Section 7.4—The Dot Product of Algebraic Vectors

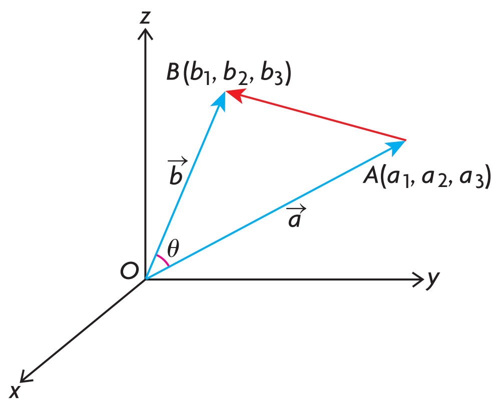
In the previous section, the dot product was discussed in geometric terms. In this section, the dot product will be expressed in terms of algebraic vectors in R^2 and R^3. Recall that a vector expressed as $\vec{a} = (-1, 4, 5)$ is referred to as an algebraic vector. The geometric properties of the dot product developed in the previous section will prove useful in understanding the dot product in algebraic form. The emphasis in this section will be on developing concepts in R^3, but these ideas apply equally well to R^2 or to higher dimensions.

Defining the Dot Product of Algebraic Vectors

Theorem: In R^3, if $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = a_1b_1 + a_2b_2 + a_3b_3$.

Proof: Draw $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, as shown in the diagram.



In ΔOAB, $|\overrightarrow{AB}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 - 2|\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta$ (Cosine law)

So, $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$ and
$|\overrightarrow{AB}|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$

We know that $|\overrightarrow{OA}|^2 = a_1^2 + a_2^2 + a_3^2$ and $|\overrightarrow{OB}|^2 = b_1^2 + b_2^2 + b_3^2$.

It should also be noted that $\vec{a} \cdot \vec{b} = |\overrightarrow{OA}||\overrightarrow{OB}|\cos\theta$. (Definition of dot product)

We substitute each of these quantities in the expression for the cosine law.

This gives $(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 =$
$$a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\vec{a} \cdot \vec{b}$$

Expanding, we get

$$b_1^2 - 2a_1b_1 + a_1^2 + b_2^2 - 2a_2b_2 + a_2^2 + b_3^2 - 2a_3b_3 + a_3^2$$
$$= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - 2\vec{a}\cdot\vec{b} \qquad \text{(Simplify)}$$
$$-2a_1b_1 - 2a_2b_2 - 2a_3b_3 = -2\vec{a}\cdot\vec{b} \qquad \text{(Divide by } -2\text{)}$$
$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Observations about the Algebraic Form of the Dot Product

There are some important observations to be made about this expression for the dot product. First and foremost, the quantity on the right-hand side of the expression, $a_1b_1 + a_2b_2 + a_3b_3$, is evaluated by multiplying corresponding components and then adding them. Each of these quantities, a_1b_1, a_2b_2, and a_3b_3, is just a real number, so their sum is a real number. This implies that $\vec{a}\cdot\vec{b}$ is itself just a real number, or a scalar product. Also, since the right side is an expression made up of real numbers, it can be seen that $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = \vec{b}\cdot\vec{a}$. This is a restatement of the commutative law for the dot product of two vectors. All the other rules for computation involving dot products can now be proven using the properties of real numbers and the basic definition of a dot product.

In this proof, we have used vectors in R^3 to calculate a formula for $\vec{a}\cdot\vec{b}$. It is important to understand, however, that this procedure could be used in the same way for two vectors, $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$, in R^2, to obtain the formula $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2$.

EXAMPLE 1

Proving the distributive property of the dot product in R^3

Prove that the distributive property holds for dot products in R^3—that is, $\vec{a}\cdot(\vec{b} + \vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$.

Solution

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, and $\vec{c} = (c_1, c_2, c_3)$.

