Mathematics for Data Science

Lecture 4

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Previously covered topics

- (Lecture 1) Vector spaces, subspaces, linear transormations. Rank, image, kernel
- (Lecture 2) Matrices, link with linear transformations, linear systems
- (Lecture 3) Determinant, diagonalization, Eigendecomposition (part 1)

In this lecture: Eigendecomposition (part 2). Diagonalization. Triangularisability.

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Endomorphisms

Definition

An endomorphism f of a vector space E is a linear map from the space to itself:

$$f : E \to E.$$

NB: we denote by $\mathcal{L}(E)$ the set of endomorphisms on E.

Finite-dimensional case

- When E is finite-dimensional, the study of f reduces to the study of its matrix with respect to a chosen basis.
- The resulting matrix is square; often the same basis of E is used at the source and at the target.

Reduction of endomorphisms

In linear algebra, the reduction of an endomorphism aims to express matrices and endomorphisms in a simpler form, for example to make computations easier.

Method

Reduction essentially consists in decomposing the vector space as a direct sum of invariant subspaces on which the induced endomorphism is simpler:

$$V = U_1 \oplus \cdots \oplus U_k \quad \text{with } f(U_i) \subseteq U_i.$$

NB: Geometrically, this amounts to choosing a basis of the space in which the endomorphism has a simple expression (e.g., block or diagonal form).

Diagonalization

- In finite dimension, diagonalizing an endomorphism means finding a basis in which the matrix of f is diagonal.
- Not every endomorphism is diagonalizable; in some cases one can at best triangularize it, i.e. put it in upper-triangular form.
- Diagonalization is useful for analyzing f, computing powers f^k, searching for square roots of f, etc.

Eigenvalue, eigenvector

Let E be a vector space defined over a field $\mathbb{K}$. For an endomorphism $f \in \mathcal{L}(E)$, there *may* be vectors which, when f is applied to them, are simply scaled by some constant.

Definition

We say that a nonzero vector $\mathbf{x} \in E$ is an **eigenvector** of f corresponding to the **eigenvalue** λ if

$$f(\mathbf{x}) = \lambda \mathbf{x}$$

NB: The zero vector is excluded from this definition because $f(\mathbf{0}) = \mathbf{0} = \lambda \mathbf{0}$ for every $\lambda \in \mathbb{K}$.

Eigenspace and eigenspectrum

Definition (Eigenspectrum)

The **eigenspectrum** (or **spectrum**) of an endomorphism is the list of its eigenvalues, repeated according to their multiplicity. We denote it $sp(f)$.

$$sp(f) = \{\lambda \in \mathbb{K}, \exists v \in E, v \neq 0_E \text{ and } f(v) = \lambda v\}$$

Notations : m_λ the multiplicity of eigenvalue λ.
NB: usually, we write the eigenspectrum as a set and specify the multiplicity of each eigenvalue separately.

Eigenspace

Definition (Eigenspace)

The set of all eigenvectors of f corresponding to the same eigenvalue λ, together with the zero vector, is called an **eigenspace**.

Notation: Let $\lambda \in \mathbb{K}$ be an eigenvalue of f. We denote the eigenspace of f associated to the eigenvalue λ by $E_\lambda(f)$ or $Eig(f, \lambda)$.

$$E_\lambda(f) = \{v \in E, f(v) = \lambda v\}$$

Can you express $E_\lambda(f)$ as the kernel of a linear application ?

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$$E_\lambda(f) = Ker(f - \lambda Id)$$

What can you say about the dimension of this subspace ?

