Lecture 5 - Normed spaces, Orthogonality

1 Short computations

Question 1 Show that for any $u, v \in \mathbb{R}^n$:

$$\|u+v\|_2^2 = \|u\|_2^2 + 2\langle u, v\rangle + \|v\|_2^2$$

where $\|\cdot\|_2$ is the Euclidean norm and $\langle \cdot, \cdot \rangle$ is the dot product.

Question 2 Show that for any $u, v \in \mathbb{R}^n$ and any matrix $A \in \mathbb{R}^{m\times n}$:

$$\langle u, Av\rangle = \langle A^T u, v\rangle.$$

Question 3 Let $V = \text{span}\{v\} \subset \mathbb{R}^2$ with $v = (1,1)^T$. Find the orthogonal complement $V^\perp$.

Question 1 — Solution.

$$\|u+v\|_2^2 = \langle u+v,\, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle.$$

In $\mathbb{R}^n$ the dot product is symmetric, so $\langle u, v\rangle = \langle v, u\rangle$. Hence

$$\|u+v\|_2^2 = \|u\|_2^2 + 2\langle u, v\rangle + \|v\|_2^2.$$

Question 2 — Solution. Using matrix–vector notation for the Euclidean inner product,

$$\langle u, Av\rangle = u^\top (Av) = (u^\top A)\, v = (A^\top u)^\top v = \langle A^\top u,\, v\rangle.$$

Question 3 — Solution. By definition,

$$V^\perp = \{\, w \in \mathbb{R}^2 :\ \langle w, v\rangle = 0 \,\} = \{(x,y)^\top \in \mathbb{R}^2 :\ x+y=0\}.$$

Thus $y = -x$ and every such vector is of the form $x(1,-1)^\top$. Therefore

$$V^\perp = \text{span}\{(1,-1)^\top\}.$$

2 Short proofs

Question 4 Show that the dot product in $\mathbb{R}^n$ satisfies the following properties:

1. Symmetry: $\langle x, y\rangle = \langle y, x\rangle$.
2. Homogeneity: $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$.
3. Linearity in the first argument: $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
4. Linearity in the second argument: $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

Solution.

For $x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n),\ z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, define $\langle x, y\rangle := \sum_{i=1}^n x_i y_i$.

1. **Symmetry.**
$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} y_i x_i = \langle y, x\rangle,$$
by commutativity of multiplication in $\mathbb{R}$.

2. **Homogeneity (in the first argument).**
$$\langle \lambda x, y\rangle = \sum_{i=1}^{n} (\lambda x_i) y_i = \sum_{i=1}^{n} \lambda (x_i y_i) = \lambda \sum_{i=1}^{n} x_i y_i = \lambda \langle x, y\rangle.$$

3. **Linearity in the first argument.**
$$\langle x + y,\ z\rangle = \sum_{i=1}^{n} (x_i + y_i) z_i = \sum_{i=1}^{n} x_i z_i + \sum_{i=1}^{n} y_i z_i = \langle x, z\rangle + \langle y, z\rangle.$$

4. **Linearity in the second argument.**
$$\langle x,\ y + z\rangle = \sum_{i=1}^{n} x_i (y_i + z_i) = \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i z_i = \langle x, y\rangle + \langle x, z\rangle.$$

Question 5 Let F be a subspace of a finite-dimensional inner product space E. Show the following properties of the orthogonal complement on F:

(i) $F^\perp$ is a subspace of E.

(ii) $F \cap F^\perp = \{\vec{0}_E\}$.

(iii) $\dim(F) + \dim(F^\perp) = \dim(E)$.

(iv) $(F^\perp)^\perp = F$.

Question 5 – Solution.

We use the standard properties of the inner product shown in Question 4: linearity in the first argument, symmetry, and positive-definiteness.

(i) $F^\perp$ is a subspace. By definition,
$$F^\perp := \{x \in E :\ \langle x, f\rangle = 0 \ \ \forall f \in F\}.$$

First, $0 \in F^\perp$ since $\langle 0, f\rangle = 0$ for all $z \in F$.

If $x, y \in F^\perp$ and $\lambda \in \mathbb{K}$, then for all $z \in F$,
$$\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle = 0 + 0 = 0$$
and
$$\langle \lambda x, z\rangle = \lambda \langle x, z\rangle = \lambda \cdot 0 = 0.$$
Hence $x + y \in F^\perp$ and $\lambda x \in F^\perp$. Thus $F^\perp$ is a subspace.

(ii) $F \cap F^\perp = \{\vec{0}_E\}$. If $x \in F \cap F^\perp$, then $x \in F$ and $\langle x, z\rangle = 0$ for all $z \in F$. In particular, taking $z = x$ (since $x \in F$) gives $\langle x, x\rangle = 0$. By positive-definiteness, this implies $x = \vec{0}_E$.

