Lecture 4 - Eigendecomposition, Diagonalization

Exercise 1

(a) Determine the **eigenvalues** and **eigenspaces** of the square matrix

$$A = \begin{pmatrix} -5 & 4 \\ -6 & 5 \end{pmatrix} \in M_2(\mathbb{R}).$$

Is the matrix A diagonalizable in $M_2(\mathbb{R})$?

(b) Determine the **eigenvalues** and **eigenspaces** of the square matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{R}).$$

Exercise 2

Let

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R}),$$

and consider the map

$$f : M_2(\mathbb{R}) \to M_2(\mathbb{R}), \qquad M \mapsto AMB.$$

Verify that f is a linear endomorphism of the vector space $M_2(\mathbb{R})$ and determine the eigenvalues and eigenspaces of f.

Exercise 3

Let $A \in M_5(\mathbb{C})$ satisfy

$$A^2 - 4A + 3I_5 = 0 \qquad \text{and} \qquad \mathrm{tr}(A) = 9.$$

Determine the eigenvalues of A and their multiplicities.

Exercise 4

Let $n \in \mathbb{N}^*$ and $A \in M_n(\mathbb{R})$ satisfy

$$A^2 - 5A + 6I_n = 0.$$

Show that $\quad \mathrm{tr}(A) \leqslant 3n.$

Exercise 5

Is the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

diagonalizable in $M_3(\mathbb{R})$? If so, diagonalize it.

Exercise 6

Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Is A diagonalizable in $M_3(\mathbb{R})$?

Exercise 7

Is the matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

diagonalizable in $M_3(\mathbb{R})$?

Exercise 8

Consider

$$A = \begin{pmatrix} -4 & 6 & -3 \\ -1 & 3 & -1 \\ 4 & -4 & 3 \end{pmatrix}.$$

Is A diagonalizable over $M_3(\mathbb{R})$? If so, diagonalize it.

Bonus Exercises

Exercise A Let $A \in \mathbb{R}^{n\times n}$ and suppose (λ_i, v_i), $i = 1, \ldots, n$, are eigenpairs of A with the eigenvectors $v_1, \ldots, v_n$ linearly independent. Set $V = [v_1 \ \cdots \ v_n] \in \mathbb{R}^{n\times n}$ and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n\times n}$.

1. Prove that for every integer $k \geq 1$ and each i,

   $$A^k v_i = \lambda_i^k v_i,$$

   and deduce the matrix identity

   $$A^k = VD^kV^{-1}.$$

2. Assume A is invertible (equivalently every $\lambda_i \neq 0$). Prove that for each i

   $$A^{-1} v_i = \lambda_i^{-1} v_i,$$

   and deduce

   $$A^{-1} = VD^{-1}V^{-1}.$$

3. Conclude that A^k (for $k \geq 1$) and A^{-1} (when defined) are diagonalizable; give a rigorous justification.

Exercise B Let $A \in \mathbb{R}^{m\times n}$. Prove that

$$\operatorname{rank}(A^TA) = \operatorname{rank}(A).$$

(Hint: show first that $\ker(A^TA) = \ker(A)$, then use rank–nullity.)

Exercise C Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite (i.e. $A^T = A$ and $x^TAx > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$).

1. Prove that every eigenvalue of A is strictly positive.

2. Deduce that A is invertible and that the eigenvalues of A^{-1} are λ_i^{-1} (hence strictly positive).

Lecture 5 - Normed spaces, Orthogonality

Exercise 1 Show that for any $u, v \in \mathbb{R}^n$:

$$\|u + v\|_2^2 = \|u\|_2^2 + 2\langle u, v\rangle + \|v\|_2^2$$

where $\|\cdot\|_2$ is the Euclidean norm and $\langle \cdot, \cdot \rangle$ is the dot product.

Exercise 2 Show that for any $u, v \in \mathbb{R}^n$ and any matrix $A \in \mathbb{R}^{m\times n}$:

$$\langle u, Av\rangle = \langle A^T u, v\rangle.$$

Exercise 3 Let $V = \text{span}\{v\} \subset \mathbb{R}^2$ with $v = (1, 1)^T$. Find the orthogonal complement $V^\perp$.