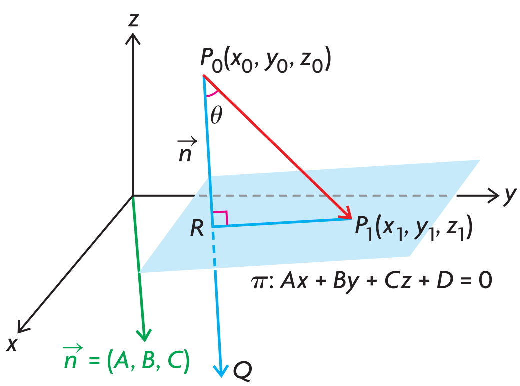
# *Section 9.6*—The Distance from a Point to a Plane

In the previous section, we developed a formula for finding the distance from a point $P_0(x_0, y_0)$ to the line $Ax + By + C = 0$ in $R^2$. In this section, we will use the same kind of approach to develop a formula for the distance from a point $P_0(x_0, y_0, z_0)$ to the plane with equation $Ax + By + Cz + D = 0$ in $R^3$.

## Determining a Formula for the Distance between a Point and a Plane in $R^3$

We start by considering a general plane in $R^3$ that has $Ax + By + Cz + D = 0$ as its equation. The point $P_0(x_0, y_0, z_0)$ is a point whose coordinates are known. A line from $P_0$ is drawn perpendicular to $Ax + By + Cz + D = 0$ and meets this plane at $R$. The point $P_1(x_1, y_1, z_1)$ is a point on the plane, with coordinates different from $R$, and $Q$ is chosen so that $\overrightarrow{P_0Q} = n = (A, B, C)$ is the normal to the plane. The objective is to find a formula for $|\overrightarrow{PR}|$—the perpendicular distance from $P_0$ to the plane. To develop this formula, we are going to use the fact that $|\overrightarrow{PR}|$ is the scalar projection of $\overrightarrow{P_0P_1}$ on the normal $\vec{n}$.




In $\triangle P_0RP_1$, $\cos\theta = \dfrac{|\overrightarrow{PR}|}{|\overrightarrow{P_0P_1}|}$

$|\overrightarrow{PR}| = |\overrightarrow{P_0P_1}|\cos\theta$, where $\theta$ is the angle between the vectors $\vec{n}$ and $\overrightarrow{P_0P_1}$

Since $\vec{n} \cdot \overrightarrow{P_0P_1} = |\vec{n}||\overrightarrow{P_0P_1}|\cos\theta$,

$$|\overrightarrow{P_0P_1}|\cos\theta = \frac{\vec{n} \cdot \overrightarrow{P_0P_1}}{|\vec{n}|}$$

Therefore, $d = |\overrightarrow{PR}| = \dfrac{\vec{n} \cdot \overrightarrow{P_0P_1}}{|\vec{n}|}$, where $d$ is the distance from the point to the plane.

Since $\vec{n} = (A, B, C)$, $\overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$ and

$$|\vec{n}| = \sqrt{A^2 + B^2 + C^2}$$

$$d = |\overrightarrow{PR}| = \frac{(A, B, C) \cdot (x_1 - x_0, y_1 - y_0, z_1 - z_0)}{\sqrt{A^2 + B^2 + C^2}}$$

or

$$d = \frac{Ax_1 - Ax_0 + By_1 - By_0 + Cz_1 - Cz_0}{\sqrt{A^2 + B^2 + C^2}}$$

Since $P_1(x_1, y_1, z_1)$ is a point on $Ax + By + Cz + D = 0$, $Ax_1 + By_1 + Cz_1 + D = 0$ and $Ax_1 + By_1 + Cz_1 = -D$.

Rearranging the formula,

$$d = \frac{-Ax_0 - By_0 - Cz_0 + Ax_1 + By_1 + Cz_1}{\sqrt{A^2 + B^2 + C^2}}$$

Therefore, $d = \dfrac{-Ax_0 - By_0 - Cz_0 - D}{\sqrt{A^2 + B^2 + C^2}}$

$$d = \frac{-(Ax_0 + By_0 + Cz_0 + D)}{\sqrt{A^2 + B^2 + C^2}}$$

Since the distance $d$ is always positive, the formula is written as

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

**Distance from a Point $P_0(x_0, y_0, z_0)$ to the Plane with Equation $Ax + By + Cz + D = 0$**

In $R^3$, $d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$, where $d$ is the required distance between the point and the plane.

