3.6 The Zeros of a Quadratic Function

GOAL

Use a variety of strategies to determine the number of zeros of a quadratic function.

LEARN ABOUT the Math

Samantha has been asked to predict the number of zeros for each of three quadratic functions without using a graphing calculator. Samantha knows that quadratics have 0, 1, or 2 zeros. The three functions are:

$$f(x) = -2x^2 + 12x - 18$$
$$g(x) = 2x^2 + 6x - 8$$
$$h(x) = x^2 - 4x + 7$$

? How can Samantha predict the number of zeros each quadratic has without graphing?

EXAMPLE 1 Connecting functions to their graphs

Determine the properties of each function that will help you determine the number of x-intercepts each has.

Tara's Solution: Using Properties of the Quadratic Function

$$f(x) = -2x^2 + 12x - 18$$
$$f(x) = -2(x^2 - 6x + 9)$$
$$= -2(x - 3)^2$$

I decided to find the vertex of the first function. I factored -2 out as a common factor. The trinomial that was left was a perfect square, so I factored it. This put the function in vertex form. Because the vertex is on the x-axis, there is only one zero.

Vertex is (3, 0) and the parabola opens down. This function has one zero.

$$g(x) = 2x^2 + 6x - 8$$
$$g(x) = 2(x^2 + 3x - 4)$$
$$= 2(x + 4)(x - 1)$$

I factored 2 out as a common factor in the second function, then factored the trinomial inside the brackets. I used the factors to find the zeros, so this function has two.

This function has two zeros, at $x = -4$ and $x = 1$.

$$h(x) = x^2 - 4x + 7$$
$$= (x^2 - 4x + 4 - 4) + 7$$
$$= (x^2 - 4x + 4) - 4 + 7$$
$$= (x - 2)^2 + 3$$

The vertex is (2, 3) and the parabola opens up. This function has no zeros.

This function would not factor, so I found the vertex by completing the square. The vertex is above the *x*-axis, and the parabola opens up because *a* is positive. Therefore, the function has no zeros.

Asad's Solution: Using the Quadratic Formula

$$f(x) = -2x^2 + 12x - 18$$
$$0 = -2x^2 + 12x - 18$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-12 \pm \sqrt{(12)^2 - 4(-2)(-18)}}{2(-2)}$$
$$= \frac{-12 \pm \sqrt{144 - 144}}{-4}$$
$$= \frac{-12 \pm \sqrt{0}}{-4}$$
$$= \frac{-12}{-4}$$
$$= 3$$

The zeros or *x*-intercepts occur when each function equals 0. I set $f(x) = 0$, then solved the resulting equation using the quadratic formula with $a = -2$, $b = 12$, and $c = -18$.

The first function has only one value for the *x*-intercept, so there is only one zero. This can be seen from the value of the **discriminant** (the quantity under the radical sign), which is zero.

This function has one zero.

$$g(x) = 2x^2 + 6x - 8$$
$$0 = 2x^2 + 6x - 8$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-6 \pm \sqrt{(6)^2 - 4(2)(-8)}}{2(2)}$$
$$= \frac{-6 \pm \sqrt{36 + 64}}{4}$$

I used the quadratic formula again with $a = 2$, $b = 6$, and $c = -8$. There were two solutions, since the discriminant was positive. So the function has two zeros.

$$= \frac{-6 \pm \sqrt{100}}{4}$$

$$x = \frac{-6 - 10}{4} \quad \text{or} \quad x = \frac{-6 + 10}{4}$$

$$x = -4 \quad \text{or} \quad x = 1$$

This function has two zeros.

$$h(x) = x^2 - 4x + 7$$

I used the quadratic formula with $a = 1$, $b = -4$, and $c = 7$. The discriminant was negative, so there were no real-number solutions. The function has no zeros.

$$0 = x^2 - 4x + 7$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{(-4)^2 - 4(1)(7)}}{2}$$

$$= \frac{4 \pm \sqrt{16 - 28}}{2}$$

$$= \frac{4 \pm \sqrt{-12}}{2}$$

This function has no zeros.

Reflecting

A. Describe the possibilities for the number of zeros of a quadratic function.

B. How can finding the vertex help determine the number of zeros?

C. Why is the factored form useful in determining the number of zeros of a quadratic function?

D. Explain how the quadratic formula can be used to predict the number of zeros of a quadratic function.

APPLY the Math

EXAMPLE 2 Using the discriminant to determine the number of zeros

Find the value of the discriminant to determine the number of zeros of each quadratic function.

a) $f(x) = 2x^2 - 3x - 5$
b) $g(x) = 4x^2 + 4x + 1$
c) $h(x) = -5x^2 + x - 2$

Larry's Solution

a) $b^2 - 4ac = (-3)^2 - 4(2)(-5)$
$= 9 + 40$
$= 49$

Since $49 > 0$, there are two distinct zeros.

