AST3220 Project 1

Kandidatnr: 3

March 2023

Problem 1

$$\dot{\rho}_\phi = \frac{d\rho_\phi}{dt} = -3H\left(\rho_\phi + p_\phi\right) = -3H\left(1+\omega_\phi\right)\rho_\phi$$

$$d\rho_\phi = -3H\left(1+\omega_\phi\right)\rho_\phi dt$$

$$H = \frac{\dot{a}}{a} = \frac{da}{dt}\frac{1}{a} \quad \Rightarrow \quad dt = \frac{da}{H}\frac{1}{a}$$

Using the last two equations, we get:

$$\Rightarrow \frac{d\rho_\phi}{\rho_\phi} = -3(1+\omega_\phi)\frac{1}{a}da$$

$$(1+z) = \frac{a_0}{a} \quad \Rightarrow \quad z = \frac{a_0}{a} - 1$$

$$\frac{dz}{da} = -\frac{a_0}{a^2} \quad \Rightarrow \quad da = -\frac{a^2}{a_0}dz$$

$$\Rightarrow \frac{d\rho_\phi}{\rho_\phi} = -3(1+\omega_\phi)\frac{a}{a_0}dz = \frac{3(1+\omega_\phi)}{1+z}dz$$

We can then integrate from today's values to an unspecific time:

$$\int_{\rho_{\phi 0}}^{\rho_\phi} \frac{1}{\rho}d\rho = \int_0^z \frac{3[1+\omega_\phi(z')]}{1+z'}dz' \quad \Rightarrow \quad ln(\frac{\rho_\phi}{\rho_{\phi 0}}) = \int_0^z \frac{3[1+\omega_\phi(z')]}{1+z'}dz'$$

We then get:

$$\rho_\phi = \rho_{\phi 0} e^{\int_0^z \frac{3[1+\omega_\phi(z')]}{1+z'}dz'}$$

Problem 2

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{2}$$

$$\frac{d\rho_\phi}{dt} = -3H(\rho_\phi + p_\phi) \tag{3}$$

Deriving (1)

$$\frac{d\rho_\phi}{dt} = \frac{d\phi}{dt}\frac{d^2\phi}{dt^2} + \frac{dV}{d\phi}\frac{d\phi}{dt}$$

Setting 1 & 2 together, and setting that in 3:

$$\frac{d\rho_\phi}{dt} = -3H(\dot{\phi}^2)$$

Setting that and the derivation together:

$$\frac{d\phi}{dt}\frac{d^2\phi}{dt^2} + \frac{dV}{d\phi}\frac{d\phi}{dt} = -3H(\dot{\phi}^2)$$

Or in another form:

$$\dot{\phi}\ddot{\phi}^2 + V'\dot{\phi} + 3H(\dot{\phi}^2) = 0 \quad \Rightarrow \quad \ddot{\phi} + V' + 3H\dot{\phi} = 0$$

Problem 4

$$\Omega_i = \frac{\rho_i(t)}{\rho_c(t)} \tag{4}$$

$$\rho_c(t) = \frac{3H^2}{\kappa^2} \tag{5}$$

Where $\kappa^2 = 8\pi G$

$$\Omega_\phi = \frac{\rho_\phi \kappa^2}{3H^2} = (\frac{1}{2}\dot{\phi}^2 + V)(\frac{\kappa^2}{3H^2}) = \frac{\dot{\phi}^2\kappa^2}{6H^2} + \frac{V\kappa^2}{3H^2} = {x_1}^2 + {x_2}^2$$

$$\Omega_r = \frac{\rho_r \kappa^2}{3H^2} = {x_3}^2$$

The cosmological principle states:

$$1 = \Omega_\phi + \Omega_r + \Omega_m$$

And therefore:

$$\Omega_m = 1 - {x_1}^2 - {x_2}^2 - {x_3}^2$$

Problem 5

$$\dot{H} = -\frac{\kappa^2}{2}[\rho_m + \rho_r(1+\omega_r) + \dot{\phi}^2] \tag{6}$$

Using equation 1 and dividing by H^2 on both sides:

$$\frac{\dot{H}}{H^2} = -\frac{\kappa^2}{2H^2}[\rho_m + \rho_r(1+\omega_r) + 2\rho_\phi - 2V]$$

Using equation 4 and since $\omega_r = \frac{1}{3}$:

$$\frac{\dot{H}}{H^2} = -\frac{\kappa^2}{2H^2}[\Omega_m\rho_c + \frac{4}{3}\Omega_r\rho_c + 2\Omega_\phi\rho_c - 2V$$

