AST3220 Project 3

Kandidatnr: 3

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Task a

H_i is the Hubble constant and therefore $\left[\frac{1}{s}\right]$, so τ must be:

$$[\tau] = H_i t = \left[\frac{1}{s}s\right] = 1$$

Then h is as such:

$$[h] = \frac{H}{H_i} = \left[\frac{\frac{1}{s^2}}{\frac{1}{s}}\right] = 1$$

Similarly, if ψ is dimensionless, then ϕ must be in Joules.

$$[\psi] = \frac{\phi}{E_p} = \left[\frac{J}{J}\right] = 1$$

Using Equation 4 from the Problem Set, we have:

$$V = \frac{3c^2}{8\pi G}H_i^2$$

$$[V] = \left[\frac{H_i^2 c^2}{G}\right] = \left[\frac{1}{s^2}\frac{m^2}{s^2}\frac{kg^2}{Nm^2}\right] = \left[\frac{kg^2}{s^4}\frac{1}{N}\right]$$

We then multiply and divide by m^2, set in SI units for N, and get:

$$= \left[\frac{kg^2}{s^4}\frac{s^2}{kgm}\frac{m^2}{m^2}\right] = \left[\frac{kgm^2}{s^2}\frac{1}{m^3}\right] = \left[\frac{J}{m^3}\right]$$

So V is an energy density, and with that information we have:

$$[\nu] = \frac{\hbar c^3}{H_i^2 E_p^2}V = \left[\frac{Js^3m^3}{s^3J^2}\frac{J}{m^3}\right] = 1$$

Task b

We will show that $h(\tau)$ can be expressed by:

$$h(\tau) = \frac{\partial}{\partial \tau}\left[ln\left(\frac{a}{a_i}\right)\right]$$

First, we remind that:

$$\tau = H_i t$$

And therefore:

$$\frac{\partial}{\partial \tau} = \frac{1}{H_i}\frac{\partial}{\partial t}$$

We then set that in and rearrange the logarithm:

$$h(\tau) = \frac{1}{H_i}\frac{\partial}{\partial t}\left[ln(a(t)) - ln(a_i)\right]$$

We then take the derivative using the chain rule, and remind that a_i is only a constant:

$$h(\tau) = \frac{1}{H_i}\frac{\dot{a}}{a} = \frac{H}{H_i}$$

Which is indeed how we defined h, and therefor the expression is correct. Next, we start with Equation 2 from the Problem set.

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + \hbar c^3 V'(\phi) = 0$$

We can start by dividing by H_i^2 (also a reminder that $\tau = H_i t$) :

$$\frac{d^2\phi}{d\tau^2} + 3\frac{H}{H_i}\frac{d\phi}{d\tau} + \frac{\hbar c^3 V'(\phi)}{H_i^2} = 0$$

Then divide by E_p (a reminder that $\psi = \frac{\phi}{E_p}$ and $h = \frac{H}{H_i}$):

$$\frac{d^2\psi}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{\hbar c^3}{H_i^2 E_p}\frac{dV}{d\phi} = 0$$

We then multiply and divide the last term with E_p:

$$\frac{d^2\psi}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{\hbar c^3}{H_i^2 E_p^2}\frac{dV}{d\psi} = 0$$

With $V = \frac{H_i^2 E_p^2}{\hbar c^3}\nu$, we have:

$$\frac{d\psi^2}{d\tau^2} + 3h\frac{d\psi}{d\tau} + \frac{d\nu}{d\psi} = 0 \tag{1}$$

Then starting from Equation 3 in the Problem set:

$$H^2 = \frac{8\pi G}{3c^2}\left[\frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi)\right]$$

We then divide by H_i^2:

$$h^2 = \frac{8\pi}{3}\left[\frac{G}{2\hbar H_i^2 c^5}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{G}{H_i^2 c^2}V(\phi)\right]$$

We set $G/\hbar c^5 = E_p^2$ and multiply that and H_i^2 into the derivative:

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{\partial\psi}{\partial\tau}\right)^2 + \frac{G}{H_i^2 c^2}V(\phi)\right]$$

We now need to "massage" the last term a bit, we have:

$$V(\phi) = \frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi^2 \quad , \quad v = \frac{\hbar c^3}{H_i^2 E_p^2}V$$

