Section 4.4—Concavity and Points of Inflection

In Chapter 3, you saw that the second derivative of a function has applications in problems involving velocity and acceleration or in general rates-of-change problems. Here we examine the use of the second derivative of a function in curve sketching.

INVESTIGATION 1 The purpose of this investigation is to examine the relationship between slopes of tangents and the second derivative of a function.

A. Sketch the graph of $f(x) = x^2$.

B. Determine $f'(x)$. Use $f'(x)$ to calculate the slope of the tangent to the curve at the points with the following x-coordinates: $x = -4, -3, -2, -1, 0, 1, 2, 3,$ and 4. Sketch each of these tangents.

C. Are these tangents above or below the graph of $y = f(x)$?

D. Describe the change in the slopes as x increases.

E. Determine $f''(x)$. How does the value of $f''(x)$ relate to the way in which the curve opens? How does the value of $f''(x)$ relate to the way $f'(x)$ changes as x increases?

F. Repeat parts B, C, and D for the graph of $f(x) = -x^2$.

G. How does the value of $f''(x)$ relate to the way in which the curve opens?

INVESTIGATION 2 The purpose of this investigation is to extend the results of Investigation 1 to other functions.

A. Sketch the graph of $f(x) = x^3$.

B. Determine all the values of x for which $f'(x) = 0$.

C. Determine intervals on the domain of the function such that $f''(x) < 0$, $f''(x) = 0$, and $f''(x) > 0$.

D. For values of x such that $f''(x) < 0$, how does the shape of the curve compare with your conclusions in Investigation 1?

E. Repeat part D for values of x such that $f''(x) > 0$.

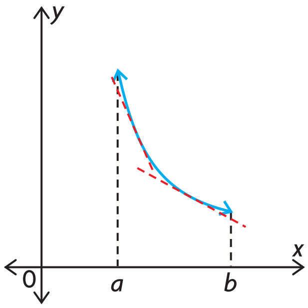
F. What happens when $f''(x) = 0$?

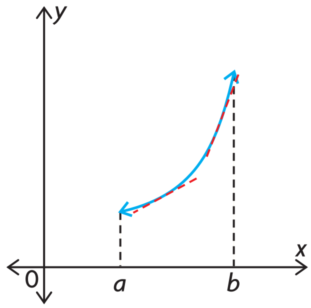
G. Using your observations from this investigation, sketch the graph of $y = x^3 - 12x$.

From these investigations, we can make a summary of the behaviour of the graphs.

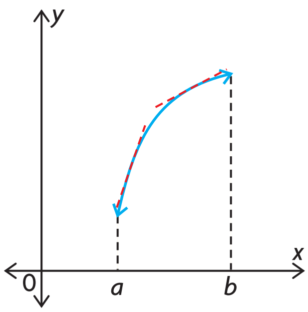
Concavity and the Second Derivative

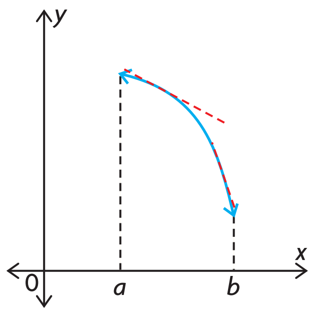
1. The graph of $y = f(x)$ is **concave up** on an interval $a \leq x \leq b$ in which the slopes of $f(x)$ are increasing. On this interval, $f''(x)$ exists and $f''(x) > 0$. The graph of the function is above the tangent at every point on the interval.



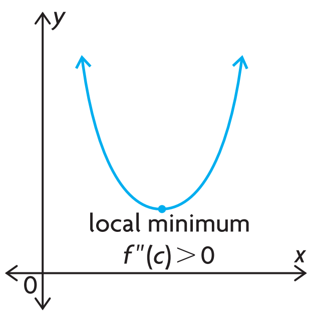


2. The graph of $y = f(x)$ is **concave down** on an interval $a \leq x \leq b$ in which the slopes of $f(x)$ are decreasing. On this interval, $f''(x)$ exists and $f''(x) < 0$. The graph of the function is below the tangent at every point on the interval.

