Quiz — Mathematics for Data Science

Courses 1 and 2: Vector spaces, linear applications, matrices

True/False questions

Work over $\mathbb{R}$ unless otherwise stated. For each statement, indicate whether it is **True** or **False** and justify briefly.

1. If $(G, \star)$ is a commutative group, then $x \star y = y \star x$ for all $x, y \in G$.
2. If $(V, +, \cdot)$ is a vector space over $\mathbb{R}$, then $(V, +)$ is a commutative group.
3. In a vector space $(V, +, \cdot)$, the operation $+$ need not be commutative.
4. A nonempty subset $S \subseteq V$ that is closed under addition and scalar multiplication is always a subspace, even if $0 \notin S$.
5. The span of any two nonzero vectors in $\mathbb{R}^2$ is always $\mathbb{R}^2$, even if the vectors are parallel.
6. In any vector space $(V, +, \cdot)$ over $\mathbb{R}$, there exists $1 \in \mathbb{R}$ such that $1 \cdot x = x$ for all $x \in V$.
7. In a vector space, subtraction is always defined by $x - y := x + (-y)$.
8. $(\mathbb{R}, +, \cdot)$ is a vector space over $\mathbb{Q}$ (with scalar multiplication by rationals).
9. If E and F are vector spaces, then $E \times F$ with $(x, y) + (x', y') = (x + y,\, y' + x')$ is a vector space.
10. If $L : U \to V$ is linear, then $L(x + y) = L(x)\, L(y)$ for all $x, y \in U$.
11. $(\mathbb{C}, +, \cdot)$ is a vector space over $\mathbb{R}$.
12. The set of all real $n \times p$ matrices $\mathcal{M}_{n,p}(\mathbb{R})$ is a vector space over $\mathbb{R}$.
13. If $L : U \to V$ is linear and $\mathrm{range}(L) = \{0_V\}$, then $L \neq 0$ (the zero map).
14. A linear map $L : U \to V$ is injective if and only if $\ker(L) = U$.
15. A linear map $L : U \to V$ is surjective if and only if $\mathrm{range}(L) = \{0_V\}$.
16. The set of real-valued sequences that converge is a subspace of the set of all real-valued sequences.
17. (Rank–Nullity) For linear $L : V \to W$ with V finite-dimensional, one has $\mathrm{rank}(L) + \dim\ker(L) = \dim(W)$.
18. If $A \in \mathbb{R}^{m \times n}$, then $\mathrm{rank}(A^\top) \geq \mathrm{rank}(A) + 1$ unless $A = 0$.
19. For any matrix A, $\mathrm{rank}(AA^\top) = \mathrm{rank}(A)^2$.
20. If $A, B \in \mathbb{R}^{n \times n}$ and AB is invertible, then exactly one of A or B is invertible.
21. The set of polynomials of degree at most k (with $k < n$) is a subspace of the set of polynomials of degree at most n.
22. Let $S \subseteq V$. If S is closed under addition and scalar multiplication but $0 \notin S$, then S is a subspace of V.
23. If $A \in \mathbb{R}^{n \times n}$ satisfies $A^2 = 0$, then necessarily $A = 0$.

24. If $A, B \in \mathbb{R}^{n\times n}$ are similar, then $A = B$.

25. If the columns of $A \in \mathbb{R}^{m\times n}$ span $\mathbb{R}^m$, then $\ker(A) = \{0\}$ for every $n \geq m$.

26. If E and F are $\mathbb{F}$-vector spaces, then $E\times F$ endowed with $(x, y)+(x', y') = (x+x', y+y')$ and $\lambda \cdot (x, y) = (\lambda x, \lambda y)$ is an $\mathbb{F}$-vector space.

27. In $\mathbb{R}^n$, vector addition and scalar multiplication are defined component-wise.

28. If $L : U \to V$ is linear, then $\ker(L)$ is a subspace of U and $\text{range}(L)$ is a subspace of V.

29. If $Q \in \mathbb{R}^{n\times n}$ satisfies $Q^\top Q = I_n$, then its columns are orthogonal but need not have unit norm.

