3.7 Families of Quadratic Functions

GOAL

Determine the properties of families of quadratic functions.

INVESTIGATE the Math

Equations that define quadratic functions can look quite different, yet their graphs can have similar characteristics.

Group 1	Group 2	Group 3
$f(x) = x^2 - 3x - 10$	$m(x) = -2x^2 + 4x + 1$	$r(x) = -3x^2 + 5x - 2$
$g(x) = -2x^2 + 6x + 20$	$n(x) = 0.5x^2 - 1x + 3.5$	$s(x) = 2x^2 + x - 2$
$h(x) = 4x^2 - 12x - 40$	$p(x) = -6x^2 + 12x - 3$	$t(x) = 7x^2 - 2x - 2$
$k(x) = -0.5x^2 + 1.5x + 5$	$q(x) = 10x^2 - 20x + 13$	$u(x) = -4x^2 - 4x - 2$

? What characteristics do the graphs in each of these groups have in common?

A. Graph each of the functions in Group 1 on a graphing calculator. Use the window settings shown. How are the graphs the same? How are they different?

B. Write each of the functions in Group 1 in factored form. What do you notice?

C. Clear all functions, and then graph each of the functions for Group 2 on a graphing calculator. Use the window settings shown.

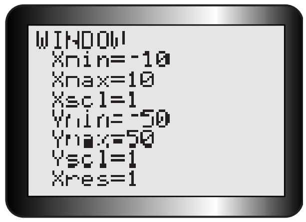
D. How are the graphs the same? How are they different?

E. Write each of the functions in Group 2 in vertex form. What do you notice?

F. Clear all functions, and then graph each of the functions in Group 3 on a graphing calculator. Use the window settings shown. What do these functions have in common?

G. Summarize your findings for each group.

A.



C.

F.

family of parabolas

a group of parabolas that all share a common characteristic

Reflecting

H. Each of the three groups of functions forms a **family of parabolas.** Describe the common characteristics of each of the groups.

I. In each of Groups 1 and 2, what single value was varied to create the family? What transformation is this parameter associated with?

J. What common characteristic appears in all quadratic functions in the same family if the equation is in factored form? vertex form? standard form?

APPLY the Math

EXAMPLE 1 Looking for quadratics that share a vertex

Given the function $f(x) = -3(x + 2)^2 - 1$, determine another quadratic function with the same vertex.

Ian's Solution

$f(x) = -3(x + 2)^2 - 1$

I identified the vertex.

Vertex is $(-2, -1)$.

Family of parabolas is of the form

$f(x) = a(x + 2)^2 - 1$

To get another quadratic function with the same vertex, I needed to change the value of *a* because parabolas with the same vertex are vertically stretched or compressed, but not horizontally or vertically translated.

So another quadratic in the family is $g(x) = 2(x + 2)^2 - 1$.

EXAMPLE 2 Determining a specific equation of a member of the family

Determine the equation of the quadratic function that passes through $(-3, 20)$ if its zeros are 2 and -1.

Preet's Solution

$$\begin{aligned} f(x) &= a(x - 2)[x - (-1)] \\ &= a(x - 2)(x + 1) \\ &= a[x^2 - 2x + x - 2] \\ &= a(x^2 - x - 2) \end{aligned}$$

I wrote the general function of all parabolas that have zeros at 2 and -1, then simplified by expanding.

$$f(x) = a(x^2 - x - 2)$$
$$20 = a[(-3)^2 - (-3) - 2)]$$
$$20 = 10a$$
$$a = 2$$

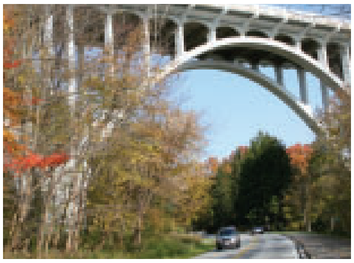
To determine the equation passing through $(-3, 20)$, I had to find the correct value of *a*.

I substituted the point into the equation and solved for *a*.

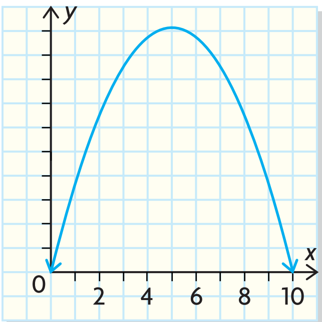
Therefore, $f(x) = 2(x^2 - x - 2)$.

