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CogSci Methods 2 – F24

Solutions to class 5 exercises

Exercises

0. **Come up with examples of the following matrices and discuss and explain why they are called so:**
   - Square matrix

     A square matrix is a matrix with the same number of rows and columns. The dimension of a square matrix is often denoted as n×n, where n represents the number of rows or columns.

     Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

     This matrix would be called a square matrix because it has 3 rows and 3 columns = 3x3.

   - Symmetric matrix

     A symmetric matrix is a square matrix that is equal to its transpose. In math terms this means that it holds that $A = A^T$.

     In other words, the element at the i'th row and j'th column is equal to the element at the i'th row and j'th column, for all i,j.

     Example:

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

     Matrix B is symmetric because $B = B^T$. You can see it if you try to transpose B that it becomes the same as before.

   - Diagonal matrix

A diagonal matrix is a square matrix in which the elements outside the main diagonal are all zero. The main diagonal itself may have non-zero elements.

Example:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

Matrix C is diagonal because only the diagonal elements from the top left to bottom right are non-zero.

- Identity matrix

  An identity matrix is a special form of a diagonal matrix where all the elements on the main diagonal are 1, and all other elements are 0. It acts as the multiplicative identity in matrix multiplication, meaning any matrix multiplied by the identity matrix is unchanged.

  Example:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  This 3x3 matrix can be called an identity matrix because multiplying it by any 3x3 matrix will return the original matrix.

- J Matrix and 0 matrix

  (I haven't actually heard about J-matrix before this course ☺ why is it called "j"? lol) But yeah, it's just a matrix where all elements are 1. So:

  Example:

$$J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

  The 0-matrix is similarly just one where all elements are 0. It acts as the additive identity in matrix addition which is fancy for saying that if you add a matrix to the zero matrix then the matrix remains unchanged = nothing happens when you add 0. So:

Example:

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thereafter, exercises in the GILL book:

For these, remember that X prime (X') is the j*i transpose matrix of matrix i*j X:

$$\mathbf{X}' = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

a) $[1 \quad 1 \quad 1] * [a \quad b \quad c]'$

$$[1 \quad 1 \quad 1] * [a \quad b \quad c]'$$

$$= [1 \quad 1 \quad 1] * \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= (1 * a + 1 * b + 1 * c)$$

$$= a + b + c$$

—

b) $[1 \quad 1 \quad 1] \times [a \quad b \quad c]'$ (not multiplying (*) but taking the cross-product by ×)

$$[1 \quad 1 \quad 1] \times [a \quad b \quad c]'$$

$$= [1 \quad 1 \quad 1] \times \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$= \begin{bmatrix} 1 * c - 1 * b \\ 1 * a - 1 * c \\ 1 * b - 1 * a \end{bmatrix}$$

$$= \begin{bmatrix} c - b \\ a - c \\ b - a \end{bmatrix}$$

RULES USED

(Approximately on the line where it has been applied)

Using the cross-product formula of two 1x3 vectors (1 row, 3 columns) **a** and **b**:

$$\boldsymbol{a} \times \mathbf{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

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c) $[-1 \quad 1 \quad -1] * [4 \quad 3 \quad 12]'$

$$[-1 \quad 1 \quad -1] * [4 \quad 3 \quad 12]'$$

$$= [-1 \quad 1 \quad -1] * \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$$

$$= (-1) * 4 + 1 * 3 + (-1) * 12$$

$$= -4 + 3 - 12$$

$$= -13$$

Sidenote: How do you know how many values / the dimensions should be in the final matrix when multiplying matrices?

When multiplying matrices or vectors, the dimensions of the matrices involved determine the size of the resulting matrix. The rule for matrix multiplication compatibility is that the number of **columns** in the first matrix must be equal to the number of rows in the second matrix. If you have a matrix A of size $m \times n$ (where m is the number of rows and n is the number of columns) and another matrix B of size $n \times p$, the resulting matrix C, obtained from multiplying A by B (A * B), will have a size of $m \times p$.

