Lecture 4 - Eigendecomposition, Diagonalization

NB: notation for the eigenspace of an endomorphism f associated to an eigenvalue λ: either $E_\lambda(f)$ or $Eig(f, \lambda)$.

Exercise 1

(a) Determine the **eigenvalues** and **eigenspaces** of the square matrix

$$A = \begin{pmatrix} -5 & 4 \\ -6 & 5 \end{pmatrix} \in M_2(\mathbb{R}).$$

Form the characteristic polynomial χ_A of A:

$$\chi_A(\lambda) = \begin{vmatrix} -5-\lambda & 4 \\ -6 & 5-\lambda \end{vmatrix} = (\lambda^2 - 25) + 24 = \lambda^2 - 1 = (\lambda+1)(\lambda-1).$$

Hence the eigenvalues of A are -1 and 1, i.e.

$$\mathrm{Sp}_{\mathbb{R}}(A) = \{-1, 1\},$$

and each of these two eigenvalues has multiplicity 1.

For any $X = \binom{x}{y} \in M_{2,1}(\mathbb{R})$ we have:

$$X \in \mathrm{Eig}(A, -1) \iff AX = -X \iff \begin{cases} -5x + 4y = -x, \\ -6x + 5y = -y \end{cases} \iff x = y,$$

so

$$\mathrm{Eig}(A, -1) = \mathrm{Span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \dim \mathrm{Eig}(A, -1) = 1.$$

$$X \in \mathrm{Eig}(A, 1) \iff AX = X \iff \begin{cases} -5x + 4y = x, \\ -6x + 5y = y \end{cases} \iff -6x + 4y = 0,$$

so

$$\mathrm{Eig}(A, 1) = \mathrm{Span}\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \qquad \dim \mathrm{Eig}(A, 1) = 1.$$

Remark. The square matrix A is diagonalizable in $M_2(\mathbb{R})$.

(b) Determine the **eigenvalues** and **eigenspaces** of the square matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R}).$$

Method. To study a square matrix that satisfies an equation, *remember to use the notion of an annihilating polynomial.*

Let $A \in M_5(\mathbb{C})$ be such that

$$A^2 - 4A + 3I_5 = 0 \qquad \text{and} \qquad \mathrm{tr}(A) = 9.$$

Determine the eigenvalues of A and their multiplicities.

The polynomial $P = X^2 - 4X + 3$ annihilates A, and

$$P = (X-1)(X-3),$$

hence, $\mathrm{Sp}_{\mathbb{C}}(A) \subset \{1, 3\}$. Let α (resp. β) be the multiplicity of the eigenvalue 1 (resp. 3) of A, with the convention $\alpha = 0$ if 1 is not an eigenvalue of A. Since χ_A splits over $\mathbb{C}$, we have, $\alpha + \beta = 5$ (the size of A) and moreover $\alpha \cdot 1 + \beta \cdot 3 = \mathrm{tr}(A) = 9$. It follows that $\alpha = 3$, $\beta = 2$. Conclusion: the eigenvalues of A are 1 (of multiplicity 3) and 3 (of multiplicity 2).

Method. To obtain information—for instance about the trace or determinant—of a matrix $A \in M_n(K)$ when an annihilating polynomial P of A is known, use that

$$\text{the spectrum of } A \text{ is contained in the zero set of } P \text{ in } K.$$

Exercise 2

Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R}),$$

and consider the map

$$f : M_2(\mathbb{R}) \to M_2(\mathbb{R}), \qquad M \mapsto AMB.$$

Verify that f is a linear endomorphism of the vector space $M_2(\mathbb{R})$ and determine the eigenvalues and eigenspaces of f.

For all $\alpha \in \mathbb{R}$ and $M, N \in M_2(\mathbb{R})$,

$$f(\alpha M + N) = A(\alpha M + N)B = \alpha AMB + ANB = \alpha f(M) + f(N),$$

hence f is linear on $M_2(\mathbb{R})$.

Method 1 (from the definition of eigenvalue/eigenspace). Let $\lambda \in \mathbb{R}$ and $M = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \neq 0$. Then

$$f(M) = \lambda M \iff \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x & y \\ z & t \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \lambda \begin{pmatrix} x & y \\ z & t \end{pmatrix} \iff \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \lambda x & \lambda y \\ \lambda z & \lambda t \end{pmatrix}.$$

Equivalently, $t = \lambda x$ and $\lambda y = \lambda z = \lambda t = 0$. If $\lambda \neq 0$, then $y = z = t = 0$ and next $\lambda x = 0$, hence $x = 0$, a contradiction with $M \neq 0$. Therefore

$$f(M) = \lambda M \iff (\lambda = 0 \text{ and } t = 0),$$

so

$$\operatorname{Sp}(f) = \{0\}, \qquad E_0(f) = \left\{ \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} ;\ (x, y, z) \in \mathbb{R}^3 \right\},$$

and $\dim E_0(f) = 3$.