Set $V(\phi)$ into the expression for v and you get $v(\phi)$. After simplifying and setting in ψ^2 for ϕ^2/E_p^2, you then get:

$$v(\psi) = \frac{1}{2}\left(\frac{mc^2}{H_i\hbar}\right)^2\psi^2$$

If you multiply $G/H_i^2c^2$ into $V(\phi)$, you get our expression for $v(\psi)$:

$$\frac{G}{H_i^2c^2}V(\phi) = \frac{G}{H_i^2c^2}\frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi^2 = \frac{1}{2}\left(\frac{mc^2}{H_i\hbar}\right)^2\psi^2 = v(\psi)$$

We then end up with the correct expression for h:

$$h^2 = \frac{8\pi}{3}\left[\frac{1}{2}\left(\frac{d\psi}{d\tau}\right)^2 + \nu(\psi)\right] \tag{2}$$

Task c

The reason why the initial value for $d\psi/d\tau$ is irrelevant when using the slow-roll approximation is that the value must be, by definition, so small that we can assume it to be 0, and therefore irrelevant.

Task d

We have the given function for the potential and its derivative:

$$V = \frac{m^2c^4}{2(\hbar c)^3}\phi^2 \quad , \quad V' = \frac{m^2c^4}{(\hbar c)^3}\phi$$

The formula for the total number of e-folds in during inflation:

$$N(t) = \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i}\frac{V}{V'}d\phi$$

In order to find ϕ_{end} we set it into ϵ, one of the slow-roll conditions, and set it equal to 1:

$$\epsilon = \frac{E_p^2}{16\pi}\left(\frac{V'}{V}^2\right) = \frac{E_p^2}{16\pi}\frac{4}{\phi_{end}^2} = 1$$

$$\rightarrow \quad \phi_{end} = \sqrt{\frac{4E_p^2}{16\pi}} = \frac{E_p}{2}\frac{1}{\sqrt{\pi}}$$

We set $N(t) = 500$ and solve for ϕ_i:

$$N(t) = 500 = \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i}\frac{V}{V'}d\phi = \frac{8\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i}\frac{\phi}{2}d\phi = \frac{4\pi}{E_p^2}\int_{\phi_{end}}^{\phi_i}\phi d\phi$$

$$= \frac{4\pi}{E_p^2}\left[\frac{1}{2}\phi_i^2 - \frac{1}{2}\phi_{end}^2\right] = \frac{2\pi}{E_p^2}\left[\phi_i^2 - \phi_{end}^2\right]$$

$$\rightarrow \quad \frac{E_p^2}{2\pi}500 = \phi_i^2 - \frac{E_p^2}{4\pi}$$

$$\rightarrow \quad \phi_i^2 = 500\frac{E_p^2}{2\pi} + \frac{E_p^2}{4\pi} = \frac{E_p^2}{2\pi}\left(500 + \frac{1}{2}\right)$$

$$\rightarrow \quad \phi_i = \sqrt{\frac{E_p^2}{4\pi}1001} = \frac{E_p}{2}\sqrt{\frac{1001}{\pi}} \quad , \quad \psi_i = \frac{1}{2}\sqrt{\frac{1001}{\pi}}$$

Task e & f

In order to solve the differential equation, we need a usable expression for h, which means that we need a functional expression for $v(psi)$, which we don't have as of now due to us not having a mass or a value for H_i to use in calculations. Hence, we need to find an expression for H_i as well.

$$V(\phi_i) = \frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\phi_i^2 \quad \rightarrow \quad V(\psi_i) = \frac{1}{2}\frac{m^2c^4}{(\hbar c)^3}\psi_i^2 E_p^2$$

ψ_i can be expressed using ϕ_i, which we found in task d. We then set $V(\psi_i)$ into the given expression for H_i:

$$H^2 = \frac{8\pi G}{3c^2}V(\psi_i) = \frac{1}{2}\frac{8\pi G}{3c^2}\frac{m^2c^4}{(\hbar c)^3}\psi_i^2 E_p^2$$

Which can be simplified as such:

$$H^2 = \frac{4\pi}{3}\frac{m^2c^4}{\hbar^2}\psi_i^2$$

We can now set that into our expression for $v(\psi)$ from Task b:

$$v(\psi) = \frac{1}{2}\left(\frac{mc^2}{H_i\hbar}\right)^2\psi^2 = \frac{1}{2}\left(\frac{mc^2}{\hbar}\right)^2\frac{3\hbar^2}{4\pi m^2c^4}\frac{\psi^2}{\psi_i^2}$$

Which, again, can be simplified to:

$$v(\psi) = \frac{3}{8\pi}\frac{\psi}{\psi_i^2}$$

We can then use this in our expression for h and take the derivative to use in the differential equation. We then use scipy's *odeint*() and split the second-order differential equation into two first-order differential equations. We also solve for $ln(a/a_i)$ using equation 9 from the Problem set. *odeint*() takes initial values for every variable.

