Section 2.3—The Product Rule

In this section, we will develop a rule for differentiating the product of two functions, such as $f(x) = (3x^2 - 1)(x^3 + 8)$ and $g(x) = (x - 3)^3(x + 2)^2$, without first expanding the expressions.

You might suspect that the derivative of a product of two functions is simply the product of the separate derivatives. An example shows that this is not so.

EXAMPLE 1 Reasoning about the derivative of a product of two functions

Let $p(x) = f(x)g(x)$, where $f(x) = (x^2 + 2)$ and $g(x) = (x + 5)$.

Show that $p'(x) \neq f'(x)g'(x)$.

Solution

The expression $p(x)$ can be simplified.

$$\begin{aligned} p(x) &= (x^2 + 2)(x + 5) \\ &= x^3 + 5x^2 + 2x + 10 \\ p'(x) &= 3x^2 + 10x + 2 \end{aligned}$$

$f'(x) = 2x$ and $g'(x) = 1$, so $f'(x)g'(x) = (2x)(1) = 2x$.

Since $2x$ is not the derivative of $p(x)$, we have shown that $p'(x) \neq f'(x)g'(x)$.

The correct method for differentiating a product of two functions uses the following rule.

The Product Rule

If $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g(x) + f(x)g'(x)$.

If u and v are functions of x, $\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$.

In words, the product rule says, "the derivative of the product of two functions is equal to the derivative of the first function times the second function plus the first function times the derivative of the second function."

Proof:

$p(x) = f(x)g(x)$, so using the definition of the derivative,

$$p'(x) = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}.$$

To evaluate $p'(x)$, we subtract and add the same term in the numerator.

$$\text{Now, } p'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\left\{\left[\frac{f(x+h)-f(x)}{h}\right]g(x+h) + f(x)\left[\frac{g(x+h)-g(x)}{h}\right]\right\}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]\lim_{h\to 0}g(x+h) + \lim_{h\to 0}f(x)\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= f'(x)g(x) + f(x)g'(x)$$

EXAMPLE 2

Applying the product rule

Differentiate $h(x) = (x^2 - 3x)(x^5 + 2)$ using the product rule.

Solution

$h(x) = (x^2 - 3x)(x^5 + 2)$

Using the product rule, we get

$$h'(x) = \frac{d}{dx}[x^2 - 3x] \cdot (x^5 + 2) + (x^2 - 3x)\frac{d}{dx}[x^5 + 2]$$

$$= (2x - 3)(x^5 + 2) + (x^2 - 3x)(5x^4)$$

$$= 2x^6 - 3x^5 + 4x - 6 + 5x^6 - 15x^5$$

$$= 7x^6 - 18x^5 + 4x - 6$$

We can, of course, differentiate the function after we first expand. The product rule will be essential, however, when we work with products of polynomials such as $f(x) = (x^2 + 9)(x^3 + 5)^4$ or non-polynomial functions such as $f(x) = (x^2 + 9)\sqrt{x^3 + 5}$.

It is not necessary to simplify an expression when you are asked to calculate the derivative at a particular value of x. Because many expressions obtained using differentiation rules are cumbersome, it is easier to substitute, then evaluate the derivative expression.

The next example could be solved by finding the product of the two polynomials and then calculating the derivative of the resulting polynomial at $x = -1$. Instead, we will apply the product rule.

EXAMPLE 3 Selecting an efficient strategy to determine the value of the derivative

Find the value $h'(-1)$ for the function $h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$.

Solution

$h(x) = (5x^3 + 7x^2 + 3)(2x^2 + x + 6)$

Using the product rule, we get

$$\begin{aligned} h'(x) &= (15x^2 + 14x)(2x^2 + x + 6) + (5x^3 + 7x^2 + 3)(4x + 1) \\ h'(-1) &= [15(-1)^2 + 14(-1)][2(-1)^2 + (-1) + 6] \\ &\quad + [5(-1)^3 + 7(-1)^2 + 3][4(-1) + 1] \\ &= (1)(7) + (5)(-3) \\ &= -8 \end{aligned}$$

The following example illustrates the extension of the product rule to more than two functions.

EXAMPLE 4 Connecting the product rule to a more complex function

Find an expression for $p'(x)$ if $p(x) = f(x)g(x)h(x)$.

