Mathematics for Data Science

Lecture 1

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M1[AI], Fall 2025

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Vector spaces

Vector spaces: the basic setting in which linear algebra happens
Elements of V are called **vectors**.

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Vector space

A $\mathbb{F}-$vector space $(V, +, \cdot)$ is a set endowed with two operations:

- $+ : V \times V \to V$ that allows to sum two vectors: $(x, y) \mapsto x + y$
- $\cdot : \mathbb{R} \times V \to V$ that is the multiplication of a vector by a **scalar**: $(a, x) \mapsto a \cdot x$

Vector spaces can be defined over any **field** $\mathbb{F}$. We take $\mathbb{F} = \mathbb{R}$ in this course.

Group

Let G be a non-empty set. G is a group if there exists an operation $\star$ such that:

Group

Let G be a non-empty set. G is a group if there exists an operation $\star$ such that:

1. **Closure.** For any $\mathbf{x}, \mathbf{y} \in G$, $\mathbf{x} \star \mathbf{y}$ also belongs to G.
2. **Associativity.** For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$, $(\mathbf{x} \star \mathbf{y}) \star \mathbf{z} = \mathbf{x} \star (\mathbf{y} \star \mathbf{z})$.
3. **Neutral element/Identity.** There exists $e \in G$ such that $\mathbf{x} \star \mathbf{e} = \mathbf{e} \star \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in G$.
4. **Symmetric element/Inverse.** For each $\mathbf{x} \in G$, there exists an element $\mathbf{x}' \in G$, such that $\mathbf{x}' \star \mathbf{x} = \mathbf{x} \star \mathbf{x}' = e$.

Vector spaces

A number of axioms must be satisfied such that $(V, +)$ is an **additive group**.

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1. **Additive identity.** There exists an element in V, denoted $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$.
2. **Additive inverse.** For each $\mathbf{x} \in V$, there exists an element in V, denoted $-\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
3. **Multiplicative identity.** There exists an element in $\mathbb{R}$, denoted 1, such that $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.
4. **Commutativity.** For all $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
5. **Associativity.** For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{R}$, $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ and $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$.
6. **Distributivity.** For all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha, \beta \in \mathbb{R}$ $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ and $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$.

Examples of Vector Spaces

Examples of Vector Spaces

Example

- $(\mathbb{R}, +, \cdot)$ is a vector space where $\mathbb{R}$ is the set of real numbers.
- $\ldots$

Can we substract vectors ?

Examples of Vector Spaces

Example

- $(\mathbb{R}, +, \cdot)$ is a vector space where $\mathbb{R}$ is the set of real numbers.
- ...

Can we substract vectors ? We can subtract vectors because a subtraction is the addition of the opposite vector.

Product of vector spaces

Cartesian product of two sets A and B, denoted $A \times B$:

$$\{(a, b), a \in A, b \in B\}$$

.

Definition

Let E and F be two $\mathbb{F}$-vector spaces. The sum of $(x, y) \in E \times F$ and $(x', y') \in E \times F$ is defined as $(x, y) + (x', y') = (x + x', y + y')$. The multiplication of (x, y) by a scalar $\lambda \in \mathbb{F}$ is defined as $\lambda(x, y) = (\lambda x, \lambda y)$.

Proposition

Endowed with the above operations, $E \times F$ is a $\mathbb{F}$-vector space. It is the called the product vector space of E by F.

Euclidean space

Let n be a positive integer.

Example

$(\mathbb{R}^n, +, \cdot)$ is a vector space, called the **Euclidean space**.

Euclidean space

The vectors in the Euclidean space consist of n-tuples of real numbers, i.e. for $x \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n)$. The x_i are the components or entries of the vector.

NB: It will be useful to think of vectors of $\mathbb{R}^n$ as $n \times 1$ matrices, or **column vectors**.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Examples of Operations

Example

- Summation of two vectors in $\mathbb{R}^2$.
- Multiplication by a scalar in $\mathbb{R}^n$.

