Mathematics for Data Science

Lecture 1

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Group

Definition

Let G be a non-empty set. G is a group if there exists an operation $\star$ such that:

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Let G be a non-empty set. G is a group if there exists an operation $\star$ such that:

1. **Closure.** For any $\mathbf{x}, \mathbf{y} \in G$, $\mathbf{x} \star \mathbf{y}$ also belongs to G.
2. **Associativity.** For any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in G$, $(\mathbf{x} \star \mathbf{y}) \star \mathbf{z} = \mathbf{x} \star (\mathbf{y} \star \mathbf{z})$.
3. **Neutral element/Identity.** There exists $e \in G$ such that $\mathbf{x} \star \mathbf{e} = \mathbf{e} \star \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in G$.
4. **Symmetric element/Inverse.** For each $\mathbf{x} \in G$, there exists an element $\mathbf{x}' \in G$, such that $\mathbf{x}' \star \mathbf{x} = \mathbf{x} \star \mathbf{x}' = e$.

Commutative group

Definition

Let G be a group, If for any $x, y \in G$, $x \star y = y \star x$ then we say that the group is commutative.

NB: "Abelian" group is another name for commutative groups

Vector spaces

Vector spaces: the basic setting in which linear algebra happens
Elements of V are called **vectors**.

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Elements of V are called **vectors**.

Vector space

A vector space $(V, +, \cdot)$ on $\mathbb{R}$ is a set endowed with two operations:

- $+ : V \times V \to V$ that allows to sum two vectors: $(x, y) \mapsto x + y$
- $\cdot : \mathbb{R} \times V \to V$ that is the multiplication of a vector by a **scalar**: $(a, x) \mapsto a \cdot x$

Vector spaces can be defined over any **field** $\mathbb{F}$. We take $\mathbb{F} = \mathbb{R}$ in this course.

Vector spaces

$(V, +, \cdot)$ must follow a number of axioms:

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- $(V, +)$ is a commutative group.
  - **Associativity.** For any $x, y, z \in V$, $(x + y) + z = x + (y + z)$
  - **Additive identity.** There exists an element in V, denoted $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x} \in V$.
  - **Additive inverse.** For any $\mathbf{x} \in V$, there exists an element in V, denoted $-\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
  - **Commutativity.** For any $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.

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  - **Commutativity.** For any $\mathbf{x}, \mathbf{y} \in V$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
- The law $\cdot$ verifies:
  - **Distributivity (left).** For all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha, \beta \in \mathbb{R}$, $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$.
  - **Distributivity (right).** For all $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{R}$, $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.
  - **"Mixed" associativity.** For all $\mathbf{x} \in V$ and $\alpha, \beta \in \mathbb{R}$, $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$.
  - **Multiplicative identity.** There exists an element in $\mathbb{R}$, denoted 1, such that $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.

Remark

Can we substract vectors ?

Remark

Can we substract vectors ?

We can subtract vectors because a subtraction is the addition of the opposite vector.

Examples of Vector Spaces

Example

- $(\mathbb{R}, +, \cdot)$ is a vector space where $\mathbb{R}$ is the set of real numbers
- $(\mathbb{R}, +, \cdot)$ is also a vector space with the $\cdot$ law defined on $\mathbb{Q}$
- $(\mathbb{C}, +, \cdot)$ with the $\cdot$ law defined on $\mathbb{R}$ or $\mathbb{Q}$
- The set of matrices with $n \times p$ real-valued coefficients $\mathcal{M}_{n,p}(\mathbb{R})$
- The set of functions from an interval $I \subset \mathbb{R}$ to $\mathbb{R}$
- The set of sequences in $\mathbb{R}$
- The set of polynoms in $\mathbb{R}$, denoted $\mathbb{R}_n[X] = \{a_0 + a_1 X + \cdot + a_n X^n \mid a_0, a_1, \dots a_n \in \mathbb{R}\}$, with the following laws: for $P_1, P_2 \in \mathbb{R}_n[X]$, $P_1(X) + P_2(X) = a_0 + a_1 X + \cdot + a_n X^n + a'_0 + a'_1 X + \cdot + a'_n X^n$ $= (a_0 + a'_0) + (a_1 + a'_1)X + \cdots + (a_n + a'_n)X^n$ and for $\lambda \in \mathbb{R}$, $\lambda P_1[X] = \sum \lambda a_i X^i$
- $(\mathbb{R}^n, +, \cdot)$ is a vector space, called the **Euclidean space**.

