Section 3.1—Higher-Order Derivatives, Velocity, and Acceleration

Derivatives arise in the study of motion. The velocity of a car is the rate of change of displacement at a specific point in time. We have already developed the rules of differentiation and learned how to interpret the derivative at a point on a curve. We can now extend the applications of differentiation to higher-order derivatives. This will allow us to discuss the applications of the first and second derivatives to rates of change as an object moves in a straight line, either vertically or horizontally, such as a space shuttle taking off into space or a car moving along a straight section of road.

Higher-Order Derivatives

The function $y = f(x)$ has a first derivative $y = f'(x)$. The **second derivative** of $y = f(x)$ is the derivative of $y = f'(x)$.

The derivative of $f(x) = 10x^4$ with respect to x is $f'(x) = 40x^3$. If we differentiate $f'(x) = 40x^3$, we obtain $f''(x) = 120x^2$. This new function is called the second derivative of $f(x) = 10x^4$, and is denoted $f''(x)$.

For $y = 2x^3 - 5x^2$, the first derivative is $\frac{dy}{dx} = 6x^2 - 10x$ and the second derivative is $\frac{d^2y}{dx^2} = 12x - 10$.

Note the appearance of the superscripts in the second derivative. The reason for this choice of notation is that the second derivative is the derivative of the first derivative. That is, we write $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

Other notations that are used to represent first and second derivatives of $y = f(x)$ are $\frac{dy}{dx} = f'(x) = y'$ and $\frac{d^2y}{dx^2} = f''(x) = y''$.

EXAMPLE 1 **Selecting a strategy to determine the second derivative of a rational function**

Determine the second derivative of $f(x) = \frac{x}{1 + x}$ when $x = 1$.

Solution

Write $f(x) = \frac{x}{1 + x}$ as a product, and differentiate.

$$f(x) = x(x + 1)^{-1}$$

$$f'(x) = (1)(x + 1)^{-1} + (x)(-1)(x + 1)^{-2}(1)$$

(Product and power of a function rule)

$$= \frac{1}{x + 1} - \frac{x}{(x + 1)^2}$$

$$= \frac{1(x+1)}{(x+1)^2} - \frac{x}{(x+1)^2} \quad \text{(Simplify)}$$

$$= \frac{x+1-x}{(x+1)^2}$$

$$= \frac{1}{(1+x)^2} \quad \text{(Rewrite as a power function)}$$

$$= (1+x)^{-2}$$

Differentiating again to determine the second derivative,

$$f''(x) = -2(1+x)^{-3}(1) \quad \text{(Power of a function rule)}$$

$$= \frac{-2}{(1+x)^3}$$

When $x = 1$, $f''(1) = \frac{-2}{(1+1)^3}$ (Evaluate)

$$= \frac{-2}{8}$$

$$= -\frac{1}{4}$$

Velocity and Acceleration—Motion on a Straight Line

One reason for introducing the derivative is the need to calculate rates of change. Consider the motion of an object along a straight line. Examples are a car moving along a straight section of road, a ball dropped from the top of a building, and a rocket in the early stages of flight.

When studying motion along a line, we assume that the object is moving along a number line, which gives us an origin of reference, as well as positive and negative directions. The position of the object on the line relative to the origin is a function of time, t, and is commonly denoted by $s(t)$.

The rate of change of $s(t)$ with respect to time is the object's **velocity**, $v(t)$, and the rate of change of the velocity with respect to time is its **acceleration**, $a(t)$. The absolute value of the velocity is called **speed**.

Motion on a Straight Line

An object that moves along a straight line with its position determined by a function of time, $s(t)$, has a velocity of $\boldsymbol{v(t) = s'(t)}$ and an acceleration of $\boldsymbol{a(t) = v'(t) = s''(t)}$ at time t.
In Leibniz notation,
$$\boldsymbol{v = \frac{ds}{dt}} \text{ and } \boldsymbol{a = \frac{dv}{dt} = \frac{d^2s}{dt^2}}.$$
The speed of the object is $|\boldsymbol{v(t)}|$.

The units of velocity are displacement divided by time. The most common units are m/s.

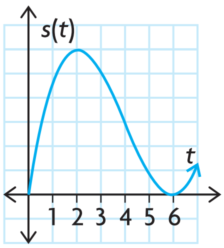
The units of acceleration are displacement divided by $(\text{time})^2$. The most common units are metres per second per second, or metres per second squared, or m/s^2.

