6.1 Periodic Functions and Their Properties

YOU WILL NEED

- graph paper

GOAL

Interpret and describe graphs that repeat at regular intervals.

LEARN ABOUT the Math

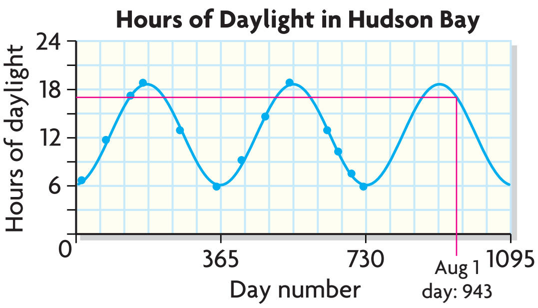
The number of hours of daylight at any particular location changes with the time of year. The table shows the average number of hours of daylight for approximately a two-year period at Hudson Bay, Nunavut. *Note*: Day 15 is January 15 of year 1. Day 74 is March 15 of year 1. Day 411 is February 15 of year 2.

Day	15	74	135	166	258	349	411	470	531	561	623	653	684	714
Hours of Daylight	6.7	11.7	17.2	18.8	12.9	5.9	9.2	14.6	18.8	18.1	12.9	10.2	7.5	5.9

? How many hours of daylight will there be on August 1 of year 3?

EXAMPLE 1 Representing data in a graph to make predictions

Jacob's Solution



I drew a scatter plot with the day as the independent variable and the hours of daylight as the dependent variable. I drew a smooth curve to connect the points.

The data and graph repeat every 365 days. I can tell because the greatest number of hours of daylight occurs on days 166 and 531, and 531 − 166 = 365.

The least number of hours of daylight occurs on days 349 and 714, also 365 days apart.

I used the pattern to extend the graph to year 3. That would be 1095 days.

The number of hours of daylight for day 943 is about 17 h.

I used the graph to estimate the number of hours of daylight for day 943.

Reflecting

A. Why does it make sense to call the graph of the hours of daylight a periodic function?

B. How does the table help you predict the period of the graph?

C. Which points on the graph could you use to determine the range of this function?

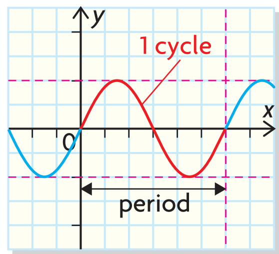
D. How does knowing the period of a periodic graph help you predict future events?

periodic function

a function whose graph repeats at regular intervals; the *y*-values in the table of values show a repetitive pattern when the *x*-values change by the same increment

period

the change in the independent variable (typically *x*) corresponding to one cycle; a cycle of a periodic function is a portion of the graph that repeats

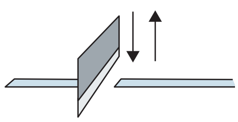


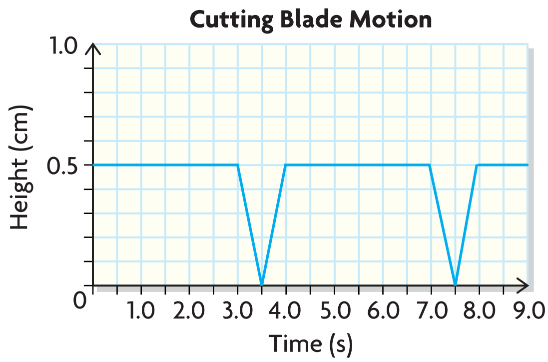
APPLY the Math

EXAMPLE 2 — Interpreting periodic graphs and connecting them to real-world situations

Part A: Analyzing a Cutting Blade's Motion

Tanya's mother works in a factory that produces tape measures. One day, Tanya and her brother Norman accompany their mother to work. During manufacturing, a metal strip is cut into 6 m lengths and is coiled within the tape measure holder. A cutting machine chops the strips into their appropriate lengths. Tanya's mother shows a graph that models the motion of the cutting blade on the machine in terms of time.





How can Norman interpret the graph and relate its characteristics to the manufacturing process?

