Simulating the Solar Core

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1 Introduction

In the core of every star, there is fusion taking place. At certain temperatures or energy levels, two particles can get excited enough to the point where they could fuse, if it wasn't for the weak force, or the Coulomb force, that repels atoms from each other. What we are missing is quantum tunneling, where particles at certain energy levels can simply disappear and reappear somewhere else. This is how fusion takes place. Particles get excited enough to tunnel through the Coulomb barrier, and then the strong force, the binding force, takes place and fuses the nucleons. As a side note, which we will get into further, a very small percent of the mass is released in the form of neutrinos which escape the star almost immediately, unlike everything else which is stuck inside the star for a very long time due to the high density and no free path.

Most of the fusion processes that take place in the core of every star take place within the Proton-Proton chain (PP-chain) and the most dominant sub-branch of the CNO cycle. These reactions are what we will focus on and simulate to calculate the amount of energy the sun produces (See Figure 1).

There are a couple of assumptions and simplifications that we take. Some of the reactions take much longer to occur than others. For example, the proton-proton reaction takes much longer than the proton-deuterium reaction, roughly an order of ten times longer to be specific. Therefore, when calculating the reaction rate for PP0, we can safely only use r_pp. The same is also true for r_p14 in the CNO-cycle, that reaction takes so much longer than all of the other reactions in the cycle, and we can therefore use that as a limiter, and only use that reaction rate to calculate the energy production from the CNO-cycle.

2 Method

We wished to calculate the total energy being produced in a star ϵ $[J/kgs]$, which is dependent on the temperature T $[K]$ and the mass density ρ $[kg/m^3]$. To do so, we need to know what reactions are taking place, and for each reaction, we must know how often they occur (their reaction rate r_{ik} $[1/kgs]$), and how much energy is released Q_{ik} $[J]$.

Nuclear reactions of the PP chain.

Branch	Reaction	Q' [MeV]	Q_ν [MeV]	Rate symbol
all	${}^1_1\mathrm{H} + {}^1_1\mathrm{H} \rightarrow {}^2_1\mathrm{D} + \mathrm{e}^+ + \nu_\mathrm{e}$	1.177	0.265	λ_{pp}
	${}^2_1\mathrm{D} + {}^1_1\mathrm{H} \rightarrow {}^3_2\mathrm{He} + \gamma$	5.494		λ_{pd}
I	${}^3_2\mathrm{He} + {}^3_2\mathrm{He} \rightarrow {}^4_2\mathrm{He} + 2\,{}^1_1\mathrm{H}$	12.860		λ_{33}
II & III	${}^3_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^7_4\mathrm{Be} + \gamma$	1.586		λ_{34}
II	${}^7_4\mathrm{Be} + \mathrm{e}^- \rightarrow {}^7_3\mathrm{Li} + \nu_\mathrm{e} + \gamma$	0.049	0.815	λ_{e7}
	${}^7_3\mathrm{Li} + {}^1_1\mathrm{H} \rightarrow 2\,{}^4_2\mathrm{He}$	17.346		λ'_{17}
III	${}^7_4\mathrm{Be} + {}^1_1\mathrm{H} \rightarrow {}^8_5\mathrm{B} + \gamma$	0.137		λ_{17}
	${}^8_5\mathrm{B} \rightarrow {}^8_4\mathrm{Be} + \mathrm{e}^+ + \nu_\mathrm{e}$	8.367	6.711	λ_8
	${}^8_4\mathrm{Be} \rightarrow 2\,{}^4_2\mathrm{He}$	2.995		λ'_8

Nuclear reactions of the CNO cycle.

Reaction	Q' [MeV]	Q_ν [MeV]	Rate symbol
${}^{12}_6\mathrm{C} + {}^1_1\mathrm{H} \rightarrow {}^{13}_7\mathrm{N} + \gamma$	1.944		λ_{p12}
${}^{13}_7\mathrm{N} \rightarrow {}^{13}_6\mathrm{C} + \mathrm{e}^+ + \nu_e$	1.513	0.707	λ_{13}
${}^{13}_6\mathrm{C} + {}^1_1\mathrm{H} \rightarrow {}^{14}_7\mathrm{N} + \gamma$	7.551		λ_{p13}
${}^{14}_7\mathrm{N} + {}^1_1\mathrm{H} \rightarrow {}^{15}_8\mathrm{O} + \gamma$	7.297		λ_{p14}
${}^{15}_8\mathrm{O} \rightarrow {}^{15}_7\mathrm{N} + \mathrm{e}^+ + \nu_\mathrm{e}$	1.757	0.997	λ_{15}
${}^{15}_7\mathrm{N} + {}^1_1\mathrm{H} \rightarrow {}^{12}_6\mathrm{C} + {}^4_2\mathrm{He}$	4.966		λ_{p15}