In showing this statement to be true, the right side will be expressed in component form and then rearranged to be the same as the left side.

$$\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} = (a_1, a_2, a_3)\cdot(b_1, b_2, b_3) + (a_1, a_2, a_3)\cdot(c_1, c_2, c_3)$$
$$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \qquad \text{(Definition of dot product)}$$
$$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3 \qquad \text{(Rearranging terms)}$$
$$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \qquad \text{(Factoring)}$$
$$= \vec{a}\cdot(\vec{b} + \vec{c})$$

This example shows how to prove the distributive property for the dot product in R^3. The value of writing the dot product in component form is that it allows us to combine the geometric form with the algebraic form, and create the ability to do calculations that would otherwise not be possible.

Computation of the Dot Product of Algebraic Vectors

In R^2, $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta = x_1y_1 + x_2x_2$, where $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$.

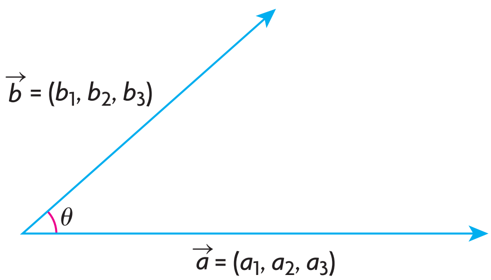
In R^3, $\vec{x} \cdot \vec{y} = |\vec{x}||\vec{y}|\cos\theta = x_1y_1 + x_2y_2 + x_3y_3$, where $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$.

In both cases q is the angle between $\vec{x}$ and $\vec{y}$.

The dot product expressed in component form has significant advantages over the geometric form from both a computational and theoretical point of view. At the outset, the calculation appears to be somewhat artificial or contrived, but as we move ahead, we will see its applicability to many situations.

A useful application of the dot product is to calculate the angle between two vectors. Solving for $\cos\theta$ in the formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ gives the following result.

Formula for the Angle between Two Vectors



When two vectors are placed tail to tail, as shown, $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$.

EXAMPLE 2 Selecting a strategy to determine the angle between two algebraic vectors

a. Given the vectors $\vec{a} = (-1, 2, 4)$ and $\vec{b} = (3, 4, 3)$, calculate $\vec{a} \cdot \vec{b}$.

b. Calculate, to the nearest degree, the angle between $\vec{a}$ and $\vec{b}$.

Solution

a. $\vec{a} \cdot \vec{b} = (-1)(3) + (2)(4) + (4)(3) = 17$

b. $|\vec{a}|^2 = (-1)^2 + (2)^2 + (4)^2 = 21,\ |\vec{a}| = \sqrt{21}$
$|\vec{b}|^2 = (3)^2 + (4)^2 + (3)^2 = 34,\ |\vec{b}| = \sqrt{34}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\theta = \frac{17}{\sqrt{21}\sqrt{34}} \quad \text{(Substitution)}$$

$$\cos\theta \doteq 0.6362$$

$$\theta \doteq \cos^{-1}(0.6362)$$

$$\theta \doteq 50.5°$$

Therefore, the angle between the two vectors is approximately 50.5°.

In the previous section, we showed that when two nonzero vectors are perpendicular, their dot product equals zero—that is, $\vec{a}\cdot\vec{b} = 0$.

EXAMPLE 3 Using the dot product to solve a problem involving perpendicular vectors

a. For what values of k are the vectors $\vec{a} = (-1, 3, -4)$ and $\vec{b} = (3, k, -2)$ perpendicular?

b. For what values of m are the vectors $\vec{x} = (m, m, -3)$ and $\vec{y} = (m, -3, 6)$ perpendicular?

