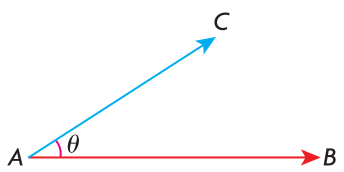
Section 7.3—The Dot Product of Two Geometric Vectors

In Chapter 6, the concept of multiplying a vector by a scalar was discussed. In this section, we introduce the dot product of two vectors and deal specifically with geometric vectors. When we refer to geometric vectors, we are referring to vectors that do not have a coordinate system associated with them. The dot product for any two vectors is defined as the product of their magnitudes multiplied by the cosine of the angle between the two vectors when the two vectors are placed in a tail-to-tail position.

Dot Product of Two Vectors



$\overrightarrow{AC} \cdot \overrightarrow{AB} = |\overrightarrow{AC}||\overrightarrow{AB}|\cos\theta, 0 \leq \theta \leq 180°$

Observations about the Dot Product

There are some elementary but important observations that can be made about this calculation. First, the result of the dot product is always a scalar. Each of the quantities on the right side of the formula above is a scalar quantity, and so their product must be a scalar. For this reason, the dot product is also known as the **scalar product**. Second, the dot product can be positive, zero, or negative, depending upon the size of the angle between the two vectors.

Sign of the Dot Product

For the vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta, 0 \leq \theta \leq 180°$:

- for $0 \leq \theta < 90°$, $\cos\theta > 0$, so $\vec{a} \cdot \vec{b} > 0$
- for $\theta = 90°$, $\cos\theta = 0$, so $\vec{a} \cdot \vec{b} = 0$
- for $90° < \theta \leq 180°$, $\cos\theta < 0$, so $\vec{a} \cdot \vec{b} < 0$

The dot product is only calculated for vectors when the angle θ between the vectors is to 0° to 180°, inclusive. (For convenience in calculating, the angle between the vectors is usually expressed in degrees, but radian measure is also correct.)

Perhaps the most important observation to be made about the dot product is that when two nonzero vectors are perpendicular, their dot product is always 0. This will have many important applications in Chapter 8, when we discuss lines and planes.

EXAMPLE 1 Calculating the dot product of two geometric vectors

Two vectors, $\vec{a}$ and $\vec{b}$, are placed tail to tail and have magnitudes 3 and 5, respectively. There is an angle of 120° between the vectors. Calculate $\vec{a} \cdot \vec{b}$.

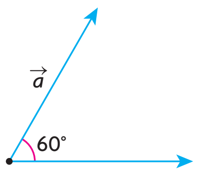
Solution

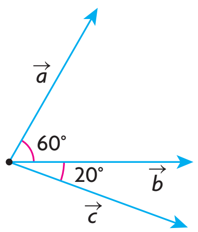
Since $|\vec{a}| = 3$, $|\vec{b}| = 5$, and $\cos 120° = -0.5$,

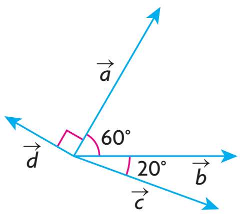
$$\begin{aligned}\vec{a} \cdot \vec{b} &= (3)(5)(-0.5)\\ &= -7.5\end{aligned}$$

Notice that, in this example, it is stated that the vectors are tail to tail when taking the dot product. This is the convention that is always used, since this is the way of defining the angle between any two vectors.

INVESTIGATION







A. Given vectors $\vec{a}$ and $\vec{b}$ where $|\vec{a}| = 5$, $|\vec{b}| = 8$ and the angle between the vectors is 60°, calculate $\vec{a} \cdot \vec{b}$.

B. For the vectors given in part A calculate $\vec{b} \cdot \vec{a}$. What do you notice? Will this relationship always hold regardless of the two vectors used and the measure of the angle between them? Explain.

C. For the vectors given in part A calculate $\vec{a} \cdot \vec{a}$ and $\vec{b} \cdot \vec{b}$. Based on your observations, what can you conclude about $\vec{u} \cdot \vec{u}$ for any vector $\vec{u}$?

D. Using the vectors given in part A and a third vector $\vec{c}$, $|\vec{c}| = 4$, as shown in the diagram, calculate each of the following without rounding:

i. $|\vec{b} + \vec{c}|$

ii. the angle between $\vec{b} + \vec{c}$ and $\vec{a}$.

iii. $\vec{a} \cdot (\vec{b} + \vec{c})$

iv. $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

E. Compare your results from part iii and iv in part D. What property does this demonstrate? Write an equivalent expression for $\vec{c} \cdot (\vec{a} + \vec{b})$ and confirm it using the appropriate calculations with the vectors given in part D.

