5.1

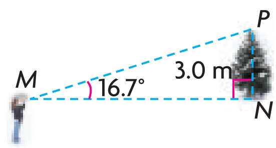
Trigonometric Ratios of Acute Angles

GOAL

Evaluate reciprocal trigonometric ratios.

LEARN ABOUT the Math

From a position some distance away from the base of a tree, Monique uses a clinometer to determine the angle of elevation to a treetop. Monique estimates that the height of the tree is about 3.0 m.



? How far, to the nearest tenth of a metre, is Monique from the base of the tree?

Communication Tip

A clinometer is a device used to measure the angle of elevation (above the horizontal) or the angle of depression (below the horizontal).

EXAMPLE 1 Selecting a strategy to determine a side length in a right triangle

In $\triangle MNP$, determine the length of MN.

Clive's Solution: Using Primary Trigonometric Ratios

$$\tan 16.7° = \frac{3.0}{MN}$$

I knew the opposite side but I needed to calculate the adjacent side *MN*. So I used tangent.

$$MN(\tan 16.7°) = 3.0$$

$$MN = \frac{3.0}{\tan 16.7°}$$

I multiplied both sides of the equation by *MN*, then divided by tan 16.7°.

I used my calculator to evaluate.

$$MN \doteq 10.0 \text{ m}$$

Monique is about 10.0 m away from the base of the tree.

Communication Tip

The symbol ≐ means "approximately equal to" and indicates that a result has been rounded.

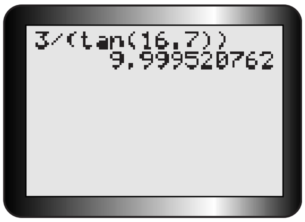
Tony's Solution: Using Reciprocal Trigonometric Ratios

$$\cot 16.7° = \frac{MN}{3.0}$$

NP is opposite the 16.7° angle, and *MN* is adjacent. I used the **reciprocal trigonometric ratio** cot 16.7°. This gave me an equation with the unknown in the numerator, making the equation easier to solve.

$$(3.0)\cot 16.7° = MN$$

To solve for *MN*, I multiplied both sides by 3.0.



I evaluated $\frac{1}{\tan 16.7°}$ to get cot 16.7°.

$$10.0 \text{ m} \doteq MN$$

Monique is about 10.0 m away from the base of the tree.

reciprocal trigonometric ratios

the reciprocal ratios are defined as 1 divided by each of the primary trigonometric ratios

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\text{adjacent}}{\text{opposite}}$$

Cot θ is the short form for the cotangent of angle θ, sec θ is the short form for the secant of angle θ, and csc θ is the short form for the cosecant of angle θ.

Tech *Support*

Most calculators do not have buttons for evaluating the reciprocal ratios. For example, to evaluate

- csc 20°, use $\frac{1}{\sin 20°}$
- sec 20°, use $\frac{1}{\cos 20°}$
- cot 20°, use $\frac{1}{\tan 20°}$

Reflecting

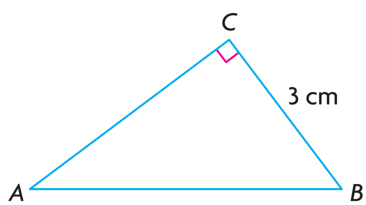
A. What was the advantage of using a reciprocal trigonometric ratio in Tony's solution?

B. Suppose Monique wants to calculate the length of *MP* in $\triangle MNP$. State the two trigonometric ratios that she could use based on the given information. Which one would be better? Explain.

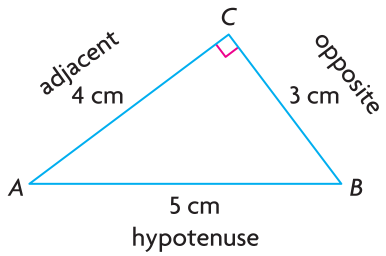
APPLY the Math

EXAMPLE 2 Evaluating the six trigonometric ratios of an angle

$\triangle ABC$ is a right triangle with side lengths of 3 cm, 4 cm, and 5 cm. If $CB = 3$ cm and $\angle C = 90°$, which trigonometric ratio of $\angle A$ is the greatest?