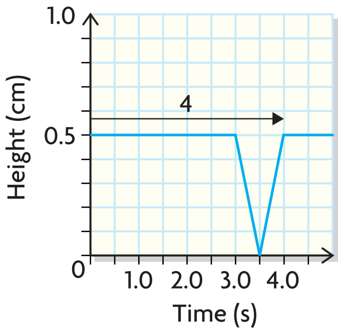
Norman's Solution

This is a periodic function.

It's a periodic function because the graph repeats in exactly the same way at regular intervals.

The period of this function is 4 s.

The cutting blade cuts a new section of metal strip every 4 s because the graph has a pattern that repeats every 4 s.



The maximum height of the blade is 0.5 cm. The minimum height is 0 cm.

The *y*-value is always 0.5 cm or less, so the blade can't be higher than this. When the height is 0 cm, the blade is hitting the cutting surface.

The blade stops for 3 s intervals.

Flat sections, like the ones from 0 to 3.0 and 4.0 to 7.0, must mean that the blade stops for these intervals. The machine is probably pulling the next 6 m section of metal strip through before it's cut.

The blade takes 1 s to go up and down.

Parts of the graph, like from $t = 3.0$ to $t = 3.5$, show that the blade takes 0.5 s to go down.

Other parts of the graph, like from $t = 3.5$ to $t = 4.0$, show that the blade takes 0.5 s to go up.

The blade will strike the cutting surface again at 11.5 s and every 4 s after that.

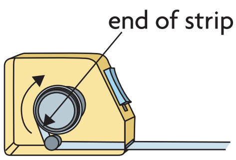
Since the graph repeats every 4 s and the blade hits the surface at 3.5 s and 7.5 s, I can figure out the next time it will hit the surface.

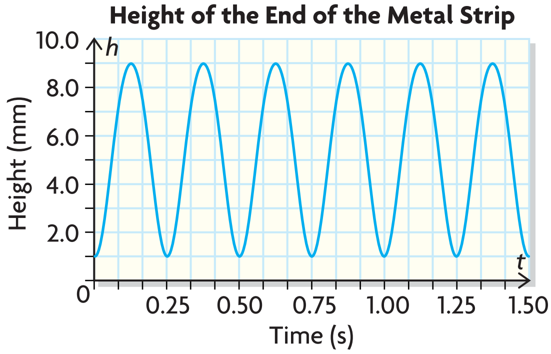
Part B: Analyzing the Motion of the Tape as It Is Spooled

Farther down the assembly line, the metal strip is raised and spooled onto a rotating cylinder contained within the tape measure.

Tanya notices that the height of the end of the metal strip that attaches to the spool goes up and down as the rest of the strip is pulled onto the cylinder.

Tanya's mother shows them a graph that models the height of the end of the strip in terms of time.





How can Tanya interpret the graph and relate its characteristics to the manufacturing process?

Tanya's Solution

This is a periodic function.

It's a periodic function because the graph repeats in exactly the same way at regular intervals. This time the action is smooth.

The range for this function is $\{h \in \mathbf{R} \mid 1 \leq h \leq 9\}$.

The highest the graph goes is 9 mm, and the lowest is 1 mm. The heights are always at or between these two values.

The period of this function is 0.25 s.

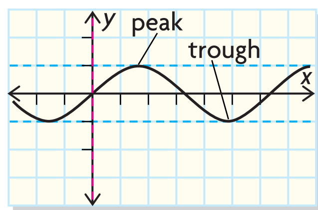
The first **trough** is at $t = 0$. The next trough is at $t = 0.25$. The distance between the two troughs gives the period.

I could also have measured the distance between the first two **peaks** to get that value.

The period represents the time it takes for the rotating cylinder to make one complete revolution.

trough
the minimum point on a graph

peak
the maximum point on a graph



equation of the axis

the equation of the horizontal line halfway between the maximum and the minimum; it is determined by

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

amplitude

half the difference between the maximum and minimum values; it is also the vertical distance from the function's axis to the maximum or minimum value

$$\frac{9 + 1}{2} = 5$$

I calculated the halfway point between the maximum and minimum values of the graph, giving me the **equation of the axis**.

The equation of the axis for this function is $h = 5$.

$$9 - 5 = 4$$

The **amplitude** of a function is the vertical distance from its axis ($h = 5$) to its maximum value (9 mm).

The amplitude of this function is 4 mm.

