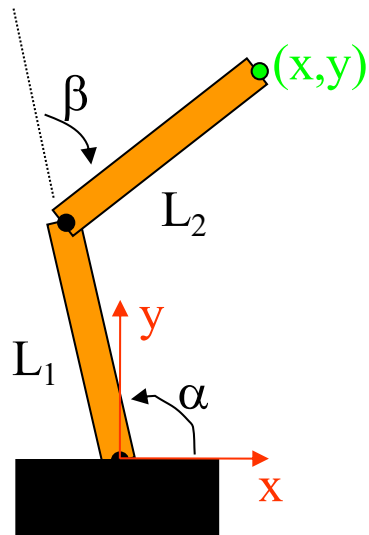
It is used to evaluate the robot for a specific application

What are this arm's forward kinematics?



(How does its position depend on its joint angles?)

What are this arm's forward kinematics?



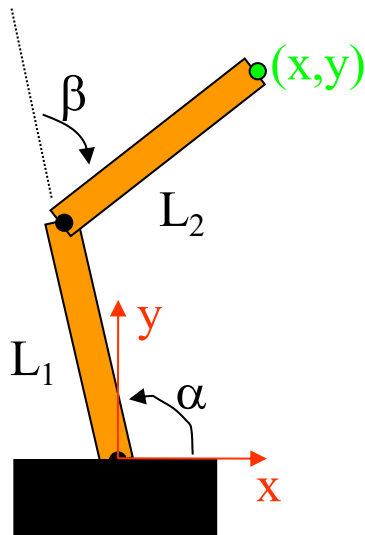
Find (x, y) in terms of α and β ...

Keeping it “simple”

$$c_{\alpha} = \cos(\alpha) \quad , \quad s_{\alpha} = \sin(\alpha)$$

$$c_{\beta} = \cos(\beta) \quad , \quad s_{\beta} = \sin(\beta)$$

$$c_{\square} = \cos(\alpha \square \beta) \quad , \quad s_{\square} = \sin(\alpha \square \beta)$$



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 c_\alpha \\ L_1 s_\alpha \end{pmatrix} + \begin{pmatrix} L_2 c_{\alpha+\beta} \\ L_2 s_{\alpha+\beta} \end{pmatrix} \quad \text{Position}$$

Keeping it “simple”

$$c_\alpha = \cos(\alpha) \quad , \quad s_\alpha = \sin(\alpha)$$

$$c_\beta = \cos(\beta) \quad , \quad s_\beta = \sin(\beta)$$

$$c_{\alpha+\beta} = \cos(\alpha+\beta) \quad , \quad s_{\alpha+\beta} = \sin(\alpha+\beta)$$

In general, a point in n-D space transforms by

$$P' = \text{rotate}(\text{point}) + \text{translate}(\text{point})$$

In 2-D space, this can be written as a matrix equation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} tx \\ ty \end{pmatrix}$$

In 3-D space (or n-D), this can be generalized as a matrix equation:

