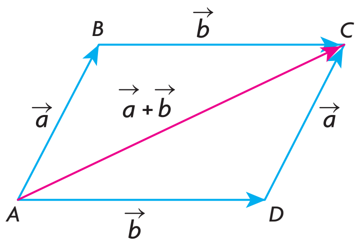
Section 6.4—Properties of Vectors

In previous sections, we developed procedures for adding and subtracting vectors and for multiplying a vector by a scalar. In carrying out these computations, certain assumptions were made about how to combine vectors without these rules being made explicit. Although these rules will seem apparent, they are important for understanding the basic structure underlying vectors, and for their use in computation. Initially, three specific rules for dealing with vectors will be discussed, and we will show that these rules are similar to those used in dealing with numbers and basic algebra. Later, we demonstrate an additional three rules.

Properties of Vector Addition

1. Commutative Property of Addition: When we are dealing with numbers, the order in which they are added does not affect the final answer. For example, if we wish to add 2 and 3, the answer is the same if it is written as $2 + 3$ or as $3 + 2$. In either case, the answer is 5. This property of being able to add numbers, in any chosen order, is called the commutative property of addition for real numbers. This property also works for algebra, because algebraic expressions are themselves numerical in nature. We make this assumption when simplifying in the following example: $2x + 3y + 3x = 2x + 3x + 3y = 5x + 3y$. Being able to switch the order like this allows us to carry out addition without concern for the order of the terms being added.

   This property that we have identified also holds for vectors, as can be seen in the following diagram:

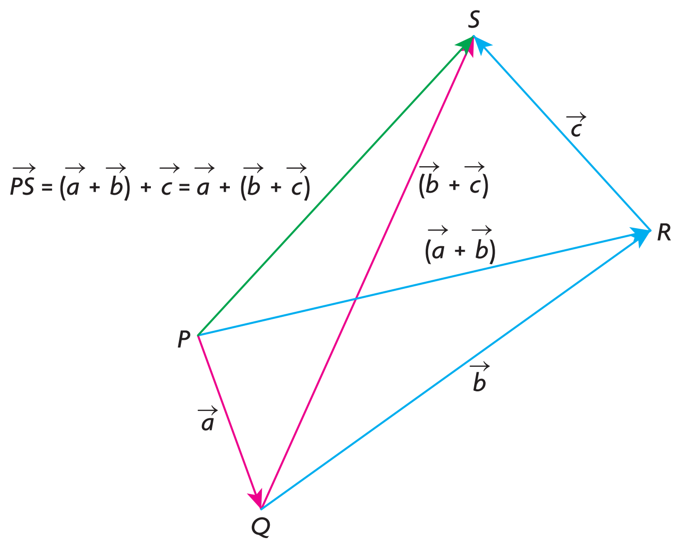
   From triangle ABC, using the vector addition rules,
   $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \vec{a} + \vec{b}$
   From triangle ADC,
   $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \vec{b} + \vec{a}$
   So, $\overrightarrow{AC} = \vec{a} + \vec{b} = \vec{b} + \vec{a}$.

   Although vector addition is commutative, certain types of vector operations are not always commutative. We will see this when dealing with cross products in Chapter 7.

2. Associative Property of Addition: When adding numbers, the associative property is used routinely. If we wish to add 3, 5 and 8, for example, we can do this as $(3 + 5) + 8$ or as $3 + (5 + 8)$. Doing it either way, we get the answer 16. In doing this calculation, we are free to associate the numbers however we choose. This property also holds when adding algebraic expressions, such as $2x + 3x - 7x = (2x + 3x) - 7x = 2x + (3x - 7x) = -2x$.

   In adding vectors, we are free to associate them in exactly the same way as we do for numbers or algebraic expressions. For vectors, this property is stated as $\vec{a} + \vec{b} + \vec{c} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

   We will use the following diagram and addition of vectors to demonstrate the associative property.

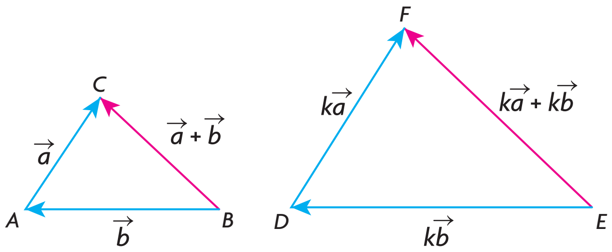


   In the diagram, $\overrightarrow{PQ} = \vec{a}$, $\overrightarrow{QR} = \vec{b}$, and $\overrightarrow{RS} = \vec{c}$. From triangle PRQ, $\overrightarrow{PR} = \vec{a} + \vec{b}$, and then from triangle PSR, $\overrightarrow{PS} = \overrightarrow{PR} + \overrightarrow{RS} = (\vec{a} + \vec{b}) + \vec{c}$. Similarly, from triangle SQR, $\overrightarrow{QS} = \vec{b} + \vec{c}$ and then from triangle PQS, $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{QS} = \vec{a} + (\vec{b} + \vec{c})$. So $\overrightarrow{PS} = (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

   It is interesting to note, just as we did with the commutative property, that the associative property holds for the addition of vectors but does not hold for certain kinds of multiplication.

