Lecture 3 - Determinants, Linear Maps and Matrices

Exercise 1

Compute the following determinants.

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a+2)^2 & (b+2)^2 & (c+2)^2 \end{vmatrix}. \qquad \text{and} \qquad \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Exercise 2

Compute the following determinants.

$$\begin{vmatrix} 1 & \cos(a) & \cos(2a) \\ 1 & \cos(b) & \cos(2b) \\ 1 & \cos(c) & \cos(2c) \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} \cos(a-b) & \cos(b-c) & \cos(c-a) \\ \cos(a+b) & \cos(b+c) & \cos(c+a) \\ \sin(a+b) & \sin(b+c) & \sin(c+a) \end{vmatrix}.$$

Exercise 3

Compute the determinants:

$$\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix} \qquad \text{and} \qquad \begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{vmatrix}.$$

Exercise 4

Let $T : \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ be the linear map defined by $T(M) = M^\top$. Find $\det(T)$, the determinant of this linear operator (acting on the n^2-dimensional space $\mathcal{M}_n(\mathbb{R})$).

Exercise 5

Determine all matrices $A \in \mathcal{M}_3(\mathbb{C})$ such that

$$\forall\, M \in \mathcal{M}_3(\mathbb{C}), \qquad \det(A+M) = \det(A) + \det(M).$$

Exercise 6

Compute the following determinants.

$$\Delta_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ -1 & 0 & 3 & \cdots & n \\ -1 & -2 & 0 & \cdots & n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -2 & -3 & \cdots & 0 \end{vmatrix}, \qquad \Delta_n = \begin{vmatrix} 1 & x_1 & x_2 & \cdots & x_{n-1} & x_n \\ 1 & x & x_2 & \cdots & x_{n-1} & x_n \\ 1 & x_1 & x & \cdots & x_{n-1} & x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_1 & x_2 & \cdots & x & x_n \\ 1 & x_1 & x_2 & \cdots & x_{n-1} & x \end{vmatrix}.$$