Solution

a. Since $\vec{a}\cdot\vec{b} = 0$ for perpendicular vectors,

$$-1(3) + 3(k) - 4(-2) = 0$$

$$3k = -5$$

$$k = \frac{-5}{3}$$

In calculations of this type involving the dot product, the calculation should be verified as follows:

$$\begin{aligned}(-1, 3, -4)\cdot\left(3, \frac{-5}{3}, -2\right) &= -1(3) + 3\left(\frac{-5}{3}\right) - 4(-2)\\ &= -3 - 5 + 8\\ &= 0\end{aligned}$$

This check verifies that the calculation is correct.

b. Using the conditions for perpendicularity of vectors,

$$(m, m, -3)\cdot(m, -3, 6) = 0$$

$$m^2 - 3m - 18 = 0$$

$$(m - 6)(m + 3) = 0$$

$$m = 6 \text{ or } m = -3$$

Check:
For $m = 6$, $(6, 6, -3) \cdot (6, -3, 6) = 36 - 18 - 18 = 0$

For $m = -3$, $(-3, -3, -3) \cdot (-3, -3, 6) = 9 + 9 - 18 = 0$

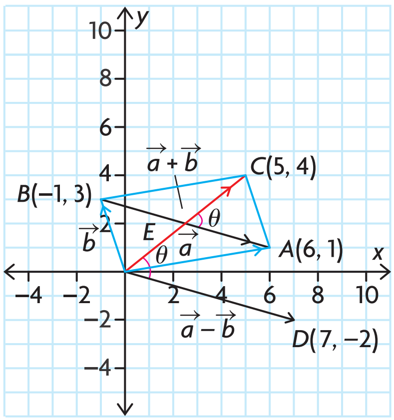
We can combine various operations that we have learned for calculation purposes in R^2 and R^3.

EXAMPLE 4 Using the dot product to solve a problem involving a parallelogram

A parallelogram has its sides determined by $\vec{a} = (6, 1)$ and $\vec{b} = (-1, 3)$. Determine the angle between the diagonals of the parallelogram formed by these vectors.

Solution

The diagonals of the parallelogram are determined by the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, as shown in the diagram. The components of these vectors are
$\vec{a} + \vec{b} = (6 + (-1), 1 + 3) = (5, 4)$ and
$\vec{a} - \vec{b} = (6 - (-1), 1 - 3) = (7, -2)$, as shown in the diagram



At this point, the dot product is applied directly to find θ, the angle between the vectors $\overrightarrow{OD}$ and $\overrightarrow{OC}$.

$$\text{Therefore, } \cos\theta = \frac{(5, 4) \cdot (7, -2)}{|(5, 4)||(7, -2)|}$$

$$\cos\theta = \frac{27}{\sqrt{41}\,\sqrt{53}}$$

$$\cos\theta \doteq 0.5792$$

$$\text{Therefore, } \theta \doteq 54.61°$$

The angle between the diagonals is approximately 54.6°. The answer given is 54.6°, but its supplement, 125.4°, is also correct.

One of the most important properties of the dot product is its application to determining a perpendicular vector to two given vectors, which will be demonstrated in the following example.

EXAMPLE 5 Selecting a strategy to determine a vector perpendicular to two given vectors

Find a vector (or vectors) perpendicular to each of the vectors $\vec{a} = (1, 5, -1)$ and $\vec{b} = (-3, 1, 2)$.

Solution

Let the required vector be $\vec{x} = (x, y, z)$. Since $\vec{x}$ is perpendicular to each of the two given vectors, $(x, y, z) \cdot (1, 5, -1) = 0$ and $(x, y, z) \cdot (-3, 1, 2) = 0$.

Multiplying gives $x + 5y - z = 0$ and $-3x + y + 2z = 0$, which is a system of two equations in three unknowns.

① $x + 5y - z = 0$

② $-3x + y + 2z = 0$

③ $3x + 15y - 3z = 0$ (Multiplying equation ① by 3)

④ $16y - z = 0$ (Adding equations ② and ③)

$z = 16y$

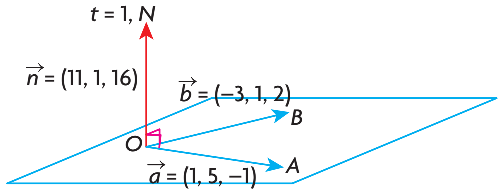
Now, we substitute $z = 16y$ into equation ① to solve for x in terms of y. We obtain $x + 5y - 16y = 0$, or $x = 11y$.

