AST3220

Kandidatnr: 3

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Task a

We will show that these two equations hold:

$$\frac{\mathrm{d}Y_n}{\mathrm{d}(\ln T)} = -\frac{1}{H}\left[Y_p\Gamma_{p\to n} - Y_n\Gamma_{n\to p}\right] \tag{1}$$

$$\frac{\mathrm{d}Y_p}{\mathrm{d}(\ln T)} = -\frac{1}{H}\left[Y_n\Gamma_{n\to p} - Y_p\Gamma_{p\to n}\right] \tag{2}$$

By using these two equations:

$$\frac{\mathrm{d}n_n}{\mathrm{d}t} + 3Hn_n = n_p\Gamma_{p\to n} - n_n\Gamma_{n\to p} \tag{3}$$

$$\frac{\mathrm{d}n_p}{\mathrm{d}t} + 3Hn_p = n_n\Gamma_{n\to p} - n_p\Gamma_{p\to n} \tag{4}$$

We will be proving both of the equations at once. First:

$$\frac{\mathrm{d}Y_n}{\mathrm{d}(\ln T)} = \left(\frac{\mathrm{d}Y_n}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}(\ln T)}\right)$$

Some important definitions are:

$$Y_n = \frac{n_n}{n_b} = \frac{n_n a^3}{n_{b0}}, \quad T = \frac{T_0}{a}, \quad H = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt}$$

We then take the derivative of Y_n and lnT with respect to t:

$$\frac{\mathrm{d}Y_n}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{n_n a^3}{n_{b0}}\right) = \frac{1}{n_{b0}}\left[\frac{\mathrm{d}n_n}{\mathrm{d}t}a^3 + \frac{\mathrm{d}a^3}{\mathrm{d}t}n_n\right] = \left[\frac{\partial n_n}{\partial t}\frac{a^3}{n_{b0}} + 3\frac{a^2}{n_{b0}}\frac{\partial a}{\partial t}n_n\right]$$

Using the defintion of Y_n, this is then equal to:

$$\left[\frac{\partial n_n}{\partial t}\frac{Y_n}{n_n} + 3\frac{Y_n}{n_n}Hn_n\right] = \frac{1}{n_b}\left[\frac{\partial n_n}{\partial t} + 3Hn_n\right] \tag{5}$$

Setting in equation 3, we get:

$$\frac{\mathrm{d}Y_n}{\mathrm{d}t} = \frac{1}{n_b}\left[n_p\Gamma_{p\to n} - n_n\Gamma_{n\to p}\right]$$

We then must solve for $\frac{dt}{dlnT}$, we have from the definition of T that:

$$ln(T) = ln(T_0) - ln(a)$$

The time derivative is then:

$$\frac{dlnT}{dt} = -\frac{1}{a}\dot{a} = -H \quad , \quad \frac{dt}{dlnT} = -\frac{1}{H}$$

With that we have:

$$\frac{\mathrm{d}Y_n}{\mathrm{d}(\ln T)} = \left(\frac{\mathrm{d}Y_n}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}(\ln T)}\right) = -\frac{1}{H}\left[\frac{1}{n_b}\left(n_p\Gamma_{p\to n} - n_n\Gamma_{n\to p}\right)\right]$$

$$= -\frac{1}{H}\left[Y_p\Gamma_{p\to n} - Y_n\Gamma_{n\to p}\right]$$

And with that we have proved equation 1 and 2 at the same time.

Task b

The conservation of entropy says that:

$$g^*(aT)^3 = constant$$

Where g^* is:

$$g^* = \sum_{i=bosons} g_i\left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i=fermions} g_i\left(\frac{T_i}{T}\right)^3$$

Where T is photon temperature, g_i is the degrees of freedom, and as of right now, all we're dealing with are photons(bosons), positrons, and electrons(fermions). All have 2 degrees of freedom, so for before electrons and positrons annihilate, we have :

$$g^*_{before} = 2 + \frac{7}{8}22 = \frac{11}{2}$$

And after:

$$g^*_{after} = 2$$

We then can set those equal to each other and get:

$$(aT)_{after} = \left(\frac{11}{4}\right)^{1/3}(aT)_{before}$$

Since the nuetrinos are thermally decoupled from the gas and have no temperature change (i.e):

$$(aT_\nu)_{before} = (aT_\nu)_{\text{after}}$$

The temperature of neutrinos today T_ν is lower than the cosmic photons T by this relation:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T$$

Task c

We wish to prove this equation:

$$\Omega_{r0} = \frac{8\pi^3}{45}\frac{G}{H_0^2}\frac{(k_B T_0)^4}{\hbar^3 c^5}\left[1 + N_{\text{eff}}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right]$$

To start we first define Ω_{r0}:

$$\Omega_{r0} = \frac{\rho_{r0}}{\rho_{c0}} \quad , \quad \rho_{r0} = \frac{\pi^2}{30}\frac{(k_b T)^4}{\hbar^3 c^5} g^* \quad , \quad \rho_{c0} = \frac{3H_0^2}{8\pi G}$$