F. Using the 3 vectors given above, explain why $(\vec{a} \cdot \vec{b}) \cdot \vec{c} \neq \vec{a} \cdot (\vec{b} \cdot \vec{c})$.

G. A fourth vector $\vec{d}$, $|\vec{d}| = 3$, is given as shown in the diagram. Explain why $\vec{a} \cdot \vec{d} = \vec{a} \cdot (-\vec{d}) = (-\vec{a}) \cdot \vec{d} = (-\vec{a}) \cdot (-\vec{d})$

H. Using the vectors given, calculate $\vec{a} \cdot \vec{0}$, $\vec{b} \cdot \vec{0}$ and $\vec{c} \cdot \vec{0}$. What does this imply?

Properties of the Dot Product

It should also be noted that the dot product can be calculated in whichever order we choose. In other words, $\vec{p} \cdot \vec{q} = |\vec{p}||\vec{q}|\cos\theta = |\vec{q}||\vec{p}|\cos\theta = \vec{q} \cdot \vec{p}$. We can change the order in the multiplication because the quantities in the formula are just scalars (that is, numbers) and the order of multiplication does not affect the final answer. This latter property is known as the *commutative* property for the dot product.

Another property that proves to be quite important for both computation and theoretical purposes is the dot product between a vector $\vec{p}$ and itself. The angle between $\vec{p}$ and itself is $0°$, so $\vec{p} \cdot \vec{p} = |\vec{p}||\vec{p}|(1) = |\vec{p}|^2$ since $\cos(0°) = 1$.

EXAMPLE 2 **Calculating the dot product between a vector and itself**

a. If $|\vec{a}| = \sqrt{7}$, calculate $\vec{a} \cdot \vec{a}$.

b. Calculate $\vec{i} \cdot \vec{i}$.

Solution

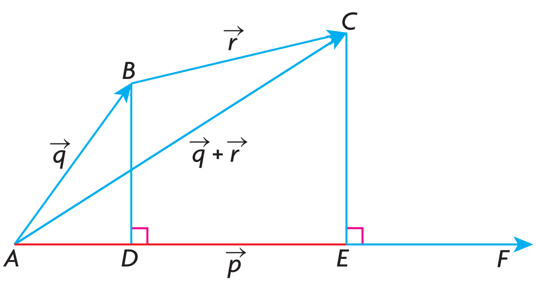
a. This is an application of the property just shown. So, $\vec{a} \cdot \vec{a} = (\sqrt{7})(\sqrt{7}) = 7$.

b. Since we know that $\vec{i}$ is a unit vector (along the positive x-axis), $\vec{i} \cdot \vec{i} = (1)(1) = 1$. In general, for any vector $\vec{x}$ of unit length, $\vec{x} \cdot \vec{x} = |\vec{x}|^2 = 1$. Thus, $\vec{j} \cdot \vec{j} = 1$ and $\vec{k} \cdot \vec{k} = 1$, where $\vec{j}$ and $\vec{k}$ are the unit vectors along the positive y- and z-axes, respectively.

Another important property that the dot product follows is the *distributive* property. In elementary algebra, the distributive property states that $p(q + r) = pq + pr$. We will prove that the distributive property also holds for the dot product. We will prove this geometrically below and algebraically in the next section.

Theorem: For the vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, $\vec{p}(\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$.

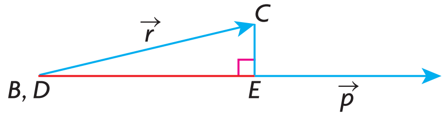
Proof: The vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$, are drawn, and the diagram is labelled as shown with $\overrightarrow{AC} = \vec{q} + \vec{r}$. To help visualize the dot products, lines from B and C have been drawn perpendicular to $\vec{p}$ (which is $\overrightarrow{AF}$).



Using the definition of a dot product, we write $\vec{q} \cdot \vec{p} = |\vec{q}||\vec{p}|\cos BAF$.

