Section 6.3—Multiplication of a Vector by a Scalar

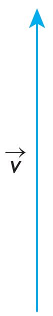
In this section, we will demonstrate the effect of multiplying a vector, $\vec{a}$, by a number k to produce a new vector, $k\vec{a}$. The number k used for multiplication is called a scalar and can be any real number. Previously, the distinction was made between scalars and vectors by saying that scalars have magnitude and not direction, whereas vectors have both. In this section, we are giving a more general meaning to the word *scalar* so that it means any real number. Since real numbers have magnitude (size) but not direction, this meaning is consistent with our earlier understanding.

Examining Scalar Multiplication

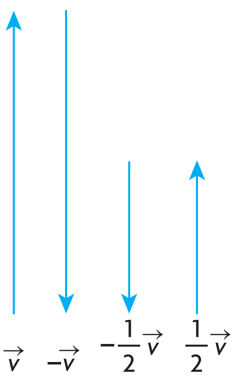
Multiplying $\vec{a}$ by different values of k can affect the direction and magnitude of a vector, depending on the values of k that are chosen. The following example demonstrates the effect on a velocity vector when it is multiplied by different scalars.

EXAMPLE 1 **Reasoning about the meaning of scalar multiplication**

An airplane is heading due north at 1000 km/h. The airplane's velocity is represented by $\vec{v}$. Draw the vectors $-\vec{v}$, $\frac{1}{2}\vec{v}$, and $-\frac{1}{2}\vec{v}$ and give an interpretation for each.



Scale: 1 cm is equivalent to 250 km/h



Solution

We interpret the vectors in the following way:

$\vec{v}$: the velocity vector for an airplane heading due north at 1000 km/h

$-\vec{v}$: the velocity vector for an airplane heading due south at 1000 km/h

$\frac{1}{2}\vec{v}$: the velocity vector for an airplane heading due north at 500 km/h

$-\frac{1}{2}\vec{v}$: the velocity vector for an airplane heading due south at 500 km/h

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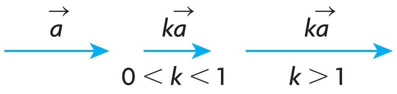
The previous example illustrates how multiplication of a vector by different values of a scalar k can change the magnitude and direction of a vector. The effect of multiplying a vector by a scalar is summarized as follows.

Multiplication of a Vector by a Scalar

For the vector $k\vec{a}$, where k is a scalar and $\vec{a}$ is a nonzero vector:

1. If $k > 0$, then $k\vec{a}$ is in the same direction as $\vec{a}$ with magnitude $k|\vec{a}|$.

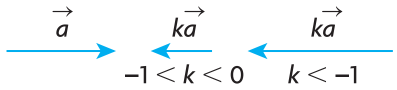
   For $k > 0$, two different possibilities will be considered and are illustrated in the following diagram:

   For $0 < k < 1$, the vector is shortened, and the direction stays the same. If $\vec{a}$ is as shown above, then $\frac{1}{2}\vec{a}$ is half the length of the original vector and in the same direction, i.e., $\left|\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$.

   For $k > 1$, the vector is lengthened, and the direction stays the same. If $\vec{a}$ is as shown above, then $\frac{3}{2}\vec{a}$ is one and a half times as long as $\vec{a}$ and in the same direction, i.e., $\left|\frac{3}{2}\vec{a}\right| = \frac{3}{2}|\vec{a}|$.

2. If $k < 0$, then $k\vec{a}$ is in the opposite direction as $\vec{a}$ with magnitude $|k||\vec{a}|$. Again, two situations will be considered for $k < 0$.

   For $-1 < k < 0$, the vector is shortened and changes to the opposite direction. If $\vec{a}$ is as shown above, then $-\frac{1}{2}\vec{a}$ is half the length of the original vector $\vec{a}$ but in the opposite direction, i.e., $\left|-\frac{1}{2}\vec{a}\right| = \frac{1}{2}|\vec{a}|$. In the situation where $k < -1$, the vector is lengthened and changes to the opposite direction. If $\vec{a}$ is as shown above, then $-\frac{3}{2}\vec{a}$ is one and a half times as long as $\vec{a}$ but in the opposite direction, i.e., $\left|-\frac{3}{2}\vec{a}\right| = \frac{3}{2}|\vec{a}|$.