One would expect ψ to at first decrease at a constant rate, since the inflation is happening so slowly, we assume it to be linear, which is practically the definition of SRA. Once ψ hits 0 (the bottom of the potential well), it should then begin to oscillate (roll up and down the sides of the well).

The analytical slow-roll solution is given as:

$$\psi = \psi_i - \frac{mc^2}{\hbar\sqrt{12\pi}}\frac{\tau}{H_i}$$

$$H_i = \sqrt{\frac{4\pi}{3}}\frac{mc^2}{\hbar}\psi_i$$

If we set H_i into ψ, we get:

$$\psi = \psi_i - \frac{mc^2}{\hbar\sqrt{12\pi}}\tau\sqrt{\frac{3}{4\pi}\frac{\hbar}{mc^2}\frac{1}{\psi_i}} = \psi_i - \frac{\tau}{\psi_i}\sqrt{\frac{3}{48\pi}} = \psi_i - \frac{\tau}{\psi_i}\frac{1}{4\pi}$$

We can then use the ψ_i that we calculated in Task d.

As shown in Figure 1, the slow-roll solution deviates significantly at zero. The numerical solution acts as one would expect, ψ begins to oscillate once it hits the bottom of the potential well, while the analytical slow-roll solution continues linearly and stops having any real meaning. The number of e-folds (N), expressed as $ln(a/a_i)$, can be seen in Figure 2.

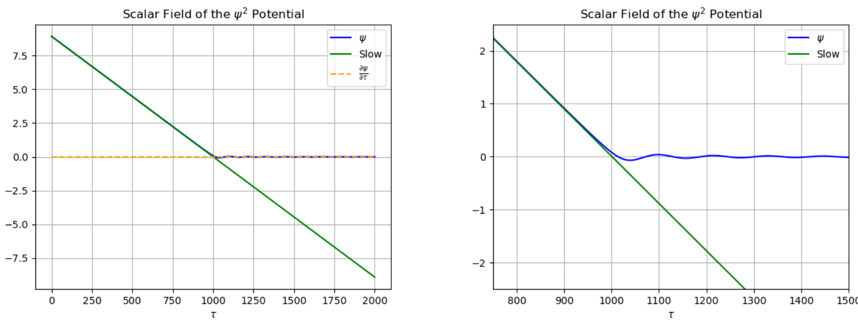

Figure 1: The figure on the left shows the entire numerical solution and the analytical solution, while the right figure is zoomed in to get a better idea of what is happening when ψ hits 0.

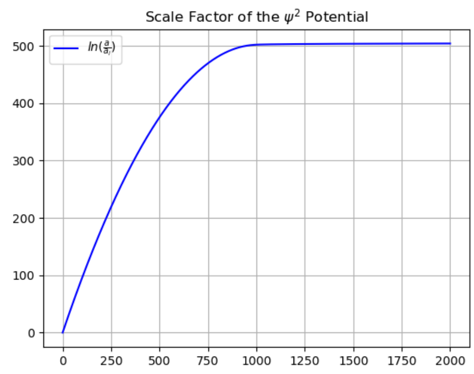

Figure 2: The number of e-folds at a given time τ.

Task g

Numerically we found the total number of e-folds during inflation to be 501.42, compared to the 500 we used in SRA, meaning the numerical answer is very similar, but as always, more accurate. Looking at Figure 3, one can quickly see that ϵ exponentially grows until after ϵ becomes greater than 1.

