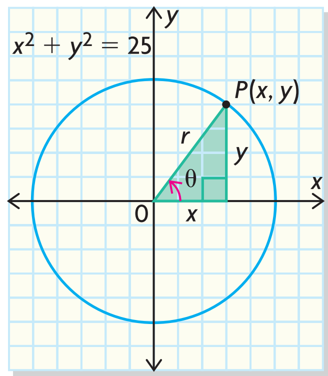
5.4 Evaluating Trigonometric Ratios for Any Angle Between 0° and 360°

GOAL

Use the Cartesian plane to evaluate the primary trigonometric ratios for angles between 0° and 360°.

YOU WILL NEED

- graph paper
- protractor
- dynamic geometry software (optional)



LEARN ABOUT the Math

Miriam knows that the equation of a circle of radius 5 centred at $(0, 0)$ is $x^2 + y^2 = 25$. She also knows that a point $P(x, y)$ on its circumference can rotate from 0° to 360°.

? For any point on the circumference of the circle, how can Miriam determine the size of the corresponding principal angle?

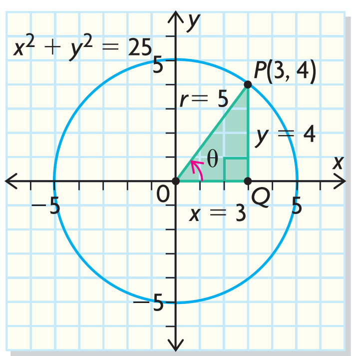
EXAMPLE 1 Relating trigonometric ratios to a point in quadrant 1 of the Cartesian plane

a) If Miriam chooses the point $P(3, 4)$ on the circumference of the circle, determine the primary trigonometric ratios for the principal angle.

b) Determine the principal angle to the nearest degree.

Flavia's Solution

a)



I drew a circle centred about the origin in the Cartesian plane and labelled the point $P(3, 4)$ on the circumference. Then I formed a right triangle with the x-axis. Angle θ is the principal angle and is in standard position. In $\triangle OPQ$, I noticed that the side opposite θ has length $y = 4$ units and the adjacent side has length $x = 3$ units. The hypotenuse is equal to the radius of the circle, so I labelled it r. In this case, $r = 5$ units. From the Pythagorean theorem, I also knew that $r^2 = x^2 + y^2$. Since r is the radius of the circle, it will always be positive.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$= \frac{y}{r} \qquad\qquad = \frac{x}{r} \qquad\qquad = \frac{y}{x}$$

$$= \frac{4}{5} \qquad\qquad = \frac{3}{5} \qquad\qquad = \frac{4}{3}$$

I used the definitions of sine, cosine, and tangent to write each ratio in terms of x, y, and r in the Cartesian plane.

b) $\sin\theta = \frac{4}{5}$

$$\theta = \sin^{-1}\left(\frac{4}{5}\right)$$

I used the inverse sine function on my calculator to determine angle θ.

$$\theta \doteq 53°$$

The principal angle is about 53°.

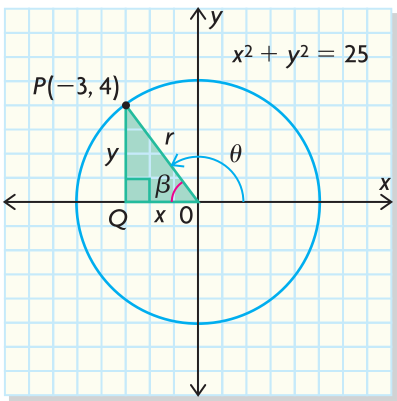
EXAMPLE 2 Relating trigonometric ratios to a point in quadrant 2 of the Cartesian plane

a) If Miriam chooses the point $P(-3, 4)$ on the circumference of the circle, determine the primary trigonometric ratios for the principal angle to the nearest hundredth.

b) Determine the principal angle to the nearest degree.