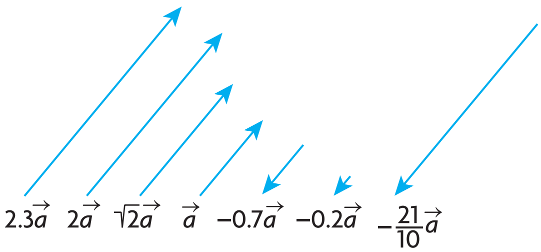
Collinear Vectors

A separate comment should be made about the cases $k = 0$ and $k = -1$.

If we multiply any vector $\vec{a}$ by the scalar 0, the result is always the zero vector, i.e., $0\vec{a} = \vec{0}$. Note that the right side of this equation is a vector, not a scalar.

When we multiply a vector by -1, i.e., $(-1)\vec{a}$, we normally write this as $-\vec{a}$. When any vector is multiplied by -1, its magnitude is unchanged but the direction changes to the opposite. For example, the vectors $-4\vec{a}$ and $4\vec{a}$ have the same magnitude (length) but are opposite.

The effect of multiplying a vector, $\vec{a}$, by different scalars is shown below.



When two vectors are parallel or lie on the same straight line, these vectors are described as being **collinear**. They are described as being collinear because they can be translated so that they lie in the same straight line. Vectors that are not collinear are not parallel. All of the vectors shown above are scalar multiples of $\vec{a}$ and are collinear. When discussing vectors, the terms *parallel* and *collinear* are used interchangeably.

Two vectors u and v are collinear if and only if it is possible to find a nonzero scalar k such that $\vec{u} = k\vec{v}$.

In the following example, we combine concepts learned in the previous section with those introduced in this section.

EXAMPLE 2 **Selecting a strategy to determine the magnitude and direction of a vector**

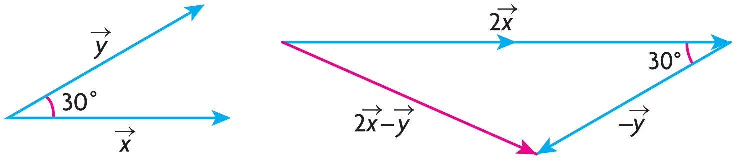
The vectors $\vec{x}$ and $\vec{y}$ are unit vectors (vectors with magnitude 1) that make an angle of 30° with each other.

a. Calculate the value of $|2\vec{x} - \vec{y}|$.

b. Determine the direction of $2\vec{x} - \vec{y}$.

Solution

a.



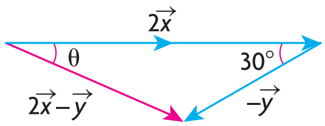
To calculate the value of $|2\vec{x} - \vec{y}|$, construct $2\vec{x} - \vec{y}$ by drawing $2\vec{x}$ and $-\vec{y}$ head-to-tail and then adding them.

Using the cosine law, $|2\vec{x} - \vec{y}|^2 = |2\vec{x}|^2 + |-\vec{y}|^2 - 2|2\vec{x}||-\vec{y}|\cos 30°$

$$|2\vec{x} - \vec{y}|^2 = 2^2 + 1^2 - 2(2)(1)\frac{\sqrt{3}}{2}$$
$$|2\vec{x} - \vec{y}|^2 = 5 - 2\sqrt{3}$$
$$|2\vec{x} - \vec{y}| = \sqrt{5 - 2\sqrt{3}}$$

Therefore, $|2\vec{x} - \vec{y}| \doteq 1.24$.

b. To determine the direction of $2\vec{x} - \vec{y}$, we will calculate θ using the sine law and describe the direction relative to the
the
direction of $\vec{x}$.



$$\frac{\sin\theta}{|-\vec{y}|} = \frac{\sin 30°}{|2\vec{x} - \vec{y}|}$$

$$\frac{\sin\theta}{1} \doteq \frac{\sin 30°}{1.24}$$

$$\theta = \sin^{-1}\left(\frac{\sin 30°}{1.24}\right)$$

$$\theta \doteq 23.8°$$

Therefore, $2\vec{x} - \vec{y}$ has a direction of 23.8° rotated clockwise relative to $\vec{x}$.