Method 2 (matrix of f in the canonical basis of $M_2(\mathbb{R})$). In the canonical basis $\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$ with

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain

$$f(E_{11}) = 0, \qquad f(E_{12}) = 0, \qquad f(E_{21}) = 0, \qquad f(E_{22}) = E_{11}.$$

Thus the matrix of f in $\mathcal{B}$ is

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since M is upper triangular, its eigenvalues (hence those of f) are the diagonal entries:

$$\operatorname{Sp}(M) = \{0\}.$$

Finally $\operatorname{Eig}(M, 0) = \ker M$ is spanned by the first three canonical vectors of $M_{4,1}(\mathbb{R})$, hence

$$E_0(f) = \operatorname{Span}(E_{11}, E_{12}, E_{21}).$$

Exercise 3

Let $A \in M_5(\mathbb{C})$ satisfy

$$A^2 - 4A + 3I_5 = 0 \qquad \text{and} \qquad \operatorname{tr}(A) = 9.$$

Determine the eigenvalues of A and their multiplicities.

The polynomial $P = X^2 - 4X + 3$ annihilates A, and

$$P = (X-1)(X-3),$$

hence $\operatorname{Sp}_{\mathbb{C}}(A) \subset \{1, 3\}$. Let α (resp. β) be the multiplicity of the eigenvalue 1 (resp. 3) of A, with the convention $\alpha = 0$ if 1 is not an eigenvalue of A. Since χ_A splits over $\mathbb{C}$, we have $\alpha + \beta = 5$ (the size of A) and also

$$\alpha \cdot 1 + \beta \cdot 3 = \operatorname{tr}(A) = 9.$$

Therefore $\alpha = 3$ and $\beta = 2$.

Conclusion: the eigenvalues of A are 1 (multiplicity 3) and 3 (multiplicity 2).

Exercise 4

Let $n \in \mathbb{N}^*$ and $A \in M_n(\mathbb{R})$ satisfy

$$A^2 - 5A + 6I_n = 0.$$

Show that $\quad \operatorname{tr}(A) \leqslant 3n$.

The polynomial $P = X^2 - 5X + 6$ annihilates A, and

$$P = (X-2)(X-3),$$

hence, $\operatorname{Sp}_{\mathbb{C}}(A) \subset \{2, 3\}$. Let α (resp. β) be the multiplicity of the eigenvalue 2 (resp. 3) of A, with the convention $\alpha = 0$ if 2 is not an eigenvalue of A. Since χ_A splits over $\mathbb{C}$, we have

$$\alpha + \beta = n \qquad \text{and} \qquad \operatorname{tr}(A) = 2\alpha + 3\beta.$$

Therefore,

$$\operatorname{tr}(A) = 2\alpha + 3\beta \leqslant 3\alpha + 3\beta = 3(\alpha + \beta) = 3n.$$

Exercise 5

Is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

diagonalizable in $M_3(\mathbb{R})$? If so, diagonalize it.

The matrix A is upper triangular, so its eigenvalues are read on the diagonal:

$$\operatorname{Sp}_{\mathbb{R}}(A) = \{1, 2, 3\}.$$

Since A is 3×3 and has three pairwise distinct eigenvalues, we have a sufficient condition for A to be diagonalizable in $M_3(\mathbb{R})$ and each eigenspace has dimension 1.

Let us find the eigenspaces of A. For any $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in M_{3,1}(\mathbb{R})$:

- $X \in \mathrm{Eig}(A,1) \iff AX = X \iff \begin{cases} x+y+z = x \\ 2y+2z = y \\ 3z = z \end{cases} \Rightarrow \begin{cases} y = 0 \\ z = 0 \end{cases}$

  Hence $\mathrm{Eig}(A,1)$ has basis $V_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

- $X \in \mathrm{Eig}(A,2) \iff AX = 2X \iff \begin{cases} x+y+z = 2x \\ 2y+2z = 2y \\ 3z = 2z \end{cases} \Rightarrow \begin{cases} x = y \\ z = 0 \end{cases}$

  Hence $\mathrm{Eig}(A,2)$ has basis $V_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

- $X \in \mathrm{Eig}(A,3) \iff AX = 3X \iff \begin{cases} x+y+z = 3x \\ 2y+2z = 3y \\ 3z = 3z \end{cases} \Rightarrow \begin{cases} 2x = 3z \\ y = 2z \end{cases}$

  Hence $\mathrm{Eig}(A,3)$ has basis $V_3 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$.

  So A is diagonalizable and we can write it as

$$A = PDP^{-1}$$

  with

$$P = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}, \quad D = Diag(1,2,3)$$

Exercise 6

Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is A diagonalizable in $M_3(\mathbb{R})$?

The matrix A is upper triangular, hence its eigenvalues are the diagonal entries; therefore

$$\mathrm{Sp}_{\mathbb{R}}(A) = \{1\}.$$

If A were diagonalizable over $\mathbb{R}$, there would exist an invertible matrix $P \in \mathcal{M}_3(\mathbb{R})$ and a diagonal matrix D with eigenvalues on the diagonal such that $A = PDP^{-1}$. Since the only eigenvalue is 1, we must have $D = \mathrm{diag}(1,1,1) = I_3$, hence

$$A = PI_3P^{-1} = I_3,$$

a contradiction (because $A \neq I_3$).

Conclusion: A is not diagonalizable in $M_3(\mathbb{R})$.