Product of vector spaces

Definition

Cartesian product of two sets A and B, denoted $A \times B$:

$$A \times B = \{(a, b), a \in A, b \in B\}$$

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Definition

Let E and F be two $\mathbb{F}$-vector spaces. The sum of $(x, y) \in E \times F$ and $(x', y') \in E \times F$ is defined as $(x, y) + (x', y') = (x + x', y + y')$. The multiplication of (x, y) by a scalar $\lambda \in \mathbb{F}$ is defined as $\lambda \cdot (x, y) = (\lambda \cdot x, \lambda \cdot y)$.

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Proposition

Endowed with the above operations, $(E \times F, +, \cdot)$ is a $\mathbb{F}$-vector space. It is called the **product vector space** *of E by F.*

Euclidean space

Let n be a positive integer.

Example

$(\mathbb{R}^n, +, \cdot)$ is a vector space, called the **Euclidean space**.

Euclidean space

The vectors in the Euclidean space consist of n-tuples of real numbers, i.e. for $x \in \mathbb{R}^n, x = (x_1, x_2, \ldots, x_n)$. The x_i are the components or entries of the vector.

NB: you can think of vectors of $\mathbb{R}^n$ as $n \times 1$ matrices, or **column vectors**.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Examples of Operations

Example

- Summation of two vectors in $\mathbb{R}^2$.
- Multiplication by a scalar in $\mathbb{R}^n$.

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- Multiplication by a scalar in $\mathbb{R}^n$.

Addition and scalar multiplication are defined component-wise on vectors in $\mathbb{R}^n$:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}, \quad \alpha\mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Geometrical interpretation

In $\mathbb{R}^n$, a vector $\mathbf{x}$ has a **direction** and an **amplitude**.

Scalar multiplication changes the amplitude but keeps the same direction.

Parallel vectors: We say that two vectors $v, w \in \mathbb{R}^n$ are **parallel** if there is a non-zero scalar $r \in \mathbb{R}$ such that

$$w = rv.$$

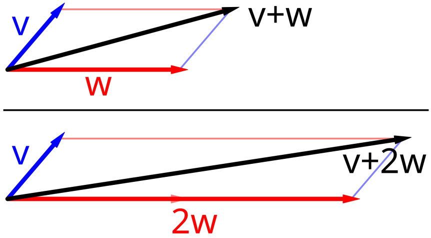

Figure 1: Vector addition and scalar multiplication: a vector $\mathbf{v}$ (blue) is added to another vector $\mathbf{w}$ (red, upper illustration). Below, $\mathbf{w}$ is multiplied by a factor of 2, yielding the sum $\mathbf{v} + 2\mathbf{w}$. (image from Wikipedia)

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Subspaces

Vector spaces can contain other vector spaces.

Subspaces

If V is a vector space, then $S \subseteq V$ is said to be a **subspace** of V if

1. $\mathbf{0} \in S$
2. S is closed under addition: $\mathbf{x}, \mathbf{y} \in S$ implies $\mathbf{x} + \mathbf{y} \in S$
3. S is closed under scalar multiplication: $\mathbf{x} \in S, \alpha \in \mathbb{R}$ implies $\alpha\mathbf{x} \in S$

Subspaces

Example

- V is always a subspace of V.
- The trivial vector space which contains only $\mathbf{0}$.
- A line passing through the origin is a subspace of the Euclidean space.
- The set of real-valued sequences that converge is a subspace of the set of real-valued sequences.
- The set of continuous fonctions from an interval $I \subset \mathbb{R}$ into $\mathbb{R}$ is a subspace of the set of fonctions from I to $\mathbb{R}$.
- The set of polynomials from degree at most $k < n$ is a subspace of the set of polynomials of degree at most n.