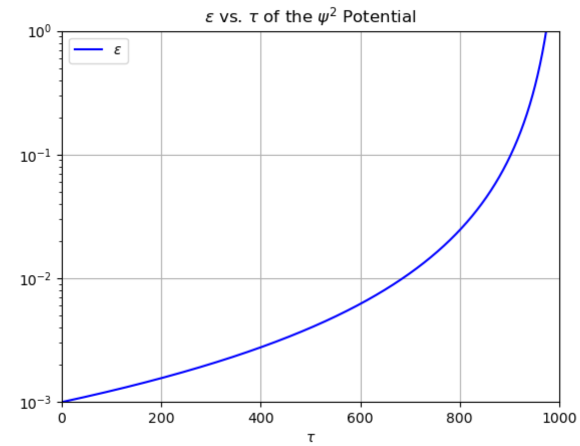

Figure 3: Slow roll condition ϵ over time.

Task h

We have:

$$p_\phi = \frac{1}{2\hbar c^3}\dot{\phi}^2 - V(\phi) \quad , \quad \rho_\phi c^2 = \frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi)$$

$$\rightarrow \quad \frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2\hbar c^3}\dot{\phi}^2 - V(\phi)}{\frac{1}{2\hbar c^3}\dot{\phi}^2 + V(\phi)}$$

Reminder that $V = \frac{H_i^2 E_p^2}{\hbar c^3} v$:

$$\frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{1}{2\hbar c^3}\dot{\phi}^2 - \frac{H_i^2 E_p^2}{\hbar c^3} v}{\frac{1}{2\hbar c^3}\dot{\phi}^2 + \frac{H_i^2 E_p^2}{\hbar c^3} v} = \frac{\frac{H_i^2 E_p^2}{2\hbar c^3}\left(\frac{\partial \psi}{\partial \tau}\right)^2 - \frac{H_i^2 E_p^2}{\hbar c^3} v}{\frac{H_i^2 E_p^2}{2\hbar c^3}\left(\frac{\partial \psi}{\partial \tau}\right)^2 + \frac{H_i^2 E_p^2}{\hbar c^3} v}$$

$$\rightarrow \quad \frac{p_\phi}{\rho_\phi c^2} = \frac{\frac{H_i^2 E_p^2}{\hbar c^3}\left[\frac{1}{2}\left(\frac{\partial \psi}{\partial \tau}\right)^2 - v\right]}{\frac{H_i^2 E_p^2}{\hbar c^3}\left[\frac{1}{2}\left(\frac{\partial \psi}{\partial \tau}\right)^2 + v\right]} = \frac{\frac{1}{2}\left(\frac{\partial \psi}{\partial \tau}\right)^2 - v}{\frac{1}{2}\left(\frac{\partial \psi}{\partial \tau}\right)^2 + v}$$

Task i

In the slow-roll regime, the change in ψ is so small that we assume it to be 0, and so the equation of state (EoS) will be $-v/v$ or -1. Once we come out of the slow roll regime and begin to oscillate the EoS should oscillate between -1 and 1 due to the v's holding it in that range. From a non-mathematical perspective, if we are talking about inflation, it also would make sense that the EoS would represent dark energy with $\omega = -1$, and once the inflation period is over, start to oscillate (See Figure 4).

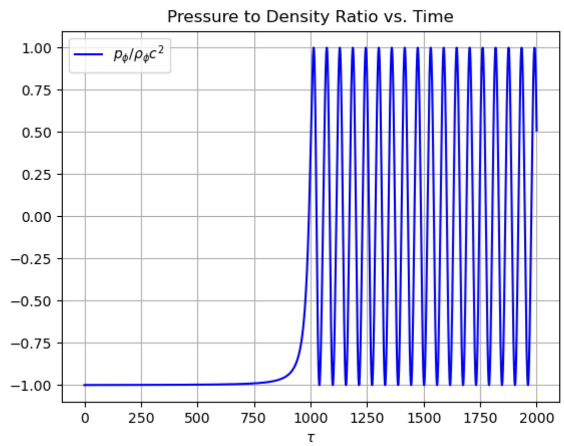

Figure 4: Equation of state of the model universe over time.