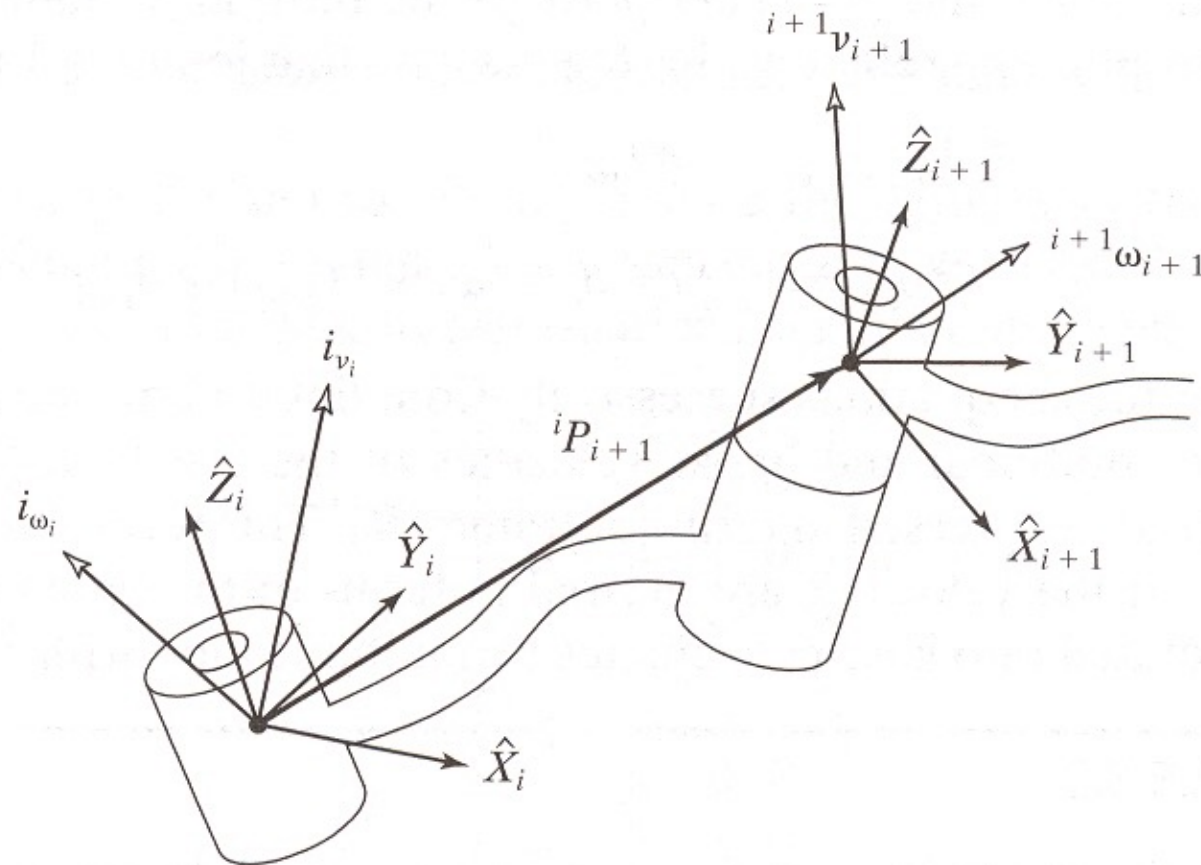
$$p' = R p + T \quad \text{or} \quad p = R^t (p' - T)$$

Now, using the idea of homogeneous transforms,
we can write:

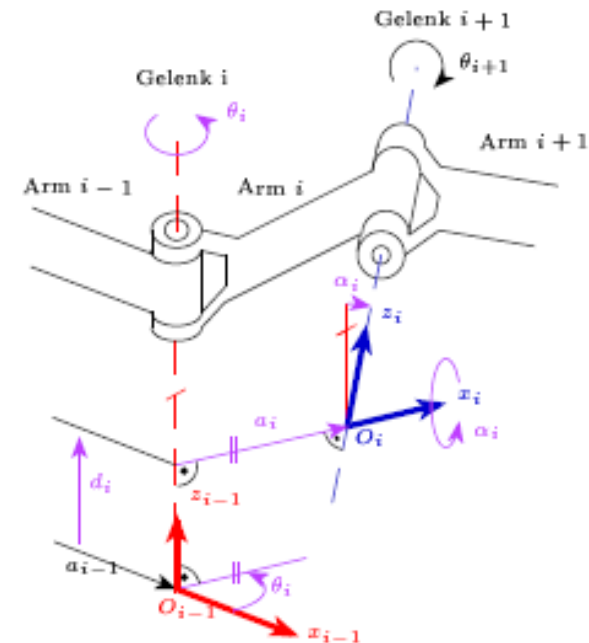
$$p' = \begin{pmatrix} R & T \\ 0 & 0 & 0 & 1 \end{pmatrix} p$$

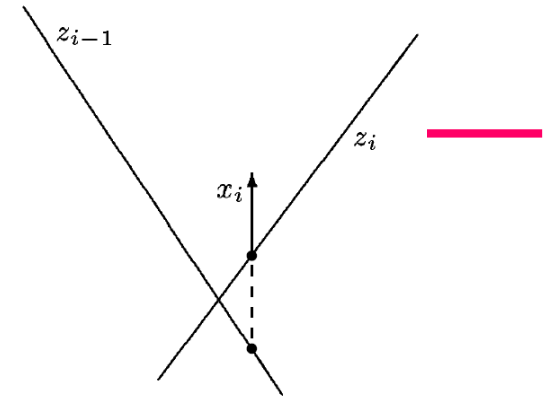
The group of rigid body rotations $SO(3) \times \mathbb{R}(3)$ is
denoted $SE(3)$ (for special Euclidean group)