3. Distributive Property of Addition: The distributive property is something we have used implicitly from the first day we thought about numbers or algebra. In calculating the perimeter of a rectangle with width w and length l, we write the perimeter as $P = 2(w + l) = 2w + 2l$. In this case, the 2 has been distributed across the brackets to give $2w$ and $2l$.

Demonstrating the distributive law for vectors depends on being able to multiply vectors by scalars and on the addition law for vectors.



In this diagram, we started with $\vec{a}$ and $\vec{b}$ and then multiplied each of them by k, a positive scalar, to give the vectors $k\vec{a}$ and $k\vec{b}$, respectively. In $\triangle ABC$, $\overrightarrow{BC} = \vec{a} + \vec{b}$, and in $\triangle DEF$, $\overrightarrow{EF} = k\vec{a} + k\vec{b}$. However, the two triangles are similar, so $\overrightarrow{EF} = k(\vec{a} + \vec{b})$. Since $\overrightarrow{EF} = k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, we have shown that the distributive law is true for $k > 0$ and any pair of vectors.

Although we chose k to be a positive number, we could have chosen any real number for k.

Properties of Vector Addition

1. Commutative Property of Addition: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
2. Associative Property of Addition: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
3. Distributive Property of Addition: $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, $k \in \mathbf{R}$

EXAMPLE 1 Selecting appropriate vector properties to determine an equivalent vector

Simplify the following expression: $3(2\vec{a} + \vec{b} + \vec{c}) - (\vec{a} + 3\vec{b} - 2\vec{c})$.

Solution

$3(2\vec{a} + \vec{b} + \vec{c}) - (\vec{a} + 3\vec{b} - 2\vec{c})$

$= 6\vec{a} + 3\vec{b} + 3\vec{c} - \vec{a} - 3\vec{b} + 2\vec{c}$ (Distributive property)

$= 6\vec{a} - \vec{a} + 3\vec{b} - 3\vec{b} + 3\vec{c} + 2\vec{c}$ (Commutative property)

$= (6 - 1)\vec{a} + (3 - 3)\vec{b} + (3 + 2)\vec{c}$ (Distributive property for scalars)

$= 5\vec{a} + \vec{0} + 5\vec{c}$

$= 5\vec{a} + 5\vec{c}$

In doing the calculation in Example 1, assumptions were made that are implicit but that should be stated.

Further Laws of Vector Addition and Scalar Multiplication

1. Adding $\vec{0}$: $\vec{a} + \vec{0} = \vec{a}$
2. Associative Law for Scalars: $m(n\vec{a}) = (mn)\vec{a} = mn\vec{a}$
3. Distributive Law for Scalars: $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

It is important to be aware of all these properties when calculating, but the properties can be assumed without having to refer to them for each simplification.

EXAMPLE 2 Selecting appropriate vector properties to create new vectors

If $\vec{x} = 3\vec{i} - 4\vec{j} + \vec{k}$, $\vec{y} = \vec{j} - 5\vec{k}$, and $\vec{z} = -\vec{i} - \vec{j} + 4\vec{k}$, determine each of the following in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.

a. $\vec{x} + \vec{y}$ b. $\vec{x} - \vec{y}$ c. $\vec{x} - 2\vec{y} + 3\vec{z}$

Solution

a. $$\begin{aligned}\vec{x} + \vec{y} &= (3\vec{i} - 4\vec{j} + \vec{k}) + (\vec{j} - 5\vec{k})\\ &= 3\vec{i} - 4\vec{j} + \vec{j} + \vec{k} - 5\vec{k}\\ &= 3\vec{i} - 3\vec{j} - 4\vec{k}\end{aligned}$$

b. $$\begin{aligned}\vec{x} - \vec{y} &= (3\vec{i} - 4\vec{j} + \vec{k}) - (\vec{j} - 5\vec{k})\\ &= 3\vec{i} - 4\vec{j} - \vec{j} + \vec{k} + 5\vec{k}\\ &= 3\vec{i} - 5\vec{j} + 6\vec{k}\end{aligned}$$

c. $$\begin{aligned}\vec{x} - 2\vec{y} + 3\vec{z} &= (3\vec{i} - 4\vec{j} + \vec{k}) - 2(\vec{j} - 5\vec{k}) + 3(-\vec{i} - \vec{j} + 4\vec{k})\\ &= 3\vec{i} - 4\vec{j} + \vec{k} - 2\vec{j} + 10\vec{k} - 3\vec{i} - 3\vec{j} + 12\vec{k}\\ &= -9\vec{j} + 23\vec{k}\end{aligned}$$

As stated previously, it is not necessary to state the rules as we simplify, and furthermore, it is better to try to simplify without writing in every step.

The rules that were developed in this section will prove useful as we move ahead. They are necessary for our understanding of linear combinations, which will be dealt with later in this chapter.

IN SUMMARY

Key Idea

- Properties used to evaluate numerical expressions and simplify algebraic expressions also apply to vector addition and scalar multiplication.

