Math for Data Science – Exam – Fall 2025

NB: There are five sections in the exam. Indicative point scale, currently computed out of 40 points. *No point without justification.* ***Please give all documents at the end of the exam (including your draft paper and this document).*** *Good luck!*

1 Proofs – Vector spaces (5 pts)

1. **Sum of subspaces.** Let E be a vector space, F and G be two subspaces of E. We recall that their sum is defined by

$$F + G = \{u \in E, \exists (v, w) \in F \times G, u = v + w\}.$$

   (a) Is $F + G$ a subspace of E? Prove it.

   (b) Let suppose that $F \cap G = \{0_E\}$ (direct sum). Show that for every $u \in F + G$, there exists a unique pair $(v, w) \in F \times G$ such that $u = v + w$. (Hint: prove existence, then uniqueness).

2. **Injectivity and kernel.** Let E and F be two vector spaces and $f : E \to F$ be a linear map.

   (a) Prove that $f(0_E) = 0_F$.

   (b) Prove that f is injective *if and only if* $\mathrm{Ker}(f) = \{0_E\}$. (Hint: prove one implication, then the other).

2 Exercise – Determinant (6 pts)

Let $n \in \mathbb{N}^*$. We consider the determinant Δ_n defined by:

$$\Delta_n = \begin{vmatrix} 1 & 2 & 3 & \cdots & (n-1) & n \\ -1 & 0 & 3 & \cdots & (n-1) & n \\ -1 & -2 & 0 & \cdots & (n-1) & n \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & -3 & \cdots & 0 & n \\ -1 & -2 & -3 & \cdots & -(n-1) & 0 \end{vmatrix}$$

For example, we have:

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 3 \\ -1 & -2 & 0 \end{vmatrix} \qquad \Delta_4 = \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 3 & 4 \\ -1 & -2 & 0 & 4 \\ -1 & -2 & -3 & 0 \end{vmatrix} \qquad \Delta_5 = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ -1 & 0 & 3 & 4 & 5 \\ -1 & -2 & 0 & 4 & 5 \\ -1 & -2 & -3 & 0 & 5 \\ -1 & -2 & -3 & -4 & 0 \end{vmatrix}$$

1. Compute Δ_1, Δ_2, Δ_3 and Δ_4. (Hint: use elementary operations on the rows and/or columns of the matrix.)

2. Make a hypothesis on the value of Δ_n for every $n \in \mathbb{N}^*$ (hint: use $n!$.). Explain your reasoning.

3. Prove your hypothesis by induction. (Hint: use your previous answers for initializing the proof. Then prove that if the property is true for a given n, then it is also true for $n+1$.)

3 Proofs – Eigenvalues and eigenvectors (6 pts)

Let $A \in \mathbb{R}^{n\times n}$ and suppose (λ_i, v_i), $i = 1, \ldots, n$, are eigenpairs of A with the eigenvectors $v_1, \ldots, v_n$ linearly independent. We set $V = [v_1 \ \cdots \ v_n] \in \mathbb{R}^{n\times n}$ (columns of V are the v_i vectors) and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n\times n}$.

1. (a) Prove by induction that for every integer $k \geq 1$, we have:
$$\forall i \in \mathbb{N}, 1 \leq i \leq n, A^k v_i = \lambda_i^k v_i,$$
   (b) Deduce the matrix identity:
$$A^k = VD^kV^{-1}.$$

2. Now assume A is invertible.
   (a) What does it mean for the eigenvalues of A? and for D?
   (b) Prove that for each $i, 1 \leq i \leq n$, v_i is an eigenvector of A^{-1} associated with the eigenvalue λ_i^{-1}, i.e.,
$$A^{-1}v_i = \lambda_i^{-1}v_i,$$
   (c) Deduce the matrix identity:
$$A^{-1} = VD^{-1}V^{-1}.$$

3. Are A^k (for $k \geq 1$) and A^{-1} (when defined) diagonalizable? Justify based on the previous questions.

4 Computations – Diagonalization (9 pts)

We consider the matrix
$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$
We want to know whether it is diagonalizable in $\mathcal{M}_3(\mathbb{R})$. We recall that the algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial of A, and the geometric multiplicity of λ is the dimension of the eigenspace associated to λ:
$$\mathrm{Eig}(A, \lambda) = \{X \in \mathbb{R}^3 : \ AX = \lambda X\}.$$

1. (a) *Using the characteristic polynomial of* A, determine the eigenvalues of A and their algebraic multiplicities. (Hint: the characteristic polynomial that you obtain should be divisible by $P(\lambda) = (\lambda - 2)(\lambda - 4)$.)
   (b) Using a property of the trace, check your previous answer.