Therefore:

$$\Omega_{r0} = \frac{8\pi^3 G}{90 H_0^2}\frac{(k_b T)^4}{\hbar c^5} g^*$$

Photons have 2 degrees of freedom, and neutrinos have 1. We have defined the number of different neutrino species as N_{eff}, but one must remember the antiparticles, so the sum of the fermions ends up being:

$$\frac{7}{8}\sum_{i=fermions} g_i \left(\frac{T_i}{T}\right)^3 = \frac{7}{8} 2N_{eff}\left(\frac{T_i}{T}\right)^3$$

We can then set that in:

$$g^* = 2[1 + N_{eff}\frac{7}{8}\left(\frac{T_i}{T}\right)^3]$$

And we have showed that $T_\nu = (\frac{4}{11})^{1/3} T$:

$$g^* = 2[1 + N_{eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}]$$

We can then set this into the equation for Ω_{r0}:

$$\Omega_{r0} = \frac{8\pi^3 G}{45 H_0^2}\frac{(k_b T)^4}{\hbar c^5}\left[1 + N_{eff}\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\right]$$

Task d

In order to find a function for $a(t)$ and then from that, $t(T)$, we start with the Friedman equation:

$$H = \frac{H_0}{a^2}\sqrt{\Omega_{r0}} \quad \rightarrow \quad \frac{\dot{a}}{a} = \frac{H_0}{a^2}\sqrt{\Omega_{r0}} \quad \rightarrow \quad da = \frac{H_0}{a}\sqrt{\Omega_{r0}}dt$$

We then integrate on both sides:

$$\int_0^{a(t)} da = \int_0^t \frac{H_0}{a}\sqrt{\Omega_{r0}}dt \quad \rightarrow \quad \frac{a(t)^2}{2} = H_0\sqrt{\Omega_{r0}}t$$

$$\rightarrow \quad a(t) = \sqrt{2H_0 t}(\Omega_r 0)^{1/4}$$

In order to find the expression for $t(T)$, we then set in our expression for $a(t)$ into $T = \frac{T_0}{a}$:

$$T = \frac{T_0}{\sqrt{2H_0 t}(\Omega_r 0)^{1/4}}$$

And solve for t:

$$\sqrt{t}T = \frac{T_0}{\sqrt{2H_0}(\Omega_r 0)^{1/4}} \quad \rightarrow \quad t = \left(\frac{T_0}{T\sqrt{2H_0}(\Omega_r 0)^{1/4}}\right)^2$$

$$\rightarrow \quad t = \left(\frac{T_0}{T}\right)^2 \frac{1}{2H_0}\frac{1}{\sqrt{\Omega_r 0}}$$

When setting in temperatures $T = 10^{10}K, 10^9K$, and 10^8K we get that the age of the universe was $1.78s, 177.74s(2.96min)$, and $17,773.69s(12.3days)$ respectfully.

Task e

Since we're assuming that all the baryonic mass is in neutrons and protons, it follows that:

$$Y_n + Y_p = 1$$

We are also using $m_p \approx m_n$ outside of exponents, and with that we have:

$$\frac{n_n}{n_p} = \frac{Y_n}{Y_p} = e^{-(m_n - m_p)c^2/k_B T}$$

Setting $(m_n - m_p)c^2/k_B T = X$ we can solve that for Y_p:

$$Y_p = \frac{Y_n}{e^{-X}} \quad \rightarrow \quad Y_n + \frac{Y_n}{e^{-X}} = 1 \quad \rightarrow \quad Y_n\left(1 + \frac{1}{e^{-X}}\right) = 1$$

$$\rightarrow \quad Y_n(t) = \frac{1}{\left(1+\frac{1}{e^{-X}}\right)} = \frac{1}{1+e^X} = \left[1+e^X\right]^{-1} = \left[1+e^(m_n - m_p)c^2/k_B T\right]^{-1}$$

And from that it follows that:

$$Y_p(t) = 1 - Y_n(t)$$

Task f

As one can see in Figure 1, as T approaches 10^8, the neutron abundance quickly drops off and protons always dominate.

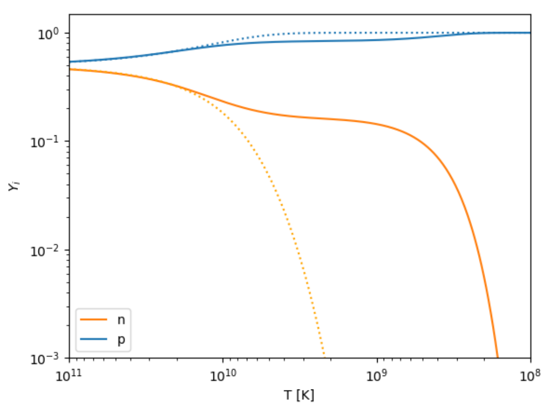

Figure 1: The dotted lines are the equilibrium curves.