What does the inverse transformation look like?



z-axis along the axis of motion
 x-axis perpendicular on the two
 consecutive motion axes
 y in a direction defined by a right hand
 system





1. Translation along z_i
2. Rotation around z_i
3. Translation along to the origin of the next frame
4. Rotation between the coordinate frames

$$\mathbf{T}(0, 0, d_i)$$

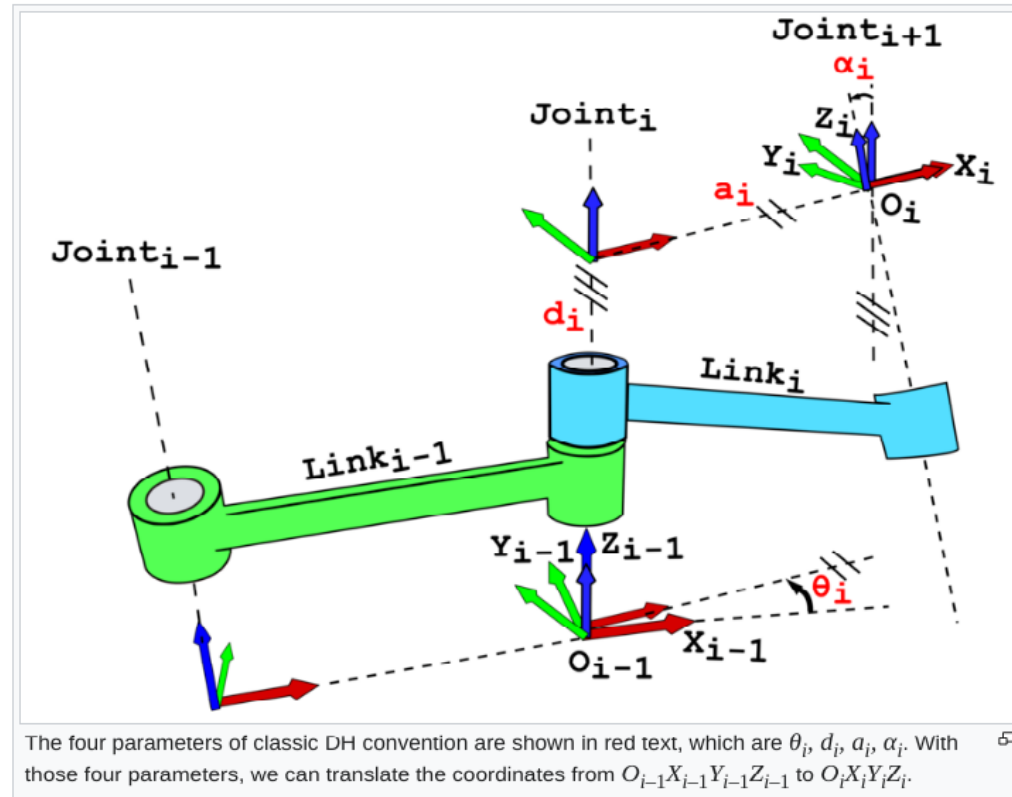
$$\mathbf{R}(z, \theta_i)$$

$$\mathbf{T}(a_i, 0, 0)$$

$$\mathbf{R}(x, \alpha_i)$$

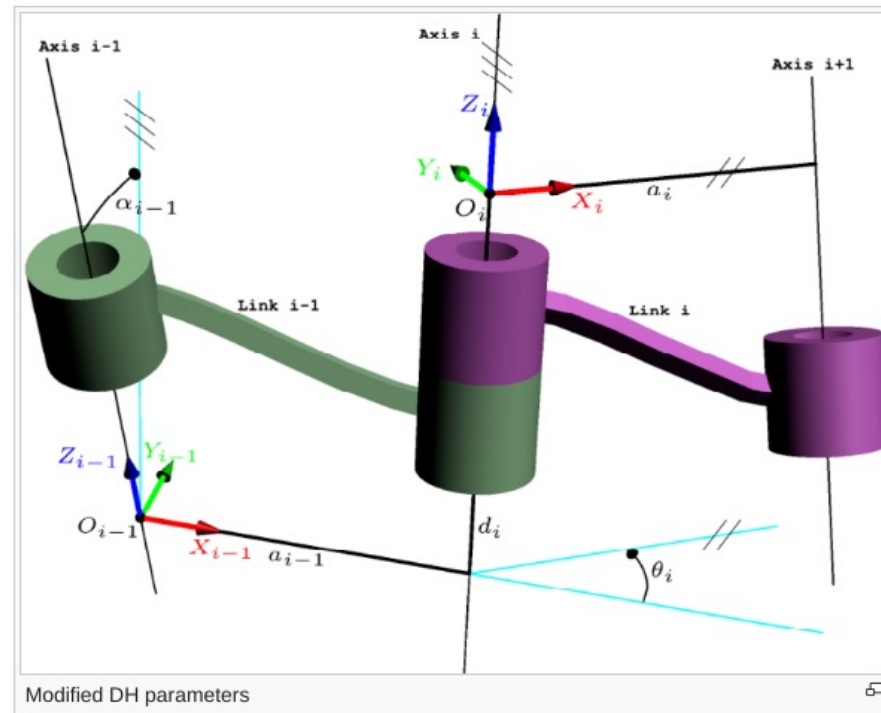
$${}^{i-1}\mathbf{A}_i = \mathbf{T}(0, 0, d_i) \cdot \mathbf{R}(z, \theta_i) \cdot \mathbf{T}(a_i, 0, 0) \cdot \mathbf{R}(x, \alpha_i)$$

$${}^{i-1}\mathbf{A}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_i & -S\alpha_i & 0 \\ 0 & S\alpha_i & C\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