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Addition and scalar multiplication are defined component-wise on vectors in $\mathbb{R}^n$:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}, \quad \alpha \mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Geometrical interpretation

In $\mathbb{R}^n$, a vector $\mathbf{x}$ has a **direction** and an **amplitude**.
Scalar multiplication changes the amplitude but keeps the same direction.
Parallel vectors: We say that two vectors $v, w \in \mathbb{R}^n$ are **parallel** if there is a non-zero scalar $r \in \mathbb{R}$ such that

$$w = rv.$$

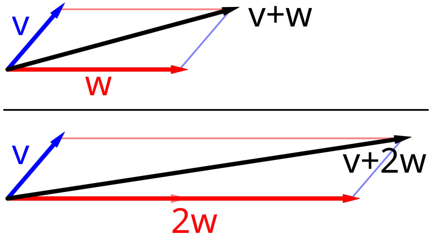

Figure 1: Vector addition and scalar multiplication: a vector $\mathbf{v}$ (blue) is added to another vector $\mathbf{w}$ (red, upper illustration). Below, $\mathbf{w}$ is multiplied by a factor of 2, yielding the sum $\mathbf{v} + 2\mathbf{w}$. (image from Wikipedia)

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Subspaces

Vector spaces can contain other vector spaces.

Subspace

If V is a vector space, then $S \subseteq V$ is said to be a **subspace** of V if

1. $\mathbf{0} \in S$
2. S is closed under addition: $\mathbf{x}, \mathbf{y} \in S$ implies $\mathbf{x} + \mathbf{y} \in S$
3. S is closed under scalar multiplication: $\mathbf{x} \in S, \alpha \in \mathbb{R}$ implies $\alpha\mathbf{x} \in S$

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Example

- V is always a subspace of V.
- The trivial vector space which contains only $\mathbf{0}$.
- A line passing through the origin is a subspace of the Euclidean space.

Subspace

Proposition

Let V be a vector space on $\mathbb{F}$. Any subspace U of V is a vector space on $\mathbb{F}$.

Proof.

By definition, the null element belongs to U. Verify the composition rules. $\square$

Linear Combination

Linear combination

Given vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$ we say that a vector of the form

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

is a **linear combination** of the vectors $v_1, v_2, \ldots, v_k$ with scalar coefficients $c_1, c_2, \ldots, c_k$.

NB : The scalars c_i are sometimes called **weights**.

Propositions

Span (*Sous-espace vectoriel engendré*)

Span

The **span** of a set of vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ is the set of all the vectors which can be written as a linear combination of the $v_1, v_2, \ldots, v_k$, i.e.

$$\operatorname{span}\{v_1, v_2, \ldots, v_k\} = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

NB: equivalent to the French notation $Vect(v_1, v_2, \ldots, v_k)$

Example

- $\operatorname{span}((1,0),(0,1)) = \mathbb{R}^2$
- What is the span of $((2,0),(0,1))$?
- What is the span of $((1,0),(3,0))$?

Geometric interpretation

Given p_0, fixed in $\mathbb{R}^n$, a line in $\mathbb{R}^n$ is given by (cf affine equation in $\mathbb{R}^2$):

$$\ell = \{x \in \mathbb{R}^n \mid x = p_0 + tv,\ t \in \mathbb{R}\}.$$

A special case is $\mathrm{span}\{v\} = \{tv \mid t \in \mathbb{R}\}$, the line through the origin ($p_0 = 0$) in direction v.

Example

Let $p_1 = (1, 2)$ and $p_2 = (3, 1)$ be on

$$\ell = \{(1, 2) + (2, -1)t \mid t \in \mathbb{R}\}.$$

Is $p_1 + p_2$ on the line ?

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Is $p_1 + p_2$ on the line ?

NB : If $u, w \in \mathrm{span}\{v_1, \dots, v_k\}$, then $u + w \in \mathrm{span}\{v_1, \dots, v_k\}$.
However, the sum of two points on a line need not be on the line.

Geometric interpretation

A plane in $\mathbb{R}^n$ through p_0 with directions $\mathbf{u}, \mathbf{v}$ is

$$P = \{x \in \mathbb{R}^n \mid x = p_0 + s\mathbf{u} + t\mathbf{v},\ \ s, t \in \mathbb{R}\}.$$

If $p_0 = 0$, then $P = \mathrm{span}\{\mathbf{u}, \mathbf{v}\}$.