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$$dim(E_\lambda(f)) \geq 1$$

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Can you express $E_\lambda(f)$ as the kernel of a linear application ?

$$E_\lambda(f) = Ker(f - \lambda Id)$$

What can you say about the dimension of this subspace ?

$$dim(E_\lambda(f)) \geq 1$$

We will see that

$$dim(E_\lambda(f)) \leq m_\lambda$$

Manipulating eigenvalues

Proposition

Let $\mathbf{x}$ *be an eigenvector of* $\mathbf{A} \in \mathcal{M}_n(\mathbb{R})$ *with corresponding eigenvalue* λ. *Then*

1. *For any* $\gamma \in \mathbb{R}$, $\mathbf{x}$ *is an eigenvector of* $\mathbf{A} + \gamma\mathbf{I}$ *with eigenvalue* $\lambda + \gamma$.
2. *If* $\mathbf{A}$ *is invertible, then* $\mathbf{x}$ *is an eigenvector of* $\mathbf{A}^{-1}$ *with eigenvalue* λ^{-1}.
3. *If* $\mathbf{A}$ *is invertible,* $\mathbf{A}^k\mathbf{x} = \lambda^k\mathbf{x}$ *for any* $k \in \mathbb{Z}$ *(where* $\mathbf{A}^0 = \mathbf{I}$ *by definition).*

Manipulating eigenvalues

Proof

(i) By computing

$$(\mathbf{A} + \gamma \mathbf{I})\mathbf{x} = \mathbf{A}\mathbf{x} + \gamma \mathbf{I}\mathbf{x} = \lambda \mathbf{x} + \gamma \mathbf{x} = (\lambda + \gamma)\mathbf{x}$$

(ii) Suppose $\mathbf{A}$ is invertible. Then

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}(\lambda \mathbf{x}) = \lambda \mathbf{A}^{-1}\mathbf{x}$$

Dividing by λ, which is valid because the invertibility of $\mathbf{A}$ implies $\lambda \neq 0$, gives $\lambda^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}$.

(iii) Cf previous lecture. The case $k \geq 0$ follows by induction on k. Then the general case $k \in \mathbb{Z}$ follows by combining the $k \geq 0$ case with (ii).

Properties

Proposition

Let $f \in \mathcal{L}(E)$ an endomorphism. Then 0 is an eigenvalue of f if and only if $Ker(f) \neq \{0_E\}$ i.e. f is not injective.

$$0 \in \mathrm{Sp}(f) \iff \ker(f) \neq \{0\} \iff f \text{ is not injective}$$

NB: directly follows from expressing $E_\lambda(f)$ as $Ker(f - \lambda Id)$.

Proposition (Determinant and eigenspace)

Let $f \in \mathcal{L}(E)$ an endomorphism, with E a finite-dimensional vector space. Then

$$\lambda \in sp(f) \text{ iff } det(\lambda Id_E - f) = 0$$

Proof: exercise

Properties

Let's prove the property on the determinant of the linear map $\lambda Id_E - f$.
($\Longrightarrow$) Let $\lambda \in sp(f)$. Assume there exists $u \in E$ with $u \neq 0$ such that $f(u) = \lambda u$. Then

$$\lambda u - f(u) = 0 \iff (\lambda Id_E - f)(u) = 0 \iff u \in \ker(\lambda Id_E - f).$$

Hence $\lambda Id_E - f$ is not bijective (i.e. its kernel contains a nonzero vector), and therefore

$$\det(\lambda Id_E - f) = 0.$$

($\Longleftarrow$) If $\det(\lambda Id_E - f) = 0$, then $\lambda Id_E - f$ is not bijective, hence not injective (since E is finite-dimensional).
Thus there exists $u \in \ker(\lambda Id_E - f)$ with $u \neq 0$, which means $f(u) = \lambda u$.
Therefore λ is an eigenvalue of f (i.e., $\lambda \in \mathrm{Sp}(f)$).

Properties

Proposition

Let $f \in \mathcal{L}(E)$ and let $\lambda_1, \dots \lambda_p$ be p pairwise-distinct eigenvalues of f.

1. *The sum of subspaces $E_{\lambda_1}(f) + \cdots + E_{\lambda_p}(f)$ is a direct sum.*
2. *If $(u_1, u_2, \dots u_p) \in E^p$ such that*

$$\forall i \in [\![1, p]\!], u_i \neq 0, \text{ and } f(u_i) = \lambda_i u_i$$

Then the family $(u_1, u_2, \dots u_p)$ is linearly independent.