The following four transformation parameters are known as D-H parameters: [4]

- d : offset along previous z to the common normal
- θ : angle about previous z , from old x to new x
- r : length of the common normal (aka a , but if using this notation, do not confuse with α). Assuming a revolute joint, this is the radius about previous z .
- α : angle about common normal, from old z axis to new z axis



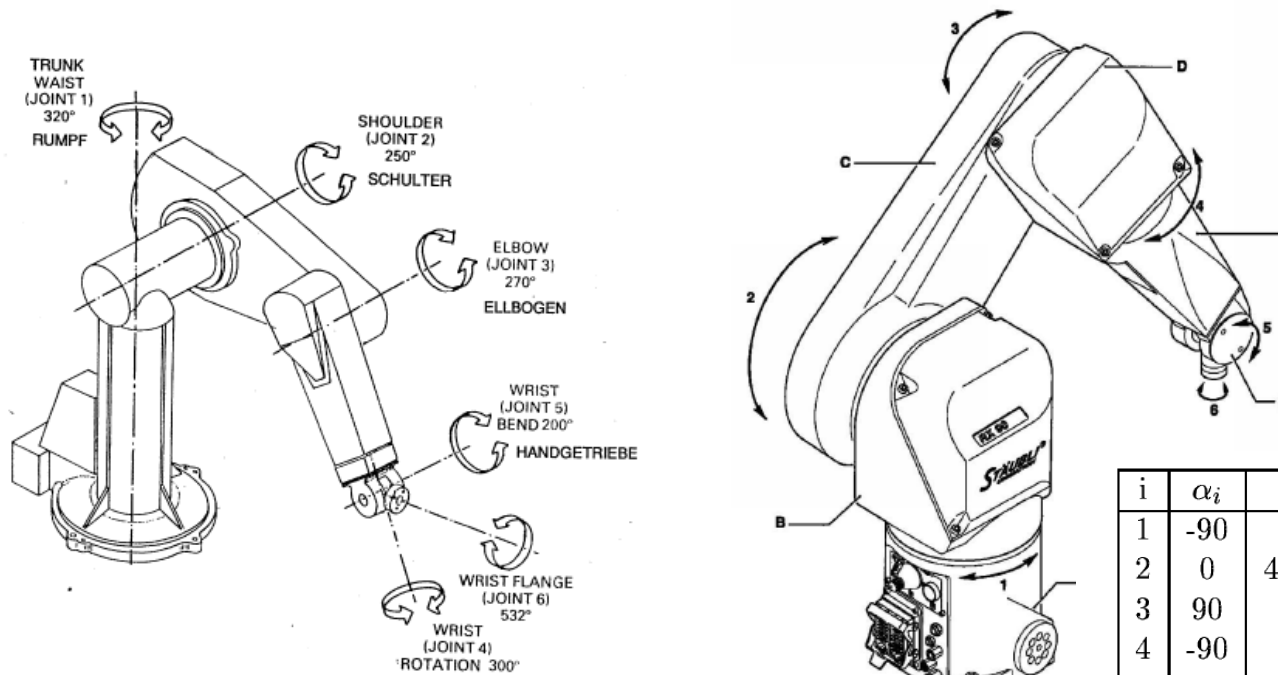
Compared with the classic DH parameters, the coordinates of frame O_{i-1} is put on axis $i-1$, not the axis i in classic DH convention. The coordinates of O_i is put on the axis i , not the axis $i+1$ in classic DH convention.