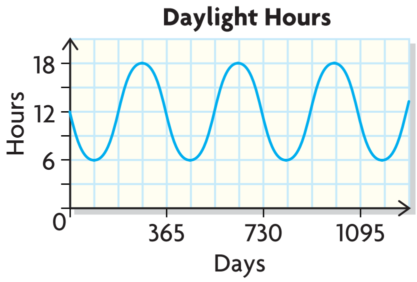
EXAMPLE 3 Identifying a periodic function from its graph

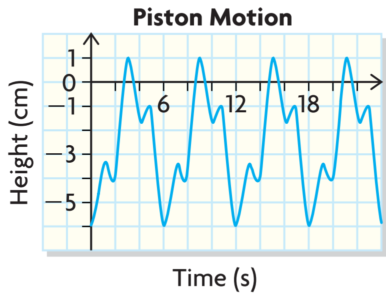
Determine whether the term *periodic* can be used to describe the graph for each situation. If so, state the period, equation of the axis, and amplitude.

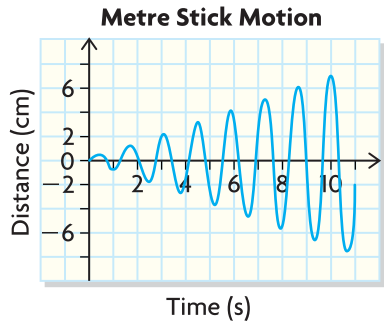
a) the average number of hours of daylight over a three-year period

b) the motion of a piston on an automated assembly line

c) a student is moving a metre stick back and forth with progressively larger movements







Tina's Solution

a) periodic

The graph looks like a series of waves that are the same size and shape. The waves repeat at regular intervals, so the function is periodic.

period = 1 year

The graph repeats its pattern every 365 days. That is the period of the function.

$$\frac{18 + 6}{2} = 12$$

equation of the axis: $h = 12$

To get the equation of the axis, I calculated the halfway point between the maximum and minimum values of the height.

$$18 - 12 = 6$$

amplitude = 6 h

The amplitude is the vertical distance from its axis ($h = 12$) to the maximum value (18 h) or minimum value (6 h).

b) periodic

The shape of the graph repeats over the same interval, so the function is periodic.

period $= 6$ s

The graph repeats every 6 s, so that's the period of the function.

$$\frac{1 + (-6)}{2} = -2.5$$

The equation of the axis is halfway between the maximum of 1 and the minimum of -6.

equation of the axis: $h = -2.5$

$$1 - (-2.5) = 3.5$$

The distance between the maximum and the axis is 3.5.

amplitude $= 3.5$ cm

c) nonperiodic

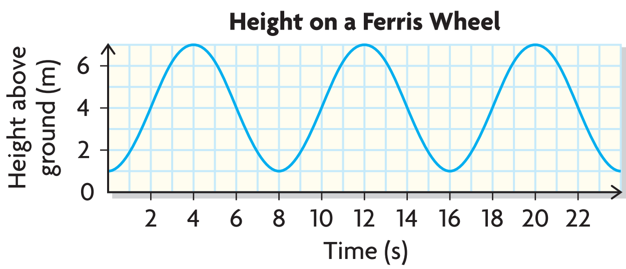
The shape of the graph does not repeat over the same interval, so the function is not periodic.

This means that the function does not have a period, amplitude, or equation of the axis.

In Summary

Key Ideas

- A function that produces a graph that has a regular repeating pattern over a constant interval is called a periodic function. It describes something that happens in a cycle, repeating in the same way over and over.



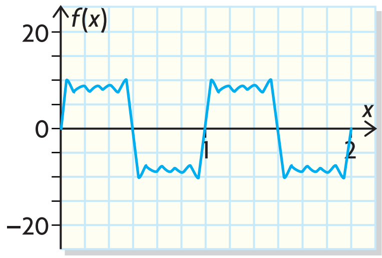
- A function that produces a graph that does not have a regular repeating pattern over a constant interval is called a nonperiodic function.

