Chapter 8

EQUATIONS OF LINES AND PLANES

In this chapter, you will work with vector concepts you learned in the preceding chapters and use them to develop equations for lines and planes. We begin with lines in R^2 and then move to R^3, where lines are once again considered along with planes. The determination of equations for lines and planes helps to provide the basis for an understanding of geometry in R^3. All of these concepts provide the foundation for the solution of systems of linear equations that result from intersections of lines and planes, which are considered in Chapter 9.

CHAPTER EXPECTATIONS

In this chapter, you will

- determine the vector and parametric equations of a line in two-space, **Section 8.1**
- make connections between Cartesian, vector, and parametric equations of a line in two-space, **Section 8.2**
- determine the vector, parametric, and symmetric equations of a line in 3-space, **Section 8.3**
- determine the vector, parametric, and Cartesian equations of a plane, **Sections 8.4, 8.5**
- determine some geometric properties of a plane, **Section 8.5**
- determine the equation of a plane in Cartesian, vector, or parametric form, given another form, **Section 8.5**
- sketch a plane in 3-space, **Section 8.6**

Review of Prerequisite Skills

In this chapter, we will develop the equation of a line in two- and three-dimensional space and the equation of a plane in three-dimensional space. You will find it helpful to review the following concepts:

- geometric and algebraic vectors
- the dot product
- the cross product
- plotting points and vectors in three-space

We will begin this chapter by examining equations of lines. Lines are not vectors, but vectors are used to describe lines. The table below shows their similarities and differences.

Lines	Vectors
Lines are bi-directional. A line defines a direction, but there is nothing to distinguish forward from backwards.	Vectors are unidirectional. A vector defines a direction with a clear distinction between forward and backwards.
A line is infinite in extent in both directions. A line segment has a finite length.	Vectors have a finite magnitude.
Lines and line segments have a definite location. The opposite sides of a parallelogram are two different line segments.	A vector has no fixed location. The opposite sides of a parallelogram are described by the same vector.
Two lines are the same when they have the same direction and same location. These lines are said to be coincident.	Two vectors are the same when they have the same direction and the same magnitude. These vectors are said to be equal.

Exercise

1. Determine a single vector that is equivalent to each of the following expressions:

a. $(3, -2, 1) - (1, 7, -5)$

b. $5(2, -3, -4) + 3(1, 1, -7)$

2. Determine if the following sets of points are collinear:

a. $A(1, -3), B(4, 2), C(-8, -18)$

b. $J(-4, 3), K(4, 5), L(0, 4)$

c. $A(1, 2, 1), B(4, 7, 0), C(7, 12, -1)$

d. $R(1, 2, -3), S(4, 1, 3), T(2, 4, 0)$

3. Determine if $\triangle ABC$ is a right-angled triangle, given $A(1, 6, -2)$, $B(2, 5, 3)$, and $C(5, 3, 2)$.

4. Given $\vec{u} = (t, -1, 3)$ and $\vec{v} = (2, t, -6)$, for what values of t are the vectors perpendicular?

5. State a vector perpendicular to each of the following:

a. $\vec{a} = (1, -3)$ b. $\vec{b} = (6, -5)$ c. $\vec{c} = (-7, -4, 0)$

6. Calculate the area of the parallelogram formed by the vectors $(4, 10, 9)$ and $(3, 1, -2)$.

7. Use the cross product to determine a vector perpendicular to each of the following pairs of vectors. Check your answer using the dot product.