Subspaces

Proposition

Let V be a vector space on $\mathbb{F}$. Any subspace U of V is a vector space on $\mathbb{F}$.

Exercise: Proof. By definition, the null element belongs to U. Verify that the composition rules $+, \cdot$ defined on V also hold for U.

Linear Combination

Linear combination

Given vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ and scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$ we say that a vector of the form

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

is a **linear combination** of the vectors $v_1, v_2, \ldots, v_k$ with scalar coefficients $c_1, c_2, \ldots, c_k$.

NB : The scalars c_i are sometimes called **weights**.

Span (*Sous-espace vectoriel engendré*)

Span

The **span** of a set of vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ is the set of all the vectors which can be written as a linear combination of the $v_1, v_2, \ldots, v_k$, i.e.

$$\mathrm{span}\{v_1, v_2, \ldots, v_k\} = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \mid c_1, c_2, \ldots, c_k \in \mathbb{R}\}.$$

NB: equivalent to the French notation $Vect(v_1, v_2, \ldots, v_k)$

Example

- $\mathrm{span}((1,0),(0,1)) = \mathbb{R}^2$
- What is the span of $((2,0),(0,1))$?
- What is the span of $((1,0),(3,0))$?

Properties of subspaces

Proposition

Let $n \in \mathbb{N}, a_1, a_2, \dots a_n \in \mathbb{R}$. The following set F is a vector subspace of $\mathbb{R}^n$.

$$\{(x_1, x_2, \dots x_n), \sum_{i=1}^{n} a_i x_i = 0\}$$

Proposition

Let F and G be subspaces of a vector space E. Then $F \cap G$ is a subspace of E.

Geometric interpretation

Given p_0, fixed in $\mathbb{R}^n$, a line in $\mathbb{R}^n$ is given by (cf affine equation in $\mathbb{R}^2$):

$$\ell = \{x \in \mathbb{R}^n \mid x = p_0 + tv,\ \ t \in \mathbb{R}\}.$$

A special case is $\mathrm{span}\{v\} = \{tv \mid t \in \mathbb{R}\}$, the line through the origin ($p_0 = 0$) in direction v.

Example

Let $p_1 = (1, 2)$ and $p_2 = (3, 1)$ be on

$$\ell = \{(1, 2) + (2, -1)t \mid t \in \mathbb{R}\}.$$

Is $p_1 + p_2$ on the line ?

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Is $p_1 + p_2$ on the line ?

NB : If $u, w \in \mathrm{span}\{v_1, \ldots, v_k\}$, then $u + w \in \mathrm{span}\{v_1, \ldots, v_k\}$.
However, the sum of two points on a line need not be on the line.

Geometric interpretation

A plane in $\mathbb{R}^n$ through p_0 with directions $\mathbf{u}, \mathbf{v}$ is

$$P = \{x \in \mathbb{R}^n \mid x = p_0 + s\mathbf{u} + t\mathbf{v},\ \ s, t \in \mathbb{R}\}.$$

If $p_0 = 0$, then $P = \mathrm{span}\{\mathbf{u}, \mathbf{v}\}$.

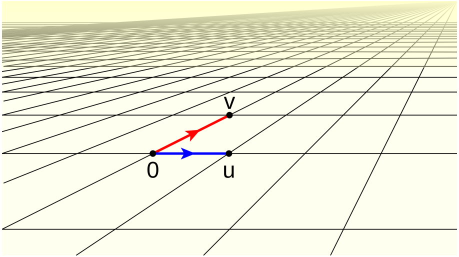

Figure 2: The cross-hatched plane is the linear span of **u** and **v** in both $\mathbb{R}^2$ and $\mathbb{R}^3$ (figure in perspective from Wikipedia).