Solution

We temporarily regard $f(x)g(x)$ as a single function.

$$p(x) = [f(x)g(x)]h(x)$$

By the product rule,

$$p'(x) = [f(x)g(x)]'h(x) + [f(x)g(x)]h'(x)$$

A second application of the product rule yields

$$\begin{aligned} p'(x) &= [f'(x)g(x) + f(x)g'(x)]h(x) + f(x)g(x)h'(x) \\ &= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x) \end{aligned}$$

This expression gives us the **extended product rule** for the derivative of a product of three functions. Its symmetrical form makes it easy to extend to a product of four or more functions.

The Power of a Function Rule for Positive Integers

Suppose that we now wish to differentiate functions such as $y = (x^2 - 3)^4$ or $y = (x^2 + 3x + 5)^6$.

These functions are of the form $y = u^n$, where n is a positive integer and $u = g(x)$ is a function whose derivative we can find. Using the product rule, we can develop an efficient method for differentiating such functions.

For $n = 2$,

$h(x) = [g(x)]^2$

$h(x) = g(x)g(x)$

Using the product rule,

$$\begin{aligned} h'(x) &= g'(x)g(x) + g(x)g'(x) \\ &= 2g'(x)g(x) \end{aligned}$$

Similarly, for $n = 3$, we can use the extended product rule.

$$\begin{aligned} \text{Thus, } h(x) &= [g(x)]^3 \\ &= g(x)g(x)g(x) \\ h'(x) &= g'(x)g(x)g(x) + g(x)g'(x)g(x) + g(x)g(x)g'(x) \\ &= 3[g(x)]^2 g'(x) \end{aligned}$$

These results suggest a generalization of the power rule.

The Power of a Function Rule for Integers

If u is a function of x, and n is an integer, then $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$.

In function notation, if $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1}g'(x)$.

The power of a function rule is a *special case* of the chain rule, which we will discuss later in this chapter. We are now able to differentiate any polynomial, such as $h(x) = (x^2 + 3x + 5)^6$ or $h(x) = (1 - x^2)^4(2x + 6)^3$, without multiplying out the brackets. We can also differentiate rational functions, such as $f(x) = \frac{2x + 5}{3x - 1}$.

EXAMPLE 5 Applying the power of a function rule

For $h(x) = (x^2 + 3x + 5)^6$, find $h'(x)$.

Solution

Here $h(x)$ has the form $h(x) = [g(x)]^6$, where the "inner" function is $g(x) = x^2 + 3x + 5$.

By the power of a function rule, we get $h'(x) = 6(x^2 + 3x + 5)^5(2x + 3)$.

EXAMPLE 6 Selecting a strategy to determine the derivative of a rational function

Differentiate the rational function $f(x) = \frac{2x + 5}{3x - 1}$ by first expressing it as a product and then using the product rule.

Solution

$$f(x) = \frac{2x + 5}{3x - 1}$$

$$= (2x + 5)(3x - 1)^{-1} \quad \text{(Express } f \text{ as a product)}$$

$$f'(x) = \frac{d}{dx}[(2x + 5)](3x - 1)^{-1} + (2x + 5)\frac{d}{dx}[(3x - 1)^{-1}] \quad \text{(Product rule)}$$

$$= 2(3x - 1)^{-1} + (2x + 5)(-1)(3x - 1)^{-2}\frac{d}{dx}(3x - 1)$$

(Power of a function rule)

$$= 2(3x - 1)^{-1} - 3(2x + 5)(3x - 1)^{-2}$$

$$= \frac{2}{(3x - 1)} - \frac{3(2x + 5)}{(3x - 1)^2} \quad \text{(Simplify)}$$

$$= \frac{2(3x - 1)}{(3x - 1)^2} - \frac{6x + 15}{(3x - 1)^2}$$

$$= \frac{6x - 2 - 6x - 15}{(3x - 1)^2}$$

$$= \frac{-17}{(3x - 1)^2}$$

EXAMPLE 7 Using the derivative to solve a problem

The position s, in centimetres, of an object moving in a straight line is given by $s = t(6 - 3t)^4$, $t \geq 0$, where t is the time in seconds. Determine the object's velocity at $t = 2$.

Solution

The velocity of the object at any time $t \geq 0$ is $v = \frac{ds}{dt}$.

$$v = \frac{d}{dt}[t(6 - 3t)^4]$$

$$= (1)(6 - 3t)^4 + (t)\frac{d}{dt}[(6 - 3t)^4] \quad \text{(Product rule)}$$

$$= (6 - 3t)^4 + (t)[4(6 - 3t)^3(-3)] \quad \text{(Power of a function rule)}$$

At $t = 2$, $v = 0 + (2)[4(0)(-3)]$

$$= 0$$

We conclude that the object is at rest at $t = 2$ s.