Task g

We have the general Boltzmann equation:

$$\frac{\mathrm{d}n_i}{\mathrm{d}t} + 3Hn_i = \sum_{j\neq i}\left[n_j\Gamma_{j\rightarrow i} - n_i\Gamma_{i\rightarrow j}\right] + \sum_{jkl}\left[n_k n_l \gamma_{kl\rightarrow ij} - n_i n_j \gamma_{ij\rightarrow kl}\right]$$

Setting that into equation 5 from task a, we get:

$$\frac{\mathrm{d}Y_n}{\mathrm{d}(\ln T)} = -\frac{1}{H}\left\{\frac{1}{n_b}\left[\sum_{j\neq i}\left[n_j\Gamma_{j\rightarrow i} - n_i\Gamma_{i\rightarrow j}\right] + \sum_{jkl}\left[n_k n_l \gamma_{kl\rightarrow ij} - n_i n_j \gamma_{ij\rightarrow kl}\right]\right]\right\}$$

Where $Y_i = \frac{n_i}{n_b}$ and $\Gamma_{ij\to kl} = n_b\gamma_{ij\to kl}$, we multiply and divide the right term by n_b and get this:

$$= -\frac{1}{H}\left\{\sum_{j\neq i}\left[Y_j\Gamma_{j\to i} - Y_i\Gamma_{i\to j}\right] + \sum_{jkl}\left[Y_kY_l\Gamma_{kl\to ij} - Y_iY_j\Gamma_{ij\to kl}\right]\right\}$$

Task h

As Deuterium starts to be created shortly after $10^9 K$, the neutron abundance drops quicker. (See Figure 2)

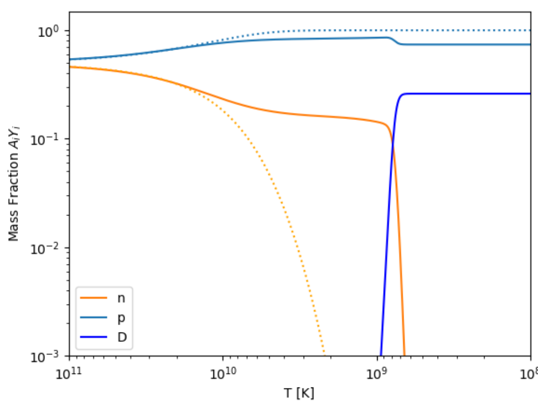

Figure 2:

Task i

We went most of the process without plotting the sum, but when we moved onto the next task, we realized shortly that our values were slightly wrong and only when we had very little time left did we plot the sum and see that lose some mass with the equations. We went through every equation at least 3 times and unfortunately could not completely debug it. (See Figure 3)

Task j

We simply attempted (in the best definition of the word) task j but had quite obvious problems. Our main problems were a, the slight difference in our values, and b, understanding how to fix the program using logarithms. (See Figure 4 & 5)

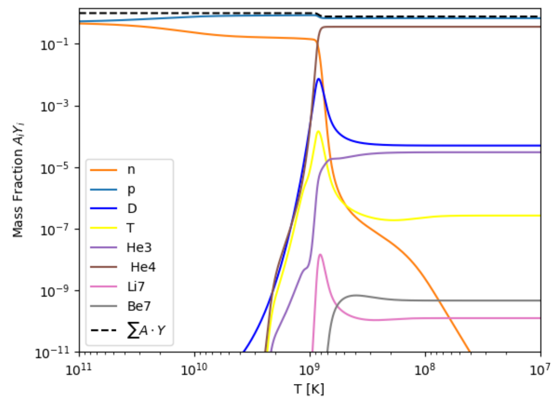

Figure 3: All of the abundances calculated using the 16 reactions.

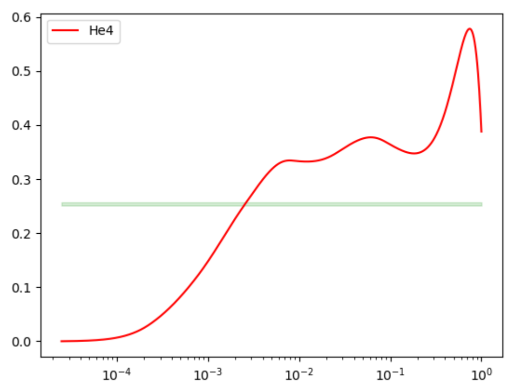

Figure 4:

Since we could not properly calculate an Ω_{b0} value, we will just use the value that was given in the assignment to make an inference. If the total matter content of the universe is around $\Omega_{m0} = 0.3$, and we assume $\Omega_{b0} = 0.05$, than that must mean that most of the matter in the universe isn't baryonic, but some other smaller particles, for example, neutrinos.

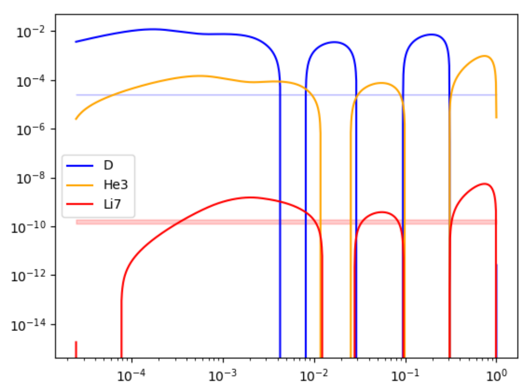

Figure 5: