Recap - Methods

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1 Diagonalization

1.1 Finding eigenvalues.

For a square matrix $A \in \mathbb{K}^{n\times n}$, the eigenvalues are the roots of the *characteristic polynomial*

$$\chi_A(\lambda) = \det(\lambda I_n - A).$$

Special cases that simplify the computation:

- If A is triangular, the eigenvalues are the diagonal entries.
- If A is block diagonal $\operatorname{diag}(B_1, \dots, B_k)$, then $\chi_A(\lambda) = \prod_{i=1}^{k} \chi_{B_i}(\lambda)$; the eigenvalues are the union of the eigenvalues of the blocks (with multiplicities).
- From the example of the course, for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\chi_A(\lambda) = \lambda^2 - (\operatorname{tr} A)\,\lambda + \det A = \lambda^2 - (a+d)\lambda + (ad - bc).$$

Example in $\mathbb{R}^2$: finding the eigenvalues Let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$$

Method 1 (triangular matrix). Since A is upper triangular, its eigenvalues are the diagonal entries: $\lambda_1 = 2$ and $\lambda_2 = 3$.

Method 2 (characteristic polynomial).

$$\lambda I_2 - A = \begin{pmatrix} \lambda - 2 & -1 \\ 0 & \lambda - 3 \end{pmatrix} \quad \Rightarrow \quad \chi_A(\lambda) = \det(\lambda I_2 - A) = (\lambda - 2)(\lambda - 3).$$

Hence the eigenvalues are $\lambda = 2$ and $\lambda = 3$. (This also matches $\lambda^2 - (\operatorname{tr} A)\lambda + \det A = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$.)

Example in $\mathbb{R}^3$: finding the eigenvalues Let

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix} 4 & 1 \\ 1 & 4 \end{matrix}} & 0 \\ 0 & \boxed{5} \end{pmatrix}.$$

Direct determinant computation.

$$\lambda I_3 - A = \begin{pmatrix} \lambda - 4 & -1 & 0 \\ -1 & \lambda - 4 & 0 \\ 0 & 0 & \lambda - 5 \end{pmatrix} \quad \Rightarrow \quad \chi_A(\lambda) = (\lambda - 5)\big((\lambda - 4)^2 - 1\big) = (\lambda - 5)\,(\lambda^2 - 8\lambda + 15),$$

Block argument. You can also recognize that A is block diagonal with a 2×2 block $B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ and a 1×1 block $[5]$.

$$\chi_A(\lambda) = \chi_B(\lambda) \cdot (\lambda - 5).$$

For the 2×2 block B,

$$\chi_B(\lambda) = \lambda^2 - (\operatorname{tr} B)\lambda + \det B = \lambda^2 - 8\lambda + (4 \cdot 4 - 1) = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5).$$

Therefore

$$\chi_A(\lambda) = (\lambda - 3)(\lambda - 5) \cdot (\lambda - 5) = (\lambda - 3)(\lambda - 5)^2,$$

so the eigenvalues are $\lambda = 3$ (multiplicity 1) and $\lambda = 5$ (multiplicity 2).

1.2 Diagonalizable matrix: Construction of P and D.

Now let suppose $A \in \mathbb{K}^{n \times n}$ is diagonalizable. Then there exist an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

If the (distinct) eigenvalues of A are $\lambda_1, \ldots, \lambda_r$ with algebraic multiplicities $m_{\lambda_1}, \ldots, m_{\lambda_r}$ (so $m_{\lambda_1} + \cdots + m_{\lambda_r} = n$), then:

1. For each λ_i, compute the eigenspace
$$E_{\lambda_i} = \ker(A - \lambda_i I_n).$$
2. Because A is diagonalizable, $\dim E_{\lambda_i} = m_{\lambda_i}$ for each i. Choose a basis $v_{i,1}, \ldots, v_{i,m_{\lambda_i}}$ of E_{λ_i}.
3. Form P by stacking these eigenvectors as columns (in any order), and put the corresponding eigenvalues on the diagonal of D in the matching order. Concretely,
$$P = \begin{bmatrix} v_{1,1} & \cdots & v_{1,m_{\lambda_1}} \mid \cdots \mid v_{r,1} & \cdots & v_{r,m_{\lambda_r}} \end{bmatrix}, \qquad D = \operatorname{diag}(\underbrace{\lambda_1, \ldots, \lambda_1}_{m_{\lambda_1}}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{m_{\lambda_r}}).$$

Non-uniqueness. If S is block-diagonal with an invertible block acting inside each eigenspace, then $P' := PS$ is also valid and $A = P'DP'^{-1}$. Scaling and mixing eigenvectors within an eigenspace changes P but not A, D.

Example in $\mathbb{R}^2$ Let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}.$$

Its eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

Eigenspaces.

$$A - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \ker(A - 2I) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \ker(A - 3I) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

Build P and D. Take

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad D = \operatorname{diag}(2, 3) = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then

$$P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad PDP^{-1} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} = A.$$

(Equivalently, $Av_1 = 2v_1$ with $v_1 = (1, 0)^\top$, and $Av_2 = 3v_2$ with $v_2 = (1, 1)^\top$.)

Example in $\mathbb{R}^3$ Let

$$A = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

The eigenvalues are $\lambda = 3$ (multiplicity 1) and $\lambda = 5$ (multiplicity 2).

Eigenspaces.

$$A - 3I = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \ker(A - 3I) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

$$A - 5I = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \ker(A - 5I) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Build P and D. Choose eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (\lambda = 5), \qquad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad (\lambda = 3).$$

Set

$$P = [v_1 \; v_2 \; v_3] = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D = \operatorname{diag}(5, 5, 3).$$

Then $A = PDP^{-1}$, since $Av_1 = 5v_1$, $Av_2 = 5v_2$, $Av_3 = 3v_3$ and the three eigenvectors are linearly independent.

2 Trigonalizability

Let $A \in \mathcal{M}_n(\mathbb{K})$ such that χ_A is split. Let us denote $sp(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with multiplicity $m_1, m_2, \ldots m_p$. NB : we know that $\sum_{i=1}^{p} \lambda_i = n$.

- For each $i \in [[1, p]]$ we look for a basis $\mathcal{B}_i$ of $ker(\lambda_i I_n - A)$.
- If $Card(\mathcal{B}_i) = m_i$ (i.e. $dim(ker(\lambda_i I_n - A)) = m_i$) then we can diagonalize.
- Otheriwse, we look for vectors to add to $\mathcal{B}_i$ to make it a basis of $ker((\lambda_i I_n - A)^2)$, etc.
- Finally, the family of vectors $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \mathcal{B}_p$ is a basis where A can be trigonalized.

2.1 Example (Exercise from the course)

Consider, for $a > 0$,

$$A = \begin{pmatrix} -1 & a & -a \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

1. Compute χ_A and show that A is not diagonalizable.
2. Find $v_1, v_2, v_3 \in \mathbb{R}^3$ such that

$$Av_1 = -v_1, \qquad Av_2 = v_1 - v_2, \qquad Av_3 = v_1 + v_2 - v_3.$$

3. Show that A is similar to $T = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$.

4. Use a nilpotent matrix to compute A^n for any $n \in \mathbb{N}$.

5. Can we compute A^n for any $n \in \mathbb{Z}$? If yes, do it.

(1) Characteristic polynomial and non-diagonalizability. We have

$$\lambda I_3 - A = \begin{pmatrix} \lambda+1 & -a & a \\ -1 & \lambda+1 & 0 \\ -1 & 0 & \lambda+1 \end{pmatrix}.$$