Geometric interpretation

Hypersurfaces: In $\mathbb{R}^n$, through p_0 in directions $v_1, \ldots, v_k$:

$$P = \{x \in \mathbb{R}^n \mid x = p_0 + s_1 v_1 + \cdots + s_k v_k,\ s_1, \ldots, s_k \in \mathbb{R}\}.$$

If $p_0 = 0$, then $P = \mathrm{span}\{v_1, \ldots, v_k\}$.

Spanning List

Spanning list

A **spanning list** of V is a list whose span is V.

NB: French speakers "famille génératrice de V"

The family of vectors $v_1, v_2, \dots v_k$ spans the $\mathbb{R}$-vector space V if for any $x \in V$, there exist coefficients $\lambda_1, \lambda_2 \dots \lambda_k$ such that $x = \sum_{i=1}^{k} \lambda_i v_i$.

Example

- Verify that $\{(2,1),(0,1)\}$ is a spanning list of $\mathbb{R}^2$.
- Let us consider $V = \{(x,0,0), x \in \mathbb{R}\}$ and $v_1 = (1,0,0)$. Verify that V is a vector space and that (v_1) is a spanning list of V.

Linear Independence

Linear independence

A list/family of vectors $v_1, \ldots, v_k$ is **linearly independent**/free if none of the vectors can be written as a linear combination of the others, i.e.

$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0 \quad \Rightarrow \quad \alpha_1 = \cdots = \alpha_k = 0$$

Example

Are the following vectors linearly independent ?
$v_1 = (1, 0, 0), v_2 = (3, 0, 0), v_3 = (0, 0, 1)$
$v_1 = (1, 0), v_2 = (0, 2)$

Basis of a vector space

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A linearly-independent spanning list of vectors from a vector space V is called a **basis** of V.

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Example

The standard basis of $\mathbb{R}^n$, called the **canonical basis**:

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, 0, \dots, 0), \quad \dots, \quad e_n = (0, \dots, 0, 1)$$

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Proposition

Given a basis, every vector has a unique coordinate representation.

Dimension of a vector space

Proposition

Let V be a finite-dimensional vector space. Then:

1. *V admits a basis.*
2. *All bases of V have the same cardinality.*
3. *Any family of linearly independant vectors from V can be completed to obtain a basis of V (FR: "théorème de la base incomplète")*
4. *A basis of V can be extracted from any spanning family of V (FR: "théorème de la base extraite").*

Dimension

All bases of a vector space have the same length, called the **dimension**.

Example

$\dim(\mathbb{R}^n) = n$

Rank of a family of vectors

Definition

Let $\mathcal{F} = (v_1, v_2, ...v_k)$ be a family of vectors from a vector space V. The **rank** of $\mathcal{F}$ is dimension of its the span, i.e. the dimension of the vector space generated by all the linear combinations of vectors from $\mathcal{F}$.

$$rank(\mathcal{F}) = dim(span(\mathcal{F}))$$

Sum of subspaces

Definition

Let F and G be subspaces of E. Their sum is defined as

$$F + G = \{x \in V, \exists u \in F, v \in G, x = u + v\}$$

Exercise: Verify that this set is also a subspace of V.

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Definition

If $U \cap W = \{0\}$, the sum is said to be a **direct sum** and written $U \oplus W$.

NB : (*sous-espaces vectoriels supplémentaires* in French)

Proposition

Every vector in $U \oplus W$ can be written uniquely as the sum of a vector from U and a vector from W.

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Function

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A **function** f from a set X to a set Y assigns to each element of X exactly one element of Y. Notation: $f : X \to Y$.

Vocabulary

Function

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Vocabulary

- The set X is called the **domain** of the function
- The set Y is called the **codomain** of the function.
- If the element $y \in Y$ is assigned to $x \in X$ by the function f, one says that f **maps** x to y, and this is commonly written $y = f(x)$.
- In this notation, x is the **argument** or **variable** of the function.
- A specific element x of X is a **value** of the variable, and the corresponding element of Y is the value of the function at x, or the **image** of x under the f.

