Linear Algebra - Exercise sheet

September 7, 2025

1 Lecture 1 - Vector spaces, linear maps

Exercise 1

For

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

describe graphically all points cv with:

(a) c being an integer, that is $c \in \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$.

(b) $c \in \mathbb{R}$, with $c \geq 0$.

Describe $cv + dw$ where $d \in \mathbb{R}$ and c is like in (a) or (b).

Exercise 2

Is

$$z = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

in the span of

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}?$$

If so, find $\alpha, \beta \in \mathbb{R}$ such that $z = \alpha x + \beta y$.

Exercise 3

1. Prove that
$$\mu_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$
are linearly independent.
2. Is $\{\mu_1, \mu_2, \mu_3\}$ a basis of $\mathbb{R}^3$?

Hint to Exercise 3

(1) solve $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0$. (2) A free family of the same size as the dimension of the vector space.

Exercise 4

Consider the following transformations:

$$L_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

$$L_2 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y \end{pmatrix}$$

$$L_3 : \mathbb{R}^3 \to \mathbb{R}^3, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z+y \\ z-y \\ 0 \end{pmatrix}$$

Are they linear? Justify your answers.

Hint to Exercise 4

Use the definition of a linear transformation.

Exercise 5

1. Prove that if $T : U \to V$ is a linear transformation, then $T(0_U) = 0_V$.
2. Prove that a linear transformation $T : U \to V$ is injective if and only if

$$\ker(T) = \{0_U\}.$$

Hint to Exercise 5

(1) Use $0_U = 0_U - 0_U$ then linearity. (2) Prove one implication, then the other.

Exercise 5bis

Prove: Let $A \in \mathbb{R}^{n\times n}$. Then the following statements are equivalent:

1. The transformation $x \mapsto Ax$ is bijective.
2. $\mathrm{Im}(A) = \mathbb{R}^n$.
3. $\ker(A) = \{0\}$.

Exercise 6

Determine whether the following functions are injective, surjective, or bijective:

1. $f_1 : \{a, b, c\} \to \{a, b, c, d\}$ defined by:

$$f_1(a) = a, \quad f_1(b) = b, \quad f_1(c) = c$$

2. $f_2 : \{a, b, c\} \to \{a, b, c, d\}$ defined by:

$$f_2(a) = a, \quad f_2(b) = a, \quad f_2(c) = c$$

3. $f_3 : \{a, b, c\} \to \{a, b, c\}$ defined by:

$$f_3(a) = b, \quad f_3(b) = c, \quad f_3(c) = a$$

Hint to Exercise 6

(1) injective by not surjective, (2) not injective, not surjective (3) bijective.