Chapter 9

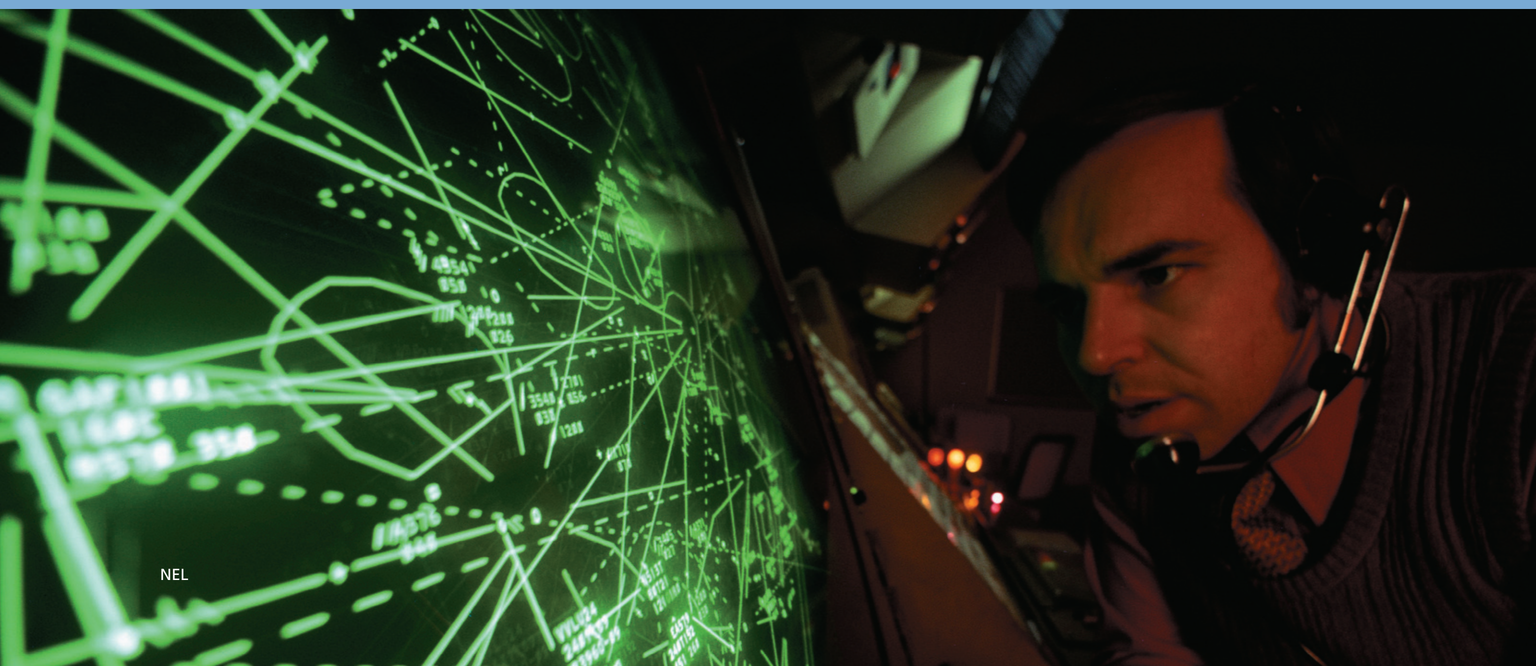
RELATIONSHIPS BETWEEN POINTS, LINES, AND PLANES

In this chapter, we will introduce perhaps the most important idea associated with vectors, the solution of systems of equations. In previous chapters, the solution of systems of equations was introduced in situations dealing with two equations in two unknowns. Geometrically, the solution of two equations in two unknowns is the point of intersection between two lines on the *xy*-plane. In this chapter, we are going to extend these ideas and consider systems of equations in R^3 and interpret their meaning. We will be working with systems of up to three equations in three unknowns, and we will demonstrate techniques for solving these systems.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine the intersection between a line and a plane and between two lines in three-dimensional space, **Section 9.1**
- algebraically solve systems of equations involving up to three equations in three unknowns, **Section 9.2**
- determine the intersection of two or three planes, **Sections 9.3, 9.4**
- determine the distance from a point to a line in two- and three-dimensional space, **Section 9.5**
- determine the distance from a point to a plane, **Section 9.6**
- solve distance problems relating to lines and planes in three-dimensional space and interpret the results geometrically, **Sections 9.5, 9.6**