Remark: functional notation, arrow notation

Vocabulary

NB: In some branches of mathematics, the term map is used to mean a function.

- A **homomorphism** is a structure-preserving map between two algebraic structures of the same type (e.g. two groups, or two vector spaces).
- Homomorphisms of vector spaces are also called **linear maps**, **linear mappings** or **linear transformations**.
- A homomorphism from a vector space V to the same vector space is called an **endomorphism**.
- A bijective homomorphism is called an **isomorphism**.

Image (range) and Preimage (inverse image)

Image/range

The image (or range) of a function is the set of the images of all the elements in the domain. Notation: $f(X)$.

$$f(X) = \{f(x) \mid x \in X\}$$

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NB: If A is a subset of X, then the image of A under f, denoted $f(A)$, is the subset of the codomain Y consisting of all images of elements of A. We have $f(A) \subset f(X)$.

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Preimage

The **inverse image** or **preimage** under f of an element y of the codomain Y is the set of all elements of the domain X whose images under f equal y. Notation : $f^{-1}(y)$.

$$f^{-1}(y) = \{x \in X \mid f(x) = y\}$$

Bijection, injection, surjection

Let $f : X \to Y$ be a function.

Injectivity

The function f is injective if $f(a) \neq f(b)$ for every two different elements a and b of X. Equivalently, f is injective iff, for every $y \in Y$, the preimage $f^{-1}(y)$ contains at most one element.

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Surjectivity

The function f is surjective if its image $f(X)$ equals its codomain Y. That is, for every element $y \in Y$, there exists an element $x \in X$ such that $f(x) = y$. Equivalently : $\forall y \in Y, f^{-1}(y) \neq \emptyset$.

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Bijectivity

The function f is bijective if it is both injective and surjective.

Bijection, injection, surjection

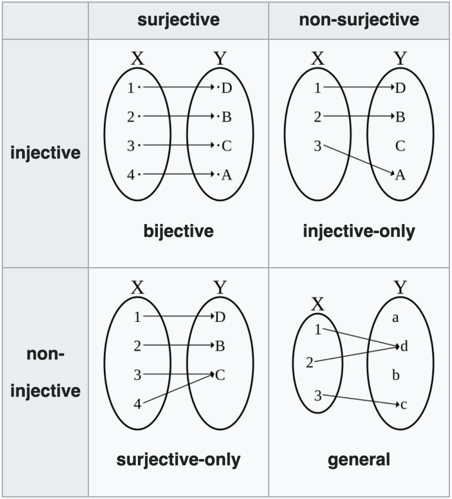

Figure 3: image source: Wikipedia

Linear transformation

Linear transformation

$L : V \to W$ is linear if for all $v_1, v_2 \in V$ and $a, b \in \mathbb{R}$,

$$L(av_1 + bv_2) = aL(v_1) + bL(v_2)$$

Example

- Show that $L : (x, y) \mapsto (x, 0)$ is linear.
- Show that $L : (x, y) \mapsto 10(x, y)$ is linear.

Kernel / Null Space

Kernel / Null space

If $L : V \to W$ is a linear map, we define the **nullspace** (also called **kernel**[a]) and denoted null(L) (or $ker(L)$) of L as

$$\text{null}(L) = \{\mathbf{x} \in V \mid L(\mathbf{x}) = \mathbf{0}\}$$

[a] Watch out, the word "kernel" has another meaning in machine learning.

Example

For $L(x, y) = x$, $\ker(L) = \{(0, y) \mid y \in \mathbb{R}\}$.

Reminder: the range of a linear map $L : U \mapsto V$ writes :

$$range(L) = \{y \in V \mid \exists x \in U, L(x) = y\}.$$

Properties of subspaces and linear maps

Proposition

If $L : U \mapsto V$ is a linear map, then

1. *the kernel of L is a subspace of U*
2. *the range of L is a subspace of V*
3. *L is injective iff $ker(L) = \{0_U\}$*
4. *L is surjective iff $range(L) = V$*

Exercise: Proof. Hint: show that the range is closed under vector addition and scalar multiplication. Use the linearity of L to show that $L(0_U) = 0_V$.