## EXAMPLE 1 Calculating the distance from a point to a plane

Determine the distance from $S(-1, 2, -4)$ to the plane with equation $8x - 4y + 8z + 3 = 0$.

### Solution

To determine the required distance, we substitute directly into the formula.

Therefore, $d = \dfrac{|8(-1) - 4(2) + 8(-4) + 3|}{\sqrt{8^2 + (-4)^2 + 8^2}} = \dfrac{|-45|}{12} = \dfrac{45}{12} = 3.75$

The distance between $S(-1, 2, -4)$ and the given plane is 3.75.

It is also possible to use the distance formula to find the distance between two parallel planes, as we show in the following example.

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## EXAMPLE 2 Selecting a strategy to determine the distance between two parallel planes

a. Determine the distance between the two planes $\pi_1$: $2x - y + 2z + 4 = 0$ and $\pi_2$: $2x - y + 2z + 16 = 0$.
b. Determine the equation of the plane that is equidistant from $\pi_1$ and $\pi_2$.

### Solution

a. The two given planes are parallel because they each have the same normal, $\vec{n} = (2, -1, 2)$. To find the distance between $\pi_1$ and $\pi_2$, it is necessary to have a point on one of the planes and the equation of the second plane. If we consider $\pi_1$, we can determine the coordinates of its $z$-intercept by letting $x = y = 0$. Substituting, $2(0) - (0) + 2z + 4 = 0$, or $z = -2$. This means that point $X(0, 0, -2)$ lies on $\pi_1$. To find the required distance, apply the formula using $X(0, 0, -2)$ and $\pi_2$: $2x - y + 2z + 16 = 0$.

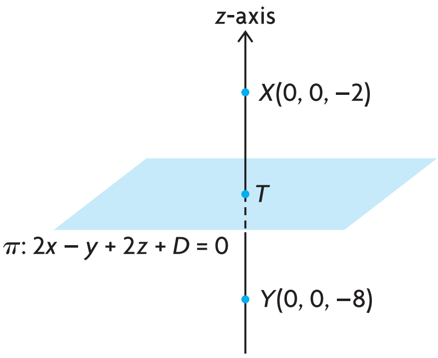
$$d = \frac{|2(0) - (0) + 2(-2) + 16|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{|-4 + 16|}{3} = \frac{12}{3} = 4$$

Therefore, the distance between $\pi_1$ and $\pi_2$ is 4.

When calculating the distance between these two planes, we used the coordinates of the $z$-intercept as our point on one of the planes. This point was chosen because it is easy to determine and leads to a simple calculation for the distance.

b. The plane that is equidistant from $\pi_1$ and $\pi_2$ is parallel to both planes and lies midway between them. Since the required plane is parallel to the two given planes, it must have the form $\pi$: $2x - y + 2z + D = 0$, with $D$ to be determined. If we follow the same procedure for $\pi_2$ that we used in part a., we can find the coordinates of the point associated with its $z$-intercept. If we substitute $x = y = 0$ into $\pi_2$, $2(0) - (0) + 2z + 16 = 0$, or $z = -8$. This means the point $Y(0, 0, -8)$ is on $\pi_2$.

The situation can be visualized in the following way:




Point $T$ is on the required plane $\pi: 2x - y + 2z + D = 0$ and is the midpoint of the line segment joining $X(0, 0, -2)$ to $Y(0, 0, -8)$, meaning that point $T$ has coordinates $(0, 0, -5)$. To find $D$, we substitute the coordinates of point $T$ into $\pi$, which gives $2(0) - (0) + 2(-5) + D = 0$, or $D = 10$.

Thus, the required plane has the equation $2x - y + 2z + 10 = 0$. We should note, in this case, that the coordinates of its $z$-intercept are $(0, 0, -5)$.

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We have shown how to use the formula to find the distance from a point to a plane. This formula can also be used to find the distance between two skew lines.