Proof of (1): by induction. Proof of (2): follows from (1) with the definiton of a direct sum.

Properties

Proposition (Cardinal of the eigenspectrum)

If E is a finite-dimensional vector space, with $dim(E) = n$, and $f \in \mathcal{L}(E)$, then

$$Card(sp(f)) \leq n$$

Proof:

Properties

Proposition (Cardinal of the eigenspectrum)

If E is a finite-dimensional vector space, with $dim(E) = n$, and $f \in \mathcal{L}(E)$, then

$$Card(sp(f)) \leq n$$

Proof: suppose $Card(sp(f)) \geq n + 1$. Using the previous property, we would have a linearly independent family of vectors $(u_1, u_2, \ldots u_n, u_{n+1})$ associated with eigenvalues $\lambda_1, \ldots \lambda_n, \lambda_{n+1}$. But $dim(E) = n$. Absurd.

Properties (bonus exercise)

Proposition

Let $f, g \in \mathcal{L}(E)$ two endomorphisms that commute. Then the eigenspaces of f are stable by g.

NB: Let $\lambda \in sp(f)$. This proposition writes :

$$g(E_\lambda(f)) \subseteq E_\lambda(f)$$

Since f and g are commutative, $\lambda Id_E - f$ and g are commutative too. Writing $E_\lambda(f)$ as $Ker(\lambda Id_E - f)$, the eigenspace of f associated to λ is stable by g.

Characteristic polynomial

We consider a finite-dimensional vector space E, $dim(E) = n$.

Definition

Let $f \in \mathcal{L}(E)$. We call characteristic polynomial of f the polynomial, denoted χ_f, defined as

$$\chi_f = det(\lambda Id_E - f)$$

NB: alternative notation $P_\lambda(f)$. Similarly, for a square matrix $A \in \mathcal{M}_n(\mathbb{K})$, we will denote $\chi_A = det(\lambda I_n - A)$.

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What is the degree of χ_f ? What is the value of its leading coefficient ?

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What is the degree of χ_f ? What is the value of its leading coefficient ?

The leading coefficient is 1, we say the polynomial is **monic**, of degree n.

Characteristic polynomial

- Following the property on the determinant of the linear map $\lambda Id_E - f$, a value λ is an eigenvalue of f if and only if it is a root of its characteristic polynomial.

$$\lambda \in \mathrm{Sp}(f) \iff \chi_f(\lambda) = 0, \qquad \chi_f(\lambda) = \det\big(\lambda\,\mathrm{Id}_E - [f]_{\mathcal{B}}\big).$$

- $\chi_A(\lambda)$ is a polynomial of degree n.
- Remark that the leading coefficient of $\chi_A(\lambda)$ is 1 (unitary/monic polynomial).
- If A is the matrix of f in a given basis of E, then we have the equivalence

$$\chi_A(\lambda) = \chi_f(\lambda)$$

  This means that the characteristic polynomial of a linear map does not depend on the choice of basis. Like the determinant, it does not vary under matrix similarity (NB: FR "invariant de similitude")

Multiplicity of an eigenvalue

Definition

Let $f \in \mathcal{L}(E)$, $A \in \mathcal{M}_n(\mathbb{K})$, and $\lambda \in sp(f)$. We call **multiplicity** of an eigenvalue λ its multiplicity as a root of the caracteristic polynomial of f (or of A, respectively). It is denoted m_λ.

$$m_\lambda = max(k \in \mathbb{N}^*, (X - \lambda)^k \mid \chi_A(X))$$

NB: For any $\lambda \in sp(f)$,

$$1 \leq m_\lambda \leq n$$

Proposition

Let $\lambda \in sp(f)$. We have $dim(E_\lambda(f)) \leq m_\lambda$. Similarly, for a square matrix $A \in \mathcal{M}_n(\mathbb{K})$, $dim(E_\lambda(A)) \leq m_\lambda$.