Another difference is that according to the modified convention, the transform matrix is given by the following order of operations:

$${}^{n-1}T_n = \text{Rot}_{x_{n-1}}(\alpha_{n-1}) \cdot \text{Trans}_{x_{n-1}}(a_{n-1}) \cdot \text{Rot}_{z_n}(\theta_n) \cdot \text{Trans}_{z_n}(d_n)$$

Thus, the matrix of the modified DH parameters becomes

$${}^{n-1}T_n = \begin{bmatrix} \cos \theta_n & -\sin \theta_n & 0 & a_{n-1} \\ \sin \theta_n \cos \alpha_{n-1} & \cos \theta_n \cos \alpha_{n-1} & -\sin \alpha_{n-1} & -d_n \sin \alpha_{n-1} \\ \sin \theta_n \sin \alpha_{n-1} & \cos \theta_n \sin \alpha_{n-1} & \cos \alpha_{n-1} & d_n \cos \alpha_{n-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



i	α_i	a_i	d_i	θ_i -Bereich
1	-90	0	0	$-160^\circ \dots +160^\circ$
2	0	450mm	0	$-227.5^\circ \dots +47.5^\circ$
3	90	0	0	$-52.5^\circ \dots +232.5^\circ$
4	-90	0	450mm	$-270^\circ \dots +270^\circ$
5	90	0	0	$-105^\circ \dots +120^\circ$
6	0	0	85mm	$-270^\circ \dots +270^\circ$

$${}^0A_6 = {}^0A_1 \cdot {}^1A_2 \cdot {}^2A_3 \cdot {}^3A_4 \cdot {}^4A_5 \cdot {}^5A_6$$

$${}^0A_3 = {}^0A_1 \cdot {}^1A_2 \cdot {}^2A_3 = \begin{pmatrix} \cos \theta_1 \cdot \cos(\theta_2 + \theta_3) & -\sin \theta_1 & \cos \theta_1 \cdot \cos(\theta_2 + \theta_3) & a_2 \cdot \cos \theta_1 \cdot \cos \theta_2 \\ \sin \theta_1 \cdot \cos(\theta_2 + \theta_3) & \cos \theta_1 & \sin \theta_1 \cdot \sin(\theta_2 + \theta_3) & a_2 \cdot \sin \theta_1 \cdot \cos \theta_2 \\ -\sin(\theta_2 + \theta_3) & 0 & \cos(\theta_2 + \theta_3) & -a_2 \sin \theta_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

TUM Forward Kinematics

- Given joint variables

$$q \sqsubset (q_1, q_2, \dots, q_n)$$

- End-effector position & orientation

$$Y \sqsubset (x, y, z, \phi, \theta, \psi)$$

Homogeneous matrix T_0^n

- specifies the location of the i th coordinate frame w.r.t. the base coordinate system
- chain product of successive coordinate transformation matrices of T_{i-1}^i

$$T_0^n \sqsubset T_0^1 T_1^2 \dots T_{n-1}^n$$

$$\sqsubset \begin{bmatrix} R_0^n & P_0^n \\ 0 & 1 \end{bmatrix} \sqsubset \begin{bmatrix} n & s & a & P_0^n \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orientation matrix

Position vector

Yaw-Pitch-Roll representation for orientation

$$T_0^n = \begin{bmatrix} C\phi C\theta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi & p_x \\ S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi & p_y \\ -S\theta & C\theta S\psi & C\theta C\psi & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