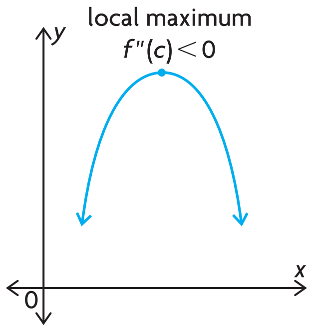




3. If $y = f(x)$ has a critical point at $x = c$, with $f'(c) = 0$, then the behaviour of $f(x)$ at $x = c$ can be analyzed through the use of the **second derivative test** by analyzing $f''(c)$, as follows:
   a. The graph is concave up, and $x = c$ is the location of a local minimum value of the function, if $f''(c) > 0$.

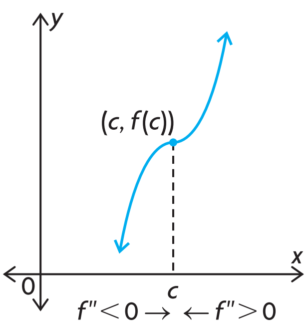


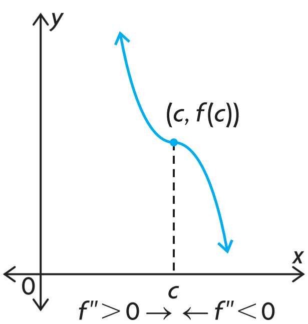
b. The graph is concave down, and $x = c$ is the location of a local maximum value of the function, if $f''(c) < 0$.



c. If $f''(c) = 0$, the nature of the critical point cannot be determined without further work.

4. A **point of inflection** occurs at $(c, f(c))$ on the graph of $y = f(x)$ if $f''(x)$ changes sign at $x = c$. That is, the curve changes from concave down to concave up, or vice versa.





5. All points of inflection on the graph of $y = f(x)$ must occur either where $\frac{d^2y}{dx^2}$ equals zero or where $\frac{d^2y}{dx^2}$ is undefined.

In the following examples, we will use these properties to sketch graphs of other functions.

EXAMPLE 1 **Using the first and second derivatives to analyze a cubic function**

Sketch the graph of $y = x^3 - 3x^2 - 9x + 10$.

Solution

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Setting $\frac{dy}{dx} = 0$, we obtain

$$3(x^2 - 2x - 3) = 0$$
$$3(x - 3)(x + 1) = 0$$
$$x = 3 \text{ or } x = -1$$

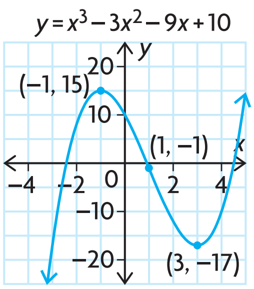
$$\frac{d^2y}{dx^2} = 6x - 6$$

Setting $\frac{d^2y}{dx^2} = 0$, we obtain $6x - 6 = 0$ or $x = 1$.

Now determine the sign of $f''(x)$ in the intervals determined by $x = 1$.

Interval	$x < 1$	$x = 1$	$x > 1$
$f''(x)$	< 0	0	> 0
Graph of $f(x)$	concave down	point of inflection	concave up
Sketch of $f(x)$			

Applying the second derivative test, at $x = 3$, we obtain the local minimum point, $(3, -17)$ and at $x = -1$, we obtain the local maximum point, $(-1, 15)$. The point of inflection occurs at $x = 1$ where $f(1) = -1$.
The graph can now be sketched.



EXAMPLE 2 Using the first and second derivatives to analyze a quartic function

Sketch the graph of $f(x) = x^4$.

Solution

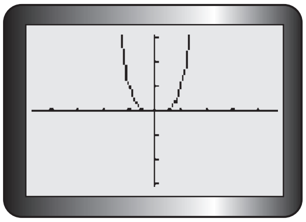
The first and second derivatives of $f(x)$ are $f'(x) = 4x^3$ and $f''(x) = 12x^2$.
Setting $f''(x) = 0$, we obtain
$12x^2 = 0$ or $x = 0$

But $x = 0$ is also obtained from $f'(x) = 0$.

Now determine the sign of $f''(x)$ on the intervals determined by $x = 0$.

Interval	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	> 0	$= 0$	> 0
Graph of $f(x)$	concave up	?	concave up
Sketch of $f(x)$			

We conclude that the point $(0, 0)$ is *not* an inflection point because $f''(x)$ does not change sign at $x = 0$. However, since $x = 0$ is a critical number and $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$, $(0, 0)$ is an absolute minimum.



EXAMPLE 3 Using the first and second derivatives to analyze a root function

Sketch the graph of the function $f(x) = x^{\frac{1}{3}}$.