Multiplicity of an eigenvalue

Not always equal

The dimension of the eigenspace associated to a given eigenvalue λ is not necessaril egal to the multiplicity of this eigenvalue in the characteristic polynomial.

Example

Example : take the matrix A with zeros everywhere, except on its diagonal and upper diagonal, filled with ones. $sp(A) = \{1\}$, $\chi_A(\lambda) = (\lambda - 1)^n$. $E_1(A) = span((1, 0, \ldots 0)^\top)$ is of dimension 1, whereas the multiplicity of 1 as root of the caracteristic polynomial of A is n.

NB: Equality holds if $m_\lambda = 1$.

Recap - Characteristic polynomial

- The **characteristic polynomial** of an endomorphism of a finite-dimensional vector space is the characteristic polynomial of the matrix of that endomorphism over any basis.
- This means that the characteristic polynomial does not depend on the choice of a basis.
- The characteristic polynomial of a square matrix is a polynomial which is invariant under matrix similarity
- It has the eigenvalues of this matrix as roots.
- The **characteristic equation** is the equation obtained by equating the characteristic polynomial to zero.

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Diagonalizable matrix

In this section we consider a finite-dimensional vector space E, $dim(E) = n$.

Definition

- We say that $f \in \mathcal{L}(E)$ is **diagonalizable** if there exists a basis $\mathcal{B}$ of E formed by eigenvectors of f, i.e. $[f]_{\mathcal{B}}$ the matrix of f in $\mathcal{B}$ is diagonal.
- We say that $A \in \mathcal{M}_n(\mathbb{K})$ is **diagonalizable** if A is similar to a diagonal matrix, i.e. there exists an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ and a diagonal matrix D such that

$$A = PDP^{-1}$$

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NB : "Diagonalizing" an endomorphism means finding a basis of E in which the matrix representaiton of f is diagonal.
"Diagonalizing" a matrix $A \in \mathcal{M}_n(\mathbb{K})$ means finding an invertible matrix $P \in \mathcal{M}_n(\mathbb{K})$ and a diagonal matrix $D \in \mathcal{M}_n(\mathbb{K})$ such that $A = PDP^{-1}$.

Properties

Let $f \in \mathcal{L}(E)$.

Proposition

- *Let $\mathcal{B}$ be a basis such that $[f]_{\mathcal{B}} = \mathrm{diag}(\lambda_1, ...\lambda_n)$ Then $sp(f) = \{\lambda_1, \ldots \lambda_n\}$.*
- *The following statements are equivalent.*
  1. *f is diagonalizable*
  2. *There exists a basis $\mathcal{B}$ of E such that $[f]_{\mathcal{B}}$ is diagonalizable.*
  3. *For any basis $\mathcal{B}$ of E, $[f]_{\mathcal{B}}$ is diagonalizable.*

NB : (1) $\chi_f(\lambda) = \chi_D(\lambda)$ with $D = \mathrm{diag}(\lambda_1, \ldots \lambda_n)$. Since $\chi_D(\lambda) = \prod_{i=1}^{n}(\lambda - \lambda_i)$ we get $sp(f) = \{\lambda_1, \ldots \lambda_n\}$.
(2) use property on similar matrices.

Not all matrices are diagonalizable

Example

$$A = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & & & \dots & & \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

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Example

$$A = \begin{bmatrix} 1 & 1 & 0 & ... & 0 & 0 \\ 0 & 1 & 1 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ ... & & & ... & & \\ 0 & 0 & 0 & ... & 1 & 1 \\ 0 & 0 & 0 & ... & 0 & 1 \end{bmatrix}$$

$sp(A) = \{1\}$. If A was diagonalizable, then A would be similar to the identity matrix. i.e. $D = \mathrm{diag}(1, \dots 1) = I_n$.
But $PI_nP^{-1} = I_n$, so I_n is the only matrix similar to I_n.
This leaves us with $A = I_n$, absurd. So A is not diagonalizable.