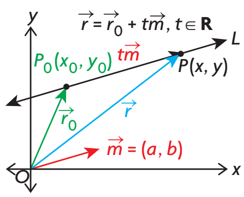
Review of Prerequisite Skills

In this chapter, you will examine how lines can intersect with other lines and planes, and how planes can intersect with other planes. Intersection problems are geometric models of linear systems. Before beginning, you may wish to review some equations of lines and planes.

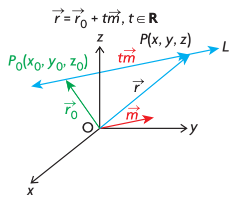
Type of Equation	Lines	Planes
Vector equation	$\vec{r} = \vec{r}_0 + t\vec{m}$	$\vec{r} = \vec{r}_0 + s\vec{a} + t\vec{b}$
Parametric equation	$x = x_0 + ta$ $y = y_0 + tb$ $z = z_0 + tc$	$x = x_0 + sa_1 + tb_1$ $y = y_0 + sa_2 + tb_2$ $z = z_0 + sa_3 + tb_3$
Cartesian equation	$Ax + By + C = 0$	$Ax + By + Cz + D = 0$ in three-dimensional space

In the table above,

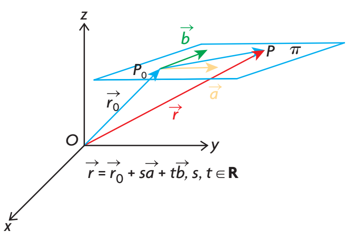
- $\vec{r}_0$ is the position vector whose tail is located at the origin and whose head is located at the point (x_0, y_0) in R^2 and (x_0, y_0, z_0) in R^3
- $\vec{m}$ is a direction vector whose components are (a, b) in R^2 and (a, b, c) in R^3
- $\vec{a}$ and $\vec{b}$ are noncollinear direction vectors whose components are (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively in R^3
- s and t are parameters where $s \in \mathbf{R}$ and $t \in \mathbf{R}$
- (A, B) is a normal to the line defined by $Ax + By + C = 0$ in R^2
- (A, B, C) is a normal to the plane defined by $Ax + By + Cz + D = 0$ in R^3



