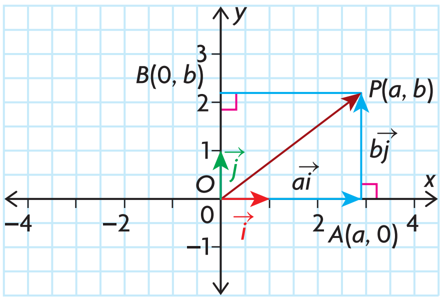
Section 6.6—Operations with Algebraic Vectors in R^2

In the previous section, we showed how to locate points and vectors in both two and three dimensions and then showed their connection to algebraic vectors. In R^2, we showed that $\overrightarrow{OP} = (a, b)$ was the vector formed when we joined the origin, $O(0, 0)$, to the point $P(a, b)$. We showed that the same meaning could be given to $\overrightarrow{OP} = (a, b, c)$, where the point $P(a, b, c)$ was in R^3 and $O(0, 0, 0)$ is the origin. In this section, we will deal with vectors in R^2 and show how a different representation of $\overrightarrow{OP} = (a, b)$ leads to many useful results.

Defining a Vector in R^2 in Terms of Unit Vectors



A second way of writing $\overrightarrow{OP} = (a, b)$ is with the use of the unit vectors $\vec{i}$ and $\vec{j}$.

The vectors $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$ have magnitude 1 and lie along the positive x- and y-axes, respectively, as shown on the graph.

Our objective is to show how $\overrightarrow{OP}$ can be written in terms of $\vec{i}$ and $\vec{j}$. In the diagram, $\overrightarrow{OA} = (a, 0)$ and, since $\overrightarrow{OA}$ is just a scalar multiple of $\vec{i}$, we can write $\overrightarrow{OA} = a\vec{i}$. In a similar way, $\overrightarrow{OB} = b\vec{j}$. Using the triangle law of addition, $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{OB} = a\vec{i} + b\vec{j}$. Since $\overrightarrow{OP} = (a, b)$, it follows that $(a, b) = a\vec{i} + b\vec{j}$.

This means that $\overrightarrow{OP} = (-3, 8)$ can also be written as $\overrightarrow{OP} = -3\vec{i} + 8\vec{j}$. Notice that this result allows us to write *all* vectors in the plane in terms of $\vec{i}$ and $\vec{j}$ and, just as before, their representation is unique.

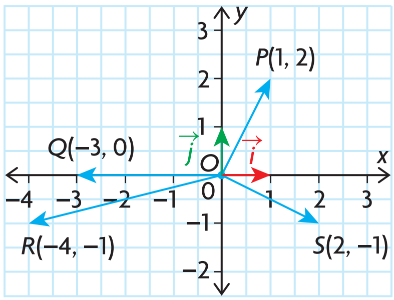
Representation of Vectors in R^2

The position vector $\overrightarrow{OP}$ can be represented as either $\overrightarrow{OP} = (a, b)$ or $\overrightarrow{OP} = a\vec{i} + b\vec{j}$, where $O(0, 0)$ is the origin, $P(a, b)$ is any point on the plane, and $\vec{i}$ and $\vec{j}$ are the standard unit vectors for R^2. Standard unit vectors, $\vec{i}$ and $\vec{j}$, are unit vectors that lie along the x- and y-axes, respectively, so $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. Every vector in R^2, given in terms of its components, can also be written uniquely in terms of $\vec{i}$ and $\vec{j}$. For this reason, vectors $\vec{i}$ and $\vec{j}$ are also called the standard basis vectors in R^2.

EXAMPLE 1 Representing vectors in R^2 in two equivalent forms

a. Four position vectors, $\vec{OP} = (1, 2)$, $\vec{OQ} = (-3, 0)$, $\vec{OR} = (-4, -1)$, and $\vec{OS} = (2, -1)$, are shown. Write each of these vectors using the unit vectors $\vec{i}$ and $\vec{j}$.

b. The vectors $\vec{OA} = -\vec{i}$, $\vec{OB} = \vec{i} + 5\vec{j}$, $\vec{OC} = -5\vec{i} + 2\vec{j}$, and $\vec{OD} = \sqrt{2}\vec{i} - 4\vec{j}$ have been written using the unit vectors $\vec{i}$ and $\vec{j}$. Write them in component form (a, b).