Solution

The derivative of $f(x)$ is

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-\frac{2}{3}} \\ &= \frac{1}{3x^{\frac{2}{3}}} \end{aligned}$$

Note that $f'(0)$ does not exist, so $x = 0$ is a critical number of $f(x)$. It is important to determine the behaviour of $f'(x)$ as $x \to 0$. Since $f'(x) > 0$ for all values of $x \neq 0$, and the denominator of $f'(x)$ is zero when $x = 0$, we have $\lim_{x\to 0} f'(x) = +\infty$. This means that there is a vertical tangent at $x = 0$. In addition, $f(x)$ is increasing for $x < 0$ and $x > 0$. As a result this graph has no local extrema.

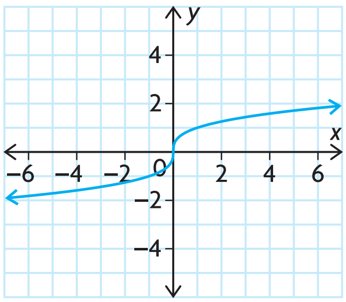
The second derivative of $f(x)$ is

$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$$
$$= -\frac{2}{9x^{\frac{5}{3}}}$$

Since $x^{\frac{5}{3}} > 0$ if $x > 0$, and $x^{\frac{5}{3}} < 0$ if $x < 0$, we obtain the following table:

Interval	$x < 0$	$x = 0$	$x > 0$
$f''(x)$	$\frac{-}{-} = +$	does not exist	$\frac{-}{+} = -$
$f(x)$			

The graph has a point of inflection when $x = 0$, even though $f'(0)$ and $f''(0)$ do not exist. Note that the curve crosses its tangent at $x = 0$.



EXAMPLE 4 Reasoning about points of inflection

Determine any points of inflection on the graph of $f(x) = \frac{1}{x^2 + 3}$.

Solution

The derivative of $f(x) = \frac{1}{x^2 + 3} = (x^2 + 3)^{-1}$ is $f'(x) = -2x(x^2 + 3)^{-2}$. The second derivative is

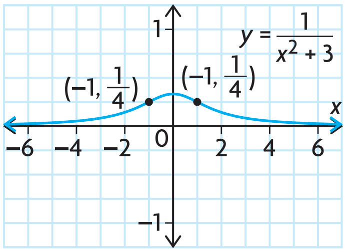
$$f''(x) = -2(x^2 + 3)^{-2} + 4x(x^2 + 3)^{-3}(2x)$$
$$= \frac{-2}{(x^2 + 3)^2} + \frac{8x^2}{(x^2 + 3)^3}$$
$$= \frac{-2(x^2 + 3) + 8x^2}{(x^2 + 3)^3}$$
$$= \frac{6x^2 - 6}{(x^2 + 3)^3}$$

Setting $f''(x) = 0$ gives $6x^2 - 6 = 0$ or $x = \pm 1$.

Determine the sign of $f''(x)$ on the intervals determined by $x = -1$ and $x = 1$.

Interval	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$f''(x)$	> 0	$= 0$	< 0	$= 0$	> 0
Graph of $f(x)$	concave up	point of inflection	concave down	point of inflection	concave up

Therefore, $\left(-1, \frac{1}{4}\right)$ and $\left(1, \frac{1}{4}\right)$ are points of inflection on the graph of $f(x)$.



IN SUMMARY

Key Ideas

- The graph of a function $f(x)$ is **concave up** on an interval if $f'(x)$ is increasing on the interval. The graph of a function $f(x)$ is **concave down** on an interval if $f'(x)$ is decreasing on the interval.
- A point of inflection is a point on the graph of $f(x)$ where the function changes from concave up to concave down, or vice versa. $f''(c) = 0$ or is undefined if $(c, f(c))$ is a point of inflection on the graph of $f(x)$.

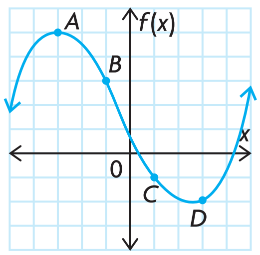
Need to Know

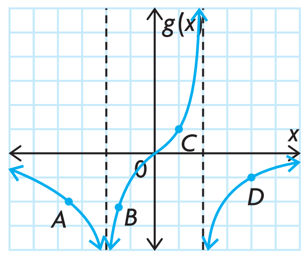
- **Test for concavity:** If $f(x)$ is a differentiable function whose second derivative exists on an open interval I, then
  - the graph of $f(x)$ is concave up on I if $f''(x) > 0$ for all values of x in I
  - the graph of $f(x)$ is concave down on I if $f''(x) < 0$ for all values of x in I
- **The second derivative test:** Suppose that $f(x)$ is a function for which $f''(c) = 0$, and the second derivative of $f(x)$ exists on an interval containing c.
  - If $f''(c) > 0$, then $f(c)$ is a local minimum value.
  - If $f''(c) < 0$, then $f(c)$ is a local maximum value.
  - If $f''(c) = 0$, then the test fails. Use the first derivative test.