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We can generalize on this example : if a square matrix M is diagonalizable and has a unique eigenvalue λ, then this matrix must be similar to λI_n. In particular, if $\lambda = 1$, it must be similar to the identity matrix, so in fact it must *be* the identity matrix.

Equivalent statements on diagonalizability

Proposition

Let $f \in \mathcal{L}(E)$. Then the following statements are equivalent.

1. *f is diagonalizable*
2. $\sum_{\lambda \in sp(f)} dim(E_\lambda(f)) = dim(E)$
3. $\sum_{\lambda \in sp(f)} E_\lambda(f) = E$

Equivalent statements on diagonalizability

Recall that the eigenspaces are in direct sum. Hence

$$dim(\sum_{\lambda \in sp(f)} E_\lambda(f)) = \sum_{\lambda \in sp(f)} dim(E_\lambda(f))$$

and the equivalence between (2) and (3).

If we suppose that (1) f is diagonalizable, then there exists a basis $\mathcal{B}$ of E formed by eigenvectors of f. So $\sum_{\lambda \in sp(f)} E_\lambda(f)$ contains a basis of E. It follows that $\sum_{\lambda \in sp(f)} E_\lambda(f) = E$ (3).

Now suppose (3): we have $E = \bigoplus_{\lambda \in sp(f)} E_\lambda(f)$. So there exists a basis of E adapted to this decomposition of E. This basis is formed by eigenvectors of f... So f is diagonalizable (1).

Split Polynomial

Definition

Let $\mathbb{K}$ be a field. A non-constant polynomial $P \in \mathbb{K}[X]$ *splits over* $\mathbb{K}$ if

$$P(X) = C \prod_{j=1}^{n} (X - a_j),$$

where each $a_j \in \mathbb{K}$ and $C \in \mathbb{K}$. Equivalently, P can be written as a product of degree-1 polynomials with coefficients in $\mathbb{K}$. In this case, C is called the **leading coefficient** of P.

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Dependence on the Field

The splitting property depends on $\mathbb{K}$. For example,

$$X^2 + 1 = (X - i)(X + i) \quad \text{splits over } \mathbb{C},$$

but it does not split over $\mathbb{R}$.

Simple Roots

Definition

A split polynomial P *has simple roots* if, in the previous factorization, the numbers $a_1, \ldots, a_n$ are pairwise distinct (i.e., each root has multiplicity 1).

Example

Over $\mathbb{K} = \mathbb{Q}$ Consider

$$P(X) = 5(X - 1)(X + 2)(X - 3) \in \mathbb{Q}[X].$$

All roots $1, -2, 3$ lie in $\mathbb{Q}$, so P splits over $\mathbb{K}$ with leading coefficient 5.
The roots are pairwise distinct, hence P is split with simple roots.
NB: Expanding,

$$P(X) = 5\left(X^3 - 2X^2 - 5X + 6\right) = 5X^3 - 10X^2 - 25X + 30.$$

Characteristic polynomial and Diagonalizability

Theorem (for endomorphisms)

Let E be a vector space defined over a field $\mathbb{K}$, $dim(E) = n$ and $f \in \mathcal{L}(E)$. The endomorphism f is diagonalizable if and only if

1. *Its characteristic polynomial χ_f is split.*
2. *For each $\lambda \in sp(f)$, $dim(E_\lambda(f)) = m_\lambda$.*

Theorem (for matrices)

Let $A \in \mathcal{M}_n(\mathbb{K})$. It is diagonalizable if and only if

1. *Its characteristic polynomial χ_A is split.*
2. *For each $\lambda \in sp(A)$, $dim(E_\lambda(A)) = m_\lambda$, or equivalently,*

$$rank(\lambda I_n - A) = n - m_\lambda$$

Annihilating Polynomials

Definition

Let E be a finite-dimensional $\mathbb{K}$-vector space with $\dim E = n$. Let $f \in \mathcal{L}(E)$ be an endomorphism and let $P \in \mathbb{K}[X]$ be a polynomial. P is called an *annihilating polynomial* for f if

$$P(f) = 0_{\mathcal{L}(E)} \quad \text{(i.e., the zero endomorphism).}$$

Notation reminder: If $P(X) = a_0 + a_1 X + \cdots + a_m X^m$, then $P(A) = a_0 I + a_1 A + \cdots + a_m A^m$.

