

# Linear Algebra - Exercise sheet

September 11, 2025

## 1 Lecture 1 - Vector spaces, linear maps

### Exercise 1

For

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

describe graphically all points  $cv$  with:

- (a)  $c$  being an integer, that is  $c \in \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$ .
- (b)  $c \in \mathbb{R}$ , with  $c \geq 0$ .

Describe  $cv + dw$  where  $d \in \mathbb{R}$  and  $c$  is like in (a) or (b).

### Exercise 2

Is

$$z = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

in the span of

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}?$$

If so, find  $\alpha, \beta \in \mathbb{R}$  such that  $z = \alpha x + \beta y$ .

### Exercise 3

1. Prove that

$$\mu_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

are linearly independent.

2. Is  $\{\mu_1, \mu_2, \mu_3\}$  a basis of  $\mathbb{R}^3$ ?

**Hint to Exercise 3**

(1) solve  $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0$ . (2) A free family of the same size as the dimension of the vector space.

**Exercise 4**

Consider the following transformations:

$$L_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

$$L_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y \end{pmatrix}$$

$$L_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z + y \\ z - y \\ 0 \end{pmatrix}$$

Are they linear? Justify your answers.

**Hint to Exercise 4**

Use the definition of a linear transformation.

**Exercise 5**

1. Prove that if  $T : U \rightarrow V$  is a linear transformation, then  $T(0_U) = 0_V$ .
2. Prove that a linear transformation  $T : U \rightarrow V$  is injective if and only if

$$\ker(T) = \{0_U\}.$$

**Hint to Exercise 5**

(1) Use  $0_U = 0_U - 0_U$  then linearity. (2) Prove one implication, then the other.

**Exercise 6**

Determine whether the following functions are injective, surjective, or bijective:

1.  $f_1 : \{a, b, c\} \rightarrow \{a, b, c, d\}$  defined by:

$$f_1(a) = a, \quad f_1(b) = b, \quad f_1(c) = c$$

2.  $f_2 : \{a, b, c\} \rightarrow \{a, b, c, d\}$  defined by:

$$f_2(a) = a, \quad f_2(b) = a, \quad f_2(c) = c$$

3.  $f_3 : \{a, b, c\} \rightarrow \{a, b, c\}$  defined by:

$$f_3(a) = b, \quad f_3(b) = c, \quad f_3(c) = a$$

**Hint to Exercise 6**

(1) injective by not surjective, (2) not injective, not surjective (3) bijective.