DETERMINANTS

Exercise 1

Let's compute $\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$ $\quad C_1 \leftarrow C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2(x+y) & y & x+y \\ 2(x+y) & x+y & x \\ 2(x+y) & x & y \end{vmatrix}$$

$$= 2(x+y) \begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix} \quad \begin{matrix} L_1 \leftarrow L_1 - L_3 \\ L_2 \leftarrow L_2 - L_3 \end{matrix}$$

$$= 2(x+y) \begin{vmatrix} 0 & y-x & x \\ 0 & y & x-y \\ 1 & x & y \end{vmatrix}$$

$$= 2(x+y) \begin{vmatrix} y-x & x \\ y & x-y \end{vmatrix}$$

$$= 2(x+y)\left[-(x-y)^2 - xy\right]$$

$$= -2(x+y)\left[x^2 - xy + y^2\right]$$

Let's compute:

$$\Delta = \begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a+2)^2 & (b+2)^2 & (c+2^2) \end{vmatrix} = \begin{vmatrix} a^2 & b^2 & c^2 \\ a^2+2a+1 & b^2+2b+1 & c^2+2c+1 \\ a^2+4a+4 & b^2+4b+4 & c^2+4c+4 \end{vmatrix}$$

$L_2 \leftarrow L_2 - L_1 \; ; \; L_3 \leftarrow L_3 - L_1$

$$\Delta = \begin{vmatrix} a^2 & b^2 & c^2 \\ 2a+1 & 2b+1 & 2c+1 \\ 4a+4 & 4b+4 & 4c+4 \end{vmatrix} \quad L_3 \leftarrow L_3 - 2L_2$$

$$= \begin{vmatrix} a^2 & b^2 & c^2 \\ 2a+1 & 2b+1 & 2c+1 \\ 2 & 2 & 2 \end{vmatrix} \quad L_2 \leftarrow L_2 - \tfrac{1}{2} L_3$$

$$= 2 \begin{vmatrix} a^2 & b^2 & c^2 \\ 2a & 2b & 2c \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 4 \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$ Vandermonde determinant $V(a,b,c)$

$$= \dots = -4(c-a)(b-a)(c-b).$$

Bonus exercise : Vandermonde determinant.

Let $(a_1, \dots a_n) \in \mathbb{R}^n$.

Let $V(a_1, \dots a_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & & \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{pmatrix}$

Prove by induction that $V(a_1, \dots a_n) = \prod_{1 \leq i < j \leq n} (a_j - a_i)$ (*)

NB: you can consider that $a_1 \neq a_2 \neq \dots \neq a_n$.

Hint : consider the polynomial $P(x) = V(a_1, a_2 \dots \cancel{a_{n-1}}\, x)$ instead of a_n

Degree of $P \leqslant n-1$

$P(a_1) = P(a_2) = \dots = P(a_{n-1}) = 0$

so $P(x) = \lambda \prod_{i=1}^{n-1} (x - a_i)$ where λ is the leading coefficient of the polynomial; i.e. the coefficient of x^{n-1}.

Hence $\lambda = \det \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_{n-1} \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_{n-1}^{n-2} \end{vmatrix} = V(a_1, \dots a_{n-1})$.

So we get $V(a_1, \dots a_n) = \prod_{i=1}^{n-1} (a_n - a_i) \; V(a_1, \dots a_{n-1})$ (*)

- Case $n = 2$

$V(a_1, a_2) = \begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = a_2 - a_1 = (a_2 - a_1) \times (1)$ ok

- Suppose (*) is true for a given $n \geqslant 2$.

Using (*) we prove that it holds for $n+1$.

- Conclusion.

$$\Delta_a = \begin{vmatrix} 1 & \cos(a) & \cos(2a) \\ 1 & \cos(b) & \cos(2b) \\ 1 & \cos(c) & \cos(2c) \end{vmatrix}$$

Reminder : $\cos(2\theta) = 2\cos^2\theta - 1$

$$\Delta a = \begin{vmatrix} 1 & \cos a & 2\cos^2 a - 1 \\ 1 & \cos b & 2\cos^2 b - 1 \\ 1 & \cos c & 2\cos^2 c - 1 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & \cos a & \cos^2 a \\ 1 & \cos b & \cos^2 b \\ 1 & \cos c & \cos^2 c \end{vmatrix}$$

$= 2\ V(\cos a, \cos b, \cos c)$ Vandermonde again or compute it manually...

$= \dots$

$= 2(\cos b - \cos a)(\cos c - \cos a)(\cos c - \cos b)$.

$$\Delta x = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix}$$ What about x ??

- If $x = 1$ then the first two rows are the same, so $\Delta_1 = 0$

($\Delta_x = 0 \Rightarrow$ Reminder : the matrix is not invertible, some rows (or cols) can be written as a linear combination of the other rows (or cols) resp.)

- If $x = 2$ then the two last rows are the same, so $\Delta_2 = 0$.

- $$\Delta x = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 0 & 1-x^2 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 4-x^2 \end{vmatrix} \quad \begin{matrix} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - 2L_1 \\ L_4 \leftarrow L_4 - L_3 \text{ (before changing } L_3) \end{matrix}$$

$$= \begin{vmatrix} 1-x^2 & 0 & 0 \\ 1 & -3 & -1 \\ 0 & 0 & 4-x^2 \end{vmatrix}$$

$$= (4-x^2) \begin{vmatrix} 1-x^2 & 0 \\ 1 & -3 \end{vmatrix}$$

$$= (4-x^2)\left[-3(1-x^2) - 0\right]$$

$= -3(4-x^2)(1-x^2)$ <u>ok</u> with cases $x=1$ and $x=2$.