2. For each eigenvalue, determine *a basis* of the corresponding eigenspace, and the geometric multiplicity of this eigenvalue.

3. Based on your previous answers, is A diagonalizable in $\mathcal{M}_3(\mathbb{R})$? Justify your answer.

5 Exercise – Orthogonality between $S_n(\mathbb{R})$ and $A_n(\mathbb{R})$ (14 pts)

We denote by $\mathcal{M}_n(\mathbb{R})$ the set of square matrices of size n with real entries, by $S_n(\mathbb{R})$ the subset of symmetric matrices and by $A_n(\mathbb{R})$ the subset of antisymmetric matrices.

Let $n \in \mathbb{N}^*$. We consider the application from $\mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_n(\mathbb{R})$ to $\mathbb{R}$ defined by:

$$(M, N) \longmapsto (M \mid N) = \operatorname{tr}\left(M^\top N\right).$$

1. **Inner product and norm.**
   (a) Show that $(\cdot \mid \cdot)$ is an inner product on $\mathcal{M}_n(\mathbb{R})$.
   (b) Deduce that the function from $\mathcal{M}_n(\mathbb{R})$ to $\mathbb{R}$ defined by $M \mapsto \|M\| = \sqrt{\operatorname{tr}(M^\top M)}$ is a norm on $\mathcal{M}_n(\mathbb{R})$.
   (c) We consider the standard basis of $\mathcal{M}_n(\mathbb{R})$ defined by the matrices E_{ij}, $i, j \in [\![1, n]\!]$, where E_{ij} is the matrix with a 1 at entry (i, j) and 0 elsewhere. Show that this basis is orthonormal for the above defined inner product.

2. **Application.**
   (a) Let
   $$W = \{A \in \mathcal{M}_3(\mathbb{R}) : \ \operatorname{tr}(A) = 0\}.$$
   We remark that W is a subspace of $\mathcal{M}_3(\mathbb{R})$. Using the above defined inner product on $\mathcal{M}_3(\mathbb{R})$, compute $W^\perp$, the orthogonal complement of W.
   (b) What are the dimensions of W and $W^\perp$? Check your answer with a property from the course.

3. **Subspaces.**
   (a) Define $S_n(\mathbb{R})$ and $A_n(\mathbb{R})$ using a property of the transpose (i.e. with a set notation).
   (b) Show that $S_n(\mathbb{R})$ and $A_n(\mathbb{R})$ are two subspaces in $\mathcal{M}_n(\mathbb{R})$.
   (c) Show that $S_n(\mathbb{R})$ and $A_n(\mathbb{R})$ are complementary and orthogonal subspaces in $\mathcal{M}_n(\mathbb{R})$.
   (d) What is the dimension of $S_n(\mathbb{R})$ and $A_n(\mathbb{R})$? Propose a basis for each of these subspaces.

4. **Distance to $S_n(\mathbb{R})$.** We recall that for a subspace F of a normed vector space $(E, \|\cdot\|)$, we define the distance from a point $x \in E$ to F by
   $$d(x, F) = \inf_{y \in F} \|x - y\|.$$
   (a) Show that for every $M \in \mathcal{M}_n(\mathbb{R})$, there exists a unique matrix $S \in S_n(\mathbb{R})$ such that $d(M, S_n(\mathbb{R})) = \|M - S\|$.
   (b) For every $M \in \mathcal{M}_n(\mathbb{R})$, compute the distance $d(M, S_n(\mathbb{R}))$ as a function of M.
   (c) **Bonus (2pt):** For $i, j \in [\![1, n]\!]$, we denote by E_{ij} the matrix of $\mathcal{M}_n(\mathbb{R})$ with a 1 at entry (i, j) and 0 elsewhere. We consider $M = \sum_{i=1}^{n} E_{i1}$. Compute $d(M, S_n(\mathbb{R}))$.