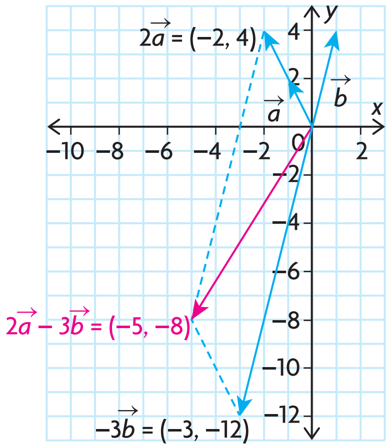
Section 6.8—Linear Combinations and Spanning Sets

We have discussed concepts involving geometric and algebraic vectors in some detail. In this section, we are going to use these ideas as a basis for understanding the notion of a **linear combination**, an important idea for understanding the geometry of three dimensions.

Examining Linear Combinations of Vectors in R^2

We'll begin by considering linear combinations in R^2. If we consider the vectors $\vec{a} = (-1, 2)$ and $\vec{b} = (1, 4)$ and write $2(-1, 2) - 3(1, 4) = (-5, -8)$, then the expression on the left side of this equation is called a linear combination. In this case, the linear combination produces the vector $(-5, -8)$. Whenever vectors are multiplied by scalars and then added, the result is a new vector that is a linear combination of the vectors. If we take the two vectors $\vec{a} = (-1, 2)$ and $\vec{b} = (1, 4)$, then $2\vec{a} - 3\vec{b}$ is a vector on the xy-plane and is the diagonal of the parallelogram formed by the vectors $2\vec{a}$ and $-3\vec{b}$, as shown in the diagram.



Linear Combination of Vectors

For noncollinear vectors, $\vec{u}$ and $\vec{v}$, a linear combination of these vectors is $a\vec{u} + b\vec{v}$, where a and b are scalars (real numbers). The vector $a\vec{u} + b\vec{v}$ is the diagonal of the parallelogram formed by the vectors $a\vec{u}$ and $b\vec{v}$.

It was shown that every vector in the xy-plane can be written uniquely in terms of the unit vectors $\vec{i}$ and $\vec{j}$. $\overrightarrow{OP} = (a, b) = a\vec{i} + b\vec{j}$, where $\vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$. This can be done in only one way. Writing $\overrightarrow{OP}$ in this way is really just writing this vector as a linear combination of $\vec{i}$ and $\vec{j}$. Because every vector in R^2 can be written as a linear combination of these two vectors, we say that $\vec{i}$ and $\vec{j}$ span R^2. Another way of stating this is to say that the set of vectors $\{\vec{i}, \vec{j}\}$ forms a **spanning set** for R^2.

Spanning Set for R^2

The set of vectors $\{\vec{i}, \vec{j}\}$ is said to form a spanning set for R^2. Every vector in R^2 can be written uniquely as a linear combination of these two vectors.

What is interesting about spanning sets is that there is not just one set of vectors that can be used to span R^2. There is an infinite number of sets, each set containing a minimum of exactly two vectors, that would serve the same purpose. The concepts of span and spanning set will prove significant for the geometry of planes studied in Chapter 8.

EXAMPLE 1 Representing a vector as a linear combination of two other vectors

Show that $\vec{x} = (4, 23)$ can be written as a linear combination of either set of vectors, $\{(-1, 4), (2, 5)\}$ or $\{(1, 0), (-2, 1)\}$.

Solution

In each case, the procedure is the same, and so we will show the details for just one set of calculations. We are looking for solutions to the following separate equations: $a(-1, 4) + b(2, 5) = (4, 23)$ and $c(1, 0) + d(-2, 1) = (4, 23)$.

Multiplying, $(-a, 4a) + (2b, 5b) = (4, 23)$ (Properties of scalar multiplication)

$(-a + 2b, 4a + 5b) = (4, 23)$ (Properties of vector addition)

Since the vector on the left side is equal to that on the right side, we can write

① $-a + 2b = 4$

② $4a + 5b = 23$

This forms a linear system that can be solved using the method of elimination.