E.g. in c) above, we multiply a matrix 1x3 (1 row, 3 cols) with another 3x1 (remember it's transposed). We can then say that m = 1, n = 3, and p = 1. Then the result will have dimensions m x p = 1x1 meaning a scalar, i.e. a single value ☺

d) $[-1 \quad 1 \quad -1] \times [4 \quad 3 \quad 12]'$ (not multiplying (*) but taking the cross-product by $\times$)

$$[-1 \quad 1 \quad -1] \times [4 \quad 3 \quad 12]'$$

$$= [-1 \quad 1 \quad -1] \times \begin{bmatrix} 4 \\ 3 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 * 12 - (-1 * 3) \\ (-1 * 4) - (-1 * 12) \\ (-1 * 3) - 1 * 4 \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 3 \\ -4 + 12 \\ -3 - 4 \end{bmatrix}$$

Using the cross-product formula of two 1x3 vectors (1 row, 3 columns) **a** and **b**:

$$\boldsymbol{a} \times \mathbf{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 \\ 8 \\ -7 \end{bmatrix}$$

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e) $[0 \quad 9 \quad 0 \quad 11] * [123.98211 \quad 6 \quad -6392.38743 \quad -5]'$

$[0 \quad 9 \quad 0 \quad 11] * [123.98211 \quad 6 \quad -6392.38743 \quad -5]$

$$= [0 \quad 9 \quad 0 \quad 11] * \begin{bmatrix} 123.98211 \\ 6 \\ -6392.38743 \\ -5 \end{bmatrix}$$

$$= 0 * 123.98211 + 9 * 6 + 0 * -6392.38743 + 11 * -5$$

$$= 9 * 6 + 11 * -5$$

$$= 54 - 55$$

$$= -1$$

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f) $[123.98211 \quad 6 \quad -6392.38743 \quad -5] * [0 \quad 9 \quad 0 \quad 11]'$

$$[123.98211 \quad 6 \quad -6392.38743 \quad -5] * [0 \quad 9 \quad 0 \quad 11]'$$

$$= [123.98211 \quad 6 \quad -6392.38743 \quad -5] * \begin{bmatrix} 0 \\ 9 \\ 0 \\ 11 \end{bmatrix}$$

$$= 123.98211 * 0 + 6 * 9 + (-6392.38743 * 0) + (-5 * 11)$$

$$= 6 * 9 + (-5 * 11)$$

$$= 54 - 55$$

$$= -1$$

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- 3.10

a)

$$\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 1 & \frac{1}{3} & 5 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 1 & \frac{1}{3} & 5 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

By the m x n and n x p rule, we can see the final matrix will have dimensions 3 x 1 (3 rows from the first matrix, 1 col from the latter matrix):

$$= \begin{bmatrix} 1 * 0.1 + \frac{1}{2} * 0.2 + 2 * 0.3 \\ 1 * 0.1 + \frac{1}{3} * 0.2 + 5 * 0.3 \\ 1 * 0.1 + 1 * 0.2 + 2 * 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0.2}{2} + 0.7 \\ \frac{0.2}{3} + 1.6 \\ 0.9 \end{bmatrix}$$

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b)

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 5 \end{bmatrix}$$

By the m x n and n x p rule, we can again see the final matrix will have dimensions 3 x 1 (3 rows from the first matrix, 1 col from the latter matrix):

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$$= \begin{bmatrix} 0*9+1*7+0*5 \\ 1*9+0*7+0*5 \\ 0*9+0*7+1*5 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 9 \\ 5 \end{bmatrix}$$

—

c)

$$\begin{bmatrix} 9 & 7 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 7 & 5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By the m x n and n x p rule, we can see the final matrix will have dimensions 1 x 3 (1 row from the first matrix, 3 cols from the latter matrix):

$$= \begin{bmatrix} 9*0+7*1+5*0 & 9*1+7*0+5*0 & 9*0+7*0+5*1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 9 & 5 \end{bmatrix}$$

—

d)

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 3 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