EXAMPLE 3 Solving a problem by applying knowledge of the vertex form of a quadratic

A highway overpass has a shape that can be modelled by the equation of a parabola. If the edge of the highway is the origin and the highway is 10 m wide, what is the equation of the parabola if the height of the overpass 2 m from the edge of the highway is 13 m?



Elizabeth's Solution



I drew a sketch. If the edge of the highway is at the origin, then one of the zeros is 0. If the highway is 10 m wide, then the other zero is at (10, 0).

$$h = ax(x - 10)$$

Since I had the zeros, I wrote the equation in factored form. The equation that would model the overpass would be in the same family, so I needed to find the value of *a*.

$$13 = a(2)(2 - 10)$$
$$13 = -16a$$
$$a = -\frac{13}{16}$$

(2, 13) is a point on the curve, so I substituted those values into the equation and solved for *a*. Once I had the value of *a*, I wrote the equation in factored form.

Therefore, the equation that models the overpass is

$$h = -\frac{13}{16}x(x - 10)$$

EXAMPLE 4 Selecting a strategy to determine the quadratic function from data

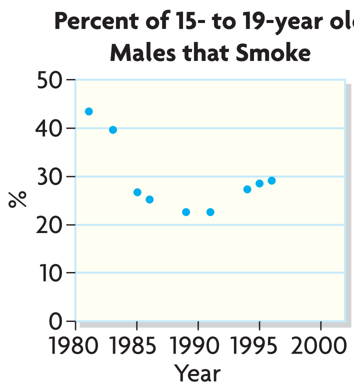
The percent of 15- to 19-year-old males who smoke has been tracked by Health Canada. The data from 1981 to 1996 are given in the table.

Year	1981	1983	1985	1986	1989	1991	1994	1995	1996
Smokers (%)	43.4	39.6	26.7	25.2	22.6	22.6	27.3	28.5	29.1

a) Draw a scatter plot of the data.
b) Draw a curve of good fit.
c) Estimate the location of the vertex.
d) Determine a quadratic function that will model the data.

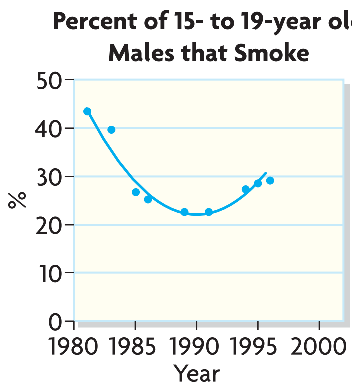
Bryce's Solution

a)



I plotted the points on a graph. They could be represented by a quadratic function.

b)



I drew a curve of good fit by hand so that I could estimate the vertex. Since the values were 22.6 in both 1989 and 1991, I used (1990, 22) as my estimated vertex.

c) The graph models a parabola with the vertex above the x-axis.
Estimated vertex: (1990, 22)

I used vertex form with the vertex I knew. I needed to find the value of a to approximate the data. I chose the point (1995, 28.5) as the point on the curve. I substituted the point into the equation and solved for a.

d)
$$f(x) = a(x - 1990)^2 + 22$$
$$28.5 = a(1995 - 1990)^2 + 22$$
$$28.5 = a(-5)^2 + 22$$
$$28.5 = 25a + 22$$
$$6.5 = 25a$$
$$\frac{65}{25} = a$$
$$a \doteq 0.26$$

Therefore, a model for the data is $f(x) = 0.26(x - 1990)^2 + 22$.

In Summary

Key Ideas

- If the value of a is varied in a quadratic function expressed in vertex form, $f(x) = a(x - h)^2 + k$, a family of parabolas with the same vertex and axis of symmetry is created.
- If the value of a is varied in a quadratic function in factored form, $f(x) = a(x - r)(x - s)$, a family of parabolas with the same x-intercepts and axis of symmetry is created.
- If the values of a and b are varied in a quadratic function expressed in standard form, $f(x) = ax^2 + bx + c$, a family of parabolas with the same y-intercept is created.

Need to Know

- The algebraic model of a quadratic function can be determined algebraically.
  - If the zeros are known, write in factored form with a unknown, substitute another known point, and solve for a.
  - If the vertex is known, write in vertex form with a unknown, substitute a known point, and solve for a.

CHECK *Your Understanding*

1. What characteristics will two parabolas in the family $f(x) = a(x - 3)(x + 4)$ share?
2. How are the parabolas $f(x) = -3(x - 2)^2 - 4$ and $g(x) = 6(x - 2)^2 - 4$ the same? How are they different?
3. What point do the parabolas $f(x) = -2x^2 + 3x - 7$ and $g(x) = 5x^2 + 3x - 7$ have in common?