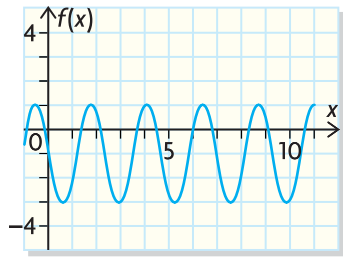
Need to Know

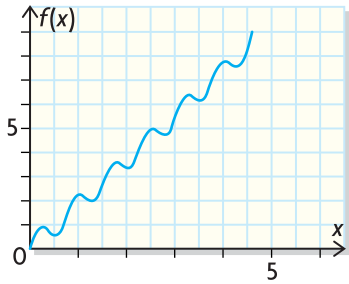
- Extending the graph of a periodic function by using the repeating pattern allows you to make reasonable predictions by extrapolating.
- The graph of a periodic function permits you to figure out the key features of the repeating pattern it represents, such as the period, amplitude, and equation of the axis.

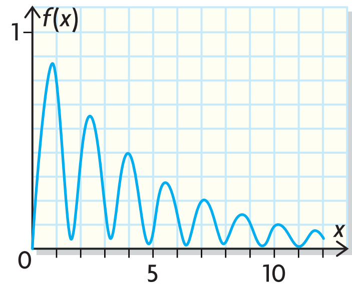
CHECK Your Understanding

1. Which of the following graphs are periodic? Explain why or why not.

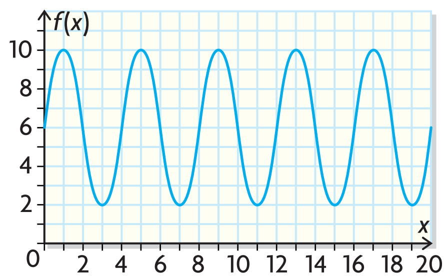
a)


c)


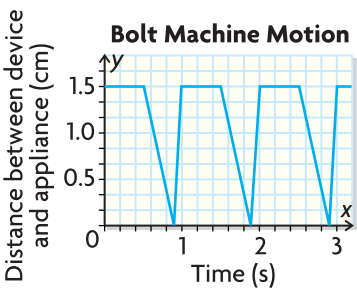
b)


d)


2. Determine the range, period, equation of the axis, and amplitude of the function shown.



Bolt Machine Motion

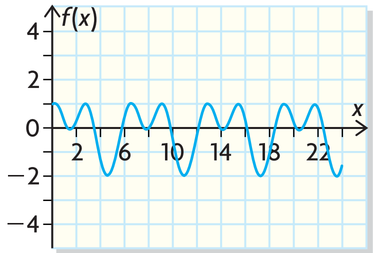


3. The motion of an automated device for attaching bolts to a household appliance on an assembly line can be modelled by the graph shown at the left.
   a) What is the period of one complete cycle?
   b) What is the maximum distance between the device and the appliance?
   c) What is the range of this function?
   d) If the device can run for five complete cycles only before it must be turned off, determine the domain of the function.
   e) Determine the equation of the axis.
   f) Determine the amplitude.
   g) There are several parts to each complete cycle of the graph. Explain what each part could mean in the context of "attaching the bolt."

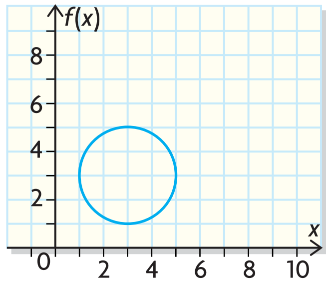
PRACTISING

4. Identify which graphs are periodic. Estimate the period of the functions that
K you identify as periodic.

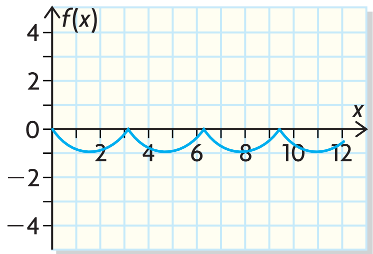
a)



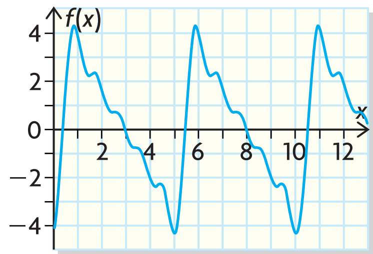
d)