If we look at the right-angled triangle ABD and use the cosine ratio, we note that $\cos BAD = \frac{AD}{|\vec{q}|}$ or $AD = |\vec{q}|\cos BAD$. The two angles BAD and BAF are identical, and so $AD = |\vec{q}|\cos BAF$.
Rewriting the formula $\vec{q} \cdot \vec{p} = |\vec{q}||\vec{p}|\cos BAF$ as $\vec{q} \cdot \vec{p} = (|\vec{q}|\cos BAF)|\vec{p}|$, and substituting $AD = |\vec{q}|\cos BAF$, we obtain, $\vec{q} \cdot \vec{p} = AD|\vec{p}|$.

We also consider the vectors $\vec{r}$ and $\vec{p}$. We translate the vector $\overrightarrow{BC}$ so that point B is moved to be coincident with D. (The vector $\overrightarrow{BC}$ maintains the same direction and size under this translation.)



Writing the dot product for $\vec{r}$ and $\vec{p}$, we obtain $\vec{r} \cdot \vec{p} = |\vec{r}||\vec{p}|\cos CDE$. If we use trigonometric ratios in the right triangle, $\cos CDE = \frac{DE}{|\vec{r}|}$ or $DE = |\vec{r}|\cos CDE$.
Substituting $DE = |\vec{r}|\cos CDE$ into $\vec{r} \cdot \vec{p} = |\vec{r}||\vec{p}|\cos CDE$, we obtain $\vec{r} \cdot \vec{p} = DE|\vec{p}|$. If we use the formula for the dot product of $\vec{q} + \vec{r}$ and $\vec{p}$, we get the following: $(\vec{q} + \vec{r}) \cdot \vec{p} = |\vec{q} + \vec{r}||\vec{p}|\cos CAE$. Using the same reasoning as before, $\cos CAE = \frac{AE}{|\vec{q} + \vec{r}|}$ and $AE = (\cos CAE)|\vec{q} + \vec{r}|$, and then, by substitution, $(\vec{q} + \vec{r}) \cdot \vec{p} = |\vec{p}|AE$.

Adding the two quantities $\vec{q} \cdot \vec{p}$ and $\vec{r} \cdot \vec{p}$,

$$\begin{aligned}\vec{q} \cdot \vec{p} + \vec{r} \cdot \vec{p} &= AD|\vec{p}| + DE|\vec{p}| \\ &= |\vec{p}|(AD + DE) \quad \text{(Factoring)} \\ &= |\vec{p}|AE \\ &= (\vec{q} + \vec{r}) \cdot \vec{p}\end{aligned}$$

Thus, $\vec{q} \cdot \vec{p} + \vec{r} \cdot \vec{p} = (\vec{q} + \vec{r}) \cdot \vec{p}$, or, written in the more usual way,

$$\vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$$

We list some of the properties of the dot product below.
This final property has not been proven, but it comes directly from the definition of the dot product and proves most useful in computation.

Properties of the Dot Product

Commutative Property: $\vec{p} \cdot \vec{q} = \vec{q} \cdot \vec{p}$,

Distributive Property: $\vec{p} \cdot (\vec{q} + \vec{r}) = \vec{p} \cdot \vec{q} + \vec{p} \cdot \vec{r}$,

Magnitudes Property: $\vec{p} \cdot \vec{p} = |\vec{p}|^2$,

Associative Property with a scalar K: $(k\vec{p}) \cdot \vec{q} = \vec{p} \cdot (k\vec{q}) = k(\vec{p} \cdot \vec{q})$

EXAMPLE 3 Selecting a strategy to determine the angle between two geometric vectors

If the vectors $\vec{a} + 3\vec{b}$ and $4\vec{a} - \vec{b}$ are perpendicular, and $|\vec{a}| = 2|\vec{b}|$, determine the angle (to the nearest degree) between the nonzero vectors $\vec{a}$ and $\vec{b}$.

Solution

Since the two given vectors are perpendicular, $(\vec{a} + 3\vec{b}) \cdot (4\vec{a} - \vec{b}) = 0$.