Since we are assuming that the motion is along the number line, it follows that when the object is moving to the right at time t, $v(t) > 0$ and when the object is moving to the left at time t, $v(t) < 0$. If $v(t) = 0$, the object is stationary at time t.

The object is accelerating when $a(t)$ and $v(t)$ are both positive or both negative. That is, the product of $a(t)$ and $v(t)$ is positive.

The object is decelerating when $a(t)$ is positive and $v(t)$ is negative, *or* when $a(t)$ is negative and $v(t)$ is positive. This happens when the product of $a(t)$ and $v(t)$ is negative.

EXAMPLE 2 Reasoning about the motion of an object along a straight line



An object is moving along a straight line. Its position, $s(t)$, to the right of a fixed point is given by the graph shown. When is the object moving to the right, when is it moving to the left, and when is it at rest?

Solution

The object is moving to the right whenever $s(t)$ is increasing, or $v(t) > 0$.

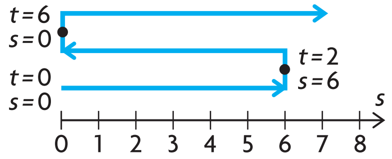
From the graph, $s(t)$ is increasing for $0 < t < 2$ and for $t > 6$.

For $2 < t < 6$, the value of $s(t)$ is decreasing, or $v(t) < 0$, so the object is moving to the left.

At $t = 2$, the direction of motion of the object changes from right to left, $v(t) = 0$, so the object is stationary at $t = 2$.

At $t = 6$, the direction of motion of the object changes from left to right, $v(t) = 0$, so the object is stationary at $t = 6$.

The motion of the object can be illustrated by the following position diagram.



EXAMPLE 3 Connecting motion to displacement, velocity, and acceleration

The position of an object moving on a line is given by $s(t) = 6t^2 - t^3$, $t \geq 0$, where s is in metres and t is in seconds.

a. Determine the velocity and acceleration of the object at $t = 2$.
b. At what time(s) is the object at rest?
c. In which direction is the object moving at $t = 5$?
d. When is the object moving in a positive direction?
e. When does the object return to its initial position?

Solution

a. The velocity at time t is $v(t) = s'(t) = 12t - 3t^2$.

At $t = 2$, $v(2) = 12(2) - 3(2)^2 = 12$.

The acceleration at time t is $a(t) = v'(t) = s''(t) = 12 - 6t$.

At $t = 2$, $a(2) = 12 - 6(2) = 0$.

At $t = 2$, the velocity is 12 m/s and the acceleration is 0 m/s^2.

We note that at $t = 2$, the object is moving at a constant velocity, since the acceleration is 0 m/s^2. The object is neither speeding up nor slowing down.

b. The object is at rest when the velocity is 0—that is, when $v(t) = 0$.

$$12t - 3t^2 = 0$$
$$3t(4 - t) = 0$$
$$t = 0 \text{ or } t = 4$$

The object is at rest at $t = 0$ s and at $t = 4$ s.

c. To determine the direction of motion, we use the velocity at time $t = 5$.

$$\begin{aligned} v(5) &= 12(5) - 3(5)^2 \\ &= -15 \end{aligned}$$

The object is moving in a negative direction at $t = 5$.

d. The object moves in a positive direction when $v(t) > 0$.

$12t - 3t^2 > 0$ (Divide by -3)

$t^2 - 4t < 0$ (Factor)

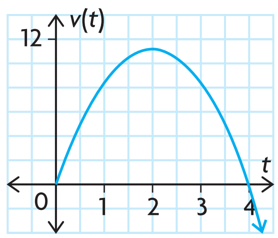
$t(t - 4) < 0$

There are two cases to consider since a product is negative when the first factor is positive and the second is negative, and vice versa.

Case 1	Case 2
$t > 0$ and $t - 4 < 0$	$t < 0$ and $t - 4 > 0$
so $t > 0$ and $t < 4$	so $t < 0$ and $t > 4$
$0 < t < 4$	no solution

Therefore, $0 < t < 4$.

The graph of the velocity function is a parabola opening downward, as shown.



From the graph and the algebraic solution above, we conclude that $v(t) > 0$ for $0 < t < 4$.

The object is moving to the right during the interval $0 < t < 4$.

e. At $t = 0$, $s(0) = 0$. Therefore, the object's initial position is at 0.