IN SUMMARY

Key Ideas

- The derivative of a product of differentiable functions is not the product of their derivatives.
- The **product rule** for differentiation:
  If $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$.
- The **power of a function rule** for integers:
  If $f(x) = [g(x)]^n$, then $f'(x) = n[g(x)]^{n-1}g'(x)$.

Need to Know

- In some cases, it is easier to expand and simplify the product before differentiating, rather than using the product rule.
  $$\begin{aligned} \text{If } f(x) &= 3x^4(5x^3 - 7) \\ &= 15x^7 - 21x^4 \\ f'(x) &= 105x^6 - 84x^3 \end{aligned}$$
- If the derivative is needed at a particular value of the independent variable, it is not necessary to simplify before substituting.

Exercise 2.3

PART A

1. Use the product rule to differentiate each function. Simplify your answers.
   a. $h(x) = x(x - 4)$
   b. $h(x) = x^2(2x - 1)$
   c. $h(x) = (3x + 2)(2x - 7)$
   d. $h(x) = (5x^7 + 1)(x^2 - 2x)$
   e. $s(t) = (t^2 + 1)(3 - 2t^2)$
   f. $f(x) = \dfrac{x - 3}{x + 3}$

K 2. Use the product rule and the power of a function rule to differentiate the following functions. Do not simplify.
   a. $y = (5x + 1)^3(x - 4)$
   b. $y = (3x^2 + 4)(3 + x^3)^5$
   c. $y = (1 - x^2)^4(2x + 6)^3$
   d. $y = (x^2 - 9)^4(2x - 1)^3$

3. When is it not appropriate to use the product rule? Give examples.

4. Let $F(x) = [b(x)][c(x)]$. Express $F'(x)$ in terms of $b(x)$ and $c(x)$.

PART B

5. Determine the value of $\frac{dy}{dx}$ for the given value of x.

   a. $y = (2 + 7x)(x - 3)$, $x = 2$

   b. $y = (1 - 2x)(1 + 2x)$, $x = \frac{1}{2}$

   c. $y = (3 - 2x - x^2)(x^2 + x - 2)$, $x = -2$

   d. $y = x^3(3x + 7)^2$, $x = -2$

   e. $y = (2x + 1)^5(3x + 2)^4$, $x = -1$

   f. $y = x(5x - 2)(5x + 2)$, $x = 3$

6. Determine the equation of the tangent to the curve $y = (x^3 - 5x + 2)(3x^2 - 2x)$ at the point $(1, -2)$.

7. Determine the point(s) where the tangent to the curve is horizontal.

   a. $y = 2(x - 29)(x + 1)$

   b. $y = (x^2 + 2x + 1)(x^2 + 2x + 1)$

8. Use the extended product rule to differentiate the following functions. Do not simplify.

   a. $y = (x + 1)^3(x + 4)(x - 3)^2$

   b. $y = x^2(3x^2 + 4)^2(3 - x^3)^4$

A

9. A 75 L gas tank has a leak. After t hours, the remaining volume, V, in litres is $V(t) = 75\left(1 - \frac{t}{24}\right)^2$, $0 \le t \le 24$. Use the product rule to determine how quickly the gas is leaking from the tank when the tank is 60% full of gas.

C

10. Determine the slope of the tangent to $h(x) = 2x(x + 1)^3(x^2 + 2x + 1)^2$ at $x = -2$. Explain how to find the equation of the normal at $x = -2$.

PART C

T

11. a. Determine an expression for $f'(x)$ if $f(x) = g_1(x)g_2(x)g_3(x) \ldots g_{n-1}(x)g_n(x)$.

    b. If $f(x) = (1 + x)(1 + 2x)(1 + 3x) \ldots (1 + nx)$, find $f'(0)$.

12. Determine a quadratic function $f(x) = ax^2 + bx + c$ if its graph passes through the point $(2, 19)$ and it has a horizontal tangent at $(-1, -8)$.

13. Sketch the graph of $f(x) = |x^2 - 1|$.

    a. For what values of x is f not differentiable?

    b. Find a formula for f', and sketch the graph of f'.

    c. Find $f'(x)$ at $x = -2$, 0, and 3.

14. Show that the line $4x - y + 11 = 0$ is tangent to the curve $y = \frac{16}{x^2} - 1$.