Mathematics for Data Science

Exercise Sheet

September 17, 2025

1 Lecture 1 - Vector spaces, linear maps

1.1 Course examples

Exercise 1 For

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

describe graphically all points cv with:

(a) c being an integer, that is $c \in \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$.

(b) $c \in \mathbb{R}$, with $c \geq 0$.

Describe $cv + dw$ where $d \in \mathbb{R}$ and c is like in (a) or (b).

Exercise 2 Is

$$z = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

in the span of

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}?$$

If so, find $\alpha, \beta \in \mathbb{R}$ such that $z = \alpha x + \beta y$.

Exercise 3

1. Prove that

$$\mu_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

are linearly independent.

2. Is $\{\mu_1, \mu_2, \mu_3\}$ a basis of $\mathbb{R}^3$?

Hint to Exercise 3 (1) solve $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0$. (2) A free family of the same size as the dimension of the vector space.

Exercise 4 Consider the following transformations:

$$L_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

$$L_2 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y \end{pmatrix}$$

$$L_3 : \mathbb{R}^3 \to \mathbb{R}^3, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z+y \\ z-y \\ 0 \end{pmatrix}$$

Are they linear? Justify your answers.

Hint to Exercise 4 Use the definition of a linear transformation.

Exercise 5 Determine whether the following functions are injective, surjective, or bijective:

1. $f_1 : \{a, b, c\} \to \{a, b, c, d\}$ defined by:

$$f_1(a) = a, \quad f_1(b) = b, \quad f_1(c) = c$$

2. $f_2 : \{a, b, c\} \to \{a, b, c, d\}$ defined by:

$$f_2(a) = a, \quad f_2(b) = a, \quad f_2(c) = c$$

3. $f_3 : \{a, b, c\} \to \{a, b, c\}$ defined by:

$$f_3(a) = b, \quad f_3(b) = c, \quad f_3(c) = a$$

Hint to Exercise 5 (1) injective but not surjective, (2) not injective, not surjective (3) bijective.

1.2 Proofs

Ex 1.

1. Prove that if $T : U \to V$ is a linear transformation, then $T(0_U) = 0_V$.
2. Prove that a linear transformation $T : U \to V$ is injective if and only if

$$\ker(T) = \{0_U\}.$$

Hint (1) Use $0_U = 0_U - 0_U$ then linearity. (2) Prove one implication, then the other.

Ex 2. (1) Let F, G be subspaces of a finite-dimensional vector space E. Prove that

$$dim(F + G) = dim(F) + dim(G) - dim(F \cap G)$$

(2) Deduce that in the case of a direct sum,

$$dim(F \oplus G) = dim(F) + dim(G)$$

Ex 3. Let $n \in \mathbb{N}^*$, $E_1, \ldots E_n$ vector spaces on $\mathbb{F}$. We suppose that $E_1, \ldots, E_n$ are finie-dimensional. Show that $E_1 \times \cdots \times E_n$ is finite-dimensional, with

$$dim(E_1 \times \cdots \times E_n) = \sum_{i=1}^{n} dim(E_i)$$

Hint: show the case $n = 2$ and then recurrence.

Ex 4. Prove the Rank-nullity theorem.

1.3 More exercices

Ex 1. We denote by E the set of fonctions from $\mathbb{R}$ to $\mathbb{R}$ We consider F and G the subsets of E corresponding to the fonctions that are symmetric and antisymmetric, respectively. Show that $E = F \oplus G$.

Hint: Show that F and G are subspaces of E. Then decompose a vector $h \in E$ as a vector from F and a vector from G

Ex 2. We denote by $\mathbb{R}[X]$ the set of polynomials on $\mathbb{R}$. Let $I \subset \mathbb{N}$, $(P_i)_{i \in I}$ a family of polynomials such that

$$\forall i, j \in I, i < j \implies deg(P_i) < deg(P_j)$$

Show that the vectors of $(P_i)_{i \in I}$ are linearly independent.

2 Lecture 2 - Matrices

2.1 Course examples

- Examples of vectors in $\mathcal{M}_{m,n}(\mathbb{R})$, operations
- Canonical basis of $\mathcal{M}_{m,n}(\mathbb{R})$
- Computing the matrix of a linear application (several versions)
- Kernel and range of a linear application with a matrix notation
- Illustration of basis change in 3D
- Similar matrices
- Systems of linear equations, geometrical interpretation

Ex 1. Compute the rank and kernel of the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine $\text{rank}(A)$ and $\dim(\ker(A))$.

Ex 2. Are the following matrices invertible? If so, compute their inverse.

1. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$
2. $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

2.2 Proofs

Ex 1. Let $A \in \mathbb{R}^{m \times n}$. Show that the map

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad x \longmapsto Ax$$

is a linear transformation.

Ex 2. Let $A \in \mathbb{R}^{m \times n}$. Show that the map

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \qquad x \longmapsto Ax$$

is injective if and only if $\ker(A) = \{0\}$.

Ex 3. Let $A \in \mathbb{R}^{n \times n}$. Prove the equivalence of the following statements:

(1) $range(A) = \mathbb{R}^n$ (surjectivity)
(2) $\ker(A) = \{0\}$ (injectivity)
(3) The map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$ is bijective.

2.3 More exercises

Exercise 1 Let $(A, B) \in \mathcal{M}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, such that

$$\lambda AB + A + B = 0$$

Show that A and B are commutative.