We consider the bloc matrix M

$$M = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right)$$

$0_{\mathcal{M}_{p,n}}(\mathbb{R})$

General case $\longrightarrow$ We work with Square matrices here

$A \in \mathcal{M}_{p,q}(\mathbb{R})$ | $A \in \mathcal{M}_{n,n}(\mathbb{R})$

$B \in \mathcal{M}_{p,n}(\mathbb{R})$ | $B \in \mathcal{M}_{n,p}(\mathbb{R})$

$C \in \mathcal{M}_{m,n}(\mathbb{R})$ | $C \in \mathcal{M}_{p,p}(\mathbb{R})$

"0" is the matrix in $\mathcal{M}_{m,q}(\mathbb{R})$ with all coefficients = zeros.

Prove that $\det(M) = \det(A) \times \det(C)$.

Case 1 : Suppose $\det(A) = 0$.

This means that there exist a column $[A]_j$ of A that can be expressed as a linear combination of the other columns of A.
Since all the columns of $0_{m,q}$ are identical (filled with zeros), the column j of M can be expressed as a linear combination of the other first q columns of M (with the same weights as the previous linear combination for $[A]_j$).
So $\det(M) = 0$ and indeed $\det(M) = \det(A)\det(C)$.

Case 2 Suppose $\det(A) \neq 0$. The inverse of A exists.

We remark that M can be written as :

$$M = \underbrace{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & I_p \end{array}\right)}_{\text{by expanding along the last column, then forelast column etc. } p \text{ times, we obtain } \det(A)} \underbrace{\left(\begin{array}{c|c} I_n & A^{-1}B \\ \hline 0 & C \end{array}\right)}_{\text{by expanding along the first column, then second column... } n \text{ times, we obtain } \det(C)}$$

, and $\det(M) = \det\left(\begin{array}{c|c} A & 0 \\ \hline 0 & I_p \end{array}\right) \det\left(\begin{array}{c|c} I_n & A^{-1}B \\ \hline 0 & C \end{array}\right)$

So $\det(M) = \det(A)\det(C)$.

Bonus exercise : Vandermonde determinant.

Let $(a_1, \dots a_n) \in \mathbb{R}^n$.

Let $V(a_1, \dots a_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & & & \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{pmatrix}$

Prove by induction that $V(a_1, \dots a_n) = \prod_{1 \leq i < j \leq n} (a_j - a_i)$ (*)

NB: you can consider that $a_1 \neq a_2 \neq \dots \neq a_n$. because otherwise, we directly have $V(a_1, \dots, a_n) = 0$ (repeated column) and the product is null too $(a_j - a_i)$

Hint: consider the polynomial $P(x) = V(a_1, a_2 \dots a_{n-1}, x)$ instead of a_n

this polynomial is obtained by "replacing a_n by x"

Degree of $P \leq n-1$ Since we elevate x to the power $(n-1)$ by definition of $V(a_1, \dots, a_{n-1}, x)$.

The values $a_1, a_2 \dots a_{n-1}$ are roots of the polynomial because if x takes any of these values, a column is repeated so the determinant is null. Using this knowledge on the roots of the polynomial, we can write P in a factorized manner.

$P(a_1) = P(a_2) = \dots = P(a_{n-1}) = 0$

so $P(x) = \lambda \prod_{i=1}^{n-1} (x - a_i)$ where λ is the leading coefficient of the polynomial;

writing P in a factorized manner

i.e. the coefficient of x^{n-1}.

Hence $\lambda = \det \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_{n-1} \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_{n-1}^{n-2} \end{vmatrix} = V(a_1, \dots a_{n-1})$. (*)

So we get $V(a_1, \dots a_n) = \underbrace{\prod_{i=1}^{n-1} (a_n - a_i)}_{\text{the factorization using the roots}} \underbrace{V(a_1, \dots a_{n-1})}_{\lambda \text{ the leading coefficient}}$ (*)

by substituting x with a_n in the polynomial

- Case $n = 2$

$V(a_1, a_2) = \begin{vmatrix} 1 & 1 \\ a_1 & a_2 \end{vmatrix} = a_2 - a_1 = (a_2 - a_1) \times (1)$ ok

- Suppose (*) is true for a given $n \geq 2$.

Using (*) we prove that it holds for $n+1$.

- Conclusion.

(*) Explanation: $V(a_1, \dots a_{n-1}, x) = \det \left[\begin{array}{cccc|c} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_{n-1} & x \\ \vdots & \vdots & & & \vdots \\ a_1^{n-2} & a_2^{n-2} & \cdots & a_{n-1}^{n-2} & x^{n-2} \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_{n-1}^{n-1} & x^{n-1} \end{array}\right]$

when applying the expansion formula on the last column, we see that the cofactor associated to each coefficient x^i of this last column corresponds to the coefficient λ_i associated with x^i in the polynomial

In particular, we recognize that the leading coefficient λ_{n-1} (denoted λ above) is $V(a_1, \dots, a_{n-1})$

Proof by induction

Proving the "hereditary" property :

Reminder : (i) we have proven the property "manually" in the case $n=2$

(ii) we suppose that the property is true for a given $n \geq 2$.

i.e for a given $n \geq 2$,

$$v(a_1, \dots, a_n) = \prod_{1 \leq i < j \leq n} (a_j - a_i). \quad \bullet$$

Now we consider $v(a_1, \dots, a_n, a_{n-1})$.

Using $(*)$: $v(a_1, \dots a_n, a_{n+1}) = v(a_1, \dots a_n) \prod_{1 \leq i \leq n} (a_{n+1} - a_i)$

Using $\bullet$ we can write

$$v(a_1, \dots a_n, a_{n+1}) = \prod_{1 \leq i < j \leq n} (a_j - a_i) \prod_{1 \leq i \leq n} (a_{n+1} - a_i)$$

$$= \prod_{1 \leq i < j \leq \underline{n+1}} (a_j - a_i)$$

So the property is true for $n+1$.

(iii) Finally, we conclude that the property is true for all $n \geq 2$.