(iii) $\dim(F) + \dim(F^\perp) = \dim(E)$. Let $(e_1, \ldots, e_k)$ be an orthonormal basis of F. It can be extend it to an orthonormal basis $(e_1, \ldots, e_k, e_{k+1}, \ldots, e_n)$ of E.

For $j > k$ and any $f \in F$ we have $\langle e_j, f\rangle = 0$, so $\operatorname{span}\{e_{k+1}, \ldots, e_n\} \subseteq F^\perp$.

Conversely, if $x \in F^\perp$, for $1 \le i \le k$, we have : $\langle x, e_i\rangle = 0$

And writing $x = \sum_{i=1}^{n} \alpha_i e_i$ we get, for $1 \leq i \leq k$,

$$\langle x, e_i \rangle = \sum_{j=1}^{n} \alpha_j \langle e_j, e_i \rangle = \alpha_i,$$

so $\alpha_1 = \cdots = \alpha_k = 0$ and $x \in \operatorname{span}\{e_{k+1}, \ldots, e_n\}$. Hence $F^\perp = \operatorname{span}\{e_{k+1}, \ldots, e_n\}$ and $\dim(F^\perp) = n - k$. Therefore $\dim(F) + \dim(F^\perp) = k + (n-k) = n = \dim(E)$.

(iv) $(F^\perp)^\perp = F$. We always have $F \subseteq (F^\perp)^\perp$ by definition of orthogonal complement. By (iii),

$$\dim\left((F^\perp)^\perp\right) = \dim(E) - \dim(F^\perp) = \dim(F).$$

Thus a subspace $(F^\perp)^\perp$ containing F has the same dimension as F, forcing equality: $(F^\perp)^\perp = F$.

Question 6 Let F be a subspace of a finite-dimensional inner product space E. Let p_F be the orthogonal projection onto F. We remind Bessel inequality: for all $x \in E$,

$$\|p_F(x)\| \leq \|x\|$$

with equality if and only if $x \in F$. Prove it.

Question 6 – Solution. Let $x \in E$. We can write

$$x = p_F(x) + (x - p_F(x))$$

with $p_F(x) \in F$ and $x - p_F(x) \in F^\perp$.

By the Pythagorean theorem, we have

$$\|x\|^2 = \|p_F(x)\|^2 + \|x - p_F(x)\|^2 \geq \|p_F(x)\|^2$$

with equality if and only if $x - p_F(x) = 0$, i.e. $x \in F$.

Question 7. Let $p \in \mathcal{L}(E)$ a projector (i.e. $p \circ p = p$). Prove that the following propositions are equivalent:

(i) p is an orthogonal projector onto $F = \operatorname{range}(p)$

(ii) For all $x, y \in E$, $\langle p(x), y - p(y) \rangle = 0$

(iii) For all $x, y \in E$, $\langle p(x), y \rangle = \langle x, p(y) \rangle$

Hint: from (i), prove first that $\operatorname{Ker}(p) = F^\perp$

Question 7 – Solution.

Preliminaries. Since $p^2 = p$, we have for every $y \in E$

$$p(y - p(y)) = p(y) - p^2(y) = 0,$$

so $y - p(y) \in \operatorname{Ker}(p)$. So in fact $\operatorname{Ker}(p) = \{y - p(y),\ y \in E\}$, meaning that if p is a projector onto $F = \operatorname{range}(p)$, then $F \perp = \operatorname{Ker}(p)$. And every $y \in E$ decomposes as

$$y = p(y) + (y - p(y)) \in \operatorname{range}(p) \oplus \operatorname{Ker}(p).$$

(i)$\Rightarrow$(ii). If p is the orthogonal projector onto F and $\operatorname{Ker}(p) = F^\perp$, then for every $x, y \in E$ we have $p(x) \in F$ and $y - p(y) \in \operatorname{Ker}(p) = F^\perp$, hence

$$\langle p(x),\ y - p(y) \rangle = 0.$$

(ii)$\Rightarrow$(i). First, we show $\mathrm{Ker}(p) \subset F^{\perp}$. If $z \in \mathrm{Ker}(p)$, then $p(z) = 0$, and by (ii),

$$\langle p(x),\, z\rangle = \langle p(x),\, z - p(z)\rangle = 0 \qquad \text{for all } x \in E.$$

Thus z is orthogonal to every element of $F = \mathrm{range}(p)$, hence $z \in F^{\perp}$ and $\mathrm{Ker}(p) \subset F^{\perp}$

Conversely, let $u \in F^{\perp}$. Using (ii) with $y = u$ gives, for all $x \in E$,

$$\langle p(x),\, u\rangle = \langle p(x),\, p(u)\rangle.$$

Since $u \in F^{\perp}$, the left-hand side is 0 for all x, hence $\langle p(x),\, p(u)\rangle = 0$ for all x.