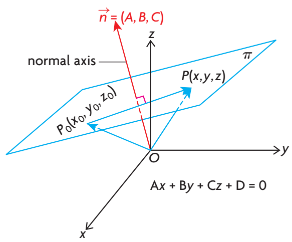
Vector Equation of a Line in R^2



Vector Equation of a Line in R^2



Vector Equation of a Plane in R^3



Scalar Equation of a Plane in R^3

Exercise

1. Determine if the point P_0 is on the given line.

   a. $P_0(2, -5), \vec{r} = (10, -12) + t(8, -7), t \in \mathbf{R}$

   b. $P_0(1, 2), 12x + 5y - 13 = 0$

   c. $P_0(7, -3, 8), \vec{r} = (1, 0, -4) + t(2, -1, 4), t \in \mathbf{R}$

   d. $P_0(1, 0, 5), \vec{r} = (2, 1, -2) + t(4, -1, 2), t \in \mathbf{R}$

2. Determine the vector and parametric equations of the line that passes through each of the following pairs of points:

   a. $P_1(2, 5), P_2(7, 3)$

   b. $P_1(-3, 7), P_2(4, -7)$

   c. $P_1(-1, 0), P_2(-3, -11)$

   d. $P_1(1, 3, 5), P_2(6, -7, 0)$

   e. $P_1(2, 0, -1), P_2(-1, 5, 2)$

   f. $P_1(2, 5, -1), P_2(12, -5, -7)$

3. Determine the Cartesian equation of the plane passing through point P_0 and perpendicular to $\vec{n}$.

   a. $P_0(4, 1, -3), \vec{n} = (2, 6, -1)$

   b. $P_0(-2, 0, 5), \vec{n} = (0, 7, 0)$

   c. $P_0(3, -1, -2), \vec{n} = (4, -3, 0)$

   d. $P_0(0, 0, 0), \vec{n} = (6, -5, 3)$

   e. $P_0(4, 1, 8), \vec{n} = (11, -6, 0)$

   f. $P_0(2, 5, 1), \vec{n} = (1, 1, -1)$

4. Determine the Cartesian equation of the plane that has the vector equation $\vec{r} = (2, 1, 0) + s(1, -1, 3) + t(2, 0, -5), s, t \in \mathbf{R}$.

5. Which of the following lines is parallel to the plane $4x + y - z = 10$? Do any of the lines lie on this plane?

   L_1: $\vec{r} = (3, 0, 2) + t(1, -2, 2), t \in \mathbf{R}$

   L_2: $x = -3t, y = -5 + 2t, z = -10t, t \in \mathbf{R}$

   L_3: $\dfrac{x - 1}{4} = \dfrac{y + 6}{-1} = \dfrac{z}{1}$

6. Determine the Cartesian equations of the planes that contain the following sets of points:

   a. $A(1, 0, -1), B(2, 0, 0), C(6, -1, 5)$

   b. $P(4, 1, -2), Q(6, 4, 0), R(0, 0, -3)$

7. Determine the vector and Cartesian equations of the plane containing $P(1, -4, 3)$ and $Q(2, -1, 6)$ and parallel to the y-axis.

8. Determine the Cartesian equation of the plane that passes through $A(-1, 3, 4)$ and is perpendicular to $2x - y + 3z - 1 = 0$ and $5x + y - 3z + 6 = 0$.

CAREER LINK Investigate

CHAPTER 9: RELATIONSHIPS BETWEEN POINTS, LINES, AND PLANES

Much of the world's reserves of fossil fuels are found in places that are not accessible to water for shipment. Due to the enormous volumes of oil that are currently being extracted from the ground in places such as northern Alberta, Alaska, and Russia, shipment by trucks would be very costly. Instead, pipelines are built to move the fuel to a place where it can be processed or loaded onto a large sea tanker for shipment. The construction of the pipelines is a costly undertaking, but, once completed, pipelines save vast amounts of time, energy, and money.

A team of pipeline construction engineers is needed to design a pipeline. The engineers have to study surveys of the land that the pipeline will cross and choose the best path. Often the least difficult path is above ground, but engineers will choose to have the pipeline go below ground. In plotting the course for the pipeline, vectors can be used to determine if the intended path of the pipeline will cross an obstruction or to determine where two different pipelines will meet.

Case Study—Pipeline Construction Engineer

New pipelines must be a certain distance away from existing pipelines and buildings, depending on the type of product that the pipeline is carrying. To calculate the distance between the two closest points on two pipelines, the lines are treated as skew lines on two different planes. (Skew lines are lines that never intersect because they lie on parallel planes.)

Suppose that an engineer wants to lay a pipeline according to the line L_1: $r = (0, 2, 1) + s(2, -1, 1)$, $s \in \mathbf{R}$. There is an existing pipeline that has a pathway determined by L_2: $r = (1, 0, 1) + t(1, -2, 0)$, $t \in \mathbf{R}$. Determine whether the proposed pathway for the new pipeline is less than 2 units away from the existing pipeline.

DISCUSSION QUESTIONS

1. Construct two parallel planes, π_1 and π_2. The first plane contains L_1 and a second intersecting line that has a direction vector of $\vec{a} = (1, -2, 0)$, the same direction vector as L_2. The second plane contains L_2 and a second intersecting line that has a direction vector of $\vec{b} = (2, -1, 1)$, the same direction vector as L_1.
2. Find the distance between π_1 and π_2.
3. Write the equation of L_1 and L_2 in parametric form.
4. Determine the point on each of the two lines in problem 3 that produces the minimal distance.