We have solved for x and z by expressing each variable in terms of y. The solution to the system of equations is $(11y, y, 16y)$ or $(11t, t, 16t)$ if we let $y = t$. The substitution of t (called a parameter) for y is not necessarily required for a correct solution and is done more for convenience of notation. This kind of substitution will be used later to great advantage and will be discussed in Chapter 9 at length.

We can find vectors to satisfy the required conditions by replacing t with any real number, $t \neq 0$. Since we can use any real number for t to produce the required vector, this implies that an infinite number of vectors are perpendicular to both $\vec{a}$ and $\vec{b}$. If we use $t = 1$, we obtain $(11, 1, 16)$.

As before, we verify the solution:
$(11, 1, 16) \cdot (1, 5, -1) = 11 + 5 - 16 = 0$ and
$(11, 1, 16) \cdot (-3, 1, 2) = -33 + 1 + 32 = 0$

It is interesting to note that the vector $(11t, t, 16t)$, $t \neq 0$, represents a general vector perpendicular to the plane in which the vectors $\vec{a} = (1, 5, -1)$ and $\vec{b} = (-3, 1, 2)$ lie. This is represented in the diagram shown, where $t = 1$.



Determining the components of a vector perpendicular to two nonzero vectors will prove to be important in later applications.

IN SUMMARY

Key Idea

- The dot product is defined as follows for algebraic vectors in R^2 and R^3, respectively:
  - If $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2,)$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$
  - If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Need to Know

- The properties of the dot product hold for both geometric and algebraic vectors.
- Two nonzero vectors, $\vec{a}$ and $\vec{b}$, are perpendicular if $\vec{a} \cdot \vec{b} = 0$.
- For two nonzero vectors $\vec{a}$ and $\vec{b}$, where θ is the angle between the vectors, $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$.

Exercise 7.4

PART A

1. How many vectors are perpendicular to $\vec{a} = (-1, 1)$? State the components of three such vectors.

2. For each of the following pairs of vectors, calculate the dot product and, on the basis of your result, say whether the angle between the two vectors is acute, obtuse, or 90°.
   a. $\vec{a} = (-2, 1), \vec{b} = (1, 2)$
   b. $\vec{a} = (2, 3, -1), \vec{b} = (4, 3, -17)$
   c. $\vec{a} = (1, -2, 5), \vec{b} = (3, -2, -2)$

3. Give the components of a vector that is perpendicular to each of the following planes:
   a. xy-plane
   b. xz-plane
   c. yz-plane

4. a. From the set of vectors $\left\{(1, 2, -1), (-4, -5, -6), (4, 3, 10), \left(5, -3, \frac{-5}{6}\right)\right\}$, select two pairs of vectors that are perpendicular to each other.
   b. Are any of these vectors collinear? Explain.

5. In Example 5, a vector was found that was perpendicular to two nonzero vectors.
   a. Explain why it would not be possible to do this in R^2 if we selected the two vectors $\vec{a} = (1, -2)$ and $\vec{b} = (1, 1)$.
   b. Explain, in general, why it is not possible to do this if we select any two vectors in R^2.