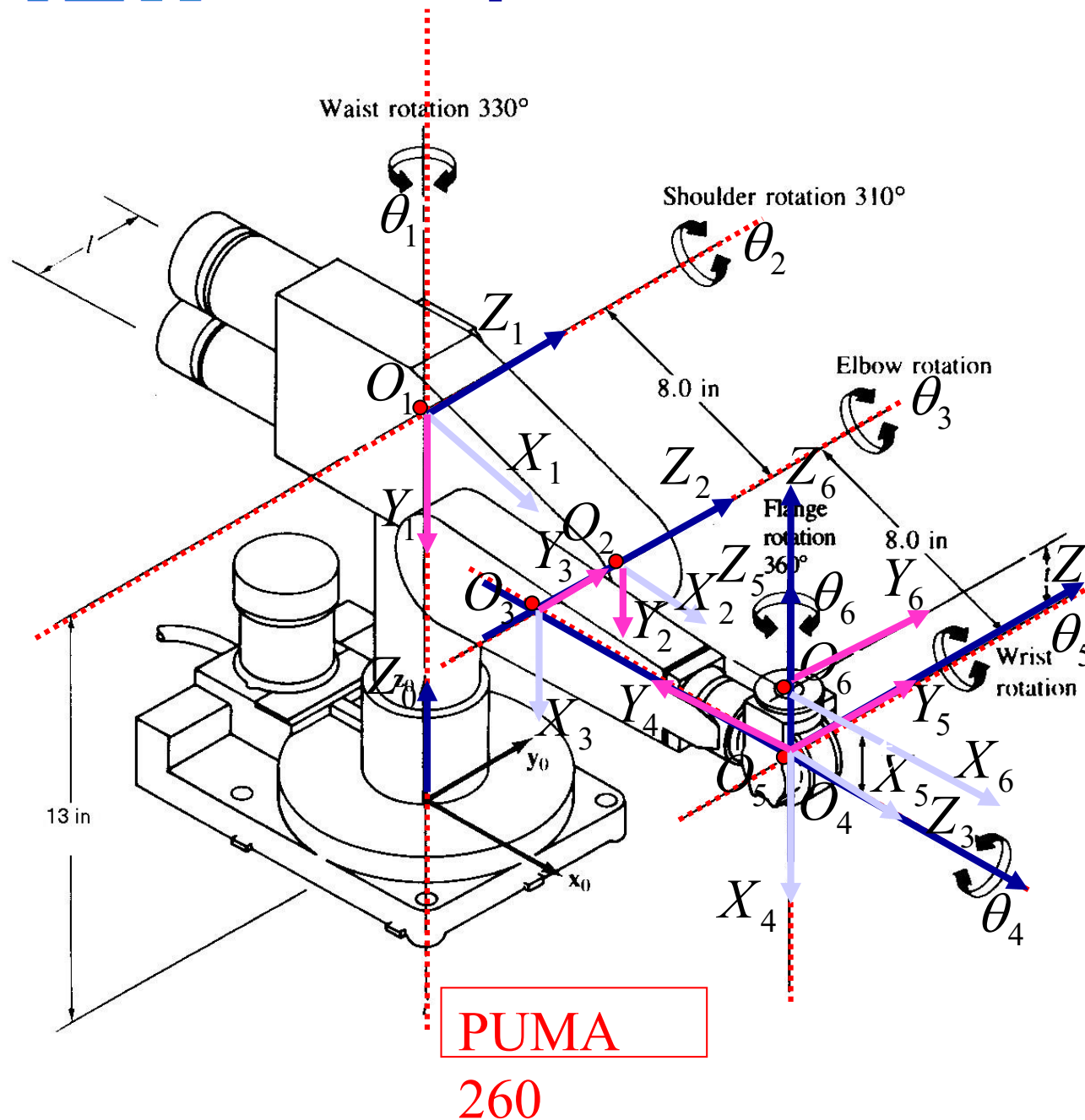
$$T_0^n = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\theta = \sin^{-1}(-n_z)$$