To find other times when the object is at this point, we solve $s(t) = 0$.

$6t^2 - t^3 = 0$ (Factor)

$t^2(6 - t) = 0$ (Solve)

$t = 0$ or $t = 6$

The object returns to its initial position after 6 s.

EXAMPLE 4 **Analyzing motion along a horizontal line**

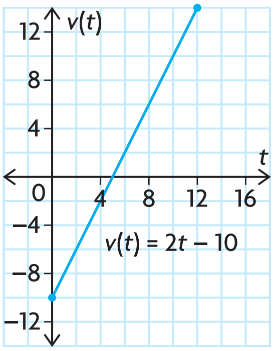
Discuss the motion of an object moving on a horizontal line if its position is given by $s(t) = t^2 - 10t$, $0 \le t \le 12$, where s is in metres and t is in seconds. Include the initial velocity, final velocity, and any acceleration in your discussion.

Solution

The initial position of the object occurs at time $t = 0$. Since $s(0) = 0$, the object starts at the origin.

The velocity at time t is $v(t) = s'(t) = 2t - 10 = 2(t - 5)$.

The object is at rest when $v(t) = 0$.



$$2(t - 5) = 0$$
$$t = 5$$

So the object is at rest after $t = 5$ s.

$v(t) > 0$ for $5 < t \le 12$, therefore the object is moving to the right during this time interval.

$v(t) < 0$ for $0 \le t < 5$, therefore the object is moving to the left during this time interval.

The initial velocity is $v(0) = -10$. So initially, the object is moving 10 m/s to the left.

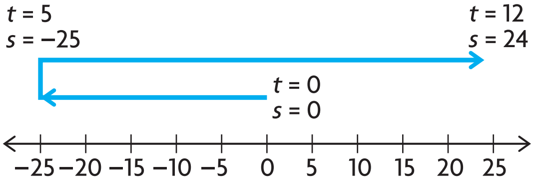
At $t = 12$, $v(12) = 2(12) - 10 = 14$.

So the final velocity is 14 m/s to the right. The velocity graph is shown.

The acceleration at time t is $a(t) = v'(t) = s''(t) = 2$. The acceleration is always 2 m/s^2. This means that the object is constantly increasing its velocity at a rate of 2 metres per second per second.

In conclusion, the object moves to the left for $0 \le t < 5$ and to the right for $5 < t \le 12$. The initial velocity is -10 m/s and the final velocity is 14 m/s.

To draw a diagram of the motion, determine the object's position at $t = 5$ and $t = 12$. (The actual path of the object is back and forth on a line.)



$s(5) = 5^2 - 10(5) = -25$

$s(12) = 12^2 - 10(12) = 24$

EXAMPLE 5 Analyzing motion under gravity near the surface of Earth

A baseball is hit vertically upward. The position function $s(t)$, in metres, of the ball above the ground is $s(t) = -5t^2 + 30t + 1$, where t is in seconds.

a. Determine the maximum height reached by the ball.

b. Determine the velocity of the ball when it is caught 1 m above the ground.

Solution

a. The maximum height occurs when the velocity of the ball is zero—that is, when the slope of the tangent to the graph is zero.

The velocity function is $v(t) = s'(t) = -10t + 30$.

Solving $v(t) = 0$, we obtain $t = 3$. This is the moment when the ball changes direction from up to down.

$$\begin{aligned} s(3) &= -5(3)^2 + 30(3) + 1 \\ &= 46 \end{aligned}$$

Therefore, the maximum height reached by the ball is 46 m.

b. When the ball is caught, $s(t) = 1$. To find the time at which this occurs, solve

$$1 = -5t^2 + 30t + 1$$

$$0 = -5t(t - 6)$$

$$t = 0 \text{ or } t = 6$$

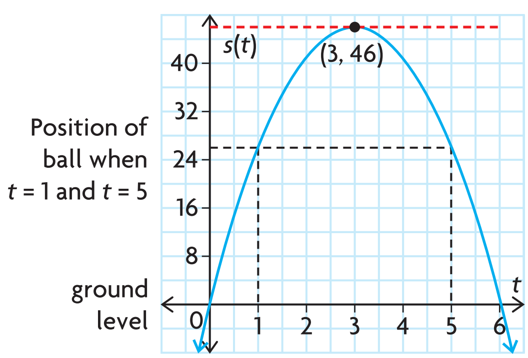
Since $t = 0$ is the time at which the ball leaves the bat, the time at which the ball is caught is $t = 6$.