Solution

a. $\vec{OP} = \vec{i} + 2\vec{j}$, $\vec{OQ} = -3\vec{i}$, $\vec{OR} = -4\vec{i} - \vec{j}$, $\vec{OS} = 2\vec{i} - \vec{j}$

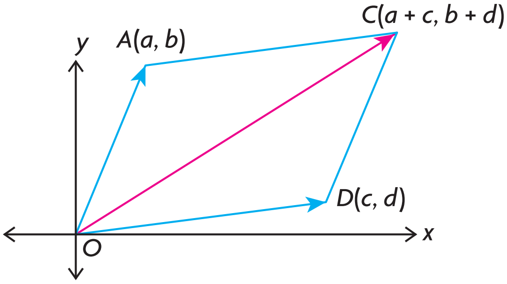
b. $\vec{OA} = (-1, 0)$, $\vec{OB} = (1, 5)$, $\vec{OC} = (-5, 2)$, and $\vec{OD} = (\sqrt{2}, -4)$

The ability to write vectors using $\vec{i}$ and $\vec{j}$ allows us to develop many of the same results with algebraic vectors that we developed with geometric vectors.

Addition of Two Vectors Using Component Form

We start by drawing the position vectors, $\vec{OA} = (a, b)$ and $\vec{OD} = (c, d)$, where A and D are any two points in R^2. For convenience, we choose these two points in the first quadrant. We rewrite each of the two position vectors, $\vec{OA} = (a, b) = a\vec{i} + b\vec{j}$ and $\vec{OD} = (c, d) = c\vec{i} + d\vec{j}$.

Adding these vectors gives
$$\begin{aligned}\vec{OA} + \vec{OD} &= a\vec{i} + b\vec{j} + c\vec{i} + d\vec{j} \\ &= a\vec{i} + c\vec{i} + b\vec{j} + d\vec{j} \\ &= (a + c)\vec{i} + (b + d)\vec{j} \\ &= (a + c, b + d) \\ &= \vec{OC}\end{aligned}$$



To find $\overrightarrow{OC}$, it was necessary to use the commutative and distributive properties of vector addition, along with the ability to write vectors in terms of the unit vectors $\vec{i}$ and $\vec{j}$.

To determine the sum of two vectors, $\overrightarrow{OA} = (a, b)$ and $\overrightarrow{OD} = (c, d)$, add their corresponding x- and y-components.
So, $\overrightarrow{OA} + \overrightarrow{OD} = (a, b) + (c, d) = (a + c, b + d) = \overrightarrow{OC}$

The process is similar for subtraction.
$\overrightarrow{OA} - \overrightarrow{OD} = (a, b) - (c, d) = (a - c, b - d)$

Scalar Multiplication of Vectors Using Components

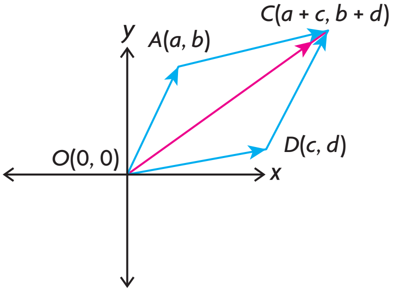
When dealing with geometric vectors, the meaning of multiplying a vector by a scalar was shown. The multiplication of a vector by a scalar in component form has the same meaning. In essence, if $\overrightarrow{OP} = (a, b)$, we wish to know how the coordinates of $m\overrightarrow{OP}$ are determined, where m is a real number. This can be determined by using various distributive properties for scalar multiplication of vectors along with the $\vec{i}$, $\vec{j}$ representation of a vector.

In algebraic form,
$$\begin{aligned} m\overrightarrow{OP} &= m(a, b) \\ &= m(a\vec{i} + b\vec{j}) \\ &= (ma)\vec{i} + (mb)\vec{j} \\ &= (ma, mb) \end{aligned}$$

To multiply an algebraic vector by a scalar, each component of the algebraic vector is multiplied by that scalar.

Adding Two Vectors in R^2

To determine the sum of two algebraic vectors, add their corresponding x- and y-components.

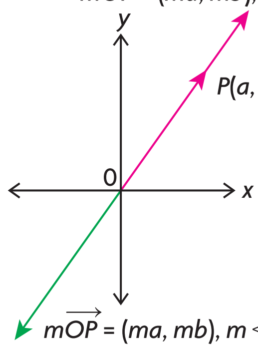


$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OD} = (a + c, b + d)$

Multiplying a Vector by a Scalar in R^2

To multiply an algebraic vector by a scalar, multiply both x- and y-components by the scalar.