Task j

One can see that when inflation is about to end, the SRA conditions both rise exponentially and drastically. They are exactly the same when using the ψ^2 potential. (See Figure 5)

Task k

(See Figure 6)

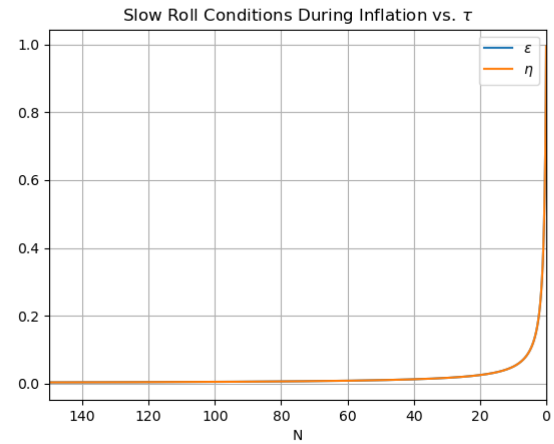

Figure 5:

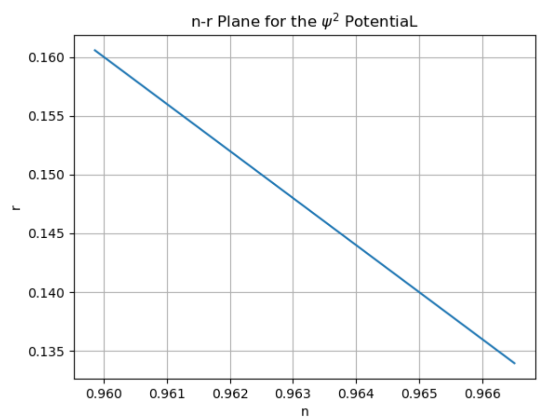

Figure 6:

Task 1

We have our potential:

$$V(\phi) = \frac{3M^2M_p^2}{4}\left(1 - e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_p}}\right)^2$$

We then, for simplicity's sake, use a temporary variable $X = \sqrt{\frac{2}{3}}\frac{1}{M_p}$, and derive the potential:

$$V'(\phi) = \frac{3M^2M_p^2}{4}2\left(1 - e^{-X\phi}\right)Xe^{-X\phi} = \frac{3M^2M_p}{2}\sqrt{\frac{2}{3}}\left(1 - e^{-X\phi}\right)e^{-X\phi}$$

And the second derivative:

$$V''(\phi) = \frac{3M^2M_p^2}{2}\left[\left(1 - e^{-X\phi}\right)\left(-X^2e^{-X\phi}\right) + \left(Xe^{-X\phi}\right)\left(Xe^{-X\phi}\right)\right]$$

$$= \frac{3M^2M_p^2}{2}\left[\left(-X^2e^{-X\phi} + X^2e^{-2X\phi}\right) + X^2e^{-2X\phi}\right] = \frac{3M^2M_p^2}{2}X^2\left(2e^{-2X\phi} - e^{-X\phi}\right)$$

$$\rightarrow \quad V''(\phi) = \frac{3M^2M_p^2}{2}X^2\left(2e^{-2X\phi} - e^{-X\phi}\right) = M^2\left(2e^{-2X\phi} - e^{-X\phi}\right)$$

Now, starting from the equation for the slow-roll condition ϵ (we will now use $y = -X\phi = -\sqrt{\frac{2}{3}}\frac{\phi}{M_p}$):

$$\epsilon = \frac{E_p^2}{16\pi}\left(\frac{V'}{V}\right)^2 = \frac{E_p^2}{16\pi}\left[\frac{\frac{3M^2M_p}{4}\sqrt{\frac{2}{3}}\left(1 - e^y\right)e^y}{\frac{3M^2M_p^2}{4}\left(1 - e^y\right)^2}\right]^2 = \frac{E_p^2}{16\pi}\left[\frac{2}{M_p}\sqrt{\frac{2}{3}}\frac{e^y}{(1 - e^y)}\right]^2$$

$$= \frac{E_p^2}{16\pi}\frac{4}{M_p^2}\frac{2}{3}\frac{e^{2y}}{(1 - e^y)^2} = \frac{E_p^2}{M_p^2}\frac{1}{6\pi}\frac{e^{2y}}{(1 - e^y)^2}$$

Using $E_p^2/M_p^2 = 8\pi$, we get:

$$\epsilon = \frac{4}{3}\frac{e^{2y}}{(1 - e^y)^2}$$

Now for η:

$$\eta = \frac{E_p^2}{8\pi}\frac{V''}{V} = \frac{E_p^2}{8\pi}\left[\frac{M^2\left(2e^{-2X\phi} - e^{-X\phi}\right)}{\frac{3M^2M_p^2}{4}\left(1 - e^{-X\phi}\right)^2}\right]$$

$$= \frac{E_p^2}{8\pi}\left[\frac{M^2\left(2e^{2y}-e^y\right)}{\frac{3M^2M_p^2}{4}\left(1-e^y\right)^2}\right] = \frac{E_p^2}{8\pi}\frac{4}{3}\frac{1}{M_p^2}\frac{\left(2e^{2y}-e^y\right)}{\left(1-e^y\right)^2}$$

$$\rightarrow \quad \eta = \frac{4}{3}\frac{\left(2e^{2y}-e^y\right)}{\left(1-e^y\right)^2}$$