Taking out ρ_c:

$$\frac{\dot{H}}{H^2} = -\frac{\kappa^2\rho_c}{2H^2}[\Omega_m + \frac{4}{3}\Omega_r + 2\Omega_\phi - \frac{2V}{\rho_c}]$$

Using the results from problem 4:

$$\frac{\dot{H}}{H^2} = -\frac{\kappa^2\rho_c}{2H^2}[(1 - {x_1}^2 - {x_2}^2 - {x_3}^2) + \frac{4}{3}({x_3}^2) + 2({x_1}^2 + {x_2}^2) - \frac{2V}{\rho_c}]$$

$$\frac{\dot{H}}{H^2} = -\frac{\kappa^2\rho_c}{2H^2}[1 - {x_1}^2 - {x_2}^2 - {x_3}^2 + \frac{4}{3}{x_3}^2 + 2{x_1}^2 + 2{x_2}^2 - \frac{2V}{\rho_c}]$$

$$\frac{\dot{H}}{H^2} = -\frac{\kappa^2\rho_c}{2H^2}[1 + {x_1}^2 + {x_2}^2 + \frac{1}{3}{x_3}^2 - \frac{2V}{\rho_c}]$$

Setting in for ρ_c using equation 5:

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}[1 + {x_1}^2 + {x_2}^2 + \frac{1}{3}{x_3}^2 - \frac{2V\kappa^2}{3H^2}]$$

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}[1 + {x_1}^2 + {x_2}^2 + \frac{1}{3}{x_3}^2 - 2{x_2}^2] = -\frac{1}{2}[3 + 3{x_1}^2 - 3{x_2}^2 + {x_3}^2]$$

Problem 6

$$\frac{1}{H}\frac{df}{dt} = \frac{df}{dN} \tag{7}$$

$$\frac{dx_1}{dt} = \frac{\kappa}{\sqrt{6}}[\frac{H\ddot{\phi} - \dot{\phi}\dot{H}}{H^2}] = \frac{\kappa}{\sqrt{6}}[\frac{\ddot{\phi}}{H} - \frac{\dot{\phi}\dot{H}}{H^2}]$$

$$\lambda = -\frac{V'}{\kappa V} \tag{8}$$

Using the results from Problem 2, and subsequently equation 8:

$$= \frac{\kappa}{\sqrt{6}}[-\frac{\dot{\phi}}{H} - \frac{V'}{H} - \frac{\dot{\phi}\dot{H}}{H^2}] = \frac{\kappa}{\sqrt{6}}[-\frac{\dot{\phi}}{H} + \frac{\kappa V \lambda}{H} - \frac{\dot{\phi}\dot{H}}{H^2}]$$

$$= -\frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H} + \frac{\kappa}{\sqrt{6}}\frac{\kappa V\lambda}{H} - \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}\dot{H}}{H^2}$$

$$\frac{df}{dN} = \frac{1}{H}\frac{df}{dt} = -\frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H^2} + \frac{\kappa^2 V\lambda}{\sqrt{6}H^2} - \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H}\frac{\dot{H}}{H^2}$$

$$= -\frac{x_1}{H} + \frac{\sqrt{6}}{2}\lambda {x_2}^2 - \frac{\kappa}{\sqrt{6}}\frac{\dot{\phi}}{H}\frac{\dot{H}}{H^2}$$

Problem 7

If λ is constant, that means that $\frac{d}{dt}(-\frac{V'}{\kappa V}) = 0$. If this is true, then the only type of function that could work is an exponential function.

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2(\Gamma - 1)x_1$$

If this equation is to be constant, that means that it must equal 0, so $\Gamma = 1$.

1 Problem 9

We created the code that calculates the density parameters and EoS equation for both the inverse power law potential and exponential potential (See figures 1-4).

In order to calculate the EoS equation, we did some "massaging" of the formulas.

$$\omega = \frac{p}{\rho}$$

Using that and equation 4, we have that:

$$\omega = \frac{p}{\rho_c \Omega}$$

And then by using equation 2 and setting in for ρ_c, we have this:

$$\omega_\phi = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\rho_c \Omega_\phi} = \frac{\dot{\phi}^2\kappa^2}{6H^2\Omega_\phi} - \frac{V\kappa^2}{3H^2\Omega_\phi}$$

And by using the dimensionless variables, this becomes:

$$\omega_\phi = \frac{{x_1}^2 - {x_2}^2}{{x_1}^2 + {x_2}^2}$$

For both quintessence fields, the omega relationships over time look quite similar, matter is dominant in the early ages of the universe, then radiation, then the quintessence. What is different is the equation of state, however, they do have a similar shape, the power-law functions drop is earlier in time.