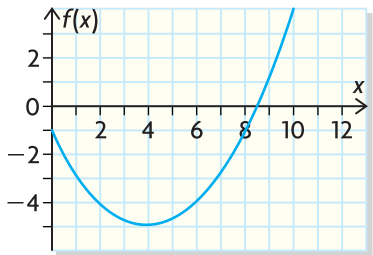
b)



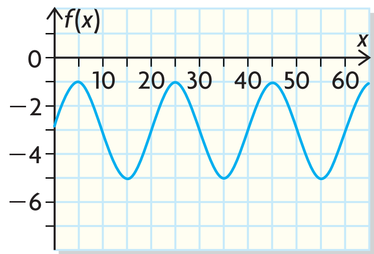
e)



c)



f)



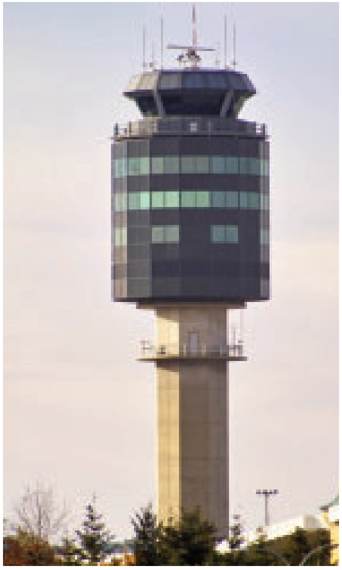
5. Which of the following situations would produce periodic graphs?

a) Sasha is monitoring the height of one of the cutting teeth on a chainsaw. The saw is on the ground, and the chain is spinning.
- independent variable: time
- dependent variable: height of tooth above the ground

b) Alex is doing jumping jacks.
- independent variable: time
- dependent variable: Alex's height above the ground

c) The cost of riding in a taxi varies, depending on how far you travel.
- independent variable: distance travelled
- dependent variable: cost

d) Brittany invested her money in a Guaranteed Investment Certificate whose return was 4% per year.
- independent variable: time
- dependent variable: value of the certificate



e) You throw a basketball to a friend, but she is so far away that the ball bounces on the ground four times.
- independent variable: distance
- dependent variable: bounce height

f) The antenna on a radar tower is rotating and emitting a signal to track incoming planes.
- independent variable: time
- dependent variable: intensity of the signal

6. Which of the tables of values might represent periodic functions? Justify.

a)

x	*y*
−5	9
−4	4
−3	1
−2	0
−1	1
0	4
1	9
2	16

b)

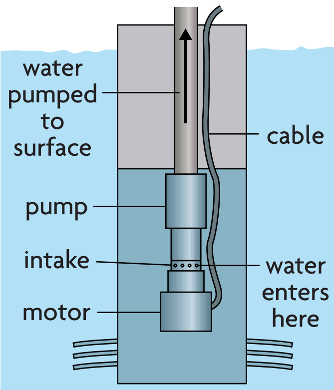
x	*y*
0.7	5
0.9	6
1.1	7
1.3	5
1.5	6
1.7	7
1.9	5
2.1	6

c)

x	*y*
23	−6
26	−6.5
29	−7
32	−7.5
35	−8
38	−8.5
41	−9
44	−9.5

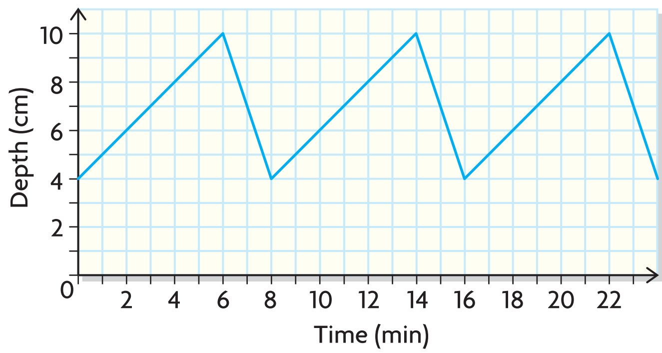
d)