By the m x n and n x p rule, we can see the final matrix will have dimensions 3 x 1 (3 rows from the first matrix, 1 col from the latter matrix):

$$= \begin{bmatrix} 3 * \frac{1}{3} + 3 * \frac{1}{3} + 1 * \frac{1}{3} \\ 3 * \frac{1}{3} + 1 * \frac{1}{3} + 3 * \frac{1}{3} \\ 1 * \frac{1}{3} + 3 * \frac{1}{3} + 3 * \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{3} \\ \frac{7}{3} \\ \frac{7}{3} \end{bmatrix}$$

—

3.7 Show that pre-multiplication and post-multiplication with the identity matrix are equivalent.

So the identity matrix is a fundamentally *cool* concept in linear algebra because multiplying any matrix by the identity matrix results in the original matrix. This property holds for both pre-multiplication (multiplying the identity matrix to the left of the original matrix) and post-multiplication (multiplying the identity matrix to the right of the original matrix).

Let's take an example. We define our matrices $\boldsymbol{A}$ and $\boldsymbol{I}$, respectively some arbitrary matrix $\boldsymbol{A}$ and the identity matrix $\boldsymbol{I}$:

$$\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Let's now do pre-multiplication, which just means we'll do **I * A**:

$$\boldsymbol{I} * \boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1*a+0*c & 1*b+0*d \\ 0*a+1*c & 0*b+1*d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \boldsymbol{A}$$

Nice. Let's show it the other way – post-multiplication (**A*I**):

$$\boldsymbol{A} * \boldsymbol{I} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a*1+b*0 & a*0+b*1 \\ c*1+d*0 & c*0+d*1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \boldsymbol{A}$$

In both pre-multiplication and post-multiplication by the identity matrix, the original matrix $\boldsymbol{A}$ remains unchanged, demonstrating that pre-multiplication and post-multiplication with the identity matrix are equivalent.

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

a)

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

By the rule of m x n and n x p, we know the final matrix will have dimensions 2 x 2 (2 rows, 2 cols):

$$= \begin{bmatrix} 3*2+(-3)*0 & 3*1+(-3)*0 \\ -3*2+3*0 & -3*1+3*0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 \\ -6 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

By the rule of m x n and n x p, we know the final matrix will have dimensions 3 x 2 (3 rows, 2 cols):

$$= \begin{bmatrix} 0*4+1*3+1*1 & 0*7+1*0+1*2 \\ 1*4+0*3+1*1 & 1*7+0*0+1*2 \\ 1*4+1*3+0*1 & 1*7+1*0+0*2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 5 & 9 \\ 7 & 7 \end{bmatrix}$$

—

$$\begin{bmatrix} 3 & 1 & -2 \\ 6 & 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

c)

$$\begin{bmatrix} 3 & 1 & -2 \\ 6 & 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

By the rule of m x n and n x p, we know the final matrix will have dimensions 2 x 2 (2 rows, 2 cols):

$$= \begin{bmatrix} 3*4+1*3+(-2)*1 & 3*7+1*0+(-2)*2 \\ 6*4+3*3+4*1 & 6*7+3*0+4*2 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 17 \\ 37 & 50 \end{bmatrix}$$

—

d) $$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

We can see that the final matrix will be of dimensions 2x2:

$$= \begin{bmatrix} 1*1+0*3 & 1*0+0*1 \\ (-3)*1+1*3 & (-3)*0+1*1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

—

e) $$\begin{bmatrix} -1 & -9 \\ -1 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -4 & -4 \\ -1 & 0 \\ -3 & -8 \end{bmatrix}'$$

$$\begin{bmatrix} -1 & -9 \\ -1 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -4 & -4 \\ -1 & 0 \\ -3 & -8 \end{bmatrix}'$$

$$= \begin{bmatrix} -1 & -9 \\ -1 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -4 & -1 & -3 \\ -4 & 0 & -8 \end{bmatrix}$$