Gabriel's Solution

a)



I drew a circle centred about the origin in the Cartesian plane and labelled the point $P(-3, 4)$ on the circumference. Then I formed a right triangle with the x-axis. Angle θ is the principal angle and is in standard position. Angle β is the related acute angle.

$$r^2 = x^2 + y^2$$
$$r^2 = 3^2 + 4^2$$
$$r^2 = 9 + 16$$
$$r^2 = 25$$
$$r = 5, \text{ since } r > 0$$

In $\triangle OPQ$, I knew that the lengths of the two perpendicular sides were $|x| = |-3| = 3$ and $y = 4$. The radius of the circle is still 5, so $r = 5$. I used the Pythagorean theorem to confirm this.

$$\sin\beta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} = \frac{4}{5} \qquad \cos\beta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{|x|}{r} = \frac{3}{5} \qquad \tan\beta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{|x|} = \frac{4}{3}$$

In $\triangle OPQ$, the side opposite β has length y and the adjacent side has length $|x|$. I used the definitions of sine, cosine, and tangent to write each ratio in terms of x, y, and r in the Cartesian plane.

$$\sin\theta = \sin\beta = \frac{4}{5} \qquad \cos\theta = -\cos\beta = -\frac{3}{5} \qquad \tan\theta = -\tan\beta = -\frac{4}{3}$$

Then I took into account the relationship among the trigonometric ratios of the related acute angle β and those of the principal angle θ. Since the terminal arm of angle θ lies in quadrant 2, the sine ratio is positive while the cosine and tangent ratios are negative.

b) $\sin\beta = \frac{4}{5}$

To determine angle β, I used a calculator to evaluate $\sin^{-1}\left(\frac{4}{5}\right)$ directly.

$$\beta = \sin^{-1}\left(\frac{4}{5}\right) \doteq 53°$$

$$\theta + \beta = 180°$$

I knew that θ and β add up to 180°. So I subtracted β from 180° to get θ.

$$\theta = 180° - \beta = 180° - 53° = 127°$$

The principal angle is about 127° because the related acute angle is about 53°.

Reflecting

A. In Example 2, explain why $\sin\theta = \sin\beta$, $\cos\theta \neq \cos\beta$, and $\tan\theta \neq \tan\beta$.

B. If Miriam chose the points $(-3, -4)$ and $(3, -4)$, what would each related acute angle be? How would the primary trigonometric ratios for the corresponding principal angles in these cases compare with those in Examples 1 and 2?

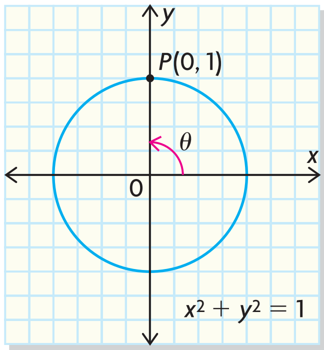
C. Given a point on the terminal arm of an angle in standard position, explain how the coordinates of that point vary from quadrants 1 to 4. How does this variation affect the size of the principal angle (and related acute angle, if it exists) and the values of the primary trigonometric ratios for that angle?

APPLY the Math

EXAMPLE 3 Determining the primary trigonometric ratios for a 90° angle

Use the point $P(0, 1)$ to determine the values of sine, cosine, and tangent for $90°$.

Charmaine's Solution



I drew a circle centred about the origin in the Cartesian plane and labelled the point $P(0, 1)$ on the circumference. Angle θ is the principal angle and is 90°.

In this case, I couldn't draw a right triangle by drawing a line perpendicular to the x-axis to P.

This meant that I couldn't use the trigonometric definitions in terms of opposite, adjacent, and hypotenuse.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$= \frac{1}{1} \qquad\qquad = \frac{0}{1} \qquad\qquad = \frac{1}{0}$$

Since $P(0, 1)$, I knew that $x = 0$, $y = 1$, and $r = 1$.

I used the definitions of sine, cosine, and tangent in terms of x, y, and r to write each ratio.

$\sin 90° = 1$ $\quad \cos 90° = 0$ $\quad \tan 90°$ is undefined

Since $x = 0$ and it is in the denominator, $\tan 90°$ is undefined.

The point $P(0, 1)$ defines a principal angle of $90°$. The sine and cosine of $90°$ are 1 and 0, respectively. The tangent of $90°$ is undefined.

EXAMPLE 4 Determining all possible values of an angle with a specific trigonometric ratio

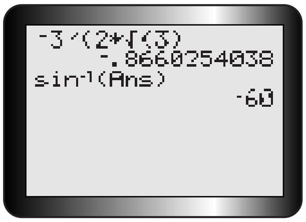
Determine the values of θ if $\csc \theta = -\frac{2\sqrt{3}}{3}$ and $0° \leq \theta \leq 360°$.

Jordan's Solution

$$\csc \theta = -\frac{2\sqrt{3}}{3}$$

$$\sin \theta = -\frac{3}{2\sqrt{3}}$$

Since $0° \leq \theta \leq 360°$, I had to use the Cartesian plane to determine θ. Cosecant is the reciprocal of sine. I found the reciprocal ratio by switching r and y. Since r is always positive, y must be -3 in this case. There were two cases where a point on the terminal arm has a negative y-coordinate: one in quadrant 3 and the other in quadrant 4.