Task m

Again, in order to solve the differential equations for the Starobinsky model, we must first solve analytically for h, and by extension, $v(\psi)$. First, we have our potential:

$$V(\phi) = \frac{3M^2M_p^2}{4}\left(1-e^{-\sqrt{\frac{2}{3}}\frac{\phi}{M_p}}\right)^2$$

We then convert ϕ to ψ using the relation $\phi = E_p\psi$, and also simplify using the relation $M_p = \frac{E_p}{\sqrt{8\pi}}$:

$$V(\psi) = \frac{3M^2M_p^2}{4}\left(1-e^{-\sqrt{\frac{16\pi}{3}}\psi}\right)^2$$

We then need an expression for H_i in order to set it into $v(\psi)$:

$$H_i^2 = \frac{8\pi G}{3c^2}V(\psi) = \frac{8\pi G}{3c^2}\left[\frac{3M^2M_p^2}{4}\left(1-e^{-\sqrt{\frac{16\pi}{3}}\psi_i}\right)^2\right]$$

We were given $\psi_i = 2$, so H_i can further be expressed as:

$$H_i^2 = \frac{8\pi G}{3c^2}\left[\frac{3M^2M_p^2}{4}\left(1-e^{-2\sqrt{\frac{16\pi}{3}}}\right)^2\right]$$

And with that, $v(\psi)$ is then:

$$v(\psi) = \frac{\hbar c^3}{H_i^2E_p^2}V(\psi) = \frac{\hbar c^3}{E_p^2}\left\{\frac{\frac{3M^2M_p^2}{4}\left(1-e^{-\sqrt{\frac{16\pi}{3}}\psi}\right)^2}{\frac{8\pi G}{3c^2}\left[\frac{3M^2M_p^2}{4}\left(1-e^{-2\sqrt{\frac{16\pi}{3}}}\right)^2\right]}\right\}$$

$$= \frac{\hbar c^3}{E_p^2}\frac{3c^2}{8\pi G}\left[\frac{\left(1-e^{-\sqrt{\frac{16\pi}{3}}\psi}\right)^2}{\left(1-e^{-2\sqrt{\frac{16\pi}{3}}\psi}\right)^2}\right] = \frac{3}{8\pi}\frac{\left(1-e^{-\sqrt{\frac{16\pi}{3}}\psi}\right)^2}{\left(1-e^{-2\sqrt{\frac{16\pi}{3}}}\right)^2}$$

$$\rightarrow \quad v(\psi) = \frac{3}{8\pi}\frac{\left(1-e^{-\sqrt{\frac{16\pi}{3}}\psi}\right)^2}{\left(1-e^{-2\sqrt{\frac{16\pi}{3}}}\right)^2}$$

We then need to derive $v(\psi)$ to set it into the second-order differential equation. For further simplification we use the term $x = -\sqrt{\frac{16\pi}{3}}$:

$$v(\psi) = \frac{3}{8\pi} \frac{1}{\left(1 - e^{2x}\right)^2} \left(1 - e^{x\psi}\right)^2$$

$$\rightarrow \quad \frac{\partial v}{\partial \psi} = \frac{3}{8\pi \left(1 - e^{2x}\right)^2} 2 \left(1 - e^{x\psi}\right) \left(-xe^{\psi}\right) = \frac{3}{4\pi \left(1 - e^{2x}\right)^2} \left(1 - e^{x\psi}\right) \left(-xe^{x\psi}\right)$$

The numerical solution for the Starobinsky model can be seen in Figure 7. N can be seen in Figure 8.