Sam's Solution



I labelled the sides of the triangle relative to $\angle A$, first in words and then with the side lengths. The hypotenuse is the longest side, so its length must be 5 cm. If the side opposite $\angle A$ is 3 cm, then the side adjacent to $\angle A$ is 4 cm.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} = 0.60 \qquad \cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} = 0.80 \qquad \tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4} = 0.75$$

First, I used the definitions of the primary trigonometric ratios to determine the sine, cosine, and tangent of $\angle A$.

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{3} \doteq 1.67 \qquad \sec A = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{4} = 1.25 \qquad \cot A = \frac{\text{adjacent}}{\text{opposite}} = \frac{4}{3} \doteq 1.33$$

Then I evaluated the reciprocal trigonometric ratios for $\angle A$. I wrote the reciprocal of each primary ratio to get the appropriate reciprocal ratio.

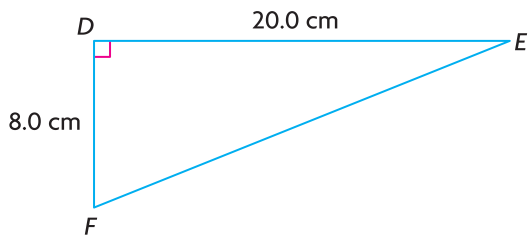
I expressed these ratios as decimals to compare them more easily.

The greatest trigonometric ratio of $\angle A$ is $\csc A$.

EXAMPLE 3 Solving a right triangle by calculating the unknown side and the unknown angles

a) Determine EF in $\triangle DEF$ to the nearest tenth of a centimetre.

b) Express one unknown angle in terms of a primary trigonometric ratio and the other angle in terms of a reciprocal ratio. Then calculate the unknown angles to the nearest degree.



Lina's Solution

a) $EF^2 = (8.0)^2 + (20.0)^2$

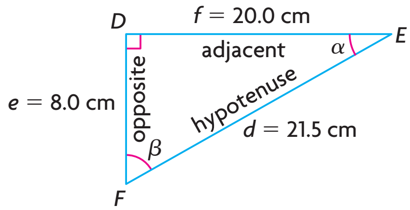
$EF^2 = 464.0 \text{ cm}^2$

$EF = \sqrt{464.0}$

$EF \doteq 21.5 \text{ cm}$

Since $\triangle DEF$ is a right triangle, I used the Pythagorean theorem to calculate the length of *EF*.

b)



I labelled $\angle E$ as α. Side *e* is opposite α and *f* is adjacent to α. So I expressed α in terms of the primary trigonometric ratio tan α.

I labelled $\angle F$ as β. Side *d* is the hypotenuse and *e* is adjacent to β.

$$\tan \alpha = \frac{\text{opposite}}{\text{adjacent}} \qquad \sec \beta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$= \frac{e}{f} \qquad = \frac{d}{e}$$

$$= \frac{8.0}{20.0} \qquad = \frac{21.5}{8.0}$$

$$= 0.40 \qquad \doteq 2.69$$

I expressed β in terms of the reciprocal trigonometric ratio sec β.

$$\alpha = \tan^{-1}(0.40)$$

$$\alpha \doteq 22°$$

To determine angle α, I used my calculator to evaluate $\tan^{-1}$ (0.40) directly.

$$\sec \beta \doteq 2.69$$

$$\cos \beta \doteq \frac{1}{2.69}$$

$$\beta \doteq \cos^{-1}\left(\frac{1}{2.69}\right)$$

Since my calculator doesn't have a $\sec^{-1}$ key, I wrote sec β in terms of the primary trigonometric ratio cos β before determining β.

$$\beta \doteq 68°$$

I determined angle β directly by evaluating $\cos^{-1}\left(\frac{1}{2.69}\right)$ with my calculator.