Eigenvalues and Annihilating Polynomials

Theorem (Spectral Mapping for Polynomials)

Let $f : E \to E$ be linear map over a field $\mathbb{K}$ and let $P \in \mathbb{K}[X]$ satisfy $P(f) = 0$, i.e. P is an annihilating polynomial for f. Then every eigenvalue λ of f is a root of P:

$$sp(f) \subseteq \{z \in \mathbb{K} : \; P(z) = 0\}.$$

Warning

The eigenspectrum of f is <u>included</u> in the set of roots of an annihilating polynomial P.

Eigenvalues and Annihilating Polynomials

Proof. Let $\lambda \in \mathbb{K}$ be an eigenvalue of f and choose $v \in E$, $v \neq 0_E$ with $f(v) = \lambda v$.
We write the polynomial as $P(X) = \sum_{k=0}^{m} a_k X^k$ (with $a_k \in \mathbb{K}$). Note that $f^0 = \mathrm{Id}_E$.
By induction, $f^k(v) = \lambda^k v$ for all $k \geq 0$. Then

$$P(f)(v) \;=\; \sum_{k=0}^{m} a_k f^k(v) \;=\; \sum_{k=0}^{m} a_k \lambda^k v \;=\; P(\lambda)\, v.$$

Since $P(f) = 0$ by hypothesis, we have $0 = P(f)(v) = P(\lambda)\, v$.
Because $v \neq 0$, it follows that $P(\lambda) = 0$. Hence every eigenvalue λ of f is a root of P.

Minimal Polynomial of a Matrix

Definition

Let $A \in M_n(\mathbb{K})$. The **minimal polynomial** of A, denoted μ_A, is the unique *monic* polynomial of least degree such that

$$\mu_A(A) = 0.$$

Corollary (corollary of the previous theorem)

In particular, all eigenvalues of f are roots of μ_f, the minimal polynomial of f.
NB: Equivalently, μ_A divides every annihilating polynomial of A (in particular, $\mu_A \mid \chi_A$).

Example

$$B = \mathrm{diag}(1, 2, 2).$$

Then $(B - I)(B - 2I) = 0$, while $B - I \neq 0$ and $B - 2I \neq 0$. Thus

$$\mu_B(X) = (X - 1)(X - 2).$$

Cayley-Hamilton Theorem

This theorem tells us that the characteristic polynomial of a given endomorphism is an annihilating polynomial for this endomorphism.

Theorem (Cayley-Hamilton)

For any $n \times n$ matrix A over a field K, let

$$\chi_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

be its characteristic polynomial. Then

$$\chi_A(A) = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I_n = 0.$$

NB: notice that the polynomial $det(\lambda I_n - A)$ is null when substituting λ with A.

Cayley-Hamilton Theorem

Note on the **Existence of an Annihilating Polynomial**
Since $\dim \mathcal{L}(E) = n^2$, the family

$$\left(\mathrm{Id}_E,\ u,\ u^2,\ \ldots,\ u^{n^2}\right)$$

of $n^2 + 1$ endomorphisms is linearly dependent. Hence there exist $a_0, \ldots, a_{n^2} \in K$, not all zero, such that

$$a_0 \,\mathrm{Id}_E + a_1 u + \cdots + a_{n^2} u^{n^2} \;=\; 0.$$

Setting $P(X) = a_0 + a_1 X + \cdots + a_{n^2} X^{n^2}$ yields a nonzero polynomial with $P(u) = 0$.

