Test — Mathematics for Data Science

Courses 1 and 2: Vector spaces, linear applications, matrices.
(*Indicative grading scale, 1 pt for presentation*).

Exercise (A): Short proofs (3pts)

Let A and B be two square matrices of size $n \times n$. Recall the definition of the trace of a square matrix $A \in \mathbb{R}^{n\times n}$ with coefficients $(a_{ij})_{1\leq i\leq n, 1\leq j\leq n}$:

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Show the following properties :

1. $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$
2. $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

Exercise (B): Short computations (10pts)

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}$$

1. (a) Compute $\mathrm{rank}(A)$ and give a basis of $\mathrm{range}(A)$. (b) Give a basis of $\ker(A)$.
2. Solve $Ax = b$ and describe all solutions.
3. Let $U = \{(x, y, z) \in \mathbb{R}^3 : \ x + y + z = 0\}$. (a) Prove U is a subspace. (b) Find a basis of U and give $\dim U$.
4. Let $f : \mathbb{R}^3 \to \mathbb{R}^3, (x, y, z) \mapsto (x, x + y, x + y + z)$. What is the matrix representation of f in the standard basis of $\mathbb{R}^3$?

Exercise (C): True/False (6pts)

Work over $\mathbb{R}$. Justify briefly whether the following assertions are True or False (*no points without justification*).

1. If $A \in \mathbb{R}^{n\times n}$ is invertible, then $\ker(A) = \{\mathbf{0}\}$ and $\mathrm{range}(A) = \mathbb{R}^n$.
2. If $v_1, \ldots, v_k$ are linearly independent in a vector space V, then $\dim(\mathrm{span}\{v_1, \ldots, v_k\}) = k$.
3. If $A \in \mathbb{R}^{m\times n}$ has $\mathrm{rank}(A) = n$, then the application $x \mapsto Ax$ is injective.
4. If the columns of $A \in \mathbb{R}^{m\times n}$ span $\mathbb{R}^m$, then $\ker(A) = \{0\}$.
5. If A and B are similar matrices, then they have the same rank.
6. $\ker(A) = \{\mathbf{0}\}$ iff the columns of A are linearly independent.

Elements of answers

Exercise (A): Short proofs

Let $A, B \in \mathbb{R}^{n\times n}$.

1. *Linearity of the trace.* Writing $A = (a_{ij})$ and $B = (b_{ij})$, we have

$$\mathrm{Tr}(A+B) = \sum_{i=1}^{n}(a_{ii}+b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = \mathrm{Tr}(A) + \mathrm{Tr}(B).$$

2. *Cyclic property of the trace for two factors.* Using $(AB)_{ii} = \sum_{j=1}^{n} a_{ij}b_{ji}$,

$$\mathrm{Tr}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n}(BA)_{jj} = \mathrm{Tr}(BA).$$

Exercise (B): Short computations

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 3}.$$

1. *Rank, bases of range and kernel of A.*

   (Method 1) : The rank of A is given by the dimension of its columnspace. Let us denote by A_j the j^{th} column of A. We remark that the third column of A, is obtained by summing the two first columns of A. Then justify that the first two columns are linearly independant. Hence

$$rank(A) = dim(span(A_1, A_2, A_3)) = dim(span(A_1, A_2)) = 2.$$

   (Method 2) : Row-reduction with elementary transformations

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{swap } R_2, R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Hence $\mathrm{rank}(A) = 2$, with pivots in columns 1 and 2.

   A basis of $\mathrm{range}(A)$ is given by the first two *original* columns:

$$\{\,(1,2,0)^\top,\ (2,4,1)^\top\,\}.$$

   To find $\ker(A)$, either directly solve $Ax = 0$ or, if you have used method 2, solve $\tilde{A}x = 0$ where $\tilde{A}$ is the row-echelon form of A. We obtain $x_1 + x_3 = 0$, $x_2 + x_3 = 0$. Let $x_3 = t$. Then the set of soluton is described by

$$\{x \in \mathbb{R}^3, x = t\,(-1,-1,1)^\top, t \in \mathbb{R}\}$$

   Finally, $\ker(A) = \mathrm{span}\{(-1,-1,1)^\top\}$.