Need to Know

- Commutative Property of Addition: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$
- Associative Property of Addition: $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
- Distributive Property of Addition: $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, $k \in \mathbf{R}$
- Adding $\vec{0}$: $\vec{a} + \vec{0} = \vec{a}$
- Associative Law for Scalars: $m(n\vec{a}) = (mn)\vec{a} = mn\vec{a}$
- Distributive Law for Scalars: $(m + n)\vec{a} = m\vec{a} + n\vec{a}$

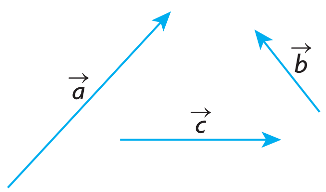
Exercise 6.4

PART A

1. If $*$ is an operation on a set, S, the element x, such that $a * x = a$, is called the identity element for the operation $*$.
   a. For the addition of numbers, what is the identity element?
   b. For the multiplication of numbers, what is the identity element?
   c. For the addition of vectors, what is the identity element?
   d. For scalar multiplication, what is the identity element?

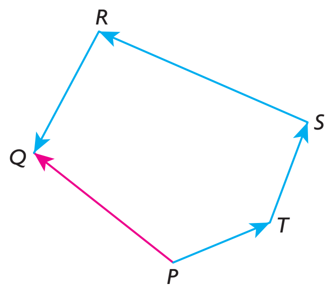
2. Illustrate the commutative law for two vectors that are perpendicular.

C 3. Redraw the following three vectors and illustrate the associative law.



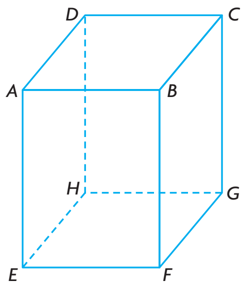
4. With the use of a diagram, show that the distributive law, $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$, holds where $k < 0$, $k \in \mathbf{R}$.

PART B



K 5. Using the given diagram, show that the following is true.

$$\begin{aligned}\overrightarrow{PQ} &= (\overrightarrow{RQ} + \overrightarrow{SR}) + \overrightarrow{TS} + \overrightarrow{PT}\\ &= \overrightarrow{RQ} + (\overrightarrow{SR} + \overrightarrow{TS}) + \overrightarrow{PT}\\ &= \overrightarrow{RQ} + \overrightarrow{SR} + (\overrightarrow{TS} + \overrightarrow{PT})\end{aligned}$$



6. $ABCDEFGH$ is a rectangular prism.
   a. Write a single vector that is equivalent to $\overrightarrow{EG} + \overrightarrow{GH} + \overrightarrow{HD} + \overrightarrow{DC}$.
   b. Write a vector that is equivalent to $\overrightarrow{EG} + \overrightarrow{GD} + \overrightarrow{DE}$.
   c. Is it true that $|\overrightarrow{HB}| = |\overrightarrow{GA}|$? Explain.

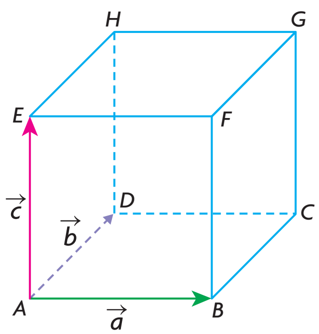
7. Write the following vector in simplified form:
   $3(\vec{a} - 2\vec{b} - 5\vec{c}) - 3(2\vec{a} - 4\vec{b} + 2\vec{c}) - (\vec{a} - 3\vec{b} + 3\vec{c})$

8. If $\vec{a} = 3\vec{i} - 4\vec{j} + \vec{k}$ and $\vec{b} = -2\vec{i} + 3\vec{j} - \vec{k}$, express each of the following in terms of $\vec{i}$, $\vec{j}$, and $\vec{k}$.
   a. $2\vec{a} - 3\vec{b}$ b. $\vec{a} + 5\vec{b}$ c. $2(\vec{a} - 3\vec{b}) - 3(-2\vec{a} - 7\vec{b})$

T 9. If $2\vec{x} + 3\vec{y} = \vec{a}$ and $-\vec{x} + 5\vec{y} = 6\vec{b}$, express $\vec{x}$ and $\vec{y}$ in terms of $\vec{a}$ and $\vec{b}$.

10. If $\vec{x} = \frac{2}{3}\vec{y} + \frac{1}{3}\vec{z}$, $\vec{x} - \vec{y} = \vec{a}$, and $\vec{y} - \vec{z} = \vec{b}$, show that $\vec{a} = -\frac{1}{3}\vec{b}$.

A 11. A cube is constructed from the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, as shown below.



   a. Express each of the diagonals $\overrightarrow{AG}$, $\overrightarrow{BH}$, $\overrightarrow{CE}$, and $\overrightarrow{DF}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.
   b. Is $|\overrightarrow{AG}| = |\overrightarrow{BH}|$? Explain.

PART C

12. In the trapezoid $TXYZ$, $\overrightarrow{TX} = 2\overrightarrow{ZY}$. If the diagonals meet at O, find an expression for $\overrightarrow{TO}$ in terms of $\overrightarrow{TX}$ and $\overrightarrow{TZ}$.