Innlevering 1A.6

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I. INTRODUCTION

The goal of this study is to be able to launch a satellite into space from any random planet, and to do so we must first "practice" with a simulation. For simplicity's sake, we chose to ignore the planet's gravitational pull on the rocket. To simulate the rocket engine we started by creating a box filled with hydrogen particles, or more specifically, H_2 particles that then shoot out of a hole on one of the sides to generate momentum. We used that momentum to find velocity and thereafter by use of Euler's method, acceleration. After we have the acceleration we can then calculate how many of our boxes we need to achieve escape velocity in under 20 minutes, and how much fuel is needed.

II. METHOD

We generated a solar system and a planet with random properties using the Ast2000 pack for python. We then used the mass of our planet and the correct formula to calculate escape velocity, the velocity needed to escape a gravitational field:

$$V_{esc} = \sqrt{\frac{2GM}{r}}$$

where G is the gravitational constant, M the mass of our planet, and r the radius of our planet. We then created a box filled with a set volume and a set number of Hydrogen particles. We used Hydrogen because it is an ideal gas, a type of gas that has properties such as the particles don't collide with each other and collisions with the box are elastic, so energy is conserved.

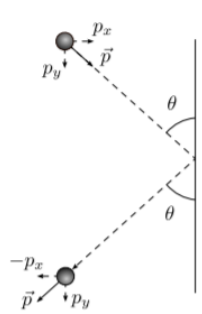

Figure 1. Illustration of an elastic collision.

The particles positions and velocities were randomly generated with the seed "234" using numpy's random functions, uniform and normal, respectively. The uniform function generates equally distributed random numbers within a given interval, in this case the length of one of the sides of the box is $L = 10^{-6}m$, so our interval is $[-L/2, L/2]$. We chose the origin to be in the middle of our cube. The normal function uses Gaussian distribution, which can be shown with the function:

$$P(\vec{v}) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-(v_x^2+v_y^2+v_z^2)/(2\sigma^2)}$$

The function requires the mean value μ and the standard deviation σ. We chose $\mu = 0$ since it's in the middle of our interval $[-L/2, L/2]$, and we used the formula:

$$\sigma = \sqrt{\frac{kT}{m}}$$

in which $k = 1.38 \times 10^{-23}$ is Boltzmann's constant, $T = 10,000K$ is the temperature of our particles, and $m = 3.347115 \times 10^{-27}kg$ is the mass of a H_2 particle.

We chose $N = 100,000$, and we are working in 3-dimensions, so we made two arrays using the previously described functions that were N elements long on the y-axis, and 3 elements long on the x-axis. Usually it would be the opposite, but python works a bit differently.

To see if we had created the velocity array correctly, we wanted to compare the numerically computed mean velocity of all the particles with the general analytical solution. To find the numerical answer, we used linalg's norm function on the velocity vector, which finds the mean speed of each particle and creates a 1-D vector. We then took the mean of that 1-D array using numpy's mean function. For the analytical solution we used the formula:

$$\langle V \rangle = \sqrt{\frac{8kT}{\pi m}}$$

which stems from Maxwell's speed distribution formula. We decided to also compare the kinetic energy. To find the numerical answer, we again used linalg's norm function on the velocity vector and then used the general formula for kinetic energy:

$$KE = \frac{1}{2}mv^2$$

with m being again, the mass of a H_2 particle. We used that formula on every particle, and took the mean of every answer. We found the analytical answer by use of the formula:

$$\langle KE \rangle = \frac{3}{2}kT.$$

In order to create the box itself, and to change the positions with time, we implemented a for-loop. In the loop we used Euler's method $x = x_0 + v\Delta t$ to update the positions. For Δt we chose $10^{-9}s$ with 1000 time-steps, so each step being $10^{-12}s$. After that we made the walls by making if-statements for each direction, if $x, y, z \geq L/2$, then we multiply the respective velocity with -1, since the collisions are elastic, the only thing that changes is the direction.

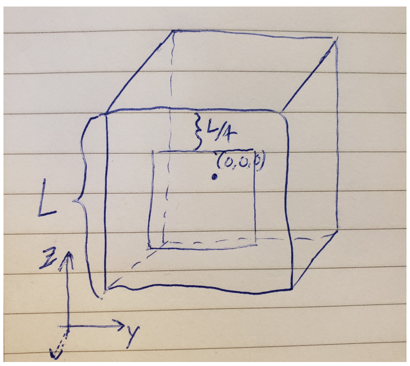

Figure 2. A visual representation of the box with the x-unit vector pointing out of the page.

We chose the wall with the hole to be the wall in the yz-plane at $x = L/2$, so if $x \geq L/2$ and $y, z \in [-L/4, L/4]$, then the particle has exited through the hole. We made an empty list before the loop and appended the x-velocities of all the escaping particles, since the rocket will be moving in the opposite direction. We added a counter to how many particles are escaping in order to keep track for later, we also added a second counter for how many times a particle would hit the wall if the hole wasn't there, that we later used in calculating the momentum. In order to keep the pressure of the box stable, we added a "new" particle each time one escaped by making the x-position of the escaping particle negative, it then seems like a new particle came in through the opposite wall of the box with the same velocity.