2. *Solve $Ax = b$.*

   Performing the same operations on the augmented vector b gives the reduced system

$$x_1 + x_3 = 1, \qquad x_2 + x_3 = 1.$$

   Let $x_3 = t$. Then

$$x = (1-t,\, 1-t,\, t)^\top = (1,1,0)^\top + t\,(-1,-1,1)^\top, \quad t \in \mathbb{R}.$$

   So the solution set is the affine line $x_p + \ker(A)$ with particular solution $x_p = (1,1,0)^\top$.

3. $U = \{(x, y, z) \in \mathbb{R}^3 : \ x + y + z = 0\}$.

   (Method 1) U contains 0, and if $u, v \in U$ and $\alpha \in \mathbb{R}$, then $(u + v)$ and αu satisfy the same linear equation (to be detailed), hence U is a subspace.

   (Method 2) Remark that U is the kernel of the linear map $\phi : \mathbb{R}^3 \to \mathbb{R}, (x, y, z) \mapsto x + y + z$. Justify that ϕ is a linear map. Then its kernel is a subspace of $\mathbb{R}^3$.

   (Method 1) For any $(x, y, z) \in U$, we have $z = -(x + y)$, so

   $$(x, y, z) = (x, 0, -x) + (0, y, -y) = x\,(1, 0, -1) + y\,(0, 1, -1).$$

   Thus a basis of U is $\{(1, 0, -1), (0, 1, -1)\}$ (any two independent solutions suffice). Finally, $\dim U = 2$.

   (Method 2) Using the standard bases, the matrix of ϕ is $A = [1, 1, 1]$. The rank of A is 1 (repeated columns). Using the rank nullity theorem, we get

   $$dim(\mathbb{R}^3) = rank(\phi) + dim(ker(\phi)) = rank(\phi) + dim(U).$$

   Hence $dim(U) = 2$.

4. *Matrix representation of a linear application.* Let $\mathcal{B} = (e_1, e_2, e_3)$ be the standard basis of $\mathbb{R}^3$, where $e_1 = (1, 0, 0)^\top$, $e_2 = (0, 1, 0)^\top$, $e_3 = (0, 0, 1)^\top$. The matrix of f in the basis $\mathcal{B}$ has columns $f(e_1), f(e_2), f(e_3)$.

   $$\begin{aligned} f(e_1) &= f(1, 0, 0) = (1,\ 1,\ 1)^\top, \\ f(e_2) &= f(0, 1, 0) = (0,\ 1,\ 1)^\top, \\ f(e_3) &= f(0, 0, 1) = (0,\ 0,\ 1)^\top. \end{aligned}$$

   Hence the matrix representation of f in the standard basis is

   $$[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

   NB: do a quick check like this

   $$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x + y \\ x + y + z \end{bmatrix} = f(x, y, z).$$

Exercise (C): True/False

Work over $\mathbb{R}$. Brief justifications are included.

1. **True.** If A is invertible, its corresponding linear map is bijective; hence $\ker(A) = \{0\}$ and $\text{range}(A) = \mathbb{R}^n$ (cf theorem of equivalent statements on invertible matrices).

2. **True.** Linearly independent $v_1, \ldots, v_k$ form a basis of their span, so the dimension is k.

3. **True.** $\text{rank}(A) = n$ means the n columns are independent; hence $\ker(A) = \{0\}$ and the map is injective (necessarily $n \leq m$).

4. **False.** If the columns span $\mathbb{R}^m$, then $\text{rank}(A) = m$, but $\dim\ker(A) = n - m$ which is > 0 when $n > m$. Example:

   $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 3}$$

   has columns spanning $\mathbb{R}^2$ and $\ker(A) = \{(0, 0, t) : t \in \mathbb{R}\} \neq \{0\}$.

5. **True.** The matrices represent the same linear application but with a different choice of basis.

6. **True.** $\ker(A) = \{0\}$ iff the only solution to $Ax = 0$ is $x = 0$, which is equivalent to the columns of A being linearly independent.