$m\overrightarrow{OP} = (ma, mb), m > 0$

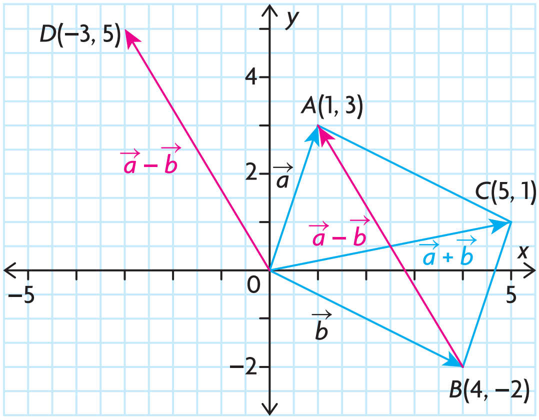


$m\overrightarrow{OP} = (ma, mb), m < 0$

EXAMPLE 2 Representing the sum and difference of two algebraic vectors in R^2

Given $\vec{a} = \overrightarrow{OA} = (1, 3)$ and $\overrightarrow{OB} = \vec{b} = (4, -2)$, determine the components of $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, and illustrate each of these vectors on the graph.

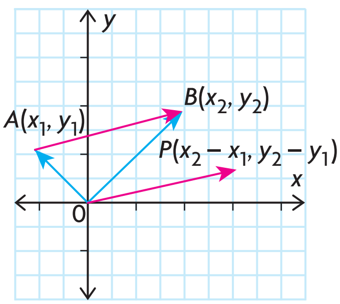
Solution



$$\vec{a} + \vec{b} = \overrightarrow{OA} + \overrightarrow{OB} = (1, 3) + (4, -2) = (1 + 4, 3 + (-2)) = (5, 1) = \overrightarrow{OC}$$
$$\vec{a} - \vec{b} = \overrightarrow{OA} - \overrightarrow{OB} = (1, 3) - (4, -2) = (1 - 4, 3 + 2) = (-3, 5) = \overrightarrow{OD}$$

From the diagram, we can see that $\vec{a} + \vec{b}$ and $\overrightarrow{BA}$ represent the diagonals of the parallelogram. It should be noted that the position vector, $\overrightarrow{OD}$, is a vector that is equivalent to diagonal $\overrightarrow{BA}$. The vector $\overrightarrow{OD} = \vec{a} - \vec{b}$ is described as a position vector because it has its tail at the origin and is equivalent to $\overrightarrow{BA}$, since their magnitudes are the same and they have the same direction.

Vectors in R^2 Defined by Two Points



$\overrightarrow{OP}$ is the position vector for $\overrightarrow{AB}$.

In considering the vector $\overrightarrow{AB}$, determined by the points $A(x_1, y_1)$ and $B(x_2, y_2)$, an important consideration is to be able to find its related position vector and to calculate $|\overrightarrow{AB}|$. In order to do this, we use the triangle law of addition. From the diagram on the left, $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, and $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$. Thus, the components of the algebraic vector are found by subtracting the coordinates of its tail from the coordinates of its head.

To determine $|\overrightarrow{AB}|$, use the Pythagorean theorem.

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The formula for determining $|\overrightarrow{AB}|$ is the same as the formula for finding the distance between two points.

Position Vectors and Magnitudes in R^2

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points, then the vector $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$ is its related position vector $\overrightarrow{OP}$, and $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

EXAMPLE 3 Using algebraic vectors to solve a problem

$A(-3, 7)$, $B(5, 22)$, and $C(8, 18)$ are three points in R^2.

a. Calculate the value of $|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CA}|$, the perimeter of triangle ABC.

b. Calculate the value of $|\overrightarrow{AB} + \overrightarrow{BC}|$.

Solution

a. Calculate a position vector for each of the three sides.

$\overrightarrow{AB} = (5 - (-3), 22 - 7) = (8, 15)$, $\overrightarrow{BC} = (8 - 5, 18 - 22) = (3, -4)$, and $\overrightarrow{CA} = (-3 - 8, 7 - 18) = (-11, -11)$

$|\overrightarrow{AB}| = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$, $|\overrightarrow{BC}| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$, and $|\overrightarrow{CA}| = \sqrt{(-11)^2 + (-11)^2} = \sqrt{121 + 121} = \sqrt{242} \doteq 15.56$

The perimeter of the triangle is approximately 37.56.

b. Since $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$, $\overrightarrow{AC} = -\overrightarrow{CA} = (11, 11)$, and $|\overrightarrow{AC}| = \sqrt{11^2 + 11^2} = \sqrt{242}$, then $|\overrightarrow{AC}| \doteq 15.56$. Note that $|\overrightarrow{AC}| = |\overrightarrow{CA}| \doteq 15.56$.