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## EXAMPLE 3 Selecting a strategy to determine the distance between skew lines

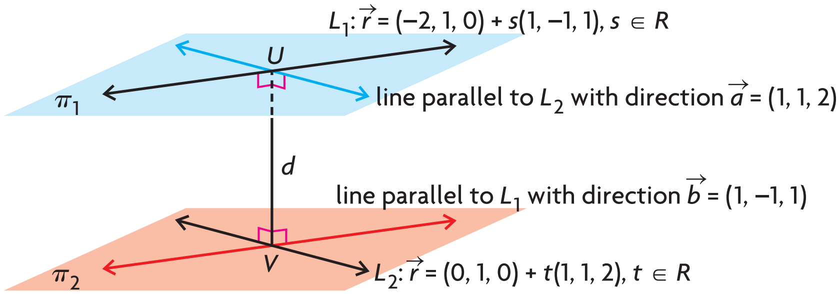
Determine the distance between $L_1: \vec{r} = (-2, 1, 0) + s(1, -1, 1), s \in \mathbf{R}$, and $L_2: \vec{r} = (0, 1, 0) + t(1, 1, 2), t \in \mathbf{R}$.

### Solution

*Method 1*:

These two lines are skew lines because they are not parallel and do not intersect (you should verify this for yourself). To find the distance between the given lines, two parallel planes are constructed. The first plane is constructed so that $L_1$ lies on it, along with a second line that has direction $\vec{a} = (1, 1, 2)$, the direction vector for $L_2$.

In the same way, a second plane is constructed containing $L_2$, along with a second line that has direction $\vec{b} = (1, -1, 1)$, the direction of $L_1$.




The two constructed planes are parallel because they have two identical direction vectors. By constructing these planes, the problem of finding the distance between the two skew lines has been reduced to finding the distance between the planes $\pi_1$ and $\pi_2$. Finding the distance between the two planes means finding the Cartesian equation of one of the planes and using the distance formula with a point from the other plane. (This method of constructing planes will not work if the two given lines are parallel because the calculation of the normal for the planes would be (0, 0, 0).)

We first determine the equation of $\pi_2$. Since this plane has direction vectors $\overrightarrow{m_2} = (1, 1, 2)$ and $\vec{b} = (1, -1, 1)$, we can choose

$\vec{n} = \overrightarrow{m_2} \times \vec{b} = (1, 1, 2) \times (1, -1, 1)$.

Thus, $\vec{n} = (1(1) - 2(-1), 2(1) - 1(1), 1(-1) - 1(1)) = (3, 1, -2)$

We must now find $D$ using the equation $3x + y - 2z + D = 0$. Since (0, 1, 0) is a point on this plane, $3(0) + 1 - 2(0) + D = 0$, or $D = -1$. This gives $3x + y - 2z - 1 = 0$ as the equation for $\pi_2$.

Using $3x + y - 2z - 1 = 0$ and the point $(-2, 1, 0)$ from the other plane $\pi_1$ the distance between the skew lines can be calculated.

$$d = \frac{|3(-2) + 1 - 2(0) - 1|}{\sqrt{3^2 + 1^2 + (-2)^2}} = \frac{6}{\sqrt{14}} = \frac{3}{7}\sqrt{14} \doteq 1.60$$

The distance between the two skew lines is approximately 1.60.

The first method gives one approach for finding the distance between two skew lines. In the second method, we will show how to determine the point on each of these two skew lines that produces this minimal distance.

*Method 2*:

In Method 1, we constructed two parallel planes and found the distance between them. Since the distance between the two planes is constant, our calculation also gave the distance between the two skew lines. There are points on each of these lines that will produce this minimal distance. Possible points, $U$ and $V$, are shown on the diagram for Method 1. To determine the coordinates of these points, we must use the fact that the vector found by joining the two points is perpendicular to the direction vector of each line.

We start by writing each line in parametric form.

For $L_1$, $x = -2 + s$, $y = 1 - s$, $z = s$, $\vec{m_1} = (1, -1, 1)$.

For $L_2$, $x = t$, $y = 1 + t$, $z = 2t$, $\vec{m_2} = (1, 1, 2)$.

The point with coordinates $U(-2 + s, 1 - s, s)$ represents a general point on $L_1$, and $V(t, 1 + t, 2t)$ represents a general point on $L_2$. We next calculate $\overrightarrow{UV}$.

