Linear Algebra - Exercise sheet

September 14, 2025

1 Lecture 1 - Vector spaces, linear maps

Exercise 1

For

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

describe graphically all points cv with:

(a) c being an integer, that is $c \in \mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$.

(b) $c \in \mathbb{R}$, with $c \geq 0$.

Describe $cv + dw$ where $d \in \mathbb{R}$ and c is like in (a) or (b).

Solution to Exercise 1

Let

$$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Write a general scalar multiple of v as $cv = \begin{pmatrix} c \\ 0 \end{pmatrix}$.

(a) c an integer, $c \in \mathbb{Z}$. The set $\{cv : c \in \mathbb{Z}\}$ is

$$\left\{ \begin{pmatrix} n \\ 0 \end{pmatrix} : n \in \mathbb{Z} \right\}.$$

Geometric description: these are all points on the x-axis whose x-coordinate is an integer. In other words, the points $(\dots, -2, 0), (-1, 0), (0, 0), (1, 0), (2, 0), \dots$. They form a discrete set of equally spaced points on the x-axis.

(b) $c \in \mathbb{R}$ with $c \geq 0$. The set $\{cv : c \in \mathbb{R},\, c \geq 0\}$ is

$$\left\{ \begin{pmatrix} t \\ 0 \end{pmatrix} : t \geq 0 \right\}.$$

Geometric description: this is the nonnegative ray of the x-axis, i.e. the half-line starting at the origin and extending to the right. In Cartesian form it is $\{(x, y) \in \mathbb{R}^2 : y = 0,\ x \geq 0\}$.

Describe $cv + dw$ with $d \in \mathbb{R}$ and c as in (a) or (b). Compute

$$cv + dw = \begin{pmatrix} c \\ d \end{pmatrix}.$$

- If $c \in \mathbb{Z}$ (case (a)) and $d \in \mathbb{R}$, then

  $$\{cv + dw : c \in \mathbb{Z},\, d \in \mathbb{R}\} = \left\{ \begin{pmatrix} n \\ y \end{pmatrix} : n \in \mathbb{Z},\ y \in \mathbb{R} \right\}.$$

  Geometric description: the union of all vertical lines whose x-coordinate is an integer. Equivalently, vertical lines located at $x = \ldots, -2, -1, 0, 1, 2, \ldots$.

- If $c \in \mathbb{R}$ with $c \geq 0$ (case (b)) and $d \in \mathbb{R}$, then

  $$\{cv + dw : c \geq 0,\, d \in \mathbb{R}\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq 0,\ y \in \mathbb{R} \right\}.$$

  Geometric description: the closed right half-plane (including the y-axis). In Cartesian form this is $\{(x, y) \in \mathbb{R}^2 : x \geq 0\}$.

NB: In the unconstrained case $c, d \in \mathbb{R}$, the set $\{cv + dw : c, d \in \mathbb{R}\}$ is all of $\mathbb{R}^2$, since $(1, 0)$ and $(0, 1)$ are the standard basis vectors.

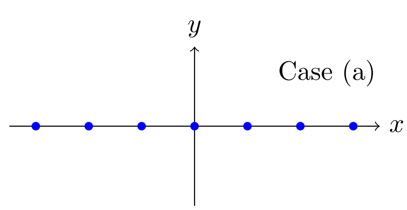

Figure 1: Set of integer multiples of $v = (1, 0)^T$: discrete points on the x-axis.

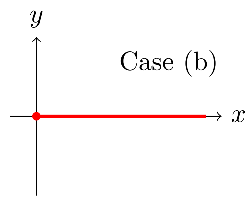

Figure 2: Nonnegative multiples of v: the ray on the positive x-axis.

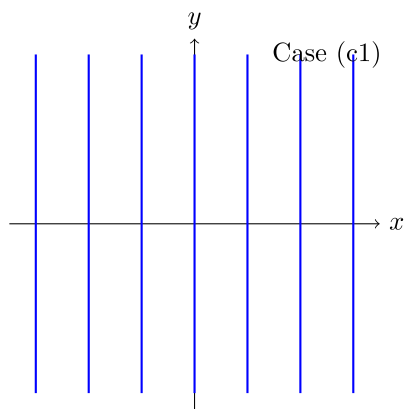

Figure 3: Points of the form $cv + dw$ with $c \in \mathbb{Z}$, $d \in \mathbb{R}$: vertical lines at integer x coordinates.

Exercise 2

Is

$$z = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$$

in the span of

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}?$$

If so, find $\alpha, \beta \in \mathbb{R}$ such that $z = \alpha x + \beta y$.

Solution to Exercise 2

$z = 3y - 2x$

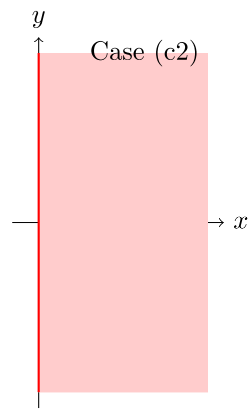

Figure 4: Points of the form $cv + dw$ with $c \geq 0$, $d \in \mathbb{R}$: the right half-plane $x \geq 0$.

Exercise 3

1. Prove that

$$\mu_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mu_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad \mu_3 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$$

   are linearly independent.

2. Is $\{\mu_1, \mu_2, \mu_3\}$ a basis of $\mathbb{R}^3$?

Hint to Exercise 3

(1) solve $\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0$. (2) A free family of the same size as the dimension of the vector space.

Exercise 4

Consider the following transformations:

$$L_1 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} y \\ x \end{pmatrix}$$

$$L_2 : \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ y \end{pmatrix}$$

$$L_3 : \mathbb{R}^3 \to \mathbb{R}^3, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} z + y \\ z - y \\ 0 \end{pmatrix}$$

Are they linear? Justify your answers.