x	*y*
1	5
2	6
4	5
7	6
11	5
16	6
22	5
29	6



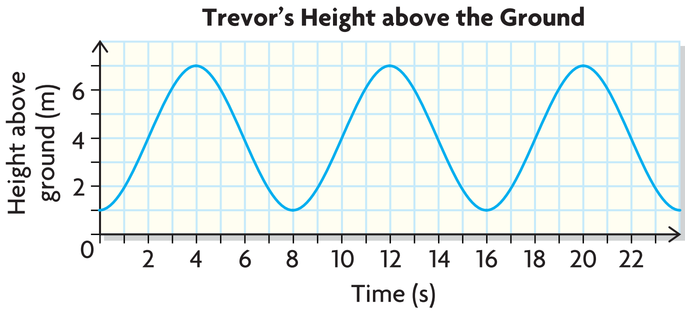
7. Chantelle has a submersible pump in her basement. During a heavy rain, the pump turned off and on to drain water collecting under her house's foundation. The graph models the depth of the water below her basement floor in terms of time. The depth of the water decreased when the pump was on and increased when the pump was off.

Depth of Water under Basement Floor



a) Is the function periodic?
b) At what depth does the pump turn on?
c) How long does the pump remain on?
d) What is the period of the function? Include the units of measure.
e) What is the range of the function?
f) What will the depth of the water be at 3 min?
g) When will the depth of the water be 10 cm?
h) What will the depth of the water be at 62 min?

8. While riding on a Ferris wheel, Trevor's height above the ground in terms of
A time can be represented by the graph shown.



- **a)** What is the period of this function, and what does it represent?
- **b)** What is the equation of the axis?
- **c)** What is the amplitude?
- **d)** What is the range of the function?
- **e)** After 24 s, when will Trevor be at the lowest height again?
- **f)** At what times is Trevor at the top of the wheel?
- **g)** When will his height be 4 m between 24 s and 30 s?

9. Sketch the graph of a periodic function with a period of 20, an amplitude of 6, and whose equation of the axis is $y = 7$.

10. Sketch the graph of a periodic function whose period is 4 and whose range is $\{y \in \mathbf{R} \mid -2 \leq y \leq 5\}$.

11. Maria's bicycle wheel has a diameter of 64 cm. As she rides at a speed of
T 21.6 km/h, she picks up a stone in her tire. Draw a graph that shows the stone's height above the ground as she continues to ride at this speed for 2 s more.

12. A spacecraft is in an elliptical orbit around Earth. The spacecraft's distance above Earth's surface in terms of time is recorded in the table.

Time (min)	0	6	12	18	24	30	36	42	48	54	60	66	72	78
Distance (km)	550	869	1000	869	550	232	100	232	550	869	1000	869	550	232

- **a)** Plot the data, and draw the resulting curve.
- **b)** Is the graph periodic?
- **c)** What is the period of the function, and what does it represent?
- **d)** What is the approximate distance between the spacecraft and Earth at 8 min?
- **e)** At what times is the spacecraft farthest from Earth?
- **f)** If the spacecraft completes only six orbits before descending to Earth, what is the domain of the function?

13. Water is stored in a cylindrical container. Sometimes water is removed from the container, and other times water is added. The table records the depth of the water at specific times.

Time (min)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Depth (cm)	10	20	30	40	40	40	25	10	20	30	40	40	40	25	10	20	30	40	40	40

a) Plot the data, and draw the resulting curve.
b) Is the graph periodic?
c) Determine the period, the equation of the axis, and the amplitude of the function.
d) How fast is the depth of the water increasing when the container is being filled?
e) How fast is the depth of the water decreasing when the container is being drained?
f) Is the container ever empty? Explain.

14. Write a definition of a periodic function. Include an example, and use your **C** definition to explain why it is periodic.

Extension

15. A Calculator-Based Ranger (CBR) is a motion detector that can attach to a graphing calculator. When the CBR is activated, it records the distance an object is in front of the detector in terms of time. The data are stored in the calculator. A scatter plot based on those recorded distances and times can then be drawn using the graphing calculator. Distance is the dependent variable, and time is the independent variable. Denis holds the paddle of the CBR at 60 cm for 3 s and then, within 0.5 s, moves the paddle so that it is 30 cm from the detector. He holds the paddle there for 2 s and then, within 0.5 s, moves the paddle back to the 60 cm location. Denis repeats this process three times.
a) Draw a sketch of the resulting graph. Include a scale.
b) What is the period of the function?
c) Determine the range and domain of the function.

16. Describe the motion of the paddle in front of a CBR that would have produced the graph shown.