In many practical situations that involve velocities, we use specialized notation to describe direction. In the following example, we use this notation along with scalar multiplication to help illustrate its meaning.

EXAMPLE 3 Representing velocity using vectors

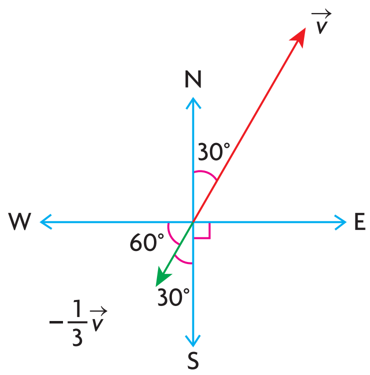
An airplane is flying in the direction $N30°E$ at an airspeed of 240 km/h. The velocity vector for this airplane is represented by $\vec{v}$.

a. Draw a sketch of $-\frac{1}{3}\vec{v}$ and state the direction of this vector.

b. For the vector $\frac{3}{2}\vec{v}$, state its direction and magnitude.

Solution

a.

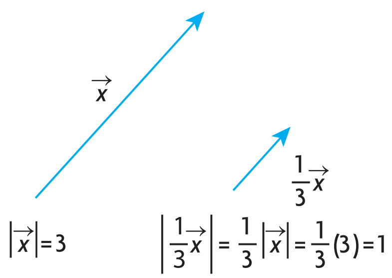


Scale: 1 cm is equivalent to 40 km/h

The vector $-\frac{1}{3}\vec{v}$ represents a speed of $\frac{1}{3}(240 \text{ km/h}) = 80$ km/h and points in the opposite direction as $\vec{v}$. The direction for this vector can be described as W60°S, S30°W, or a bearing of 210°.

b. The velocity vector $\frac{3}{2}\vec{v}$ represents a speed of $\frac{3}{2}(240 \text{ km/h}) = 360$ km/h in the same direction as $\vec{v}$.

It is sometimes useful to multiply the nonzero vector $\vec{x}$ by the scalar $\frac{1}{|\vec{x}|}$. When we multiply $\vec{x}$ by $\frac{1}{|\vec{x}|}$, we get the vector $\frac{1}{|\vec{x}|}\vec{x}$. This vector of length one and called a unit vector, which points in the same direction as $\vec{x}$.



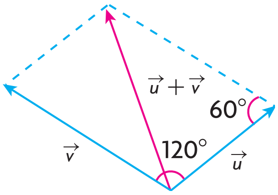
The concept of unit vector will prove to be very useful when we discuss applications of vectors.

EXAMPLE 4 **Using a scalar to create a unit vector**

Given that $|\vec{u}| = 4$ and $|\vec{v}| = 5$ and the angle between $\vec{u}$ and $\vec{v}$ is 120°, determine the unit vector in the same direction as $\vec{u} + \vec{v}$.

Solution

Draw a sketch and determine $|\vec{u} + \vec{v}|$.



Using the cosine law,

$$|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$$
$$|\vec{u} + \vec{v}|^2 = 4^2 + 5^2 - 2(4)(5)\cos 60°$$
$$|\vec{u} + \vec{v}|^2 = 21$$
$$|\vec{u} + \vec{v}| = \sqrt{21}$$

To create a unit vector in the same direction as $\vec{u} + \vec{v}$, multiply by the scalar equal to $\frac{1}{|\vec{u} + \vec{v}|}$. In this case, the unit vector is $\frac{1}{\sqrt{21}}(\vec{u} + \vec{v}) = \frac{1}{\sqrt{21}}\vec{u} + \frac{1}{\sqrt{21}}\vec{v}$.

IN SUMMARY

Key Idea

- For the vector $k\vec{a}$ where k is a scalar and $\vec{a}$ is a nonzero vector:
  - If $k > 0$, then $k\vec{a}$ is in the same direction as $\vec{a}$ with magnitude $k|\vec{a}|$.
  - If $k < 0$, then $k\vec{a}$ is in the opposite direction as $\vec{a}$ with magnitude $|k||\vec{a}|$.