Hint to Exercise 4

Use the definition of a linear transformation.

Exercise 5

1. Prove that if $T : U \to V$ is a linear transformation, then $T(0_U) = 0_V$.
2. Prove that a linear transformation $T : U \to V$ is injective if and only if

$$\ker(T) = \{0_U\}.$$

Hint to Exercise 5

(1) Use $0_U = 0_U - 0_U$ then linearity. (2) Prove one implication, then the other.

Solution to Exercise 5

(a) Let $T : U \to V$ be linear. We want to prove that $T(0_U) = 0_V$.

Since T is linear, we have

$$T(0_U - 0_U) = T(0_U) + (-1) \cdot T(0_U) = 0_V$$

(b) Let $T : U \to V$ be a linear map. We prove that T is injective if and only if $\ker(T) = \{0_U\}$.

Recall that T is injective if

$$\forall x, y \in U, \quad T(x) = T(y) \implies x = y.$$

(⇒) Suppose T is injective. Let $x \in \ker(T)$, i.e.

$$T(x) = 0_V.$$

But we have also proven that $T(0_U) = 0_V$. Since T is injective, it follows that $x = 0_U$. Therefore

$$\ker(T) = \{0_U\}.$$

(⇐) Conversely, assume $\ker(T) = \{0_U\}$. Let $x, y \in U$ such that

$$T(x) = T(y).$$

Then by linearity of T

$$T(x) - T(y) = 0_V \quad \Rightarrow \quad T(x - y) = 0_V.$$

Hence $x - y \in \ker(T)$, so $x - y = 0_U$, which implies $x = y$. Therefore T is injective.

Exercise 6

Determine whether the following functions are injective, surjective, or bijective:

1. $f_1 : \{a, b, c\} \to \{a, b, c, d\}$ defined by:
$$f_1(a) = a, \quad f_1(b) = b, \quad f_1(c) = c$$
2. $f_2 : \{a, b, c\} \to \{a, b, c, d\}$ defined by:
$$f_2(a) = a, \quad f_2(b) = a, \quad f_2(c) = c$$
3. $f_3 : \{a, b, c\} \to \{a, b, c\}$ defined by:
$$f_3(a) = b, \quad f_3(b) = c, \quad f_3(c) = a$$

Hint to Exercise 6

(1) injective by not surjective, (2) not injective, not surjective (3) bijective.

Exercise 7

Let F, G be two vector subspaces of a finite-dimensional vector space E. Prove that
$$\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G).$$

- The Cartesian product $F \times G$ is endowed with the structure of a $\mathbb{K}$-vector space. For $(x, y), (x', y') \in F \times G$ and $\lambda \in \mathbb{K}$, define:
$$(x, y) + (x', y') = (x + x', y + y'),$$
$$\lambda \cdot (x, y) = (\lambda x, \lambda y).$$
Then $(F \times G, +, \cdot)$ is a $\mathbb{K}$-vector space (NB: verify the axioms).
- Define the linear map
$$\varphi : F \times G \to E, \quad (x, y) \mapsto x + y.$$
- φ is linear, since
$$\varphi(\lambda(x, y)) = \varphi(\lambda x, \lambda y) = \lambda x + \lambda y = \lambda(x + y) = \lambda\varphi(x, y).$$
- By construction,
$$\mathrm{Im}(\varphi) = F + G.$$

- Now consider the kernel. We have

$$(x, y) \in \ker(\varphi) \iff x + y = 0 \iff x = -y.$$

So $x, y \in F \cap G$ and the map

$$F \cap G \longrightarrow \ker(\varphi), \quad x \mapsto (x, -x)$$

is an isomorphism. Hence

$$\dim(\ker(\varphi)) = \dim(F \cap G).$$

- By the rank-nullity theorem:

$$\dim(\mathrm{Im}(\varphi)) = \dim(F \times G) - \dim(\ker(\varphi)).$$

But

$$\dim(F \times G) = \dim(F) + \dim(G),$$

NB: A generating and linearly independant family of vectors $\mathcal{B}$ of $F \times G$ can be built from bases $(f_1, ...f_n)$ of F and $(g_1, g_2, \dots g_k)$ of G.

$$\mathcal{B} = ((f_1, 0), (f_2, 0), ...(f_{dim(F)}, 0), (0, g_1), (0, g_2), \dots (0, g_{dim(G)}))$$

So $\mathcal{B}$ is a basis of $F \times G$. Since $\mathcal{B}$ contains $\dim(F) + \dim(G)$ vectors, we have $dim(F \times G) = dim(F) + dim(G)$.

Therefore,

$$\dim(F + G) = \dim(F) + \dim(G) - \dim(F \cap G).$$

Exercise 7

Let $n \in \mathbb{N} \setminus \{0, 1\}$, $E_1, \dots, E_n$ vector spaces on $\mathbb{K}$. We suppose that $E_1, \dots, E_n$ are finite-dimensional. Show that $E_1 \times E_2 \times \dots \times E_n$ is of finite dimension, with

$$\dim(E_1 \times E_2 \times \dots \times E_n) = \sum_{i=1}^{n} \dim(E_i).$$

Hint: reasoning by recurrence on n.

Exercise 8

Let us denote by E the set of functions from $\mathbb{R}$ to $\mathbb{R}$, F the set of functions of E that are symmetrical, G the set of functions of E that are anti-symmetrical.

Prove that

$$E = F \oplus G.$$

Hint: decompose $h \in E$ as a vector from F and a vector from G.

Exercise 9

Prove the rank-nullity theorem.