PART B

K

6. Determine the angle, to the nearest degree, between each of the following pairs of vectors:
   a. $\vec{a} = (5, 3)$ and $\vec{b} = (-1, -2)$
   b. $\vec{a} = (-1, 4)$ and $\vec{b} = (6, -2)$
   c. $\vec{a} = (2, 2, 1)$ and $\vec{b} = (2, 1, -2)$
   d. $\vec{a} = (2, 3, -6)$ and $\vec{b} = (-5, 0, 12)$

7. Determine k, given two vectors and the angle between them.
   a. $\vec{a} = (-1, 2, -3), \vec{b} = (-6k, -1, k), \theta = 90°$
   b. $\vec{a} = (1, 1), \vec{b} = (0, k), \theta = 45°$

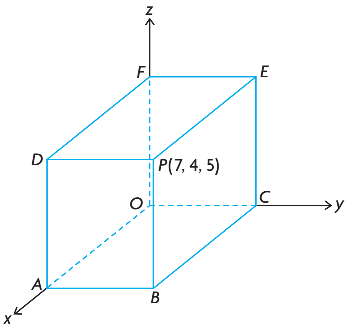
8. In R^2, a square is determined by the vectors $\vec{i}$ and $\vec{j}$.
   a. Sketch the square.
   b. Determine vector components for the two diagonals.
   c. Verify that the angle between the diagonals is 90°.

9. Determine the angle, to the nearest degree, between each pair of vectors.
   a. $\vec{a} = (1 - \sqrt{2}, \sqrt{2}, -1)$ and $\vec{b} = (1, 1)$
   b. $\vec{a} = (\sqrt{2} - 1, \sqrt{2} + 1, \sqrt{2})$ and $\vec{b} = (1, 1, 1)$

C

10. a. For the vectors $\vec{a} = (2, p, 8)$ and $\vec{b} = (q, 4, 12)$, determine values of p and q so that the vectors are
       i. collinear
       ii. perpendicular
    b. Are the values of p and q unique? Explain why or why not.

11. $\triangle ABC$ has vertices at $A(2, 5)$, $B(4, 11)$, and $C(-1, 6)$. Determine the angles in this triangle.



12. A rectangular box measuring 4 by 5 by 7 is shown in the diagram at the left.
    a. Determine the coordinates of each of the missing vertices.
    b. Determine the angle, to the nearest degree, between $\overrightarrow{AE}$ and $\overrightarrow{BF}$.

13. a. Given the vectors $\vec{p} = (-1, 3, 0)$ and $\vec{q} = (1, -5, 2)$, determine the components of a vector perpendicular to each of these vectors.
    b. Given the vectors $\vec{m} = (1, 3, -4)$ and $\vec{n} = (-1, -2, 3)$, determine the components of a vector perpendicular to each of these vectors.

14. Find the value of p if the vectors $\vec{r} = (p, p, 1)$ and $\vec{s} = (p, -2, -3)$ are perpendicular to each other.

15. a. Determine the algebraic condition such that the vectors $\vec{c} = (-3, p, -1)$ and $\vec{d} = (1, -4, q)$ are perpendicular to each other.
    b. If $q = -3$, what is the corresponding value of p?

A

16. Given the vectors $\vec{r} = (1, 2, -1)$ and $\vec{s} = (-2, -4, 2)$, determine the components of two vectors perpendicular to each of these vectors. Explain your answer.

17. The vectors $\vec{x} = (-4, p, -2)$ and $\vec{y} = (-2, 3, 6)$ are such that $\cos^{-1}\left(\frac{4}{21}\right) = \theta$, where θ is the angle between $\vec{x}$ and $\vec{y}$. Determine the value(s) of p.

PART C

18. The diagonals of a parallelogram are determined by the vectors $\vec{a} = (3, 3, 0)$ and $\vec{b} = (-1, 1, -2)$.
    a. Show that this parallelogram is a rhombus.
    b. Determine vectors representing its sides and then determine the length of these sides.
    c. Determine the angles in this rhombus.

T

19. The rectangle $ABCD$ has vertices at $A(-1, 2, 3)$, $B(2, 6, -9)$, and $D(3, q, 8)$.
    a. Determine the coordinates of the vertex C.
    b. Determine the angle between the two diagonals of this rectangle.

20. A cube measures 1 by 1 by 1. A line is drawn from one vertex to a diagonally opposite vertex through the centre of the cube. This is called a body diagonal for the cube. Determine the angles between the body diagonals of the cube.