30. If $v_1, \ldots, v_k$ are linearly independent in V, then $\text{span}\{v_1, \ldots, v_k\}$ cannot be all of V.

31. A linear map $L : U \to V$ is injective if and only if $\ker(L) = \{0_U\}$.

32. A linear map $L : U \to V$ is surjective if and only if $\text{range}(L) = V$.

33. If F and G are subspaces of E, then $F \cap G$ need not be a subspace of E.

34. For a linear map $L : U \to V$, one has $\text{range}(L) = \{0_V\}$ if and only if $L = 0$ (the zero map).

35. If U and W are subspaces with $U \cap W = \{0\}$, then $U + W$ is not a direct sum.

36. Finite-dimensional vector spaces over the same field and of the same dimension are isomorphic.

37. (Rank–Nullity) If $L : V \to W$ is linear with V finite-dimensional, then $\text{rank}(L) + \dim\ker(L) = \dim(V)$.

38. For finite-dimensional spaces $E_1, \ldots, E_n$, one has $\dim(E_1 \times \cdots \times E_n) = \prod_{i=1}^n \dim(E_i)$.

39. For subspaces $U, W \subseteq V$, $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$.

40. If $U \cap W = \{0\}$, then every $v \in U \oplus W$ can be written uniquely as $v = u + w$ with $u \in U$, $w \in W$.

41. If $U \oplus W$ is a direct sum, then $\dim(U \oplus W) = \dim(U) + \dim(W)$.

42. If $E_1, \ldots, E_n$ are finite-dimensional vector spaces over $\mathbb{F}$, then $\dim(E_1 \times \cdots \times E_n) = \sum_{i=1}^n \dim(E_i)$.

43. Let V be a vector space and $a_1, \ldots, a_n \in \mathbb{R}$. The set $F = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n a_i x_i = 0\}$ is a subspace of $\mathbb{R}^n$.

44. If F and G are subspaces of a vector space E, then $F \cap G$ is a subspace of E.

45. Let E, F be vector spaces over $\mathbb{F}$, let $\mathcal{B}_E = (e_1, \ldots, e_n)$ be a basis of E, and let $\mathcal{F} = (f_1, \ldots, f_n)$ be vectors in F. Then there exists a unique linear map $g : E \to F$ with $g(e_i) = f_i$ for all i; moreover, g is injective iff $\mathcal{F}$ is linearly independent, surjective iff $\mathcal{F}$ spans F, and bijective iff $\mathcal{F}$ is a basis of F.

Solutions

1. **True.** This is exactly the definition of a commutative (abelian) group.
2. **True.** By the vector space axioms, $(V, +)$ is a commutative group.
3. **False.** In a vector space, $(V, +)$ must be an abelian (commutative) group by axiom.
4. **False.** Any subset closed under scalar multiplication already contains 0 (since $0 \cdot x = 0$). If $0 \notin S$, it cannot be a subspace.
5. **False.** If the two vectors are parallel (colinear), their span is a line, not $\mathbb{R}^2$.
6. **True.** The scalar multiplicative identity axiom requires $1 \cdot x = x$ for all $x \in V$.
7. **True.** Additive inverses exist in $(V, +)$; define $x - y := x + (-y)$.
8. **True.** Restricting scalars from $\mathbb{R}$ to the subfield $\mathbb{Q}$ makes $\mathbb{R}$ a $\mathbb{Q}$-vector space.
9. **False.** With $(x, y) + (x', y') = (x + y,\, y' + x')$, $(0, 0)$ is not a neutral element: $(a, b) + (0, 0) = (a, 0) \neq (a, b)$ (take $b \neq 0$).
10. **False.** Linearity gives $L(x + y) = L(x) + L(y)$; in general the product $L(x)\, L(y)$ is undefined (or wrong). E.g. $U = V = \mathbb{R}$, $L = \mathrm{id}$: $L(1 + 1) = 2 \neq 1 \cdot 1$.
11. **True.** $\mathbb{C}$ is a vector space over $\mathbb{R}$ via real scalar multiplication.
12. **True.** $\mathcal{M}_{n,p}(\mathbb{R})$ is closed under entrywise addition and real scalar multiplication, and contains 0.
13. **False.** $\mathrm{range}(L) = \{0\}$ iff L is the zero map.
14. **False.** Injective $\Leftrightarrow \ker(L) = \{0\}$, not $\ker(L) = U$ (the latter means $L = 0$).
15. **False.** Surjective $\Leftrightarrow \mathrm{range}(L) = V$, not $\{0\}$ (unless $V = \{0\}$).
16. **True.** Sum and real scalar multiples of convergent sequences converge; the zero sequence converges.
17. **False.** Rank–Nullity reads $\mathrm{rank}(L) + \dim\ker(L) = \dim(V)$ (domain dimension), not $\dim(W)$.
18. **False.** Always $\mathrm{rank}(A^\top) = \mathrm{rank}(A)$, never strictly larger by 1.
19. **False.** In fact $\mathrm{rank}(AA^\top) = \mathrm{rank}(A)$ (for all A), not the square of the rank.
20. **False.** If AB is invertible, then both A and B are invertible.
21. **True.** Degree $\leq k$ polynomials are closed under addition and scalar multiplication and contain 0.
22. **False.** A subspace must contain 0; if $0 \notin S$, S cannot be a subspace.
23. **False.** There exist nonzero nilpotent matrices (e.g. $J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ with $J^2 = 0$ and $J \neq 0$).
24. **False.** Similar matrices need not be equal (e.g. $\mathrm{diag}(1, 2)$ is similar to $\mathrm{diag}(2, 1)$ via a permutation matrix).