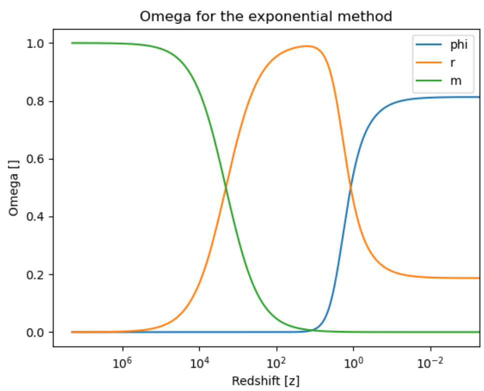

Figure 1

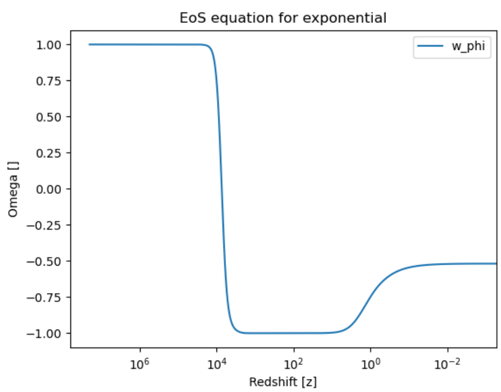

Figure 2

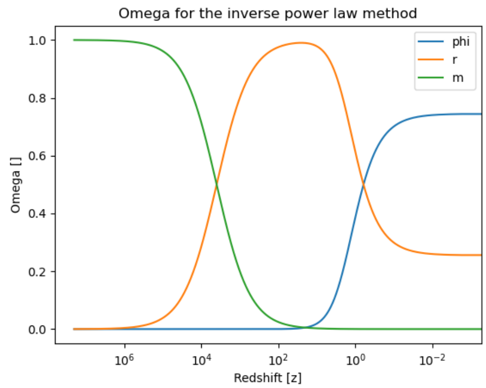

Figure 3

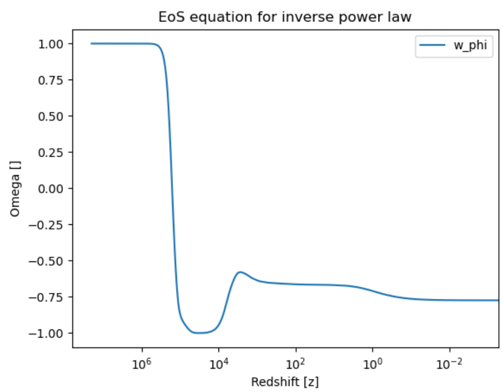

Figure 4

Problem 10

Hubble parameter for the spatially flat ΛCDM model, with $\Omega_{\mathrm{m0}} = 0.3$

$$\frac{H}{H_0} = \sqrt{\Omega_{\mathrm{m0}} \left(\frac{a_0}{a}\right)^3 + (1 - \Omega_{\mathrm{m0}})} = \sqrt{0.3\,(1+z)^3 + (0.7)}$$

Problem 11

We calculated the age of the universe based on the Hubble parameter using this formula from the lecture notes.

$$t_0 H_0 = \frac{2}{3\sqrt{1-\Omega_{m0}}} \sinh\left(\frac{1-\Omega_{m0}}{\Omega_{m0}}\right)$$

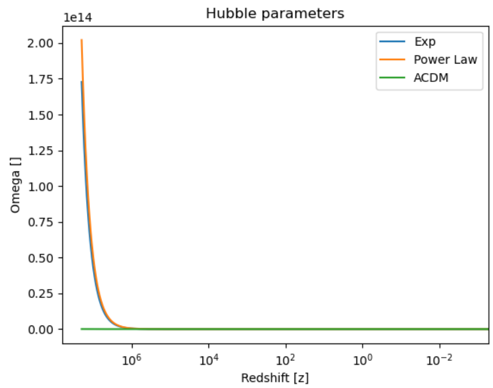

Figure 5

Our results were not as wished. We ended up getting infinity for both, and while that isn't exactly impossible, we do not think that they are the results we are looking for.