EXAMPLE 4 Selecting a strategy to combine two vectors

For the vectors $\vec{x} = 2\vec{i} - 3\vec{j}$ and $\vec{y} = -4\vec{i} - 3\vec{j}$, determine $|\vec{x} + \vec{y}|$ and $|\vec{x} - \vec{y}|$.

Solution

Method 1: (Component Form)

Since $\vec{x} = 2\vec{i} - 3\vec{j}$, $\vec{x} = (2, -3)$. Similarly, $\vec{y} = (-4, -3)$.

The sum is $\vec{x} + \vec{y} = (2, -3) + (-4, -3) = (-2, -6)$.

The difference is $\vec{x} - \vec{y} = (2, -3) - (-4, -3) = (6, 0)$.

Method 2: (Standard Unit Vectors)

The sum is
$\vec{x} + \vec{y} = (2\vec{i} - 3\vec{j}) + (-4\vec{i} - 3\vec{j}) = (2 - 4)\vec{i} + (-3 - 3)\vec{j} = -2\vec{i} - 6\vec{j}$.

The difference is
$\vec{x} - \vec{y} = (2\vec{i} - 3\vec{j}) - (-4\vec{i} - 3\vec{j}) = (2 + 4)\vec{i} + (-3 + 3)\vec{j} = 6\vec{i}$.

Thus, $|\vec{x} + \vec{y}| = \sqrt{(-2)^2 + (-6)^2} = \sqrt{40} \doteq 6.32$ and
$|\vec{x} - \vec{y}| = \sqrt{6^2} = \sqrt{36} = 6$.

EXAMPLE 5 Calculating the magnitude of a vector in R^2

If $\vec{a} = (5, -6)$, $\vec{b} = (-7, 3)$, and $\vec{c} = (2, 8)$, calculate $\left|\vec{a} - 3\vec{b} - \frac{1}{2}\vec{c}\right|$.

Solution

$$\begin{aligned}\vec{a} - 3\vec{b} - \frac{1}{2}\vec{c} &= (5, -6) - 3(-7, 3) - \frac{1}{2}(2, 8)\\ &= (5, -6) + (21, -9) + (-1, -4) = (25, -19)\end{aligned}$$

Thus, $\left|\vec{a} - 3\vec{b} - \frac{1}{2}\vec{c}\right| = \sqrt{25^2 + (-19)^2} = \sqrt{625 + 361} = \sqrt{986} \doteq 31.40$

IN SUMMARY

Key Ideas

- In R^2, $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. Both are unit vectors on the x- and y-axes, respectively.
- $\overrightarrow{OP} = (a, b) = a\vec{i} + b\vec{j}$, $|\overrightarrow{OP}| = \sqrt{a^2 + b^2}$
- The vector between two points with its tail at $A(x_1, y_1)$ and head at $B(x_2, y_2)$ is determined as follows:

  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$
- The vector $\overrightarrow{AB}$ is equivalent to the position vector $\overrightarrow{OP}$ since their directions and magnitude are the same: $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Need to Know

- If $\overrightarrow{OA} = (a, b) = a\vec{i} + b\vec{j}$ and $\overrightarrow{OD} = (c, d) = c\vec{i} + d\vec{j}$, then $\overrightarrow{OA} + \overrightarrow{OD} = (a + c, b + d)$.
- $m\overrightarrow{OP} = m(a, b) = (ma, mb)$

Exercise 6.6

PART A

1. For $A(-1, 3)$ and $B(2, 5)$, draw a coordinate plane and place the points on the graph.
   a. Draw vectors $\overrightarrow{AB}$ and $\overrightarrow{BA}$, and give vectors in component form equivalent to each of these vectors.
   b. Determine $|\overrightarrow{OA}|$ and $|\overrightarrow{OB}|$.
   c. Calculate $|\overrightarrow{AB}|$ and state the value of $|\overrightarrow{BA}|$.

2. Draw the vector $\overrightarrow{OA}$ on a graph, where point A has coordinates $(6, 10)$.

   a. Draw the vectors $m\overrightarrow{OA}$, where $m = \frac{1}{2}, \frac{-1}{2}, 2$, and -2.

   b. Which of these vectors have the same magnitude?

3. For the vector $\overrightarrow{OA} = 3\vec{i} - 4\vec{j}$, calculate $|\overrightarrow{OA}|$.

4. a. If $a\vec{i} + 5\vec{j} = (-3, b)$, determine the values of a and b.

   b. Calculate $|(-3, b)|$ after finding b.

