Test — Mathematics for Data Science

Courses 1 and 2: Vector spaces, linear applications, matrices

Exercise (A): Short proofs

1. Let $A, B \in \mathbb{R}^{n\times n}$ and let V be a real vector space. Prove that $(AB)^\top = B^\top A^\top$.

2. Justify that $\operatorname{rank}(A) = \operatorname{rank}(A^\top)$.

3. Let $A, B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times p}$, $Q \in \mathbb{R}^{n\times n}$, and $\alpha \in \mathbb{R}$. Prove each of the following properties of the transpose. Each item should be proved in a few lines.

   (a) **Linearity.** $(A+B)^\top = A^\top + B^\top$ and $(\alpha A)^\top = \alpha A^\top$.

   (b) **Product rule.** $(AC)^\top = C^\top A^\top$.

   (c) **Involution.** $(A^\top)^\top = A$.

   (d) **Invertibility and transpose.** If A is square and invertible, then $(A^{-1})^\top = (A^\top)^{-1}$.

Exercise (B): Short computations

Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 3}.$$

1. Compute $\operatorname{rank}(A)$. Give a basis of $\operatorname{range}(A)$ and a basis of $\ker(A)$.

2. Solve $Ax = b$ and describe all solutions.

3. Consider the linear map $x \mapsto Cx : \mathbb{R}^3 \to \mathbb{R}^2$. Is it injective? surjective? Compute $\operatorname{rank}(C)$, $\dim\ker(C)$, and give a basis of $\ker(C)$.

4. Let $U = \{(x, y, z) \in \mathbb{R}^3 : \; x - 2y + z = 0\}$. Prove that U is a subspace. Find $\dim U$ and a basis of U.

Exercise (C): True/False

Work over $\mathbb{R}$. Justify briefly whether the following assertions are True or False.

1. If $v_1, v_2, v_3 \in \mathbb{R}^3$ satisfy that any two of vectors are linearly independent, then $\{v_1, v_2, v_3\}$ is linearly independent.

2. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear and $\dim\ker(T) = 1$, then $\dim\operatorname{range}(T) = 2$.

3. If $Q \in \mathbb{R}^{n\times n}$ satisfies $Q^\top Q = I_n$, then the columns of Q form an orthonormal basis of $\mathbb{R}^n$.

Answer elements — Mathematics for Data Science

Exercise (A): Short proofs

1. Entrywise, $\big((AB)^\top\big)_{ij} = (AB)_{ji} = \sum_k a_{jk}b_{ki} = \sum_k b_{ki}a_{jk} = (B^\top A^\top)_{ij}$, hence $(AB)^\top = B^\top A^\top$.

2. $\operatorname{rank}(A)$ equals the dimension of the row space of A, which is the column space of $A^\top$. Thus $\operatorname{rank}(A) = \operatorname{rank}(A^\top)$.

3. Let $A, B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times p}$, and $\alpha \in \mathbb{R}$. We write M_{ij} for the (i,j)-entry of a matrix M.

   (a) **Linearity.** For all i, j,

   $$\big((A+B)^\top\big)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A^\top)_{ij} + (B^\top)_{ij} = \big(A^\top + B^\top\big)_{ij},$$

   hence $(A+B)^\top = A^\top + B^\top$. Likewise,

   $$\big((\alpha A)^\top\big)_{ij} = (\alpha A)_{ji} = \alpha A_{ji} = \alpha (A^\top)_{ij} = \big(\alpha A^\top\big)_{ij},$$

   hence $(\alpha A)^\top = \alpha A^\top$.

   (b) **Product rule.** For $A \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{n\times p}$, the product $AC \in \mathbb{R}^{m\times p}$ has entries

   $$(AC)_{ji} = \sum_{k=1}^{n} A_{jk}\, C_{ki}.$$

   Therefore, for all i, j,

   $$\big((AC)^\top\big)_{ij} = (AC)_{ji} = \sum_{k=1}^{n} A_{jk}\, C_{ki} = \sum_{k=1}^{n} (C^\top)_{ik}\, (A^\top)_{kj} = \big(C^\top A^\top\big)_{ij}.$$

   Hence $(AC)^\top = C^\top A^\top$.

   (c) **Involution.** For all i, j,

   $$\big((A^\top)^\top\big)_{ij} = (A^\top)_{ji} = A_{ij},$$

   hence $(A^\top)^\top = A$.

   (d) **Invertibility and transpose.** Let $A \in \mathbb{R}^{n\times n}$ be invertible. Then

   $$I_n = (AA^{-1})^\top = (A^{-1})^\top A^\top \qquad \text{and} \qquad I_n = (A^{-1}A)^\top = A^\top (A^{-1})^\top.$$

   Thus $(A^{-1})^\top$ is both a left and a right inverse of $A^\top$, so $A^\top$ is invertible and

   $$(A^\top)^{-1} = (A^{-1})^\top.$$

Exercise (B): Short computations

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 3 & 2 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

1. *Rank, bases of range and kernel of A.* Row-reduction gives

$$\begin{bmatrix}1 & 1 & 0\\ 2 & 1 & -1\\ 3 & 2 & -1\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1 & 0 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}.$$

   Hence $\operatorname{rank}(A) = 2$ with pivots in columns 1 and 2. A basis of $\operatorname{range}(A)$ is given by the first two *original* columns:

$$\{(1,2,3)^\top,\ (1,1,2)^\top\}.$$

   From the reduced system $x_1 - x_3 = 0$, $x_2 + x_3 = 0$, letting $x_3 = t$ we get

$$\ker(A) = \operatorname{span}\{(1,-1,1)^\top\}.$$

2. *Solve $Ax = b$.* Using the same reduction on the augmented system yields

$$x_1 - x_3 = -1, \qquad x_2 + x_3 = 2.$$

   With $x_3 = t$,

$$x = (-1,2,0)^\top + t\,(1,-1,1)^\top, \quad t \in \mathbb{R}.$$

3. *Map $x \mapsto Cx$. Injectivity/surjectivity, rank, kernel.* The columns of C span $\mathbb{R}^2$ (e.g., the first two are independent), so $\operatorname{rank}(C) = 2$. Thus $x \mapsto Cx : \mathbb{R}^3 \to \mathbb{R}^2$ is *surjective* but not injective. By rank–nullity,

$$\dim\ker(C) = 3 - \operatorname{rank}(C) = 1.$$

   Solving $Cx = 0$ gives $x = s\,(2,-1,3)^\top$, hence

$$\ker(C) = \operatorname{span}\{(2,-1,3)^\top\}.$$

4. $U = \{(x,y,z) : x - 2y + z = 0\}$. $U = \ker\phi$ where $\phi(x,y,z) = x - 2y + z$ is linear, hence U is a subspace. Writing $x = 2y - z$,

$$(x,y,z) = y\,(2,1,0) + z\,(-1,0,1).$$

   Therefore $\dim U = 2$ and a basis is $\{(2,1,0),\ (-1,0,1)\}$.

Exercise (C): True/False

1. **False.** We could have $v_1 + v_2 + v_3 = 0$.
2. **True.** By rank–nullity, $\dim\operatorname{range}(T) = 3 - \dim\ker(T) = 2$.
3. **True.** $Q^\top Q = I$ implies the columns are orthonormal and span $\mathbb{R}^n$.