We can see that the final matrix will be of dimensions 3x3:

$$= \begin{bmatrix} (-1)*(-4)+(-9)*(-4) & (-1)*(-1)+(-9)*0 & (-1)*(-3)+(-9)*(-8) \\ (-1)*(-4)+(-4)*(-4) & (-1)*(-1)+(-4)*0 & (-1)*(-3)+(-4)*(-8) \\ 1*(-4)+2*(-4) & 1*(-1)+2*0 & 1*(-3)+2*(-8) \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 1 & 75 \\ 20 & 1 & 35 \\ -12 & -1 & -19 \end{bmatrix}$$

—

f) $$\begin{bmatrix} 0 & 0 \\ 0 & \infty \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 0 & \infty \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Final matrix will be 2x2:

$$= \begin{bmatrix} 0 * 1 + 0 * (-1) & 0 * 1 + 0 * (-1) \\ 0 * 1 + \infty * (-1) & 0 * 1 + \infty * (-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -1 * \infty & -1 * \infty \end{bmatrix}$$

- 3.13 (challenging)

I didn't have time to write my solutions cleanly for this one, so I've attached Katrine's notes (TA last year in Methods 2) to this pdf. ☺

- 3.22

3.22 Vectorize the following matrix and find the vector norm. Can you think of any shortcuts that would make the calculations less repetitious?

$$\tilde{\mathbf{X}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 6 \\ 5 & 5 & 5 \\ 6 & 7 & 6 \\ 7 & 9 & 9 \\ 8 & 8 & 8 \\ 9 & 8 & 3 \end{bmatrix}.$$

I didn't have time to write my solutions cleanly for this one, so I've attached Katrine's notes (TA last year in Methods 2) to this pdf. ☺

- 3.23 (optional, extra challenging)

3.23 For two vectors in $\Re^3$ using $1 = \cos^2\theta + \sin^2\theta$ and $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \mathbf{u}^2 \cdot \mathbf{v}^2$, show that the norm of the cross product between two vectors, $\mathbf{u}$ and $\mathbf{v}$, is: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin(\theta)$.

I didn't have time to write my solutions cleanly for this one, so I've attached Katrine's notes (TA last year in Methods 2) to this pdf. ☺

3.13

Utilize that $OR \cdot PO \cdot PR = OPR$

To ensure that the dimensionalities add up the following transformation is neccessary

$$(OR' PO')' \cdot PR = OPR$$

$$OR' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad PO' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = OR' \cdot PO'$$

$(OR' \cdot PO') \cdot PR =$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \underline{\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix}}$$

3.13

Multiplying any of three matrices by a constant has the same effect on the outcome because of the properties of the inner product

3.22 Vectorization & vector norm

$$\overset{n}{\mathbf{X}} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 1 & 2 \\ 4 & 3 & 6 \\ 5 & 5 & 5 \\ 6 & 7 & 6 \\ 7 & 9 & 9 \\ 8 & 8 & 8 \\ 9 & 8 & 3 \end{bmatrix} \qquad vec(\overset{n}{\mathbf{X}}) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 2 \\ 4 \\ 1 \\ 3 \\ 5 \\ 7 \\ 9 \\ 8 \\ 8 \\ 1 \\ 3 \\ 2 \\ 6 \\ 5 \\ 6 \\ 9 \\ 8 \\ 3 \end{bmatrix}$$

Vector norm p. 93

$$\|\mathbf{v}\| = (v_1^2 + v_2^2 + \dots + v_n^2)^{\frac{1}{2}}$$

$$\|\mathbf{v}\|^2 = (1^2 + 2^2 + 3^2 \dots 9^2 + 8^2 + 3^2)^{\frac{1}{2}} = 874$$

3.23 Start with the angle between two vectors

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \iff \cos^2(\theta) = \frac{\mathbf{u}^2 \cdot \mathbf{v}^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \iff$$

$$1 - \sin^2(\theta) = \frac{\mathbf{u}^2 \cdot \mathbf{v}^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \iff \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2(\theta) = \mathbf{u}^2 \cdot \mathbf{v}^2$$

$$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \mathbf{u}^2 \cdot \mathbf{v}^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2(\theta) \iff$$

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 \sin^2(\theta)$$

$$\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta)$$