EF is about 21.5 cm long, and $\angle E$ and $\angle F$ are about 22° and 68°, respectively.

Communication *Tip*

Unknown angles are often labelled with the Greek letters θ (theta), α (alpha), and β (beta).

Communication *Tip*

Arcsine ($\sin^{-1}$), arccosine ($\cos^{-1}$), and arctangent ($\tan^{-1}$) are the names given to the inverse trigonometric functions. These are used to determine the angle associated with a given primary ratio.

In Summary

Key Idea

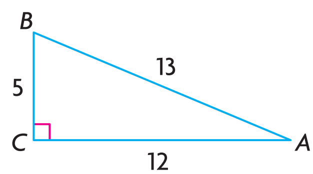
- The reciprocal trigonometric ratios are reciprocals of the primary trigonometric ratios, and are defined as 1 divided by each of the primary trigonometric ratios:
  - $\csc\theta = \dfrac{1}{\sin\theta} = \dfrac{\text{hypotenuse}}{\text{opposite}}$
  - $\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
  - $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\text{adjacent}}{\text{opposite}}$

Need to Know

- In solving problems, reciprocal trigonometric ratios are sometimes helpful because the unknown variable can be expressed in the numerator, making calculations easier.
- Calculators don't have buttons for cosecant, secant, or cotangent ratios.
- The sine and cosine ratios for an acute angle in a right triangle are less than or equal to 1 so their reciprocal ratios, cosecant and secant, are always greater than or equal to 1.
- The tangent ratio for an acute angle in a right triangle can be less than 1, equal to 1, or greater than 1, so the reciprocal ratio, cotangent, can take on this same range of values.

CHECK Your Understanding

1. Given $\triangle ABC$, state the six trigonometric ratios for $\angle A$.



2. State the reciprocal trigonometric ratios that correspond to $\sin\theta = \frac{8}{17}$, $\cos\theta = \frac{15}{17}$, and $\tan\theta = \frac{8}{15}$.

3. For each primary trigonometric ratio, determine the corresponding reciprocal ratio.

a) $\sin\theta = \dfrac{1}{2}$

b) $\cos\theta = \dfrac{3}{4}$

c) $\tan\theta = \dfrac{3}{2}$

d) $\tan\theta = \dfrac{1}{4}$

4. Evaluate to the nearest hundredth.

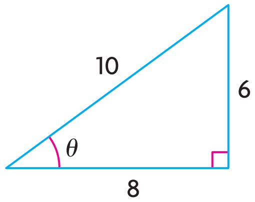
a) $\cos 34°$ **b)** $\sec 10°$ **c)** $\cot 75°$ **d)** $\csc 45°$

PRACTISING

5. **a)** For each triangle, calculate csc θ, sec θ, and cot θ.

K **b)** For each triangle, use one of the reciprocal ratios from part (a) to determine θ to the nearest degree.

i)



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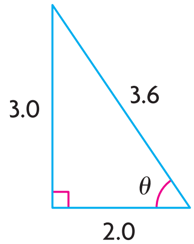

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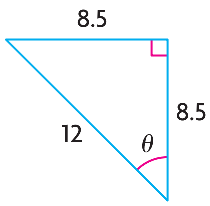

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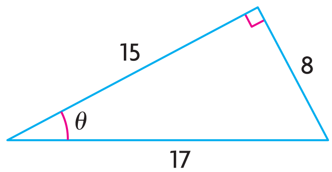


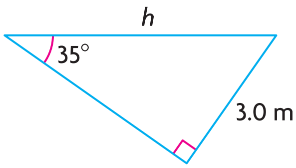
6. Determine the value of θ to the nearest degree.

a) $\cot \theta = 3.2404$ **c)** $\sec \theta = 1.4526$

b) $\csc \theta = 1.2711$ **d)** $\cot \theta = 0.5814$

7. For each triangle, determine the length of the hypotenuse to the nearest tenth of a metre.

a)