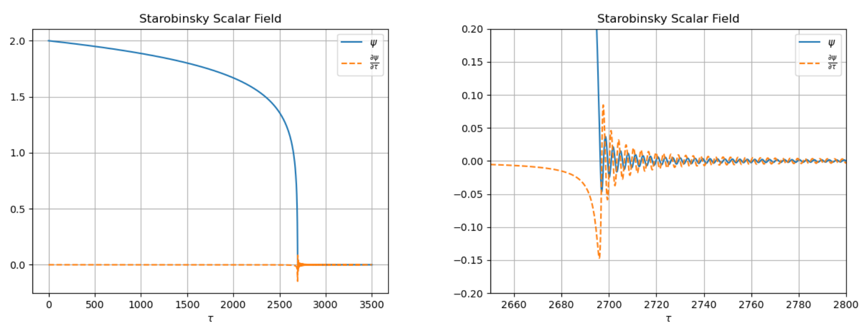

Figure 7:

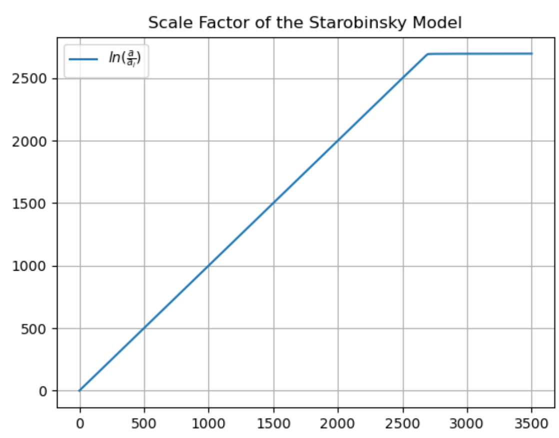

Figure 8:

Task n

The slow roll conditions for the Starobinsky model can be seen on the left in Figure 9. ϵ starts to even out at 1 as we approach 0, which makes sense, since 1 by definition, is the maximum value it can have. The n-r Plane of the Starobinsky model can be seen on the right in Figure 9.

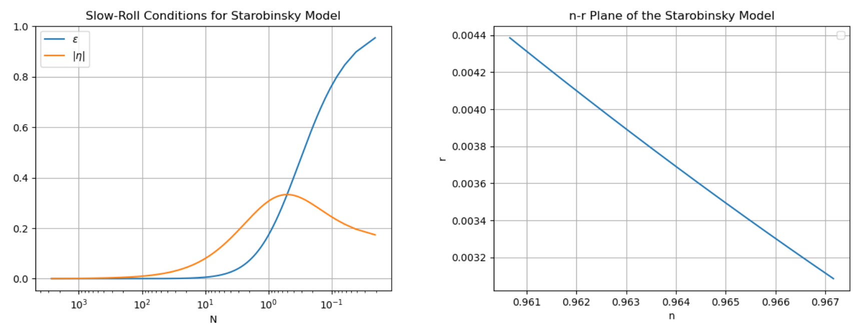

Figure 9:

Task p

The most recent n and r values measured using the Planck satallite are $n = 0.9649 \pm 0.0042$ and $r < 0.056$ [1]. As you can see in Figure 10, the ψ^2 model is inside the range of measured n-values, while the r-values are far too great, making this model not viable in our observable universe. On the other hand, the Starobinsky model falls very nice into the ranges of both the measured n and r-values, meaning it fits to our current universe, and is a viable model (See Figure 10).

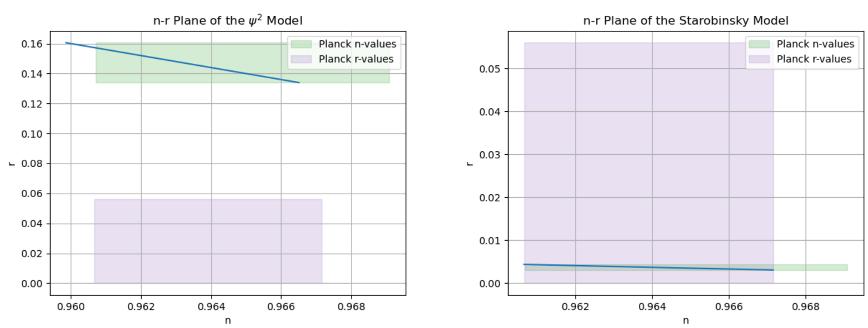

Figure 10:

References

[1] Akrami, Y. “Planck 2018 Results - X. Constraints on Inflation — Astronomy ...” AstronomyAstrophysics, 11 Sept. 2020, www.aanda.org/articles/aa/full$_html/2020/09/aa33887-18/aa33887-18.html$.