I used my calculator to evaluate $\frac{-3}{2\sqrt{3}}$. Then I took the inverse sine of the result to determine the angle.

One angle is $-60°$, which is equivalent to $360° + (-60°) = 300°$ in quadrant 4.

The angle $-60°$ corresponds to a related acute angle of 60° of clockwise rotation and has its terminal arm in quadrant 4. I added 360° to $-60°$ to get the equivalent angle using a counterclockwise rotation.

In quadrant 3, the angle is $180° + 60° = 240°$.

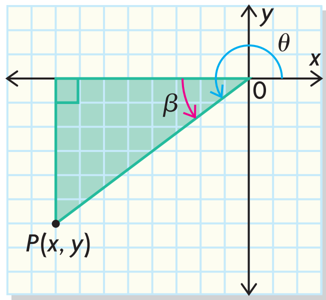
The angle in quadrant 3 must have a related acute angle of 60° as well. So I added 180° to 60° to determine the principal angle.

Given $\csc \theta = -\frac{2\sqrt{3}}{3}$ and $0° \leq \theta \leq 360°$, θ can be either 240° or 300°.

In Summary

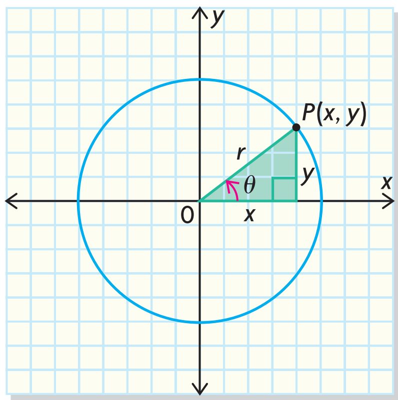
Key Idea

- The trigonometric ratios for any principal angle, θ, in standard position, where $0° \leq \theta \leq 360°$, can be determined by finding the related acute angle, β, using coordinates of any point $P(x, y)$ that lies on the terminal arm of the angle.



Need to Know

- For any point $P(x, y)$ in the Cartesian plane, the trigonometric ratios for angles in standard position can be expressed in terms of x, y, and r.



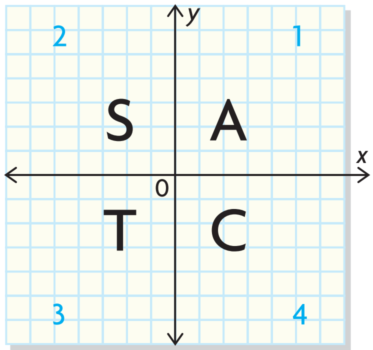
$r^2 = x^2 + y^2$ from the Pythagorean theorem and $r > 0$

$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r} \qquad \tan \theta = \frac{y}{x}$$

$$\csc \theta = \frac{r}{y} \qquad \sec \theta = \frac{r}{x} \qquad \cot \theta = \frac{x}{y}$$

(continued)

- The CAST rule is an easy way to remember which primary trigonometric ratios are positive in which quadrant. Since r is always positive, the sign of each primary ratio depends on the signs of the coordinates of the point.
  - In quadrant 1, **A**ll (A) ratios are positive because both x and y are positive.
  - In quadrant 2, only **S**ine (S) is positive, since x is negative and y is positive.
  - In quadrant 3, only **T**angent (T) is positive because both x and y are negative.
  - In quadrant 4, only **C**osine (C) is positive, since x is positive and y is negative.



CHECK Your Understanding

1. For each trigonometric ratio, use a sketch to determine in which quadrant the terminal arm of the principal angle lies, the value of the related acute angle β, and the sign of the ratio.

a) $\sin 315°$ **b)** $\tan 110°$ **c)** $\cos 285°$ **d)** $\tan 225°$

2. Each point lies on the terminal arm of angle θ in standard position.

i) Draw a sketch of each angle θ.
ii) Determine the value of r to the nearest tenth.
iii) Determine the primary trigonometric ratios for angle θ.
iv) Calculate the value of θ to the nearest degree.