Example: Characteristic polynomial in 2×2

Computation (using $\chi_A(\lambda) = \det(\lambda I_2 - A)$) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\lambda I_2 - A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix}.$$

Hence

$$\chi_A(\lambda) = \det \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} = (\lambda - a)(\lambda - d) - (-b)(-c)$$
$$= \lambda^2 - (a + d)\lambda + (ad - bc).$$

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Hence

$$\begin{aligned} \chi_A(\lambda) = \det \begin{pmatrix} \lambda - a & -b \\ -c & \lambda - d \end{pmatrix} &= (\lambda - a)(\lambda - d) - (-b)(-c) \\ &= \lambda^2 - (a + d)\lambda + (ad - bc). \end{aligned}$$

Identification
Since $\operatorname{tr}(A) = a + d$ and $\det(A) = ad - bc$, we obtain

$$\chi_A(\lambda) = \lambda^2 - (\operatorname{tr} A)\,\lambda + \det(A).$$

Example: Characteristic polynomial in 2×2

Now let us suppose A of size 2×2 is invertible. Using the previous formula, find an explicit formula for the inverse of A.

Cayley-Hamilton theorem - Examples

Example

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}, \qquad \chi_A(\lambda) = (\lambda - 2)(\lambda - 3) = \lambda^2 - 5\lambda + 6.$$

Compute $A^2 = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix}$. Then

$$A^2 - 5A + 6I_2 = \begin{pmatrix} 4 & 5 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} 10 & 5 \\ 0 & 15 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = 0.$$

Bonus: Compute the inverse from Cayley-Hamilton. $\det(A) = 6$, $\mathrm{tr}(A) = 5$, hence

$$A^{-1} = \frac{1}{6}(5I_2 - A) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} \\ 0 & \frac{1}{3} \end{pmatrix}.$$

Table of Contents

Trigonalizability

Definition

- Let $f \in \mathcal{L}(E)$. We say that f is **trigonalizable** if there exists a basis $\mathcal{B}$ of E such that $[f]_{\mathcal{B}}$ the matrix of f in $\mathcal{B}$, is triangular.
- Let $A \in \mathcal{M}_n(\mathbb{K})$. We say that A is **trigonalizable** if A is <u>similar</u> to a triangular matrix.

NB: we often consider triangular superior matrices.
If $f \in \mathcal{L}(E)$, and $\mathcal{B} = (e_1, e_2, \ldots e_n)$ is a basis of E, then $\mathcal{M}_{\mathcal{B}}(f)$ is triangular superior if and only if

$$\forall i \in [[1, n]], f(e_i) \in span(e_1, e_2, ...e_i)$$

Warning

A diagonalizable matrix is trigonalizable, but not reciprocally.

Trigonalizability

Trigonalizing a linear map f means finding a basis $\mathcal{B}$ such that $[f]_{\mathcal{B}}$ is triangular.
Trigonalizing a matrix $A \in \mathcal{M}_n(\mathbb{K})$ consists in finding P invertible and T triangular such that $A = PTP^{-1}$.

Theorem (Criterion of Triangularisability)

Let $f \in \mathcal{L}(E)$. Then f is trigonalizable if and only if its characteristic polynomial χ_f is split. (FR: scindé) Similarly, a square matrix A is trigonalizable if and only if its characteristic polynomial χ_A is split.

Proof : by induction. NB : all matrices of $\mathcal{M}_n(\mathbb{C})$ are trigonalizable because all the polynomials from $\mathbb{C}[X]$ are split, but all real valued matrices are not trigonalizable.

Trigonalizability, trace and determinant

Proposition

Let $f \in \mathcal{L}(E)$ ($A \in \mathcal{M}_n(\mathbb{K})$, respectively). Then

1. $\sum_{\lambda \in sp(f)} m_\lambda = n$
2. $det(f) = \prod_{\lambda \in sp(f)} \lambda^{m_\lambda}$
3. $tr(f) = \sum_{\lambda \in sp(f)} m_\lambda \lambda$

Method - Triangularisability

Let $A \in \mathcal{M}_n(\mathbb{K})$ such that χ_A is split. Let us denote $sp(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with multiplicity $m_1, m_2, \ldots m_p$. NB : we know that $\sum_{i=1}^{p} \lambda_i = n$.
For each $i \in [[1, p]]$ we look for a basis $\mathcal{B}_i$ of $ker(\lambda_i I_n - A)$.
If $Card(\mathcal{B}_i) = m_i$ (i.e. $dim(ker(\lambda_i I_n - A)) = m_i$) then we can diagonalize.
Otheriwse, we look for vectors to add to $\mathcal{B}_i$ to make it a basis of $ker((\lambda_i I_n - A)^2)$, etc.
Finally, the family of vectors $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots \mathcal{B}_p$ is a basis where A can be trigonalized.