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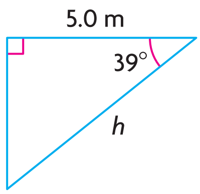
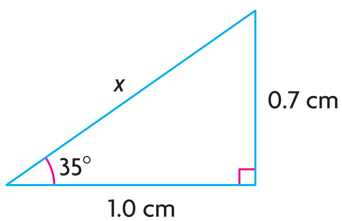


8. For each triangle, use two different methods to determine x to the nearest tenth of a unit.

a)



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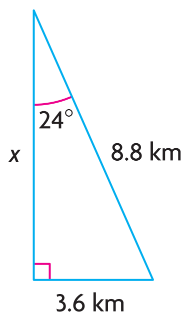
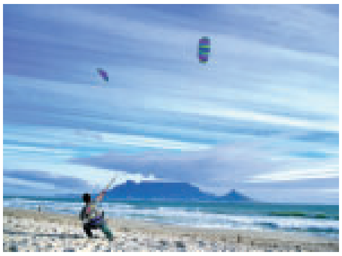


9. Given any right triangle with an acute angle θ,

a) explain why csc θ is always greater than or equal to 1

b) explain why cos θ is always less than or equal to 1



10. Given a right triangle with an acute angle θ, if $\tan \theta = \cot \theta$, describe what this triangle would look like.

11. **A** A kite is flying 8.6 m above the ground at an angle of elevation of 41°. Calculate the length of string, to the nearest tenth of a metre, needed to fly the kite using

a) a primary trigonometric ratio

b) a reciprocal trigonometric ratio

12. A wheelchair ramp near the door of a building has an incline of 15° and a run of 7.11 m from the door. Calculate the length of the ramp to the nearest hundredth of a metre.

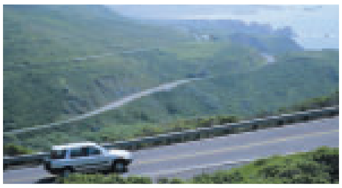
13. **T** The hypotenuse, c, of right $\triangle ABC$ is 7.0 cm long. A trigonometric ratio for angle A is given for four different triangles. Which of these triangles has the greatest area? Justify your decision.

a) $\sec A = 1.7105$ **c)** $\csc A = 2.2703$

b) $\cos A = 0.7512$ **d)** $\sin A = 0.1515$

14. The two guy wires supporting an 8.5 m TV antenna each form an angle of 55° with the ground. The wires are attached to the antenna 3.71 m above ground. Using a reciprocal trigonometric ratio, calculate the length of each wire to the nearest tenth of a metre. What assumption did you make?

15. From a position some distance away from the base of a flagpole, Julie estimates that the pole is 5.35 m tall at an angle of elevation of 25°. If Julie is 1.55 m tall, use a reciprocal trigonometric ratio to calculate how far she is from the base of the flagpole, to the nearest hundredth of a metre.



16. The maximum grade (slope) allowed for highways in Ontario is 12%.

a) Predict the angle θ, to the nearest degree, associated with this slope.

b) Calculate the value of θ to the nearest degree.

c) Determine the six trigonometric ratios for angle θ.

17. **C** Organize these terms in a word web, including explanations where appropriate.

sine	cosine	tangent	opposite
cotangent	hypotenuse	cosecant	adjacent
secant	angle of depression	angle	angle of elevation

Extending

18. In right $\triangle PQR$, the hypotenuse, r, is 117 cm and $\tan P = 0.51$. Calculate side lengths p and q to the nearest centimetre and all three interior angles to the nearest degree.

19. Describe the appearance of a triangle that has a secant ratio that is greater than any other trigonometric ratio.

20. The tangent ratio is undefined for angles whose adjacent side is equal to zero. List all the angles between 0° and 90° (if any) for which cosecant, secant, and cotangent are undefined.