Exercise 4.4

PART A

K 1. For each function, state whether the value of the second derivative is positive or negative at each of points A, B, C, and D.

a. 

b. 

2. Determine the critical points for each function, and use the second derivative test to decide if the point is a local maximum, a local minimum, or neither.

a. $y = x^3 - 6x^2 - 15x + 10$

b. $y = \dfrac{25}{x^2 + 48}$

c. $s = t + t^{-1}$

d. $y = (x - 3)^3 + 8$

3. Determine the points of inflection for each function in question 2. Then conduct a test to determine the change of sign in the second derivative.

4. Determine the value of the second derivative at the value indicated. State whether the curve lies above or below the tangent at this point.

a. $f(x) = 2x^3 - 10x + 3$ at $x = 2$

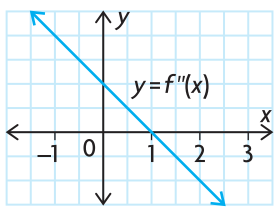
b. $g(x) = x^2 - \dfrac{1}{x}$ at $x = -1$

c. $p(w) = \dfrac{w}{\sqrt{w^2 + 1}}$ at $w = 3$

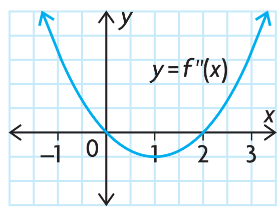
d. $s(t) = \dfrac{2t}{t - 4}$ at $t = -2$

PART B

5. Each of the following graphs represents the second derivative, $f''(x)$, of a function $f(x)$:

a. 

$f''(x)$ is a linear function.

b. 

$f''(x)$ is a quadratic function.

For each of the graphs above, answer the following questions:

i. On which intervals is the graph of $f(x)$ concave up? On which intervals is the graph concave down?

ii. List the x-coordinates of all the points of inflection.

iii. Make a rough sketch of a possible graph of $f(x)$, assuming that $f(0) = 2$.

C 6. Describe how you would use the second derivative to determine a local minimum or maximum.

7. In the algorithm for curve sketching in Section 4.3, reword step 4 to include the use of the second derivative to test for local minimum or maximum values.

8. For each of the following functions,

   i. determine any points of inflection

   ii. use the results of part i, along with the revised algorithm, to sketch each function.

   a. $f(x) = x^4 + 4x$ b. $g(w) = \dfrac{4w^2 - 3}{w^3}$

A 9. Sketch the graph of a function with the following properties:

- $f'(x) > 0$ when $x < 2$ and when $2 < x < 5$
- $f'(x) < 0$ when $x > 5$
- $f'(2) = 0$ and $f'(5) = 0$
- $f''(x) < 0$ when $x < 2$ and when $4 < x < 7$
- $f''(x) > 0$ when $2 < x < 4$ and when $x > 7$
- $f(0) = -4$

10. Find constants a, b, and c such that the function $f(x) = ax^3 + bx^2 + c$ will have a local extremum at $(2, 11)$ and a point of inflection at $(1, 5)$. Sketch the graph of $y = f(x)$.

PART C

11. Find the value of the constant b such that the function $f(x) = \sqrt{x + 1} + \frac{b}{x}$ has a point of inflection at $x = 3$.

T 12. Show that the graph of $f(x) = ax^4 + bx^3$ has two points of inflection. Show that the x-coordinate of one of these points lies midway between the x-intercepts.

13. a. Use the algorithm for curve sketching to sketch the function $y = \dfrac{x^3 - 2x^2 + 4x}{x^2 - 4}$.

    b. Explain why it is difficult to determine the oblique asymptote using a graphing calculator.

14. Find the inflection points, if any exist, for the graph of $f(x) = (x - c)^n$, for $n = 1, 2, 3$, and 4. What conclusion can you draw about the value of n and the existence of inflection points on the graph of f?