The velocity of the ball when it is caught is $v(6) = -10(6) + 30 = -30$ m/s.

This negative value is reasonable, since the ball is falling (moving in a negative direction) when it is caught.

Note, however, that the graph of $s(t)$ does not represent the path of the ball. We think of the ball as moving in a straight line along a vertical s-axis, with the direction of motion reversing when $s = 46$.

To see this, note that the ball is at the same height at time $t = 1$, when $s(1) = 26$, and at time $t = 5$, when $s(5) = 26$.



IN SUMMARY

Key Ideas

- The derivative of the derivative function is called the second derivative.
- If the position of an object, $s(t)$, is a function of time, t, then the first derivative of this function represents the velocity of the object at time t.
  $v(t) = s'(t) = \frac{ds}{dt}$
- Acceleration, $a(t)$, is the instantaneous rate of change of velocity with respect to time. Acceleration is the first derivative of the velocity function and the second derivative of the position function.
  $a(t) = v'(t) = s''(t)$, or $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

Need to Know

- Negative velocity, $v(t) < 0$ or $s'(t) < 0$, indicates that an object is moving in a negative direction (left or down) at time t.
- Positive velocity, $v(t) > 0$ or $s'(t) > 0$, indicates that an object is moving in a positive direction (right or up) at time t.
- Zero velocity, $v(t) = 0$ or $s'(t) = 0$, indicates that an object is stationary and that a possible change in direction may occur at time t.
- Notations for the second derivative are $f''(x)$, $\frac{d^2y}{dx^2}$, $\frac{d^2}{dx^2}[f(x)]$, or y'' of a function $y = f(x)''$.
- Negative acceleration, $a(t) < 0$ or $v'(t) < 0$, indicates that the velocity is decreasing.
- Positive acceleration, $a(t) > 0$ or $v'(t) > 0$, indicates that the velocity is increasing.
- Zero acceleration, $a(t) = 0$ or $v'(t) = 0$, indicates that the velocity is constant and the object is neither accelerating nor decelerating.
- An object is accelerating (speeding up) when its velocity and acceleration have the same signs.
- An object is decelerating (slowing down) when its velocity and acceleration have opposite signs.
- The speed of an object is the magnitude of its velocity at time t.
  speed = $|v(t)| = |s'(t)|$.

Exercise 3.1

PART A

C 1. Explain and discuss the difference in velocity at times $t = 1$ and $t = 5$ for $v(t) = 2t - t^2$.

2. Determine the second derivative of each of the following:

a. $y = x^{10} + 3x^6$

b. $f(x) = \sqrt{x}$

c. $y = (1 - x)^2$

d. $h(x) = 3x^4 - 4x^3 - 3x^2 - 5$

e. $y = 4x^{\frac{3}{2}} - x^{-2}$

f. $f(x) = \dfrac{2x}{x + 1}$

g. $y = x^2 + \dfrac{1}{x^2}$

h. $g(x) = \sqrt{3x - 6}$

i. $y = (2x + 4)^3$

j. $h(x) = \sqrt[3]{x^5}$

K 3. Each of the following position functions describes the motion of an object along a straight line. Find the velocity and acceleration as functions of t, $t \geq 0$.

a. $s(t) = 5t^2 - 3t + 15$

b. $s(t) = 2t^3 + 36t - 10$

c. $s(t) = t - 8 + \dfrac{6}{t}$

d. $s(t) = (t - 3)^2$

e. $s(t) = \sqrt{t + 1}$

f. $s(t) = \dfrac{9t}{t + 3}$

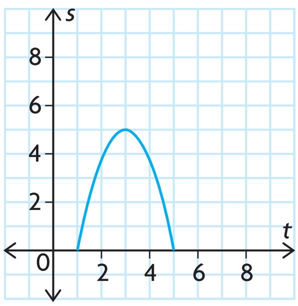
4. Answer the following questions for each position versus time graph below:

i. When is the velocity zero?

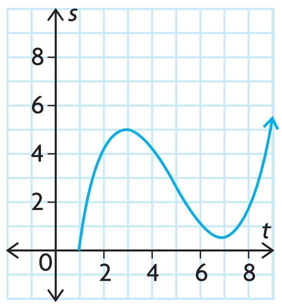
ii. When is the object moving in a positive direction?

iii. When is the object moving in a negative direction?

a.



b.