Need to Know

- If two or more vectors are nonzero scalar multiples of the same vector, then all these vectors are collinear.
- $\frac{1}{|\vec{x}|}\vec{x}$ is a vector of length one, called a unit vector, in the direction of the nonzero vector $\vec{x}$.
- $-\frac{1}{|\vec{x}|}\vec{x}$ is a unit vector in the opposite direction of the nonzero vector $\vec{x}$.

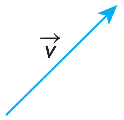
Exercise 6.3

PART A

1. Explain why the statement $\vec{a} = 2|\vec{b}|$ is not meaningful.

2. An airplane is flying at an airspeed of 300 km/h. Using a scale of 1 cm equivalent to 50 km/h, draw a velocity vector to represent each of the following:
   a. a speed of 150 km/h heading in the direction N45°E
   b. a speed of 450 km/h heading in the direction E15°S
   c. a speed of 100 km/h heading in an easterly direction
   d. a speed of 300 km/h heading on a bearing of 345°

3. An airplane's direction is E25°N. Explain why this is the same as N65°E or a bearing of 65°.

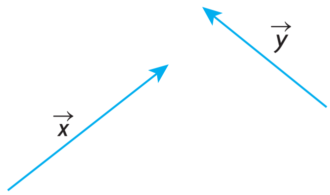


4. The vector $\vec{v}$ has magnitude 2, i.e., $|\vec{v}| = 2$. Draw the following vectors and express each of them as a scalar multiple of $\vec{v}$.
   a. a vector in the same direction as $\vec{v}$ with twice its magnitude
   b. a vector in the same direction as $\vec{v}$ with one-half its magnitude
   c. a vector in the opposite direction as $\vec{v}$ with two-thirds its magnitude
   d. a vector in the opposite direction as $\vec{v}$ with twice its magnitude
   e. a unit vector in the same direction as $\vec{v}$

PART B

K

5. The vectors $\vec{x}$ and $\vec{y}$ are shown below. Draw a diagram for each of the following.



   a. $\vec{x} + 3\vec{y}$ b. $\vec{x} - 3\vec{y}$ c. $-2\vec{x} + \vec{y}$ d. $-2\vec{x} - \vec{y}$

6. Draw two vectors, $\vec{a}$ and $\vec{b}$, that do not have the same magnitude and are noncollinear. Using the vectors you drew, construct the following:
   a. $2\vec{a}$ b. $3\vec{b}$ c. $-3\vec{b}$ d. $2\vec{a} + 3\vec{b}$ e. $2\vec{a} - 3\vec{b}$

7. Three collinear vectors, $\vec{a}$, $\vec{b}$, and $\vec{c}$, are such that $\vec{a} = \frac{2}{3}\vec{b}$ and $\vec{a} = \frac{1}{2}\vec{c}$.
   a. Determine integer values for m and n such that $m\vec{c} + n\vec{b} = \vec{0}$. How many values are possible for m and n to make this statement true?
   b. Determine integer values for d, e, and f such that $d\vec{a} + e\vec{b} + f\vec{c} = \vec{0}$. Are these values unique?

8. The two vectors $\vec{a}$ and $\vec{b}$ are collinear and are chosen such that $|\vec{a}| = |\vec{b}|$. Draw a diagram showing different possible configurations for these two vectors.

9. The vectors $\vec{a}$ and $\vec{b}$ are perpendicular. Are the vectors $4\vec{a}$ and $-2\vec{b}$ also perpendicular? Illustrate your answer with a sketch.

10. If the vectors $\vec{a}$ and $\vec{b}$ are noncollinear, determine which of the following pairs of vectors are collinear and which are not.

    a. $2\vec{a}, -3\vec{a}$ b. $2\vec{a}, 3\vec{b}$ c. $5\vec{a}, -\frac{3}{2}\vec{b}$ d. $-\vec{b}, 2\vec{b}$

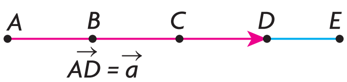
C 11. In the discussion, we defined $\frac{1}{|\vec{x}|}\vec{x}$. Using your own scale, draw your own vector to represent $\vec{x}$.

    a. Sketch $\frac{1}{|\vec{x}|}\vec{x}$ and describe this vector in your own words.

    b. Sketch $-\frac{1}{|\vec{x}|}\vec{x}$ and describe this vector in your own words.