Expanding along the first row,

$$\begin{aligned} \chi_A(\lambda) &= \det(\lambda I_3 - A) \\ &= (\lambda+1)\det\begin{pmatrix} \lambda+1 & 0 \\ 0 & \lambda+1 \end{pmatrix} - (-a)\det\begin{pmatrix} -1 & 0 \\ -1 & \lambda+1 \end{pmatrix} + a\det\begin{pmatrix} -1 & \lambda+1 \\ -1 & 0 \end{pmatrix} \\ &= (\lambda+1)^3 + a(-(\lambda+1)) + a(\lambda+1) = (\lambda+1)^3. \end{aligned}$$

Thus the only eigenvalue is $\lambda = -1$, with algebraic multiplicity 3. Next,

$$A + I_3 = \begin{pmatrix} 0 & a & -a \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \Rightarrow \quad \ker(A+I_3) = \{(0,t,t)^\top : t \in \mathbb{R}\},$$

so $\dim\ker(A+I_3) = 1$. Hence the geometric multiplicity of $\lambda = -1$ is $1 < 3$, and A is *not* diagonalizable.

(2) Vectors v_1, v_2, v_3 with the prescribed relations. Choose

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ \frac{1}{a} \\ 0 \end{pmatrix}.$$

Then, by direct computation,

$$Av_1 = -v_1, \qquad Av_2 = v_1 - v_2, \qquad Av_3 = v_1 + v_2 - v_3.$$

(We used $a > 0$ so that $1/a$ is well-defined; any $a \neq 0$ would suffice.)

(3) Similarity to the given upper triangular matrix. Let $P = [\,v_1 \ v_2 \ v_3\,]$, i.e.

$$P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \frac{1}{a} \\ 1 & 0 & 0 \end{pmatrix}.$$

The relations in (2) precisely mean that the matrix of A in the basis $\mathcal{B} = (v_1, v_2, v_3)$ is

$$T = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Equivalently, $A = PTP^{-1}$.

(4) Powers A^n for $n \in \mathbb{N}$. Write $A = -I_3 + N$ with $N := A + I_3 = \begin{bmatrix} 0 & a & -a \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. A direct check shows $N^3 = 0$ (but $N^2 \neq 0$):

$$N^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & a & -a \end{pmatrix}, \qquad N^3 = 0.$$

Since I_3 and N commute, the binomial identity gives, for $n \in \mathbb{N}$,

$$A^n = (-I_3 + N)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} N^k = (-1)^n \Big(I_3 - nN + \binom{n}{2} N^2 \Big),$$

because $N^3 = 0$ annihilates higher terms. Substituting N and N^2 yields the explicit form

$$A^n = (-1)^n \begin{pmatrix} 1 & -na & na \\ -n & 1 + \binom{n}{2}a & -\binom{n}{2}a \\ -n & \binom{n}{2}a & 1 - \binom{n}{2}a \end{pmatrix}, \qquad n \in \mathbb{N}.$$

(5) Powers A^n for $n \in \mathbb{Z}$. We have $\det A = (-1)^3 = -1 \neq 0$, so A is invertible and A^n is defined for all $n \in \mathbb{Z}$. As above,

$$A^{-n} = \big(-I_3 + N\big)^{-n} = (-1)^n \, (I_3 - N)^{-n}.$$

Using the generalized binomial formula (or $(I_3 - N)^{-1} = I_3 + N + N^2$ and induction), since $N^3 = 0$,

$$(I_3 - N)^{-n} = I_3 + nN + \binom{n+1}{2} N^2, \qquad n \in \mathbb{N}.$$

Therefore, for $n \in \mathbb{N}$,

$$A^{-n} = (-1)^n \Big(I_3 + nN + \binom{n+1}{2} N^2 \Big) = (-1)^n \begin{pmatrix} 1 & na & -na \\ n & 1 + \binom{n+1}{2}a & -\binom{n+1}{2}a \\ n & \binom{n+1}{2}a & 1 - \binom{n+1}{2}a \end{pmatrix}.$$

Combining (4) and this formula, A^m is known explicitly for every $m \in \mathbb{Z}$.