$\overrightarrow{UV} = (t - (-2 + s), (1 + t) - (1 - s), 2t - s) = (t - s + 2, t + s, 2t - s)$

$\overrightarrow{UV}$ represents a general vector with its tail on $L_1$ and its head on $L_2$.

To find the points on each of the two lines that produce the minimal distance, we must use equations $\vec{m_1} \cdot \overrightarrow{UV} = 0$ and $\vec{m_2} \cdot \overrightarrow{UV} = 0$, since $\overrightarrow{UV}$ must be perpendicular to each of the two planes.

Therefore, $(1, -1, 1)(t - s + 2, t + s, 2t - s) = 0$

$$1(t - s + 2) - 1(t + s) + 1(2t - s) = 0$$

$$\text{or } 2t - 3s = -2 \qquad \text{(Equation 1)}$$

and $(1, 1, 2) \cdot (t - s + 2, t + s, 2t - s) = 0$

$$1(t - s + 2) + 1(t + s) + 2(2t - s) = 0$$

$$\text{or } 3t - s = -1 \qquad \text{(Equation 2)}$$

This gives the following system of equations:

① $2t - 3s = -2$

② $3t - s = -1$

$$-7t = 1 \qquad -3 \times ② + ①$$

$$t = -\frac{1}{7}$$

If we substitute $t = -\frac{1}{7}$ into equation ②, $3\left(-\frac{1}{7}\right) - s = -1$ or $s = \frac{4}{7}$.

We now substitute $s = \frac{4}{7}$ and $t = -\frac{1}{7}$ into the equations for each line to find the required points.

For $L_1$, $x = -2 + \frac{4}{7} = -\frac{10}{7}$, $y = 1 - \frac{4}{7} = \frac{3}{7}$, $z = \frac{4}{7}$.

Therefore, the required point on $L_1$ is $\left(-\frac{10}{7}, \frac{3}{7}, \frac{4}{7}\right)$.

For $L_2$, $x = -\frac{1}{7}$, $y = 1 + \left(\frac{-1}{7}\right) = \frac{6}{7}$, $z = 2\left(\frac{-1}{7}\right) = -\frac{2}{7}$.

Therefore, the required point on $L_2$ is $\left(-\frac{1}{7}, \frac{6}{7}, -\frac{2}{7}\right)$.

The required distance between the two lines is the distance between these two points. This distance is $\sqrt{\left(-\frac{10}{7}+\frac{1}{7}\right)^2+\left(\frac{3}{7}-\frac{6}{7}\right)^2+\left(\frac{4}{7}+\frac{2}{7}\right)^2}$

$$= \sqrt{\left(\frac{-9}{7}\right)^2+\left(\frac{-3}{7}\right)^2+\left(\frac{6}{7}\right)^2}$$

$$= \sqrt{\frac{81}{49}+\frac{9}{49}+\frac{36}{49}}$$

$$= \sqrt{\frac{126}{49}}$$

$$= \sqrt{\frac{9 \times 14}{49}}$$

$$= \frac{3}{7}\sqrt{14}$$

Thus, the distance between the two skew lines is $\frac{3}{7}\sqrt{14}$, or approximately 1.60.

The two points that produce this distance are $\left(-\frac{10}{7}, \frac{3}{7}, \frac{4}{7}\right)$ on $L_1$ and $\left(-\frac{1}{7}, \frac{6}{7}, -\frac{2}{7}\right)$ on $L_2$.

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## INVESTIGATION

A. In this section, we showed that the formula for the distance $d$ from a point $P_0(x_0, y_0, z_0)$ to the plane with equation $Ax + By + Cz + D = 0$ is
$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$
By modifying this formula, show that a formula for finding the distance from $O(0, 0, 0)$ to the plane $Ax + By + Cz + D = 0$ is $d = \frac{|D|}{\sqrt{A^2 + B^2 + C^2}}$.

B. Determine the distance from $O(0, 0, 0)$ to the plane with equation $20x + 4y - 5z + 21 = 0$.

C. Determine the distance between the planes with equations $\pi_1$: $20x + 4y - 5z + 21 = 0$ and $\pi_2$: $20x + 4y - 5z + 105 = 0$.

D. Determine the coordinates of a point that is equidistant from $\pi_1$ and $\pi_2$.

E. Determine an equation for a plane that is equidistant from $\pi_1$ and $\pi_2$.

F. Determine two values of $D$ if the plane with equation $20x + 4y - 5z + D = 0$ is 4 units away from the plane with equation $20x + 4y - 5z = 0$.