Figure 1: The first branch in the PP-chain starts with two H^1 atoms and takes place before all other branches, we will refer to it as PP0. In the CNO cycle, the original C^{12} is likely from an earlier generation of stars, in other words, the star must have had it when it first started fusing. [1]

$$\epsilon = \sum Q'_{ik} \cdot r_{ik}$$

To calculate the reaction rate of each process, we use the appropriate formula:

$$r_{ik} = \frac{n_i n_k}{\rho\,(1 + \delta_{ik})} \lambda_{ik}$$

Where n is the number densities of the two elements fusing in the reaction, δ_{ik} is the Kronecker delta, which is either 1 when the two elements are the same, or 0 when they differ. Lastly, the proportionality function λ_{ik} $[m^3/s]$ differs based on the reduced mass of the elements in the reaction.

To calculate the number densities, we were given the general mass distribution of the major elements in the core (See Figure 2). We then used the proper formula:

$$n_i = \frac{\rho Z_i}{m_i}$$

Where Z_i is the mass distribution, and m_i, is the mass in kg. We were then given a list of proportionality functions λ using this formula:

$$
\begin{aligned}
X &= 0.7 \\
Y_{^{3}_{2}\mathrm{He}} &= 10^{-10} \\
Y &= 0.29 \\
Z_{^{7}_{3}\mathrm{Li}} &= 10^{-7} \\
Z_{^{7}_{4}\mathrm{Be}} &= 10^{-7} \\
Z_{^{14}_{7}\mathrm{N}} &= 10^{-11}
\end{aligned}
$$

Figure 2: X represents H^1 and Y represents He^4 [1]

$$\lambda_{ik} = \sqrt{\frac{8}{m\pi \left(k_{\mathrm{B}}T\right)^3}} \int_0^\infty \exp\left(-\frac{E}{k_{\mathrm{B}}T}\right) E\sigma(E)\mathrm{d}E$$

After we had calculated all of the reaction rates, we then had to make sure that more elements were being consumed than being produced, since that is impossible. We did this by using a normalization factor, which was the reaction rate of the production divided by the reactions consuming it. We did this for He^3, Be^7, and Li^7. Here is the He^3 production/consumption normalization factor for example:

$$norm_r = \frac{r_{pp}}{r_{33} \cdot r_{34}}$$

To calculate Q'_{ik}, one must subtract the mass of the inputs of the reaction from the outputs, and using $E = mc^2$, multiply the mass difference by c^2. One must also take into account any neutrino loss as well. This is how one calculates Q'_{ik}, however, we did not, we were given a list of Q'_{ik} values in Figure 1 that we simply just plugged into our program. We then used everything to calculate the relative energy production of each PP-chain and the CNO-cycle dependent on temperature.

The following formula represents how many reactions at a given energy are due to quantum tunneling.

$$\sigma(E) = E^{-1}S(E)\exp\left(-\sqrt{\frac{m}{2E}}\frac{Z_i Z_k e^2 \pi}{\epsilon_0 h}\right)$$

We are only interested in the exponential function, which comes from the electrostatic barrier between two particles. Z is the atomic number of the two particles, m is the reduced mass, e is an elementary charge, ϵ_0 is the vacuum permittivity, and h is the Planck's constant. When we multiply this with the exponential in λ_{ik}, which represents the Maxwellian energy distribution of the particles, we get a function that represents which certain energy levels the reactions are most likely to happen at. The peak in the function is referred to as the Gamow Peak.

3 Results

We then calculated the relative energy production for each of the PP branches and CNO cycle as functions of temperature (See Figure 3).