Multiplying, $\vec{a} \cdot (4\vec{a} - \vec{b}) + 3\vec{b} \cdot (4\vec{a} - \vec{b}) = 0$

$4\vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + 12\vec{b} \cdot \vec{a} - 3\vec{b} \cdot \vec{b} = 0$ (Distributive property)

Simplifying, $4|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 3|\vec{b}|^2 = 0$ (Commutative property)

Since $|\vec{a}| = 2|\vec{b}|$, $|\vec{a}|^2 = (2|\vec{b}|)^2 = 4|\vec{b}|^2$ (Squaring both sides)

Substituting, $4(4|\vec{b}|^2) + 11((2|\vec{b}|)(|\vec{b}|)\cos\theta) - 3|\vec{b}|^2 = 0$

Solving for $\cos\theta$,

$$\cos\theta = \frac{-13|\vec{b}|^2}{22|\vec{b}|^2}$$

$$\cos\theta = \frac{-13}{22}, |\vec{b}|^2 \neq 0$$

Thus, $\cos^{-1}\left(\frac{-13}{22}\right) = \theta$, $\theta \doteq 126.2°$

Therefore, the angle between the two vectors is approximately 126.2°.

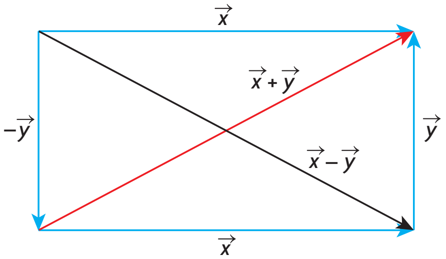
It is often necessary to square the magnitude of a vector expression. This is illustrated in the following example.

EXAMPLE 4 Proving that two vectors are perpendicular using the dot product

If $|\vec{x} + \vec{y}| = |\vec{x} - \vec{y}|$, prove that the nonzero vectors, $\vec{x}$ and $\vec{y}$, are perpendicular.

Solution

Consider the following diagram.



Since $|\vec{x}+\vec{y}| = |\vec{x}-\vec{y}|$, $|\vec{x}+\vec{y}|^2 = |\vec{x}-\vec{y}|^2$ (Squaring both sides)

$|\vec{x}+\vec{y}|^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y})$ and $|\vec{x}-\vec{y}|^2 = (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})$ (Magnitudes property)

Therefore, $(\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = (\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y})$

$$|\vec{x}|^2 + 2\vec{x}\cdot\vec{y} + |\vec{y}|^2 = |\vec{x}|^2 - 2\vec{x}\cdot\vec{y} + |\vec{y}|^2$$ (Multiplying out)

So, $4\vec{x}\cdot\vec{y} = 0$ and $\vec{x}\cdot\vec{y} = 0$

Thus, the two required vectors are shown to be perpendicular. (Geometrically, this means that if diagonals in a parallelogram are equal in length, then the sides must be perpendicular. In actuality, the parallelogram is a rectangle.)

In this section, we dealt with the dot product and its geometric properties. In the next section, we will illustrate these same ideas with algebraic vectors.

IN SUMMARY

Key Idea

- The dot product between two geometric vectors $\vec{a}$ and $\vec{b}$ is a scalar quantity defined as $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between the two vectors.

Need to Know

- If $0° \le \theta < 90°$, then $\vec{a}\cdot\vec{b} > 0$
- If $\theta = 90°$, then $\vec{a}\cdot\vec{b} = 0$
- If $90° < \theta \le 180°$, then $\vec{a}\cdot\vec{b} < 0$
- $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
- $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$
- $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
- $\vec{i}\cdot\vec{i} = 1$, $\vec{j}\cdot\vec{j} = 1$, and $\vec{k}\cdot\vec{k} = 1$
- $(k\vec{a})\cdot\vec{b} = \vec{a}\cdot(k\vec{b}) = k(\vec{a}\cdot\vec{b})$

Exercise 7.3

PART A

1. If $\vec{a} \cdot \vec{b} = 0$, why can we not necessarily conclude that the given vectors are perpendicular? (In other words, what restrictions must be placed on the vectors to make this statement true?)

C 2. Explain why the calculation $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ is not meaningful.

3. A student writes the statement $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c}$ and then concludes that $\vec{a} = \vec{c}$. Construct a simple numerical example to show that this is not correct if the given vectors are all nonzero.

4. Why is it correct to say that if $\vec{a} = \vec{c}$, then $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c}$?

5. If two vectors $\vec{a}$ and $\vec{b}$ are unit vectors pointing in opposite directions, what is the value of $\vec{a} \cdot \vec{b}$?