G. Determine the distance between the two planes $\pi_3$: $20x + 4y - 5z - 105 = 0$ and $\pi_4$: $20x + 4y - 5z + 147 = 0$.

H. Determine the distance between the following planes:

a. $2x - 2y + z - 6 = 0$ and $2x - 2y + z - 12 = 0$

b. $6x - 3y + 2z + 14 = 0$ and $6x - 3y + 2z + 35 = 0$

c. $12x + 3y + 4z - 26 = 0$ and $12x + 3y + 4z + 26 = 0$

I. If two planes have equations $Ax + By + Cz + D_1 = 0$ and $Ax + By + Cz + D_2 = 0$, explain why the formula for the distance $d$ between these planes is $d = \frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$.

## IN SUMMARY

### Key Idea

- The distance from a point $P_0(x_0, y_0, z_0)$ to the plane with equation $Ax + By + Cz + D = 0$ is $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$, where $d$ is the required distance.

### Need to Know

- The distance between skew lines can be calculated using two different methods.

  Method 1: To determine the distance between the given skew lines, two parallel planes are constructed that are the same distance apart as the skew lines. Determine the distance between the two planes.

  Method 2: To determine the coordinates of the points that produce the minimal distance, use the fact that the general vector found by joining the two points is perpendicular to the direction vector of each line.

# Exercise 9.6

## PART A

C 1. A student is calculating the distance $d$ between point $A(-3, 2, 1)$ and the plane with equation $2x + y + 2z + 2 = 0$. The student obtains the following answer:

$$d = \frac{|2(-3) + 2 + 2(1) + 2|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{0}{3} = 0$$

a. Has the student done the calculation correctly? Explain.

b. What is the significance of the answer 0? Explain.

## PART B

K 2. Determine the following distances:

   a. the distance from $A(3, 1, 0)$ to the plane with equation $20x - 4y + 5z + 7 = 0$

   b. the distance from $B(0, -1, 0)$ to the plane with equation $2x + y + 2z - 8 = 0$

   c. the distance from $C(5, 1, 4)$ to the plane with equation $3x - 4y - 1 = 0$

   d. the distance from $D(1, 0, 0)$ to the plane with equation $5x - 12y = 0$

   e. the distance from $E(-1, 0, 1)$ to the plane with equation $18x - 9y + 18z - 11 = 0$

3. For the planes $\pi_1$: $3x + 4y - 12z - 26 = 0$ and $\pi_2$: $3x + 4y - 12z + 39 = 0$, determine

   a. the distance between $\pi_1$ and $\pi_2$

   b. an equation for a plane midway between $\pi_1$ and $\pi_2$

   c. the coordinates of a point that is equidistant from $\pi_1$ and $\pi_2$

4. Determine the following distances:

   a. the distance from $P(1, 1, -3)$ to the plane with equation $y + 3 = 0$

   b. the distance from $Q(-1, 1, 4)$ to the plane with equation $x - 3 = 0$

   c. the distance from $R(1, 0, 1)$ to the plane with equation $z + 1 = 0$

A 5. Points $A(1, 2, 3)$, $B(-3, -1, 2)$, and $C(13, 4, -1)$ lie on the same plane. Determine the distance from $P(1, -1, 1)$ to the plane containing these three points.

T 6. The distance from $R(3, -3, 1)$ to the plane with equation $Ax - 2y + 6z = 0$ is 3. Determine all possible value(s) of $A$ for which this is true.

7. Determine the distance between the lines $\vec{r} = (0, 1, -1) + s(3, 0, 1)$, $s \in \mathbf{R}$, and $\vec{r} = (0, 0, 1) + t(1, 1, 0)$, $t \in \mathbf{R}$.

## PART C

8. a. Calculate the distance between the lines $L_1$: $\vec{r} = (1, -2, 5) + s(0, 1, -1)$, $s \in \mathbf{R}$, and $L_2$: $\vec{r} = (1, -1, -2) + t(1, 0, -1)$, $t \in \mathbf{R}$.

   b. Determine the coordinates of points on these lines that produce the minimal distance between $L_1$ and $L_2$.