Lecture 5 - Normed spaces, Orthogonality

1 Short computations

Question 1 Show that for any $u, v \in \mathbb{R}^n$:

$$\|u + v\|_2^2 = \|u\|_2^2 + 2\langle u, v\rangle + \|v\|_2^2$$

where $\|\cdot\|_2$ is the Euclidean norm and $\langle \cdot, \cdot \rangle$ is the dot product.

Question 2 Show that for any $u, v \in \mathbb{R}^n$ and any matrix $A \in \mathbb{R}^{m\times n}$:

$$\langle u, Av\rangle = \langle A^T u, v\rangle.$$

Question 3a (in $\mathbb{R}^2$). (a) Let $V = \text{span}\{v\} \subset \mathbb{R}^2$ with $v = (1, 1)^T$. Find the orthogonal complement $V^\perp$.

Question 3b (in $\mathbb{R}^3$). Let

$$F = \text{span}\{(1, 1, 0), (1, 0, 1)\} \subset \mathbb{R}^3.$$

Compute $F^\perp$.

Question 3c (in $\mathbb{R}^n$). Let $\mathbf{1}_n = (1, \dots, 1)^\top \in \mathbb{R}^n$ and $F = \text{span}\{\mathbf{1}_n\}$. Compute $F^\perp$ and give a basis.

2 Short proofs

Question 4 Show that the dot product in $\mathbb{R}^n$ satisfies the following properties:

1. Symmetry: $\langle x, y\rangle = \langle y, x\rangle$.
2. Homogeneity: $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$.
3. Linearity in the first argument: $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
4. Linearity in the second argument: $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

Question 5 Let F be a subspace of a finite-dimensional inner product space E. Show the following properties of the orthogonal complement on F:

(i) $F^\perp$ is a subspace of E.

(ii) $F \cap F^\perp = \{\vec{0}_E\}$.

(iii) $\dim(F) + \dim(F^\perp) = \dim(E)$.

(iv) $(F^\perp)^\perp = F$.

Question 6 Let F be a subspace of a finite-dimensional inner product space E. Let p_F be the orthogonal projection onto F. We remind Bessel inequality: for all $x \in E$,

$$\|p_F(x)\| \leq \|x\|$$

with equality if and only if $x \in F$. Prove it.

Question 7. Let $p \in \mathcal{L}(E)$ a projector (i.e. $p \circ p = p$). Prove that the following propositions are equivalent:

(i) p is an orthogonal projector onto $F = \text{range}(p)$

(ii) For all $x, y \in E$, $\langle p(x), y - p(y) \rangle = 0$

(iii) For all $x, y \in E$, $\langle p(x), y \rangle = \langle x, p(y) \rangle$

Hint: from (i), prove first that $\text{Ker}(p) = F^\perp$ and then use the decomposition $E = F \oplus F^\perp$.

3 Exercises

Exercise 1. We denote $E = C([-1;1], \mathbb{R})$.

Show that the mapping $\varphi : E^2 \to \mathbb{R}$ defined, for every $(f, g) \in E^2$, by

$$\varphi(f, g) = \int_{-1}^{1} \sqrt{1 - x^2}\, f(x)\, g(x)\, dx$$

is an inner product on E.

Exercise 2 – Use of the Cauchy–Schwarz Inequality.

Let $(E, \|\cdot\|)$ be a real normed vector space, $n \in \mathbb{N}^*$, $(x_1, \ldots, x_n) \in E^n$, $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Show that

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\|^2 \leq \left(\sum_{i=1}^{n} \alpha_i^2 \right) \left(\sum_{i=1}^{n} \|x_i\|^2 \right).$$

Bonus Exercise – Orthogonality between $S_n(\mathbb{R})$ and $A_n(\mathbb{R})$

NB: We denote by $M_n(\mathbb{R})$ the set of square matrices of size n with real entries, by $S_n(\mathbb{R})$ the subset of symmetric matrices and by $A_n(\mathbb{R})$ the subset of antisymmetric matrices.

Let $n \in \mathbb{N}^*$. We consider the following application from $M_n(\mathbb{R})$ to $\mathbb{R}$.

$$(M, N) \longmapsto (M \mid N) = \text{tr}\Big(M^\top N\Big).$$

1. Show that $(\cdot \mid \cdot)$ is an inner product on $M_n(\mathbb{R})$.
2. Deduce that the function $M \mapsto \|M\| = \sqrt{(M \mid M)}$ is a norm on $M_n(\mathbb{R})$.
3. Show that $S_n(\mathbb{R})$ and $A_n(\mathbb{R})$ are two subspaces in $M_n(\mathbb{R})$.
4. Show that $S_n(\mathbb{R})$ and $A_n(\mathbb{R})$ are complementary and orthogonal subspaces in $M_n(\mathbb{R})$.
5. What is the dimension of $S_n(\mathbb{R})$ and $A_n(\mathbb{R})$? Propose a basis for each of these subspaces.