The discriminant $b^2 - 4ac$ is the value under the square root sign in the quadratic formula.

If $b^2 - 4ac$ is a positive number, the function has two zeros.

b) $b^2 - 4ac = (4)^2 - 4(4)(1)$
$= 16 - 16$
$= 0$

Therefore, there is one zero.

This time, the discriminant is equal to zero, so the function has only one zero.

c) $b^2 - 4ac = (1)^2 - 4(-5)(-2)$
$= 1 - 40$
$= -39$

Since $-39 < 0$, there are no zeros.

In this function, $b^2 - 4ac$ is a negative number, so the function has no zeros.

EXAMPLE 3 Solving a problem involving a quadratic function with one zero

Determine the value of k so that the quadratic function $f(x) = x^2 - kx + 3$ has only one zero.

Ruth's Solution

$$b^2 - 4ac = 0$$
$$(-k)^2 - 4(1)(3) = 0$$

If there is only one zero, then the discriminant is zero. I put the values for a, b, and c into the equation $b^2 - 4ac = 0$ and solved for k.

$$k^2 - 12 = 0$$
$$k^2 = 12$$
$$k = \pm\sqrt{12}$$
$$k = \pm 2\sqrt{3}$$

Since I had to take the square root of both sides to solve for k, there were two possible values. I expressed the values of k using a mixed radical in simplest form.

EXAMPLE 4 Solving a problem by using the discriminant

A market researcher predicted that the profit function for the first year of a new business would be $P(x) = -0.3x^2 + 3x - 15$, where x is based on the number of items produced. Will it be possible for the business to break even in its first year?

Raj's Solution

$$P(x) = 0$$
$$-0.3x^2 + 3x - 15 = 0$$

At a break-even point, the profit is zero.

$$b^2 - 4ac = (3)^2 - 4(-0.3)(-15)$$
$$= 9 - 18$$
$$= -9$$

I just wanted to know if there was a break-even point and not what it was, so I only needed to know if the profit function had any zeros. I used the value of the discriminant to decide.

Since $b^2 - 4ac < 0$, there are no zeros for this function. Therefore, it is not possible for the business to break even in its first year.

In Summary

Key Idea

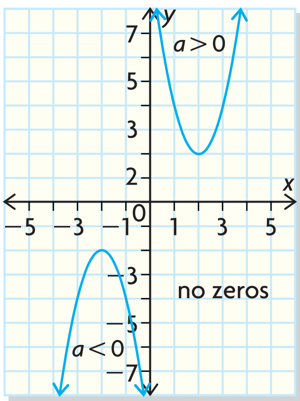
- A quadratic function can have 0, 1, or 2 zeros. You can determine the number of zeros either by graphing or by analyzing the function.

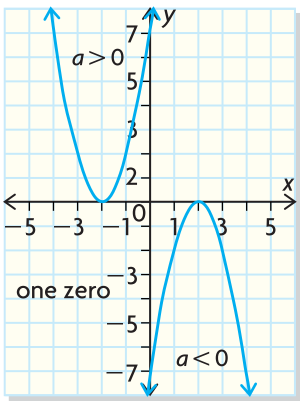
Need to Know

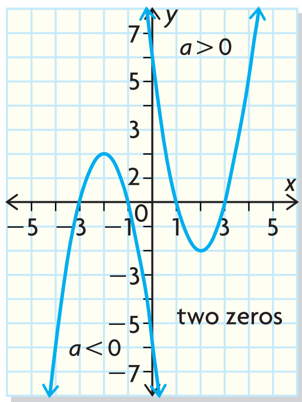
- The number of zeros of a quadratic function can be determined by looking at the graph of the function and finding the number of x-intercepts.
- For a quadratic equation $ax^2 + bx + c = 0$ and its corresponding function $f(x) = ax^2 + bx + c$, see the table below:

Value of the Discriminant	**Number of Zeros/Solutions**
$b^2 - 4ac > 0$	2
$b^2 - 4ac = 0$	1
$b^2 - 4ac < 0$	0

- The number of zeros can be determined by the location of the vertex relative to the x-axis, and the direction of opening:
  - If $a > 0$, and the vertex is above the x-axis, there are no zeros.
  - If $a > 0$, and the vertex is below the x-axis, there are two zeros.
  - If $a < 0$, and the vertex is above the x-axis, there are two zeros.
  - If $a < 0$, and the vertex is below the x-axis, there are no zeros.
  - If the vertex is on the x-axis, there is one zero.