PART B

6. If θ is the angle (in degrees) between the two given vectors, calculate the dot product of the vectors.

   a. $|\vec{p}| = 4, |\vec{q}| = 8, \theta = 60°$

   b. $|\vec{x}| = 2, |\vec{y}| = 4, \theta = 150°$

   c. $|\vec{a}| = 0, |\vec{b}| = 8, \theta = 100°$

   d. $|\vec{p}| = 1, |\vec{q}| = 1, \theta = 180°$

   e. $|\vec{m}| = 2, |\vec{n}| = 5, \theta = 90°$

   f. $|\vec{u}| = 4, |\vec{v}| = 8, \theta = 145°$

7. Calculate, to the nearest degree, the angle between the given vectors.

   a. $|\vec{x}| = 8, |\vec{y}| = 3, \vec{x} \cdot \vec{y} = 12\sqrt{3}$

   b. $|\vec{m}| = 6, |\vec{n}| = 6, \vec{m} \cdot \vec{n} = 6$

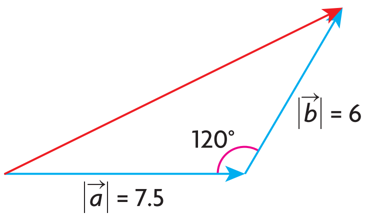
   c. $|\vec{p}| = 1, |\vec{q}| = 5, \vec{p} \cdot \vec{q} = 3$

   d. $|\vec{p}| = 1, |\vec{q}| = 5, \vec{p} \cdot \vec{q} = -3$

   e. $|\vec{a}| = 7, |\vec{b}| = 3, \vec{a} \cdot \vec{b} = 10.5$

   f. $|\vec{u}| = 10, |\vec{v}| = 10, \vec{u} \cdot \vec{v} = -50$

K 8. For the two vectors $\vec{a}$ and $\vec{b}$ whose magnitudes are shown in the diagram below, calculate the dot product.



9. Use the properties of the dot product to simplify each of the following expressions:

   a. $(\vec{a} + 5\vec{b}) \cdot (2\vec{a} - 3\vec{b})$

   b. $3\vec{x} \cdot (\vec{x} - 3\vec{y}) - (\vec{x} - 3\vec{y}) \cdot (-3\vec{x} + \vec{y})$

10. What is the value of the dot product between $\vec{0}$ and any nonzero vector? Explain.

A 11. The vectors $\vec{a} - 5\vec{b}$ and $\vec{a} - \vec{b}$ are perpendicular. If $\vec{a}$ and $\vec{b}$ are unit vectors, then determine $\vec{a} \cdot \vec{b}$.

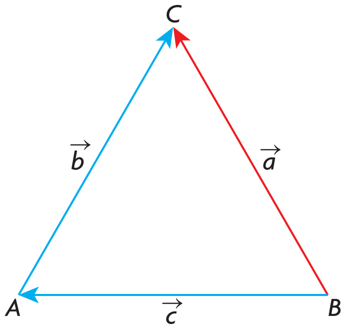
12. If $\vec{a}$ and $\vec{b}$ are any two nonzero vectors, prove each of the following to be true:
    a. $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2$
    b. $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

13. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ satisfy the relationship $\vec{a} = \vec{b} + \vec{c}$.
    a. Show that $|\vec{a}|^2 = |\vec{b}|^2 + 2\vec{b} \cdot \vec{c} + |\vec{c}|^2$.
    b. If the vectors $\vec{b}$ and $\vec{c}$ are perpendicular, how does this prove the Pythagorean theorem?

14. Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be three mutually perpendicular vectors of lengths 1, 2, and 3, respectively. Calculate the value of $(\vec{u} + \vec{v} + \vec{w}) \cdot (\vec{u} + \vec{v} + \vec{w})$.

T 15. Prove the identity $|\vec{u} + \vec{v}|^2 + |\vec{u} - \vec{v}|^2 = 2|\vec{u}|^2 + 2|\vec{v}|^2$.

16. The three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are of unit length and form the sides of equilateral triangle ABC such that $\vec{a} - \vec{b} - \vec{c} = \vec{0}$ (as shown). Determine the numerical value of $(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b} + \vec{c})$.



PART C

17. The vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. Determine the value of $\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$ if $|\vec{a}| = 1$, $|\vec{b}| = 2$, and $|\vec{c}| = 3$.

18. The vector $\vec{a}$ is a unit vector, and the vector $\vec{b}$ is any other nonzero vector. If $\vec{c} = (\vec{b} \cdot \vec{a})\vec{a}$ and $\vec{d} = \vec{b} - \vec{c}$, prove that $\vec{d} \cdot \vec{a} = 0$.