a) $(5, 11)$ **b)** $(-8, 3)$ **c)** $(-5, -8)$ **d)** $(6, -8)$

3. Use the method in Example 3 to determine the primary trigonometric ratios for each given angle.

a) $180°$ **b)** $270°$ **c)** $360°$

4. Use the related acute angle to state an equivalent expression.

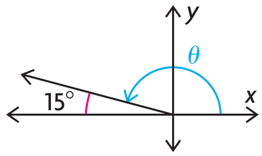
a) $\sin 160°$ **b)** $\cos 300°$ **c)** $\tan 110°$ **d)** $\sin 350°$

PRACTISING

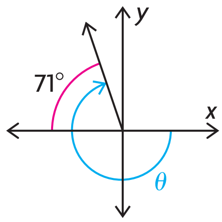
5. **i)** For each angle θ, predict which primary trigonometric ratios are positive.

ii) Determine the primary trigonometric ratios to the nearest hundredth.

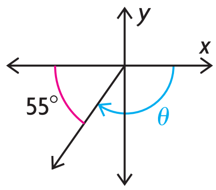
a)



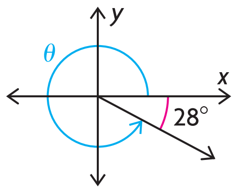
c)



b)



d)



6. Angle θ is a principal angle that lies in quadrant 2 such that $0° \leq \theta \leq 360°$.
K Given each trigonometric ratio,

i) determine the exact values of x, y, and r

ii) sketch angle θ in standard position

iii) determine the principal angle θ and the related acute angle β to the nearest degree

a) $\sin \theta = \frac{1}{3}$

b) $\cot \theta = -\frac{4}{3}$

c) $\cos \theta = -\frac{1}{4}$

d) $\csc \theta = 2.5$

e) $\tan \theta = -1.1$

f) $\sec \theta = -3.5$

7. For each trigonometric ratio in question 6, determine the smallest negative angle that has the same ratio.

8. Use each trigonometric ratio to determine all values of θ, to the nearest degree if $0° \leq \theta \leq 360°$.

a) $\sin \theta = 0.4815$

b) $\tan \theta = -0.1623$

c) $\cos \theta = -0.8722$

d) $\cot \theta = 8.1516$

e) $\csc \theta = -2.3424$

f) $\sec \theta = 0$

9. Given angle θ, where $0° \leq \theta \leq 360°$, determine two possible values of θ where each ratio would be true. Sketch both principal angles.

a) $\cos \theta = 0.6951$
b) $\tan \theta = -0.7571$
c) $\sin \theta = 0.3154$
d) $\cos \theta = -0.2882$
e) $\tan \theta = 2.3151$
f) $\sin \theta = -0.7503$

10. Given each point $P(x, y)$ lying on the terminal arm of angle θ,

i) state the value of θ, using both a counterclockwise and a clockwise rotation
ii) determine the primary trigonometric ratios

a) $P(-1, -1)$
b) $P(0, -1)$
c) $P(-1, 0)$
d) $P(1, 0)$

11. **A** Dennis doesn't like using x, y, and r to investigate angles. He says that he is going to continue using adjacent, opposite, and hypotenuse to evaluate trigonometric ratios for any angle θ. Explain the weaknesses of his strategy.

12. Given $\cos \theta = -\frac{5}{12}$, where $0° \leq \theta \leq 360°$,

a) in which quadrant could the terminal arm of θ lie?
b) determine all possible primary trigonometric ratios for θ.
c) evaluate all possible values of θ to the nearest degree.

13. **T** Given angle α, where $0° \leq \alpha < 360°$, $\cos \alpha$ is equal to a unique value. Determine the value of α to the nearest degree. Justify your answer.

14. **C** How does knowing the coordinates of a point P in the Cartesian plane help you determine the trigonometric ratios associated with the angle formed by the x-axis and a ray drawn from the origin to P? Use an example in your explanation.

Extending

15. Given angle θ, where $0° \leq \theta \leq 360°$, solve for θ to the nearest degree.

a) $\cos 2\theta = 0.6420$
b) $\sin(\theta + 20°) = 0.2045$
c) $\tan(90° - 2\theta) = 1.6443$

16. When you use the inverse trigonometric functions on a calculator, it is important to interpret the calculator result to avoid inaccurate values of θ. Using these trigonometric ratios, describe what errors might result.

a) $\sin \theta = -0.8$
b) $\cos \theta = -0.75$

17. Use sketches to explain why each statement is true.

a) $2 \sin 32° \neq \sin 64°$
b) $\sin 20° + \sin 40° \neq \sin 60°$
c) $\tan 75° \neq 3 \tan 25°$