12. Two vectors, $\vec{a}$ and $\vec{b}$, are such that $2\vec{a} = -3\vec{b}$. Draw a possible sketch of these two vectors. What is the value of m, if $|\vec{b}| = m|\vec{a}|$?

A 13. The points B, C, and D are drawn on line segment AE dividing it into four equal lengths. If $\overrightarrow{AD} = \vec{a}$, write each of the following in terms of $\vec{a}$ and $|\vec{a}|$.



    a. $\overrightarrow{EC}$ b. $\overrightarrow{BC}$ c. $|\overrightarrow{ED}|$ d. $|\overrightarrow{AC}|$ e. $\overrightarrow{AE}$

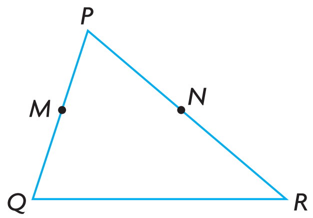
14. The vectors $\vec{x}$ and $\vec{y}$ are unit vectors that make an angle of 90° with each other. Calculate the value of $|2\vec{x} + \vec{y}|$ and the direction of $2\vec{x} + \vec{y}$.

15. The vectors $\vec{x}$ and $\vec{y}$ are unit vectors that make an angle of 30° with each other. Calculate the value of $|2\vec{x} + \vec{y}|$ and the direction of $2\vec{x} + \vec{y}$.

16. Prove that $\frac{1}{|\vec{a}|}\vec{a}$ is a unit vector pointing in the same direction as $\vec{a}$. (*Hint*: Let $\vec{b} = \frac{1}{|\vec{a}|}\vec{a}$ and then find the magnitude of each side of this equation.)

T 17. In $\triangle ABC$, a median is drawn from vertex A to the midpoint of BC, which is labelled D. If $\overrightarrow{AB} = \vec{b}$ and $\overrightarrow{AC} = \vec{c}$, prove that $\overrightarrow{AD} = \frac{1}{2}\vec{b} + \frac{1}{2}\vec{c}$.

18. Let PQR be a triangle in which M is the midpoint of PQ and N is the midpoint of PR. If $\overrightarrow{PM} = \vec{a}$ and $\overrightarrow{PN} = \vec{b}$, find vector expressions for $\overrightarrow{MN}$ and $\overrightarrow{QR}$ in terms of $\vec{a}$ and $\vec{b}$. What conclusions can be drawn about MN and QR? Explain.



19. Draw rhombus $ABCD$ where $AB = 3$ cm. For each of the following, name two vectors $\vec{u}$ and $\vec{v}$ in your diagram such that

a. $\vec{u} = \vec{v}$

b. $\vec{u} = 2\vec{v}$

c. $\vec{u} = -\vec{v}$

d. $\vec{u} = 0.5\vec{v}$

PART C

20. Two vectors, $\vec{x}$ and $\vec{y}$, are drawn such that $|\vec{x}| = 3|\vec{y}|$. Considering $m\vec{x} + n\vec{y} = \vec{0}$, determine all possible values for m and n such that

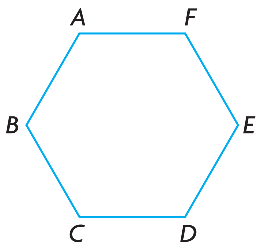
a. $\vec{x}$ and $\vec{y}$ are collinear

b. $\vec{x}$ and $\vec{y}$ are noncollinear

21. $ABCDEF$ is a regular hexagon such that $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$.

a. Express $\overrightarrow{CD}$ in terms of $\vec{a}$ and $\vec{b}$.

b. Prove that BE is parallel to CD and that $|\overrightarrow{BE}| = 2|\overrightarrow{CD}|$.



22. $ABCD$ is a trapezoid whose diagonals AC and BD intersect at the point E. If $\overrightarrow{AB} = \frac{2}{3}\overrightarrow{DC}$, prove that $\overrightarrow{AE} = \frac{3}{5}\overrightarrow{AB} + \frac{2}{5}\overrightarrow{AD}$.