③ $-4a + 8b = 16$, after multiplying equation ① by 4.

Adding equation ② and equation ③ gives, $13b = 39$, so $b = 3$ and, by substitution, $a = 2$.

Therefore, $2(-1, 4) + 3(2, 5) = (4, 23)$.
The calculations for the second linear combination are done in the same way as the first, and so $c - 2d = 4$ and $d = 23$. Substituting gives $c = 50$ and $d = 23$. Therefore, $50(1, 0) + 23(-2, 1) = (4, 23)$. You should verify the calculations on your own.

In R^2, it is possible to take any pair of noncollinear (non-parallel) vectors as a spanning set, provided that $(0, 0)$ is not one of the two vectors.

EXAMPLE 2

Reasoning about spanning sets in R^2

Show that the set of vectors $\{(2, 3), (4, 6)\}$ does not span R^2.

Solution

Since these vectors are scalar multiples of each other, i.e., $(4, 6) = 2(2, 3)$, they cannot span R^2. All linear combinations of these two vectors produce only vectors that are scalar multiples of (2, 3). This is shown by the following calculation:

$$\begin{aligned} a(2, 3) + b(4, 6) = (2a + 4b, 3a + 6b) &= (2(a + 2b), 3(a + 2b)) \\ &= (a + 2b)(2, 3) \end{aligned}$$

This result means that we cannot use linear combinations of the set of vectors $\{(2, 3), (4, 6)\}$ to obtain anything but a multiple of $(2, 3)$. As a result, the only vectors that can be created are ones in either the same or opposite direction of (2, 3). There is no linear combination of these vectors that would allow us to obtain, for example, the vector $(3, 4)$.

When we say that a set of vectors spans R^2, we are saying that every vector in the plane can be written as a linear combination of the two given vectors. In Example 1, we did not prove that either set of vectors was a spanning set. All that we showed was that the given vector could be written as a linear combination of a set of vectors. It is true in this case, however, that both sets do span R^2. In the following example, it will be shown how a set of vectors in R^2 can be proven to be a spanning set.

EXAMPLE 3

Proving that a given set of vectors spans R^2

Show that the set of vectors $\{(2, 1), (-3, -1)\}$ is a spanning set for R^2.

Solution

In order to show that the set spans R^2, we write the linear combination $a(2, 1) + b(-3, -1) = (x, y)$, where (x, y) represents *any* vector in R^2. Carrying out the same procedure as in the previous example, we obtain

① $2a - 3b = x$

② $a - b = y$

Again the process of elimination will be used to solve this system of equations.

① $2a - 3b = x$

③ $2a - 2b = 2y$, after multiplying equation ② by 2

Subtracting eliminates a, $-3b - (-2b) = x - 2y$

Therefore, $-b = x - 2y$ or $b = -x + 2y$. By substituting this value of b into equation ②, we find $a = -x + 3y$. Therefore, the solution to this system of equations is $a = -x + 3y$ and $b = -x + 2y$.

This means that, whenever we are given the components of any vector, we can find the corresponding values of a and b by substituting into the formula. Since the values of x and y are unique, the corresponding values of a and b are also unique. Using this formula to write $(-3, 7)$ as a linear combination of the two given vectors, we would say $x = -3,\ y = 7$ and solve for a and b to obtain

$$a = -(-3) + 3(7) = 24$$

and

$$b = -(-3) + 2(7) = 17$$

$$24(2, 1) + 17(-3, -1) = (-3, 7)$$

So the vector $(-3, 7)$ can be written as a linear combination of $(2, 1)$ and $(-3, -1)$. Therefore, the set of vectors $\{(2, 1), (-3, -1)\}$ spans R^2.

Examining Linear Combinations of Vectors in R^3

In the previous section, the set of unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ was introduced as unit vectors lying along the positive x-, y-, and z-axes, respectively. This set of vectors is referred to as the standard basis for R^3, meaning that every vector in R^3 can be written uniquely as a linear combination of these three vectors. (It should be pointed out that there is an infinite number of sets containing three vectors that could also be used as a basis for R^3.)