Choosing $x = p(u)$ yields

$$\langle p(u),\, p(u)\rangle = 0 \quad \Rightarrow \quad p(u) = 0,$$

so $u \in \mathrm{Ker}(p)$. Therefore $F^{\perp} \subset \mathrm{Ker}(p)$, and we conclude $\mathrm{Ker}(p) = F^{\perp}$.

Finally, for any $y \in E$ we have the decomposition

$$y = p(y) + \big(y - p(y)\big) \in F \oplus \mathrm{Ker}(p) = F \oplus F^{\perp},$$

and by (ii) with $x = y$ we get orthogonality of y and $y - p(y)$: $\langle p(y),\, y - p(y)\rangle = 0$. Thus p is the orthogonal projector onto F.

(iii)$\Rightarrow$(ii). For all $x, y \in E$,

$$\langle p(x),\, y - p(y)\rangle = \langle x,\, p(y - p(y))\rangle = \langle x,\, p(y) - p^2(y)\rangle = \langle x,\, p(y) - p(y)\rangle = 0.$$

(ii)$\Rightarrow$(iii). From (ii) we get, for all $x, y \in E$,

$$\langle p(x),\, y\rangle = \langle p(x),\, p(y)\rangle.$$

Interchanging x and y in (ii) gives $\langle p(y),\, x - p(x)\rangle = 0$, hence (by symmetry of the inner product) $\langle x - p(x),\, p(y)\rangle = 0$, i.e.

$$\langle x,\, p(y)\rangle = \langle p(x),\, p(y)\rangle.$$

Combining the last two equalities yields $\langle p(x),\, y\rangle = \langle x,\, p(y)\rangle$.

Conclusion : We have shown (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

3 Exercises

Exercise 1. We denote $E = C([-1;1], \mathbb{R})$.

Show that the mapping $\varphi : E^2 \to \mathbb{R}$ defined, for every $(f, g) \in E^2$, by

$$\varphi(f, g) = \int_{-1}^{1} \sqrt{1 - x^2}\, f(x)\, g(x)\, dx$$

is an inner product on E.

Exercise 1 – Solution.

First, for every $(f, g) \in E^2$, the function $x \mapsto \sqrt{1 - x^2}\, f(x) g(x)$ is continuous on the segment $[-1; 1]$, hence the integral defining $\varphi(f, g)$ exists.

- Symmetry and linearity with respect to the first slot are immediate.
- For every $f \in E$:

$$\varphi(f, f) = \int_{-1}^{1} \sqrt{1 - x^2}\, \underbrace{(f(x))^2}_{\geq 0}\, dx \geq 0.$$

- Let $f \in E$ such that $\varphi(f,f) = 0$, that is,

$$\int_{-1}^{1} \sqrt{1-x^2}\,(f(x))^2\,dx = 0.$$

  Since the function $x \mapsto \sqrt{1-x^2}\,(f(x))^2$ is ≥ 0 and continuous on $[-1;1]$, we deduce:

$$\forall x \in [-1;1], \quad \sqrt{1-x^2}\,(f(x))^2 = 0,$$

  hence

$$\forall x \in]-1;1[, \quad f(x) = 0.$$

  Because f is continuous at -1 and at 1, we get $f(-1) = 0$ and $f(1) = 0$, and therefore $f = 0$.

We conclude that φ is an inner product on E.

Exercise 2 – Use of the Cauchy–Schwarz Inequality.
Let $(E, \|\cdot\|)$ be a real normed vector space, $n \in \mathbb{N}^*$, $(x_1, \ldots, x_n) \in E^n$, $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Show that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 \leq \left(\sum_{i=1}^{n} \alpha_i^2 \right) \left(\sum_{i=1}^{n} \|x_i\|^2 \right).$$

Exercise 2 – Solution. Let $(E, \|\cdot\|)$ be a real normed vector space, $n \in \mathbb{N}^*$, $(x_1, \ldots, x_n) \in E^n$, and $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. By the triangle inequality and absolute homogeneity of the norm,

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \;\leq\; \sum_{i=1}^{n} \|\alpha_i x_i\| \;=\; \sum_{i=1}^{n} |\alpha_i|\,\|x_i\|.$$

Now apply the Cauchy–Schwarz inequality in $\mathbb{R}^n$ to the vectors $\big(|\alpha_1|, \ldots, |\alpha_n|\big)$ and $\big(\|x_1\|, \ldots, \|x_n\|\big)$:

$$\sum_{i=1}^{n} |\alpha_i|\,\|x_i\| \;\leq\; \Big(\sum_{i=1}^{n} \alpha_i^2 \Big)^{1/2} \Big(\sum_{i=1}^{n} \|x_i\|^2 \Big)^{1/2}.$$

Combining the two displays and squaring both sides yields the desired inequality:

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 \leq \left(\sum_{i=1}^{n} \alpha_i^2 \right) \left(\sum_{i=1}^{n} \|x_i\|^2 \right),$$

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