PRACTISING

4. Determine the equation of the parabola with x-intercepts
   a) -4 and 3, and that passes through (2, 7)
   b) 0 and 8, and that passes through $(-3, -6)$
   c) $\sqrt{7}$ and $-\sqrt{7}$, and that passes through $(-5, 3)$
   d) $1 - \sqrt{2}$ and $1 + \sqrt{2}$, and that passes through (2, 4)
5. Determine the equation of the parabola with vertex
   a) $(-2, 5)$ and that passes through $(4, -8)$
   b) (1, 6) and that passes through $(0, -7)$
   c) $(4, -5)$ and that passes through $(-1, -3)$
   d) (4, 0) and that passes through (11, 8)
6. Determine the equation of the quadratic function $f(x) = ax^2 - 6x - 7$ if $f(2) = 3$.
7. **a)** Sketch the graph of $f(x) = (x - 2)(x + 6)$.
   b) Use your graph to sketch the graph of $g(x) = -2(x - 2)(x + 6)$.
   c) Sketch the graph of $h(x) = 3(x - 2)(x + 6)$.
8. **K** Determine the equation of the parabola with x-intercepts ± 4 and passing through (3, 6).
9. Determine the equation of the quadratic function that passes through $(-4, 5)$ if its zeros are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.
10. **T** A tunnel with a parabolic arch is 12 m wide. If the height of the arch 4 m from the left edge is 6 m, can a truck that is 5 m tall and 3.5 m wide pass through the tunnel? Justify your decision.
11. **A** A projectile is launched off the top of a platform. The table gives the height of the projectile at different times during its flight.

Time (s)	0	1	2	3	4	5	6
Height (m)	11	36	51	56	51	36	11

   a) Draw a scatter plot of the data.
   b) Draw a curve of good fit.
   c) Determine the equation that will model this set of data.

12. Jason tossed a ball over a motion detector and it recorded these data.

Time (s)	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Height above Ground (m)	0	2.1875	3.75	4.6875	5	4.6875	3.75	2.1875	0

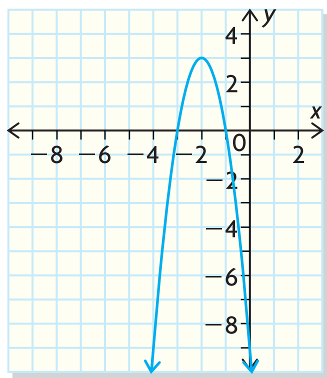
a) Draw a scatter plot of the data.
b) Draw a curve of good fit.
c) Determine an algebraic expression that models the data. Express the function in standard form.

13. Students at an agricultural school collected data showing the effect of different annual amounts of water (rainfall plus irrigation), x, in hectare-metres $(\text{ha} \cdot \text{m})$, on the yield of broccoli, y, in hundreds of kilograms per hectare (100 kg/ha).

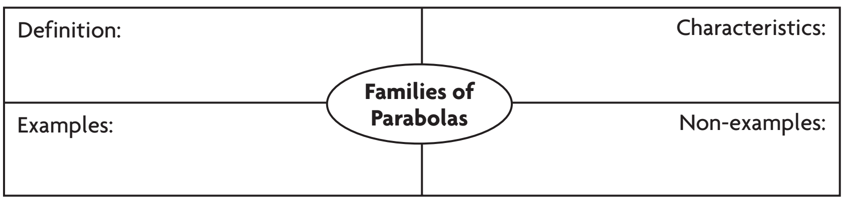
Amount of Water, x (ha • m)	0.30	0.45	0.60	0.75	0.90	1.05	1.20	1.35	1.50
Yield, y (100 kg/ha)	35	104	198	287	348	401	427	442	418

a) Draw a scatter plot and a curve of good fit.
b) Estimate the location of the vertex.
c) Determine an algebraic model for the data.

14. What is the equation of the parabola at the right if the point $(-4, -9)$ is on the graph?



15. C Complete the chart shown. Include what you know about families of parabolas in standard, vertex, and factored form.



Extending

16. A parabolic bridge is 40 m wide. Determine the height of the bridge 12 m in from the outside edge, if the height 5 m in from the outside edge is 8 m.

17. A family of cubic equations with zeros -3, 1, and 5 can be represented by the function $f(x) = a(x + 3)(x - 1)(x - 5)$. Which equation describes the cubic in the family that passes through the point (3, 6)?