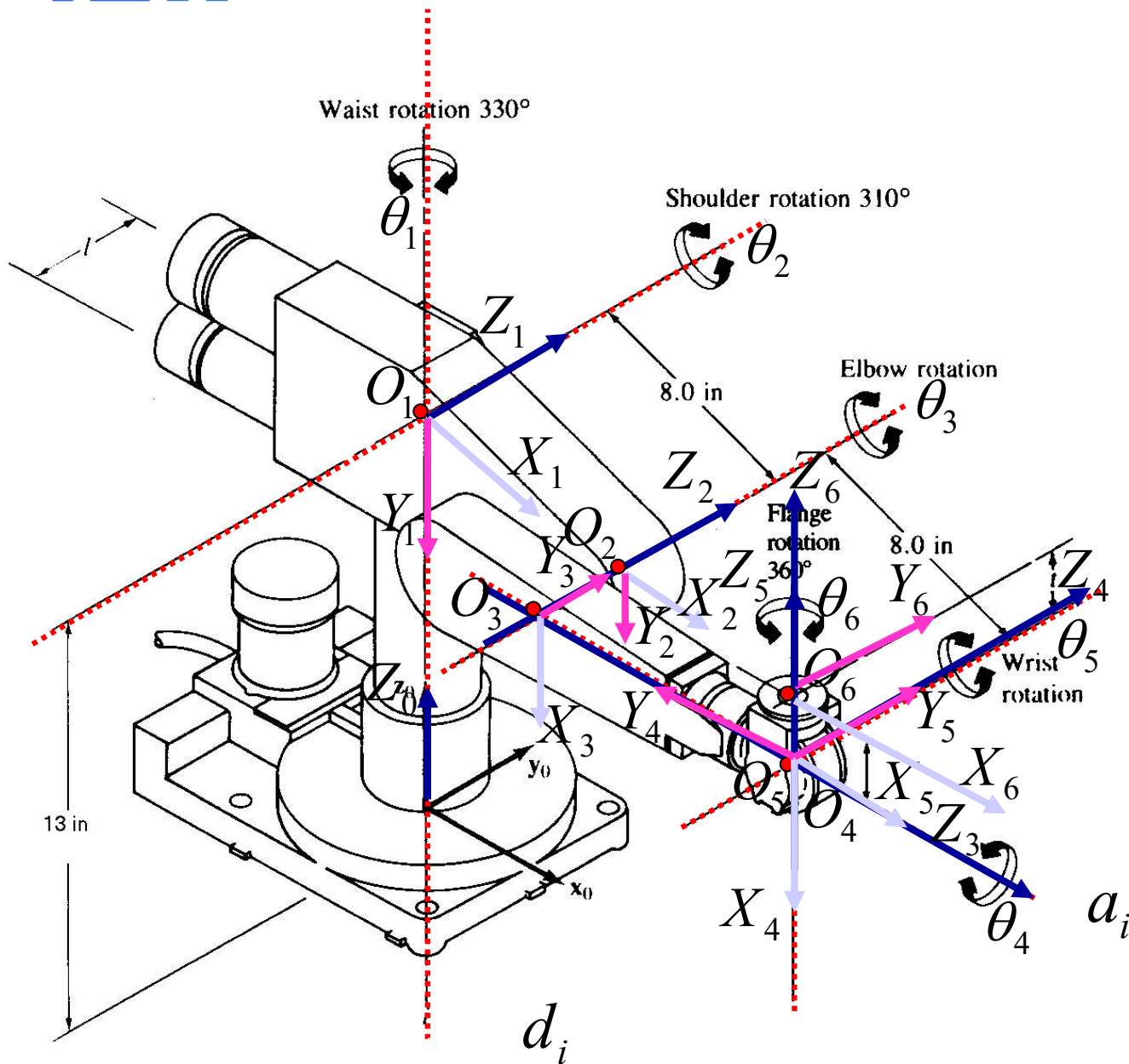
$$\psi = \cos^{-1}\left(\frac{a_z}{\cos\theta}\right)$$

$$\phi = \cos^{-1}\left(\frac{n_x}{\cos\theta}\right)$$

TUM Example: PUMA 260



1. Number the joints
2. Establish base frame
3. Establish joint axis Z_i
4. Locate origin, O_i (intersect. of Z_i & Z_{i-1}) OR (intersect of common normal & Z_i)
5. Establish X_i, Y_i



J	θ_i	α_i	a_i	d_i
1	θ_1	-90	0	13
2	θ_2	0	8	0
3	θ_3	90	0	
4	θ_4	-90	0	8
5	θ_5	90	0	0
6	θ_6	0	0	t

θ_i

α_i

a_i