Nilpotent matrices

Definition

A **nilpotent** matrix N is a square matrix such that there exist a positive integer k such that

$$N^k = 0$$

NB: the smallest power k for which N^k is null is sometimes called the **index** of N.

Example

$$N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

N is nilpotent with index 2, since $N^2 = 0$.

We can prove that any triangular matrix $T \in \mathcal{M}_n(\mathbb{R})$
with zeros along the main diagonal is nilpotent, with index $\leq n$.

Nilpotent matrices

Example

$$B = \begin{bmatrix} 0 & 2 & 1 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B^3 = \begin{bmatrix} 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore B is nilpotent, with index 4.

Nilpotent matrices

What to use nilpotent matrices for ?

Example

Triangularisability then decomposition of an upper triangular matrix as the sum of a diagonal matrix and of a nilpotent matrix.

NB: The determinant and trace of a nilpotent matrix are always zero. So a nilpotent matrix cannot be invertible.

Cayley-Hamilton theorem - Examples

Example (Jordan block)

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \chi_J(\lambda) = (\lambda - 1)^3.$$

$$\chi_J(J) = 0 \iff (J - I_3)^3 = 0 \iff J^3 - 3J^2 + 3J - I_3 = 0.$$

Writing $N = J - I_3$ (nilpotent with $N^3 = 0$), any power J^m reduces to a polynomial of degree ≤ 2 in N:

$$J^m = (I_3 + N)^m = I_3 + mN + \binom{m}{2} N^2 \quad (N^3 = 0).$$

This illustrates how we can use C–H to compute powers of a matrix even when the matrix is not diagonalizable.

Practical implications of Cayley-Hamilon theorem

- **Reduce powers.** Any A^m ($m \geq n$) can be rewritten as a linear combination of $I, A, \ldots, A^{n-1}$ using $\chi_A(A) = 0$.
- **Compute inverses.** If $\det(A) \neq 0$, C–H yields a polynomial expression for A^{-1} in terms of $I, A, \ldots, A^{n-1}$ (e.g., explicit in the 2×2 example we saw previously).
- **Minimal polynomial.** The minimal polynomial μ_A divides χ_A; in particular, every eigenvalue of A is a root of χ_A.
- **Linear recurrences.** For vectors $x_k = A^k x_0$, the sequence satisfies the recurrence with coefficients from χ_A (useful in control theory... or in deep learning!).

Example exercise

Example

Consider the following matrix, with $a > 0$:

$$A = \begin{bmatrix} -1 & a & -a \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

1. Compute χ_A and show that A is not diagonalizable.
2. Find v_1, v_2, v_3 in $\mathcal{M}_{3,1}(\mathbb{R})$ such that $Av_1 = -v_1$, $Av_2 = v_1 - v_2$, $Av_3 = v_1 + v_2 - v_3$.
3. Show that A is similar to $T = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$
4. Use a nilpotent matrix to compute A^n for any $n \in \mathbb{N}$.

Next class

Review the method !

Reminder : you can find the eigenspace $E_\lambda(A)$ of $A \in \mathcal{M}_n(\mathbb{K})$ associated with an eigenvalue λ by solving the linear system resulting from $AX = \lambda X$. If you need practice for finding the *set of solutions* of a linear system please refer to Example 2.2 from the MML book.

Next class : Inner product spaces, normed spaces, spectral theorem. You can prepare by reading the MML book, Chapter 4.