a. $\vec{a} = (2, 1, -4)$ and $\vec{b} = (3, -5, -2)$

b. $\vec{a} = (-1, -2, 0)$ and $\vec{b} = (-2, -1, 0)$

8. For each of the following, draw the x-axis, y-axis, and z-axis, and accurately draw the position vectors:

a. $A(1, 2, 3)$ b. $B(1, 2, -3)$ c. $C(1, -2, 3)$ d. $D(-1, 2, 3)$

9. Determine the position vector that passes from the first point to the second.

a. $(4, 8)$ and $(-3, 5)$

b. $(-7, -6)$ and $(3, 8)$

c. $(1, 2, 4)$ and $(3, -6, 9)$

d. $(4, 0, -4)$ and $(0, 5, 0)$

10. State the vector that is opposite to each of the vectors you found in question 9.

11. Determine the slope and y-intercept of each of the following linear equations. Then sketch its graph.

a. $y = -2x - 5$

b. $4x - 8y = 8$

c. $3x - 5y + 1 = 0$

d. $5x = 5y - 15$

12. State a vector that is collinear to each of the following and has the same direction:

a. $(4, 7)$ b. $(-5, 4, 3)$ c. $2\vec{i} + 6\vec{j} - 4\vec{k}$ d. $-5\vec{i} + 8\vec{j} + 2\vec{k}$

13. If $\vec{u} = (4, -9, -1)$ and $\vec{v} = 4\vec{i} - 2\vec{j} + \vec{k}$, determine each of the following:

a. $\vec{u} \cdot \vec{v}$

b. $-\vec{v} \cdot \vec{u}$

c. $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v})$

d. $\vec{u} \times \vec{v}$

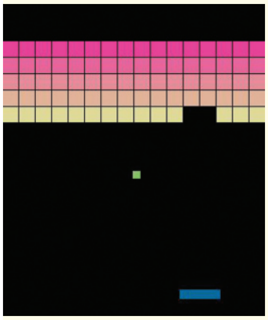
e. $\vec{v} \times \vec{u}$

f. $(2\vec{u} + \vec{v}) \times (\vec{u} - 2\vec{v})$

14. Both the dot product and the cross product are ways to multiply two vectors. Explain how these products differ.

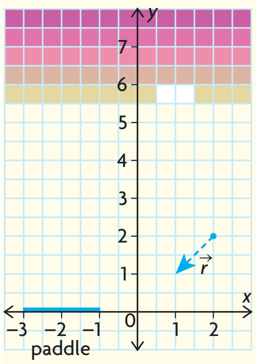
CHAPTER 8: COMPUTER PROGRAMMING WITH VECTORS

Computer programmers use vectors for a variety of graphics applications. Any time that two- and three-dimensional images are designed, they are represented in the form of vectors. Vectors allow the programmer to move the figure easily to any new location on the screen. If the figure were expressed point by point using coordinate geometry, each and every point would have to be recalculated each time the figure needed to be moved. By using vectors drawn from an anchor point to draw the figure, only the coordinates of the anchor point need to be recalculated on the screen to move the entire figure. This method is used in many different types of software, including games, flight simulators, drafting and architecture tools, and visual design tools.



Case Study—Breakout

Breakout is a classic video game, and a prime example of using vectors in computer graphics. A paddle is used to bounce a ball into a section of bricks to slowly break down a wall. Each time the ball hits the paddle, it bounces off at an angle to the paddle. The path the ball takes from the paddle can be described by a vector, which is dependent upon the angle and speed of the ball. If the wall were not in the path of the ball, the ball would continue along its path at that speed until it "fell" off of the screen.



DISCUSSION QUESTIONS

1. The coordinate plane represents the screen in the game "Breakout." The ball is travelling toward the paddle along the vector $\vec{r}$. Find the equation of the line determined by vector $\vec{r}$ in its current position. Draw a direction vector for the line.

2. Find where the line crosses the *x*-axis to show where the paddle must move in order to bounce the ball back.

3. Since the angle of entry for the ball is 45°, the ball will bounce off the paddle along a path perpendicular to $\vec{r}$. Draw a vector $\vec{s}$ perpendicular to $\vec{r}$ that emanates from the origin in the direction the ball will travel when it bounces off the paddle. Then draw a line parallel to vector $\vec{s}$ that passes through the point where $\vec{r}$ crosses the *x*-axis.