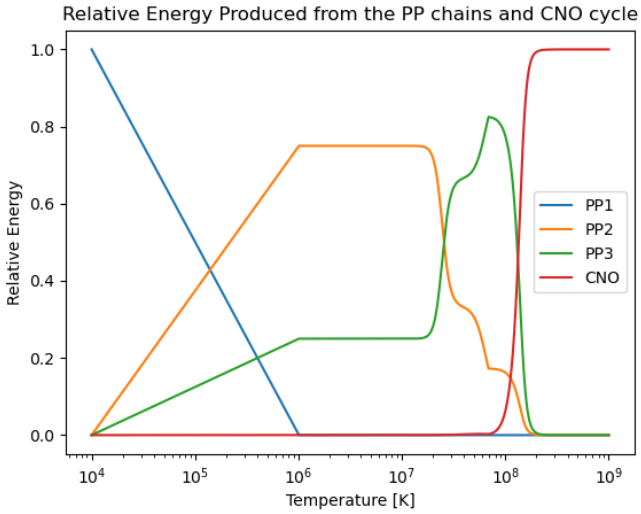

Figure 3: Calculated using a Python3 script.

Further, we used the two exponentials mentioned earlier to calculate the Gamow peaks (See Figure 4).

4 Discussion

We had a lot of confusion when it comes to how we should normalize our relative energy production and our Gamow peaks. We noticed that the PP-chains and CNO-cycle switch dominance in order. This makes sense since the elements are getting heavier with each chain, and therefore should be producing more energy at higher temperatures. The temperature of the solar core is around 10^7, so the processes that are taking place there should be predominately PP2 and PP3. We cannot explain the linear nature of the first half of the plot, nor can we explain the strange curves happening with PP2 and PP3 around 10^9. We can only assume they are due to a mistake in our code somewhere, likely in how we normalized the energy production.

When it comes to the calculated gamma peaks, the first thing we noticed was that the $e7$-reaction peak is at the lowest energy level, which would be incredibly curious if it was true, meaning that it takes place more commonly than the *pp*-reaction. We believe this to be incorrect due to how we calculate that specific reaction. We use $Z = 1$ for the electron, simply because we couldn't

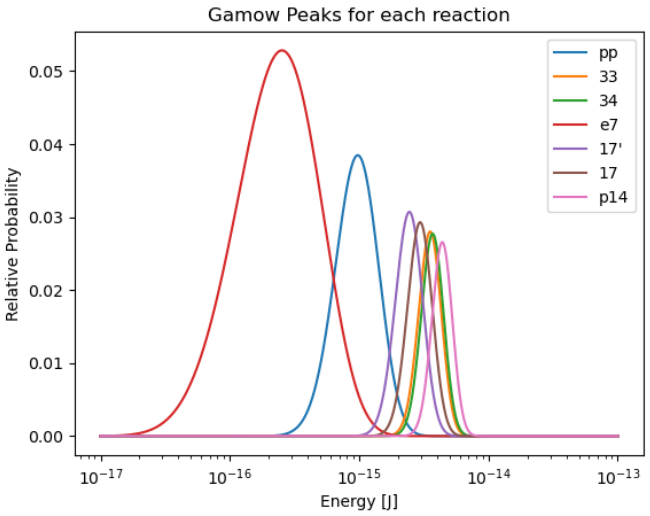

Figure 4: Descending peaks as energy increases

find anything on the atomic number of an electron, which by definition, doesn't exactly make sense. Other than that with $e7$, we noticed that the higher the energy level the peak is laying at, the larger the standard deviation, meaning that the reactions are decreasingly less likely to happen over a decreasingly extensive range of energy levels. It does make sense that those two would change proportionally with each other, the maximum and the standard deviation that is, but whether or not the maximums should increase or decrease is the most important question.

5 Conclusion

We have explained how fusion takes place and introduced the majority of the fusion processes that take place in the core of every star. Making some assumptions and simplifications, we calculated the reaction rates of each reaction and subsequently calculated the relative energy production dependent on temperature. We also found the Gamow peaks of the chains and explained why we didn't quite believe what we found to be correct. These errors were likely due to a bug in our code.

References

[1] Gudiksen, B.V. "AST3310: Astrophysical Plasma and Stellar Interiors." Institute of Theoretical Astrophysics, University of Oslo, 17 Nov. 2022.