5. If $\vec{a} = (-60, 11)$ and $\vec{b} = (-40, -9)$, calculate each of the following:

   a. $|\vec{a}|$ and $|\vec{b}|$ b. $|\vec{a} + \vec{b}|$ and $|\vec{a} - \vec{b}|$

PART B

6. Find a single vector equivalent to each of the following:

   a. $2(-2, 3) + (2, 1)$ b. $-3(4, -9) - 9(2, 3)$ c. $\frac{-1}{2}(6, -2) + \frac{2}{3}(6, 15)$

K

7. Given $\vec{x} = 2\vec{i} - \vec{j}$ and $\vec{y} = -\vec{i} + 5\vec{j}$, find a vector equivalent to each of the following:

   a. $3\vec{x} - \vec{y}$

   b. $-(\vec{x} + 2\vec{y}) + 3(-\vec{x} - 3\vec{y})$

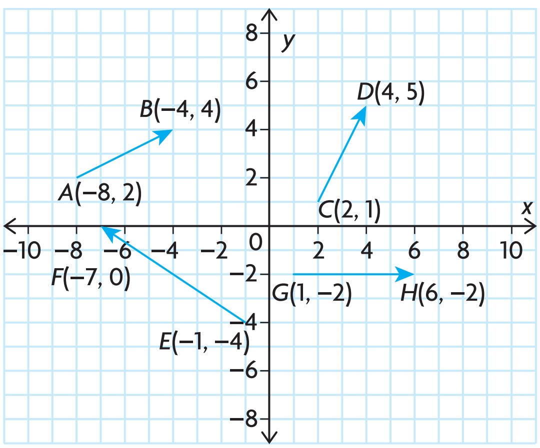
   c. $2(\vec{x} + 3\vec{y}) - 3(\vec{y} + 5\vec{x})$

8. Using $\vec{x}$ and $\vec{y}$ given in question 7, determine each of the following:

   a. $|\vec{x} + \vec{y}|$ b. $|\vec{x} - \vec{y}|$ c. $|2\vec{x} - 3\vec{y}|$ d. $|3\vec{y} - 2\vec{x}|$

9. a. For each of the vectors shown below, determine the components of the related position vector.

   b. Determine the magnitude of each vector.



A

10. Parallelogram $OBCA$ is determined by the vectors $\overrightarrow{OA} = (6, 3)$ and $\overrightarrow{OB} = (11, -6)$.
    a. Determine $\overrightarrow{OC}$, $\overrightarrow{BA}$, and $\overrightarrow{BC}$.
    b. Verify that $|\overrightarrow{OA}| = |\overrightarrow{BC}|$.

11. $\triangle ABC$ has vertices at $A(2, 3)$, $B(6, 6)$, and $C(-4, 11)$.
    a. Sketch and label each of the points on a graph.
    b. Calculate each of the lengths $|\overrightarrow{AB}|$, $|\overrightarrow{AC}|$, and $|\overrightarrow{CB}|$.
    c. Verify that triangle ABC is a right triangle.

12. A parallelogram has three of its vertices at $A(-1, 2)$, $B(7, -2)$, and $C(2, 8)$.
    a. Draw a grid and locate each of these points.
    b. On your grid, draw the three locations for a fourth point that would make a parallelogram with points A, B, and C.
    c. Determine all possible coordinates for the point described in part b.

13. Determine the value of x and y in each of the following:
    a. $3(x, 1) - 5(2, 3y) = (11, 33)$
    b. $-2(x, x + y) - 3(6, y) = (6, 4)$

C

14. Rectangle $ABCD$ has vertices at $A(2, 3)$, $B(-6, 9)$, $C(x, y)$, and $D(8, 11)$.
    a. Draw a sketch of the points A, B, and D, and locate point C on your graph.
    b. Explain how you can determine the coordinates of point C.

T

15. $A(5, 0)$ and $B(0, 2)$ are points on the x- and y-axes, respectively.
    a. Find the coordinates of point $P(a, 0)$ on the x-axis such that $|\overrightarrow{PA}| = |\overrightarrow{PB}|$.
    b. Find the coordinates of a point on the y-axis such that $|\overrightarrow{QB}| = |\overrightarrow{QA}|$.

PART C

16. Find the components of the unit vector in the direction opposite to $\overrightarrow{PQ}$, where $\overrightarrow{OP} = (11, 19)$ and $\overrightarrow{OQ} = (2, -21)$.

17. Parallelogram $OPQR$ is such that $\overrightarrow{OP} = (-7, 24)$ and $\overrightarrow{OR} = (-8, -1)$.
    a. Determine the angle between the vectors $\overrightarrow{OR}$ and $\overrightarrow{OP}$.
    b. Determine the acute angle between the diagonals $\overrightarrow{OQ}$ and $\overrightarrow{RP}$.