Proposition (Range of the zero map)

If $L : U \mapsto V$ is a linear map, then

$$range(L) = \{0_V\} \iff L = 0$$

Properties of subspaces and linear maps

Proposition

Let E, F be two vector spaces defined over $\mathbb{F}$, $\mathcal{B}_E = (e_1, e_2, \dots e_n)$ a basis of E and $\mathcal{F} = (f_1, f_2, \dots f_k)$ a family of vectors in F. Then there exists a unique linear application $g : E \mapsto F$ such that

$$\forall i, 1 \leq i \leq n, g(e_i) = f_i$$

In addition:

1. *g is injective iff $\mathcal{F}$ is free.*
2. *g is surjective iff $\mathcal{F}$ spans F.*
3. *g is bijective iff $\mathcal{F}$ is a basis of F.*

Isomorphic vector spaces

Definition

We say that two vector spaces E and F are **isomorphic** if there exists an isomorphism from E to F.

NB: equivalently, we could write “from F to E” since it is a bijection.

Proposition

Finite-dimensional vector spaces (defined over the same field) of the same dimension are isomorphic.

NB: Using the previous property, with $dim(U) = dim(V) = n$, $\mathcal{B} = (e_1, e_2, \dots e_n)$ a basis of U and $\mathcal{F} = (f_1, f_2, \dots f_n)$ a basis of V, we know that there exists a unique linear application $g : E \mapsto F$ such that

$$\forall i, 1 \leq i \leq n, g(e_i) = f_i$$

and that g is bijective.

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Rank

Rank of a linear transformation

The **rank** of a linear transformation $L : V \to W$ is:

$$\mathrm{rank}(L) = \dim(L(V))$$

where $L(V) = \{y \in W \mid y = L(x),\ x \in V\}$.

Reminder: $L(V)$ is called the **range** or **image** of L. We also write $rank(L) = \dim(\mathrm{Im}(L))$.

Rank

Rank of a linear transformation

The **rank** of a linear transformation $L : V \to W$ is:

$$\operatorname{rank}(L) = \mathsf{dim}(L(V))$$

where $L(V) = \{y \in W \mid y = L(x),\ x \in V\}$.

Reminder: $L(V)$ is called the **range** or **image** of L. We also write $rank(L) = \mathsf{dim}(\mathrm{Im}(L))$.

Example

Example: $L : \mathbb{R}^2 \to \mathbb{R}$ given by $L(x, y) = x$ has rank 1.

Properties of the rank

Proposition

Let $L : U \mapsto V$ *be a linear map,* U *and* V *of finite dimension. Then* L *is bijective iff* $rank(L) = dim(V) = dim(U)$.

Cartesian product and dimension

Proposition

Let $n > 2$ an integer, we suppose that $E_1, ...E_n$ are n finite-dimensional vector spaces on $\mathbb{F}$. Then their cardinal product is finite-dimensional, with dimension

$$dim(E_1 \times E_2 \times \cdots \times E_n) = \sum_{i=1}^{n} dim(E_i).$$

Exercise: Proof (reasoning by recurrence)

Rank-Nullity Theorem

Rank-Nullity Theorem

If V, W are finite-dimensional vector spaces and $L : V \to W$ is linear:

$$\mathrm{rank}(L) + \dim(\ker(L)) = \dim(V)$$

Example

For $L : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto (2x, 0)$, $\mathrm{rank}(L) = 1$ and $\dim(\ker(L)) = 1$.

Proof

Sum of subspaces

Proposition

For U and W subspaces of V,

$$dim(U + W) = dim(U) + dim(W) - dim(U \cap W)$$

Exercise: Proof.

Corollary

$dim(U \oplus W) = dim(U) + dim(W)$ *for a direct sum.*

Next topics

Next class : matrices (Lecture 2)