In order to check that we did in fact keep the pressure of the box stable, we needed to check our numerical answer with the analytical one. Our next step was to calculate the momentum transferred to the wall from the particles, so that we could calculate the force, and then pressure. The momentum transferred to the wall can be defined as $\Delta p = 2mv$, v being the sum of all the escaped particle's velocities in the x-direction. It's multiplied with 2 because of the momentum applied to the wall when the particle hits it, combined with when the particle bounces off it. Once we had the momentum, we could then divide it by $\Delta t = 10^{-9}s$ to find the force applied. We can now find the pressure with:

$$P = \frac{F}{A}, A = L^2$$

We can find the analytical answer with the equation of state formula for an ideal gas, $P = nkT$, where n is the number of particles divided by A, k is Boltzmann's constant, and T is temperature.

After we were sure the pressure in the box was stable, we summed up the velocities of the particles that left the box and multiplied that with m to find the momentum that "left" the system. We used that in the formula:

$$\Delta v = \frac{\Delta p}{m}$$

where in this case, m is the mass of the satellite, $1,000kg$, since Δv is the velocity of the rocket, not the particle. We then divided Δv by Δt to find a.

We had finally come to the last step, we needed to find a way to calculate how many of our boxes we needed to get our rocket to escape velocity within 20 minutes, or $\Delta t = 1200s$. To do that we chose to implement a while-loop that would go over the formula:

$$V_{rocket} = N_{boxes} a \Delta t$$

which stems from the general formula for acceleration. If the velocity wasn't greater than the escape velocity, then we added 10^{-9} boxes each time until it was.

Now that we knew the amount of boxes needed, we could then calculate the amount of fuel needed. First we divided the total amount of escaped particles by the total time to find escaped particle per second. We then multiplied that number by the mass a H_2 particle and the amount of boxes. We then have kg/s of fuel used. To wrap it all up, we multiplied that number by $\Delta t = 1200s$ to find total fuel needed.

III. RESULTS

Our home planet had a radius of $8,325km$, and a mass of $1.09 \times 10^{25}kg$. The escape velocity was $13.2km/s$. When it comes to the kinetic energy, the numerical and analytical answers were exactly the same, both at $2.07 \times 10^{-19}J$. For the avererage velocity of the particles, there was a difference of $2.91m/s$. The numerical answer was $10249.38m/s$ and the analytical $10246.47m/s$

The particles hit the wall we were measuring $136,091$ times. The answers for pressure were quite similar, but not exact, the numerical was $1.31 \times 10^4 N/m^2$ and the analytical $1.38 \times 10^4 N/m^2$.

There were $44,650$ particles that went through the hole, and with those particles, the system "lost" $1.06 \times 10^{-18} mKg/s$.

Using the lost momentum, we found the velocity, then acceleration and finally the amount of boxes needed to get the rocket to escape velocity within 20 minutes, which was 1.036×10^{13} boxes.

The rocket used $1.55kg/s$ of fuel, $1,860kg$ in total.

IV. DISCUSSION

With the fact that our home planet was 24% bigger, and 45% more massive than the earth, it then makes sense that the escape velocity of our planet would be 16% faster than earth's, since the gravitational pull is stronger.

The 0.01% relative error in the average velocity of the particles could have very likely been fixed by running the program with higher N values, and we would have done exactly that, had we have had access to a stronger computer.

When calculating pressure numerically, we realized that our answer kept being half as much as the analytical answer, so we had to accept that we must have missed a 2 somewhere in our calculations.

The fact that $44,650$ out of $100,000$, almost half, of the particles left through the hole makes sense, since the hole is relatively big compared to the whole wall. The extremely little bit of momentum "lost", at $1.06 \times 10^{-18} mKg/s$ also makes sense, since our box is so incredibly small, and that leads us to the amount of boxes needed. The fact that we need 1.036×10^{13} is due to the fact that these box's have a length of $L = 10^{-6}m$ or 1 micrometer.

The amount of fuel used seemed on the small side, but that very well maybe due to the rocket only weighing $1000kg$ and the fact that we didn't take account for our planet's gravitational pull.

V. CONCLUSION

What we have done is simulated a rocket launching off of a randomly generated planet. We "cut" the engine up into tiny boxes and calculated how many were needed to get the rocket to escape velocity within 20 minutes. We did not make the most accurate model, due to the fact that we didn't take consideration to the planet's gravitational pull, nor did we have the time or a computer powerful enough to run the program with more timesteps or particles.

REFERENCES

[1] Hansen, F. K., 2017, Forelesningsnotat 1A i kurset AST2000