EXAMPLE 4 Representing linear combinations in R^3

Show that the vector $(2, 3, -5)$ can be written as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$ and illustrate this geometrically.

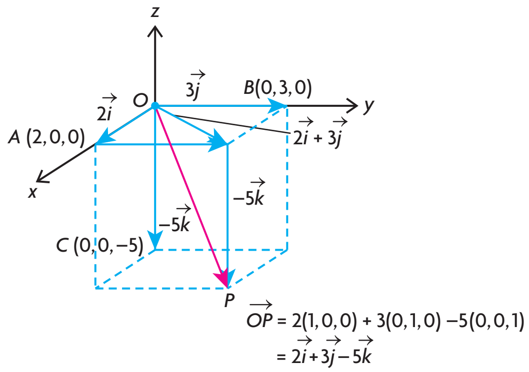
Solution

Writing the given vector as a linear combination

$$\begin{aligned}(2, 3, -5) &= 2(1, 0, 0) + 3(0, 1, 0) - 5(0, 0, 1)\\ &= 2\vec{i} + 3\vec{j} - 5\vec{k}\end{aligned}$$

This is exactly what we would expect based on the work in the previous section.

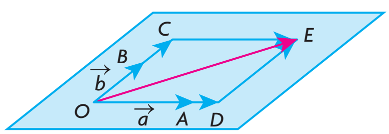
Geometrically, the linear combination of the vectors can be visualized in the following way.



The vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$ are basis vectors in R^3. This has the same meaning for R^3 that it has for R^2. As before, every vector in R^3 can be uniquely written as a linear combination of $\vec{i}$, $\vec{j}$, and $\vec{k}$. Stated simply, $\vec{OP} = (a, b, c)$ can be written as $\vec{OP} = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\vec{i} + b\vec{j} + c\vec{k}$.

The methods for working in R^3 are similar to methods we have already seen at the beginning of this section. A natural question to ask is, "Suppose you are given two noncollinear (non-parallel) vectors in R^3; what do these vectors span?"

In the diagram, the large parallelogram is meant to represent an infinite plane extending in all directions. On this parallelogram are drawn the nonzero vectors $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$ with their tails at the origin, O. When we write the linear combination $m\vec{a} + n\vec{b}$, $\vec{OE}$ is the resulting vector and is the diagonal of the smaller parallelogram, $ODEC$. Since each of the scalars, m and n, can be any real number, an infinite number of vectors, each unique, will be generated from this linear combination. All of these vectors lie on the plane determined by $\vec{a}$ and $\vec{b}$. It should be noted that if we say $\vec{OE}$ lies on the plane, the point E also lies on the plane so that when we say $\vec{OA}$, $\vec{OB}$, and $\vec{OE}$ lie on the plane, this is effectively saying that the points A, B, and E also lie on the plane. When two or more points or vectors lie on the same plane they are said to be **coplanar**.



Spanning Sets

1. Any pair of nonzero, noncollinear vectors will span R^2.
2. Any pair of nonzero, noncollinear vectors will span a plane in R^3.

EXAMPLE 5 — Selecting a linear combination strategy to determine if vectors lie on the same plane

a. Given the two vectors $\vec{a} = (-1, -2, 1)$ and $\vec{b} = (3, -1, 1)$, does the vector $\vec{c} = (-9, -4, 1)$ lie on the plane determined by $\vec{a}$ and $\vec{b}$? Explain.

b. Does the vector $(-9, -5, 1)$ lie in the plane determined by the first two vectors?

Solution

a. This question is asking whether $\vec{c}$ lies in the span of $\vec{a}$ and $\vec{b}$. Stated algebraically, are there values of m and n for which $m(-1, -2, 1) + n(3, -1, 1) = (-9, -4, 1)$?