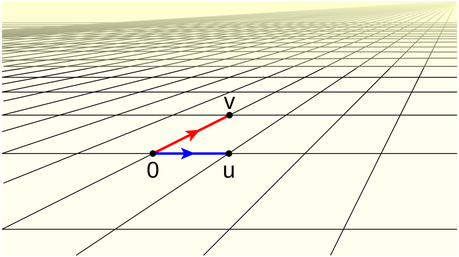

Figure 2: The cross-hatched plane is the linear span of **u** and **v** in both $\mathbb{R}^2$ and $\mathbb{R}^3$ (figure in perspective from Wikipedia).

Geometric interpretation

Hypersurfaces: In $\mathbb{R}^n$, through p_0 in directions $v_1, \dots, v_k$:

$$P = \{x \in \mathbb{R}^n \mid x = p_0 + s_1 v_1 + \cdots + s_k v_k,\ s_1, \dots, s_k \in \mathbb{R}\}.$$

If $p_0 = 0$, then $P = \mathrm{span}\{v_1, \dots, v_k\}$.

Linear Independence

Linear independence

A list of vectors $v_1, \ldots, v_k$ is **linearly independent** if none of the vectors can be written as a linear combination of the others.

Example

Are the following vectors linearly independent ?

$v_1 = (1, 0, 0), v_2 = (3, 0, 0), v_3 = (0, 0, 1)$

$v_1 = (1, 0), v_2 = (0, 2)$

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Example

Are the following vectors linearly independent ?
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$v_1 = (1,0), v_2 = (0,2)$

Proposition

$v_1, \dots, v_k$ are linearly independent iff

$$\alpha_1 v_1 + \dots + \alpha_k v_k = 0 \quad \Rightarrow \quad \alpha_1 = \dots = \alpha_k = 0$$

Spanning List

Spanning list

A **spanning list** of V is a list whose span is V.

Example

- Verify that $\{(2,1),(0,1)\}$ is a spanning list of $\mathbb{R}^2$.
- Let us consider $V = \{(x,0,0), x \in \mathbb{R}\}$ and $v_1 = (1,0,0)$. Verify that V is a vector space and that (v_1) is a spanning list of V.

Basis of a vector space

Basis

A linearly independent spanning list of vectors from a vector space V is called a **basis** of V.

Example

The standard basis of $\mathbb{R}^n$, called the **canonical basis**:

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, 0, \dots, 0), \quad \dots, \quad e_n = (0, \dots, 0, 1)$$

Proposition

Given a basis, every vector has a unique coordinate representation.

Dimension of a vector space

Dimension

All bases of a vector space have the same length, called the **dimension**.

Example

$\dim(\mathbb{R}^n) = n$

Sum of subspaces

Definition

Let U and W be subspaces of V. Their sum is defined as

$$U + W = \{u + w \mid u \in U, w \in W\}$$

Verify that this set is also a subspace of V.

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Definition

If $U \cap W = \{0\}$, the sum is said to be a **direct sum** (*sous-espaces vectoriels supplémentaires, in French*) and written $U \oplus W$.
Equivalently, every vector in $U \oplus W$ can be written uniquely as the sum of a vector from U and a vector from W.

Sum of subspaces

Proposition

For U and W subspaces of V,

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$

Corollary : $dim(U \oplus W) = dim(U) + dim(W)$ for a direct sum.

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Function

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A **function** f from a set X to a set Y assigns to each element of X exactly one element of Y. Notation: $f : X \to Y$.

Vocabulary

Function

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A **function** f from a set X to a set Y assigns to each element of X exactly one element of Y. Notation: $f : X \to Y$.

Vocabulary

- The set X is called the **domain** of the function
- The set Y is called the **codomain** of the function.
- If the element $y \in Y$ is assigned to $x \in X$ by the function f, one says that f **maps** x to y, and this is commonly written $y = f(x)$.
- In this notation, x is the **argument** or **variable** of the function.
- A specific element x of X is a **value** of the variable, and the corresponding element of Y is the value of the function at x, or the **image** of x under the f.

Remark: functional notation, arrow notation

Image and Preimage (inverse image)

Image

The image of a function is the set of the images of all the elements in the domain. Notation: $f(X)$.

$$f(X) = \{f(x) \mid x \in X\}$$

NB: If A is a subset of X, then the image of A under f, denoted $f(A)$, is the subset of the codomain Y consisting of all images of elements of A. We have $f(A) \subset f(X)$.