CHECK Your Understanding

1. Determine the vertex and the direction of opening for each quadratic function. Then state the number of zeros.
   a) $f(x) = 3x^2 - 5$
   b) $f(x) = -4x^2 + 7$
   c) $f(x) = 5x^2 + 3$
   d) $f(x) = 3(x + 2)^2$
   e) $f(x) = -4(x + 3)^2 - 5$
   f) $f(x) = 0.5(x - 4)^2 - 2$

2. Factor each quadratic function to determine the number of zeros.
   a) $f(x) = x^2 - 6x - 16$
   b) $f(x) = 2x^2 - 6x$
   c) $f(x) = 4x^2 - 1$
   d) $f(x) = 9x^2 + 6x + 1$

3. Calculate the value of $b^2 - 4ac$ to determine the number of zeros.
   a) $f(x) = 2x^2 - 6x - 7$
   b) $f(x) = 3x^2 + 2x + 7$
   c) $f(x) = x^2 + 8x + 16$
   d) $f(x) = 9x^2 - 14.4x + 5.76$

PRACTISING

4. Determine the number of zeros. Do not use the same method for all
   K four parts.
   a) $f(x) = -3(x - 2)^2 + 4$
   b) $f(x) = 5(x - 3)(x + 4)$
   c) $f(x) = 4x^2 - 2x$
   d) $f(x) = 3x^2 - x + 5$

5. For each profit function, determine whether the company can break even. If
   A the company can break even, determine in how many ways it can do so.
   a) $P(x) = -2.1x^2 + 9.06x - 5.4$
   b) $P(x) = -0.3x^2 + 2x - 7.8$
   c) $P(x) = -2x^2 + 6.4x - 5.12$
   d) $P(x) = -2.4x^2 + x - 1.2$

6. For what value(s) of k will the function $f(x) = 3x^2 - 4x + k$ have one x-intercept?

7. For what value(s) of k will the function $f(x) = kx^2 - 4x + k$ have no zeros?

8. For what values of k will the function $f(x) = 3x^2 + 4x + k = 0$ have no zeros? one zero? two zeros?

9. The graph of the function $f(x) = x^2 - kx + k + 8$ touches the x-axis at one point. What are the possible values of k?

10. Is it possible for $n^2 + 25$ to equal $-8n$? Explain.

11. Write the equation of a quadratic function that meets each of the given conditions.
    a) The parabola opens down and has two zeros.
    b) The parabola opens up and has no zeros.
    c) The parabola opens down and the vertex is also the zero of the function.

12. The demand function for a new product is $p(x) = -4x + 42.5$, where x is the quantity sold in thousands and p is the price in dollars. The company that manufactures the product is planning to buy a new machine for the plant. There are three different types of machine. The cost function for each machine is shown.

Machine A: $C(x) = 4.1x + 92.16$

Machine B: $C(x) = 17.9x + 19.36$

Machine C: $C(x) = 8.8x + 55.4$

Investigate the break-even quantities for each machine. Which machine would you recommend to the company?

13. Describe how each transformation or sequence of transformations of the function $f(x) = 3x^2$ will affect the number of zeros the function has.

a) a vertical stretch of factor 2

b) a horizontal translation 3 units to the left

c) a horizontal compression of factor 2 and then a reflection in the x-axis

d) a vertical translation 3 units down

e) a horizontal translation 4 units to the right and then a vertical translation 3 units up

f) a reflection in the x-axis, then a horizontal translation 1 unit to the left, and then a vertical translation 5 units up

14. **T** If $f(x) = x^2 - 6x + 14$ and $g(x) = -x^2 - 20x - k$, determine the value of k so that there is exactly one point of intersection between the two parabolas.

15. Determine the number of zeros of the function $f(x) = 4 - (x - 3)(3x + 1)$ without solving the related quadratic equation or graphing. Explain your thinking.

16. **C** Describe how you can determine the number of zeros of a quadratic function if the equation of the function is in

a) vertex form **b)** factored form **c)** standard form

Extending

17. Show that $(x^2 - 1)k = (x - 1)^2$ has one solution for only one value of k.

18. Investigate the number of zeros of the function $f(x) = (k + 1)x^2 + 2kx + k - 1$ for different values of k. For what values of k does the function have no zeros? one zero? two zeros?