Multiplying, $(-m, -2m, m) + (3n, -n, n) = (-9, -4, 1)$

or $(-m + 3n, -2m - n, m + n) = (-9, -4, 1)$

Equating components leads to

① $-m + 3n = -9$

② $2m + n = 4$

③ $m + n = 1$

The easiest way of dealing with these equations is to work with equations ① and ③. If we add these equations, m is eliminated and $4n = -8$, so $n = -2$. Substituting into equation ③ gives $m = 3$. We must verify that these values give a consistent answer in the remaining equation. Checking in equation ②: $2(3) + (-2) = 4$.

Since $(-9, -4, 1)$ can be written as a linear combination of $(-1, -2, 1)$ and $(3, -1, 1)$, i.e., $(-9, -4, 1) = 3(-1, -2, 1) - 2(3, -1, 1)$, it lies in the plane determined by the two given vectors.

b. If we carry out calculations identical to those in the solution for part a., the only difference would be that the second equation would now be $2m + n = 5$, and substituting $m = 3$ and $n = -2$ would give $2(3) + (-2) = 4 \neq 5$. Since we have an inconsistent result, this implies that the vector $(-9, -5, 1)$ does not lie on the same plane as $\vec{a}$ and $\vec{b}$.

In general, when we are trying to determine whether a vector lies in the plane determined by two other nonzero, noncollinear vectors, it is sufficient to solve any pair of equations and look for consistency in the third equation. If the result is consistent, the vector lies in the plane, and if not, the vector does not lie in the plane.

IN SUMMARY

Key Ideas

- In R^2, $\overrightarrow{OP} = (a, b) = a(1, 0) + b(0, 1) = a\vec{i} + b\vec{j}$. $\vec{i}$ and $\vec{j}$ *span* R^2. Every vector in R^2 can be written uniquely as a linear combination of these two vectors.
- In R^3, $\overrightarrow{OP} = (a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = a\vec{i} + b\vec{j} + c\vec{k}$. $\vec{i}$, $\vec{j}$, and $\vec{k}$ span R^3 since every vector in R^3 can be written uniquely as a linear combination of these three vectors.

Need to Know

- Any pair of nonzero, noncollinear vectors will span R^2.
- Any pair of nonzero, noncollinear vectors will span a plane in R^3. This means that every vector in the plane can be expressed as a linear combination involving this pair of vectors.

Exercise 6.8

PART A

1. A student writes $2(1, 0) + 4(-1, 0) = (-2, 0)$ and then concludes that $(1, 0)$ and $(-1, 0)$ span R^2. What is wrong with this conclusion?
2. It is claimed that $\{(1, 0, 0), (0, 1, 0), (0, 0, 0)\}$ is a set of vectors spanning R^3. Explain why it is not possible for these vectors to span R^3.
3. Describe the set of vectors spanned by $(0, 1)$. Say why this is the same set as that spanned by $(0, -1)$.
4. In R^3, the vector $\vec{i} = (1, 0, 0)$ spans a set. Describe the set spanned by this vector. Name two other vectors that would also span the same set.
5. It is proposed that the set $\{(0, 0), (1, 0)\}$ could be used to span R^2. Explain why this is not possible.
6. The following is a spanning set for R^2:
   $\{(-1, 2), (2, -4), (-1, 1), (-3, 6), (1, 0)\}$.
   Remove three of the vectors and write down a spanning set that can be used to span R^2.

PART B

7. Simplify each of the following linear combinations and write your answer in component form: $\vec{a} = \vec{i} - 2\vec{j}$, $\vec{b} = \vec{j} - 3\vec{k}$, and $\vec{c} = \vec{i} - 3\vec{j} + 2\vec{k}$
   a. $2(2\vec{a} - 3\vec{b} + \vec{c}) - 4(-\vec{a} + \vec{b} - \vec{c}) + (\vec{a} - \vec{c})$
   b. $\frac{1}{2}(2\vec{a} - 4\vec{b} - 8\vec{c}) - \frac{1}{3}(3\vec{a} - 6\vec{b} + 9\vec{c})$

K 8. Name two sets of vectors that could be used to span the xy-plane in R^3. Show how the vectors $(-1, 2, 0)$ and $(3, 4, 0)$ could each be written as a linear combination of the vectors you have chosen.