Preimage

The **inverse image** or **preimage** under f of an element y of the codomain Y is the set of all elements of the domain X whose images under f equal y. Notation : $f^{-1}(y)$.

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}$$

Bijection, injection, surjection

Let $f : X \to Y$ be a function.

Injectivity

The function f is injective if $f(a) \neq f(b)$ for every two different elements a and b of X. Equivalently, f is injective iff, for every $y \in Y$, the preimage $f^{-1}(y)$ contains at most one element.

Surjectivity

The function f is surjective if its image $f(X)$ equals its codomain Y. That is, for every element $y \in Y$, there exists an element $x \in X$ such that $f(x) = y$. Equivalently : $\forall y \in Y, f^{-1}(y) \neq \emptyset$.

Bijectivity

The function f is bijective if it is both injective and surjective.

Bijection, injection, surjection

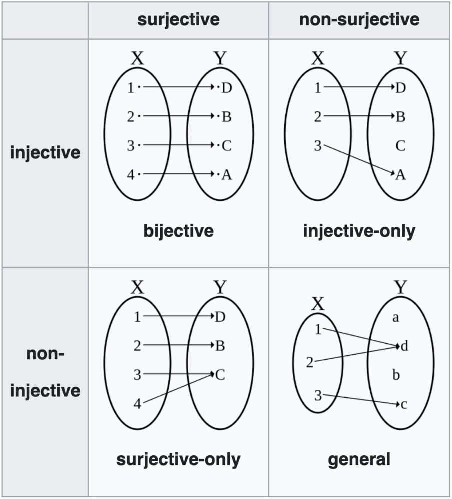

Figure 3: image source: Wikipedia

Linear maps / linear transformations

Vocabulary

- A **homomorphism** is a structure-preserving map between two algebraic structures of the same type (e.g. two groups, or two vector spaces).
- Homomorphisms of vector spaces are also called **linear maps**, **linear mappings** or **linear transformations**.
- A linear map from V to itself is called a **linear operator**.
- A homomorphism where the inverse is also a homomorphism is called an **isomorphism**.

NB: In some branches of mathematics, the term map is used to mean a function.

Linear transformation

Linear transformation

$L : V \to W$ is linear if for all $v_1, v_2 \in V$ and $a, b \in \mathbb{R}$,

$$L(av_1 + bv_2) = aL(v_1) + bL(v_2)$$

Example

- Show that $L : (x, y) \mapsto (x, 0)$ is linear.
- Show that $L : (x, y) \mapsto 10(x, y)$ is linear.
- Finite-dimensional vector spaces[a] of the same dimension are isomorphic.

[a] defined over the same field

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Rank

Rank of a linear transformation

The **rank** of a linear transformation $L : V \to W$ is:

$$\mathrm{rank}(L) = \dim(L(V))$$

where $L(V) = \{y \in W \mid y = L(x),\ x \in V\}$.

NB: $L(V)$ is called the **image** of L. We write $rank(L) = \dim(\mathrm{Im}(L))$.
Example: $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(x, y) = x$ has rank 1.

Kernel / Null Space

Kernel

If $L : V \to W$ is a linear map, we define the **nullspace**, also called **kernel**[a] of L as

$$\text{null}(T) = \{\mathbf{x} \in V \mid L(\mathbf{x}) = \mathbf{0}\}$$

[a] Watch out, the word "kernel" has another meaning in machine learning.

Example

For $L(x, y) = x$, $\ker(L) = \{(0, y) \mid y \in \mathbb{R}\}$.

Verify that the nullspace and range of a linear map are always subspaces of its domain and codomain, respectively.

Rank-Nullity Theorem

Rank-Nullity Theorem

If V is finite-dimensional and $L : V \to W$ is linear:

$$\mathrm{rank}(L) + \dim(\ker(L)) = \dim(V)$$

Example

For $L : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto (2x, 0)$, $\mathrm{rank}(L) = 1$ and $\dim(\ker(L)) = 1$.

Proof

Next topics

Next class : matrices (Lecture 2)