5. A particle moves along a straight line with the equation of motion $s = \frac{1}{3}t^3 - 2t^2 + 3t,\ t \geq 0.$
   a. Determine the particle's velocity and acceleration at any time t.
   b. When does the motion of the particle change direction?
   c. When does the particle return to its initial position?

PART B

6. Each function describes the position of an object that moves along a straight line. Determine whether the object is moving in a positive or negative direction at time $t = 1$ and at time $t = 4$.
   a. $s(t) = -\frac{1}{3}t^2 + t + 4$ b. $s(t) = t(t - 3)^2$ c. $s(t) = t^3 - 7t^2 + 10t$
7. Starting at $t = 0$, a particle moves along a line so that its position after t seconds is $s(t) = t^2 - 6t + 8$, where s is in metres.
   a. What is its velocity at time t?   b. When is its velocity zero?
8. When an object is launched vertically from ground level with an initial velocity of 40 m/s, its position after t seconds is $s(t) = 40t - 5t^2$ metres above ground level.
   a. When does the object stop rising?   b. What is its maximum height?
9. An object moves in a straight line, and its position, s, in metres after t seconds is $s(t) = 8 - 7t + t^2$.
   a. Determine the velocity when $t = 5$
   b. Determine the acceleration when $t = 5$.

A 10. The position function of a moving object is $s(t) = t^{\frac{5}{2}}(7 - t),\ t \geq 0$, in metres, at time t, in seconds.
   a. Calculate the object's velocity and acceleration at any time t.
   b. After how many seconds does the object stop?
   c. When does the motion of the object change direction?
   d. When is its acceleration positive?
   e. When does the object return to its original position?
11. A ball is thrown upward, and its height, h, in metres above the ground after t seconds is given by $h(t) = -5t^2 + 25t,\ t \geq 0$.
   a. Calculate the ball's initial velocity.
   b. Calculate its maximum height.
   c. When does the ball strike the ground, and what is its velocity at this time?

12. A dragster races down a 400 m strip in 8 s. Its distance, in metres, from the starting line after t seconds is $s(t) = 6t^2 + 2t$.
    a. Determine the dragster's velocity and acceleration as it crosses the finish line.
    b. How fast was it moving 60 m down the strip?
13. For each of the following position functions, discuss the motion of an object moving on a horizontal line, where s is in metres and t is in seconds. Make a graph similar to that in Example 4, showing the motion for $t \geq 0$. Find the velocity and acceleration, and determine the extreme positions (farthest left and right) for $t \geq 0$.
    a. $s(t) = 10 + 6t - t^2$
    b. $s(t) = t^3 - 12t - 9$
14. If the position function of an object is $s(t) = t^5 - 10t^2$, at what time, t, in seconds, will the acceleration be zero? Is the object moving toward or away from the origin at this instant?

T

15. The position–time relationship for a moving object is given by $s(t) = kt^2 + (6k^2 - 10k)t + 2k$, where k is a non-zero constant.
    a. Show that the acceleration is constant.
    b. Find the time at which the velocity is zero, and determine the position of the object when this occurs.

PART C

16. An elevator is designed to start from a resting position without a jerk. It can do this if the acceleration function is continuous.
    a. Show that the acceleration is continuous at $t = 0$ for the following position function
    $$s(t) = \begin{cases} 0, \text{ if } t < 0 \\ \dfrac{t^3}{t^2 + 1}, \text{ if } t \geq 0 \end{cases}$$
    b. What happens to the velocity and acceleration for very large values of t?
17. An object moves so that its velocity, v, is related to its position, s, according to $v = \sqrt{b^2 + 2gs}$, where b and g are constants. Show that the acceleration of the object is constant.
18. Newton's law of motion for a particle of mass m moving in a straight line says that $F = ma$, where F is the force acting on the particle and a is the acceleration of the particle. In relativistic mechanics, this law is replaced by $F = \dfrac{m_0 \frac{d}{dt} v}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$, where m_0 is the mass of the particle measured at rest, v is the velocity of the particle and c is the speed of light. Show that $F = \dfrac{m_0 a}{\left(1 - \left(\frac{v}{c}\right)^2\right)^{\frac{3}{2}}}$.