C 9. a. The set of vectors $\{(1, 0, 0), (0, 1, 0)\}$ spans a set in R^3. Describe this set.
   b. Write the vector $(-2, 4, 0)$ as a linear combination of these vectors.
   c. Explain why it is not possible to write $(3, 5, 8)$ as a linear combination of these vectors.
   d. If the vector $(1, 1, 0)$ were added to this set, what would these three vectors span in R^3?

10. Solve for a, b, and c in the following equation:
    $2(a, 3, c) + 3(c, 7, c) = (5, b + c, 15)$

11. Write the vector $(-10, -34)$ as a linear combination of the vectors $(-1, 3)$ and $(1, 5)$.

12. In Example 3, it was shown how to find a formula for the coefficients a and b whenever we are given a general vector (x, y).
    a. Repeat this procedure for $\{(2, -1), (-1, 1)\}$.
    b. Write each of the following vectors as a linear combination of the set given in part a.: $(2, -3)$, $(124, -5)$, and $(4, -11)$.

A 13. a. Show that the vectors $(-1, 2, 3)$, $(4, 1, -2)$, and $(-14, -1, 16)$ do not lie on the same plane.
    b. Show that the vectors $(-1, 3, 4)$, $(0, -1, 1)$, and $(-3, 14, 7)$ lie on the same plane, and show how one of the vectors can be written as a linear combination of the other two.

14. Determine the value for x such that the points $A(-1, 3, 4)$, $B(-2, 3, -1)$, and $C(-5, 6, x)$ all lie on a plane that contains the origin.

T 15. The vectors $\vec{a}$ and $\vec{b}$ span R^2. What values of m and n will make the following statement true: $(m - 2)\vec{a} = (n + 3)\vec{b}$? Explain your reasoning.

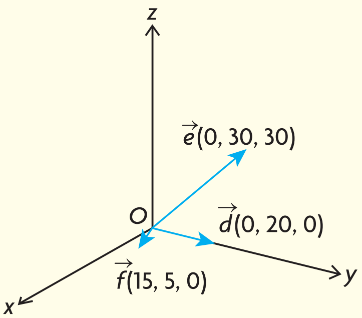
PART C

16. The vectors $(4, 1, 7)$, $(-1, 1, 6)$, and $(p, q, 5)$ are coplanar. Determine three sets of values for p and q for which this is true.

17. The vectors $\vec{a}$ and $\vec{b}$ span R^2. For what values of m is it true that $(m^2 + 2m - 3)\vec{a} + (m^2 + m - 6)\vec{b} = \vec{0}$? Explain your reasoning.

CHAPTER 6: FIGURE SKATING

A figure skater is attempting to perform a quadruple spin jump. He sets up his jump with an initial skate along vector $\vec{d}$. He then plants his foot and applies vertical force at an angle according to vector $\vec{e}$. This causes him to leap into the air and spin. After landing, his momentum will carry him into the wall if he does not apply force to stop himself. So he applies force along vector $\vec{f}$ to slow himself down and change direction.



a. Add vectors $\vec{d}$ and $\vec{e}$ to find the resulting vector ($\vec{a}$) for the skater's jump. The angle between $\vec{d}$ and $\vec{e}$ is 25°. If the xy-plane represents the ice surface, calculate the angle the skater will take with respect to the ice surface on this jump.

b. Discuss why the skater will return to the ground even though the vector that represents his leap carries him in an upward direction.

c. Rewrite the resulting vector $\vec{a}$ without the vertical coordinate. For example, if the vector has components (20, 30, 15), rewrite as (20, 30). Explain the significance of this vector.

d. Add vectors $\vec{a}$ and $\vec{f}$ to find the resulting vector ($\vec{b}$) as the skater applies force to slow himself and change direction. Explain the significance of this vector.