25. **False.** If $n > m$, columns can span $\mathbb{R}^m$ while $\ker(A) \neq \{0\}$ (rank–nullity gives $\dim\ker(A) = n - m > 0$). Example: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

26. **True.** With $(x, y) + (x', y') = (x + x', y + y')$ and $\lambda(x, y) = (\lambda x, \lambda y)$, $E \times F$ is a vector space.

27. **True.** This is the standard definition in $\mathbb{R}^n$.

28. **True.** For linear L, $\ker(L) \leq U$ and $\mathrm{range}(L) \leq V$ (closure follows from linearity).

29. **False.** $Q^\top Q = I$ implies columns are orthonormal (orthogonal and unit norm).

30. **False.** A linearly independent family can span V (e.g. any basis).

31. **True.** Injectivity $\Leftrightarrow$ only solution of $Lx = 0$ is $x = 0 \Leftrightarrow \ker(L) = \{0\}$.

32. **True.** Surjectivity means $\mathrm{range}(L) = V$ by definition of range/image.

33. **False.** Intersection of subspaces is always a subspace.

34. **True.** If $\mathrm{range}(L) = \{0\}$ then $L = 0$; conversely the zero map has that range.

35. **False.** By definition, $U \cap W = \{0\}$ implies $U + W$ is a direct sum $U \oplus W$.

36. **True.** Finite-dimensional spaces over the same field with equal dimension are isomorphic.

37. **True.** Rank–Nullity: $\mathrm{rank}(L) + \dim\ker(L) = \dim(V)$ for linear $L : V \to W$ with V finite-dimensional.

38. **False.** For finite-dimensional spaces, $\dim(E_1 \times \cdots \times E_n) = \sum_{i=1}^n \dim(E_i)$ (sum, not product).

39. **True.** Standard dimension formula for sums of subspaces.

40. **True.** Direct sum ($U \cap W = \{0\}$) gives uniqueness of the decomposition $v = u + w$.

41. **True.** In a direct sum, dimensions add: $\dim(U \oplus W) = \dim U + \dim W$.

42. **True.** $\dim(E_1 \times \cdots \times E_n) = \sum_i \dim(E_i)$ for finite-dimensional spaces.

43. **True.** It is the kernel of the linear functional $x \mapsto \sum_{i=1}^n a_i x_i$.

44. **True.** Intersection of subspaces is a subspace.

45. **True.** A linear map is uniquely determined by images of a basis; injective $\Leftrightarrow$ images are independent, surjective $\Leftrightarrow$ images span F, hence bijective $\Leftrightarrow$ they form a basis of F.