# *Section 5.4*—The Derivatives of $y = \sin x$ and $y = \cos x$

In this section, we will investigate to determine the derivatives of $y = \sin x$ and $y = \cos x$.

**INVESTIGATION 1**

A. Using a graphing calculator, graph $y = \sin x$, where $x$ is measured in radians. Use the following WINDOW settings:
- Xmin = 0, Xmax = 9.4, Xscl = $\pi \div 2$
- Ymin = −3.1, Ymax = 3.1, Yscl = 1

Enter $y = \sin x$ into Y1, and graph the function.

B. Use the CALC function (with value or $\frac{dy}{dx}$ selected) to compute $y$ and $\frac{dy}{dx}$, respectively, for $y = \sin x$. Record these values in a table like the following (correct to four decimal places):

| $x$ | $\sin x$ | $\frac{d}{dx}(\sin x)$ |
|---|---|---|
| 0 | | |
| 0.5 | | |
| 1.0 | | |
| : | | |
| : | | |
| : | | |
| 6.5 | | |

**Tech Support**

To calculate $\frac{dy}{dx}$ at a point, press 2ND TRACE 6 and enter the desired $x$-coordinate of your point. Then press ENTER.

C. Create another column, to the right of the $\frac{d}{dx}(\sin x)$ column, with $\cos x$ as the heading. Using your graphing calculator, graph $y = \cos x$ with the same window settings as above.

D. Compute the values of $\cos x$ for $x = 0, 0.5, 1.0, \ldots, 6.5$, correct to four decimal places. Record the values in the $\cos x$ column.

E. Compare the values in the $\frac{d}{dx}(\sin x)$ column with those in the $\cos x$ column, and write a concluding equation.

**Tech Support**

For help calculating a value of a function using a graphing calculator, see Technical Appendix p. 598.

**INVESTIGATION 2** A. Using your graphing calculator, graph $y = \cos x$, where $x$ is measured in radians. Use the following WINDOW settings:

- Xmin $= 0$, Xmax $= 9.4$, Xscl $= \pi \div 2$
- Ymin $= -3.1$, Ymax $= 3.1$, Yscl $= 1$

Enter $y = \cos x$ into Y1, and graph the function.

B. Use the CALC function (with value or $\frac{dy}{dx}$ selected) to compute $y$ and $\frac{dy}{dx}$, respectively, for $y = \cos x$. Record these values, correct to four decimal places, in a table like the following:

| $x$ | $\cos x$ | $\frac{d}{dx}(\cos x)$ |
|---|---|---|
| 0 | | |
| 0.5 | | |
| 1.0 | | |
| : | | |
| : | | |
| : | | |
| 6.5 | | |

C. Create another column to the right of the $\frac{d}{dx}(\cos x)$ column with $-\sin x$ as the heading. Using your graphing calculator, graph $y = -\sin x$ with the same window settings as above.

D. Compute the values of $-\sin x$ for $x = 0, 0.5, 1.0, \dots, 6.5$, correct to four decimal places. Record the values in the $-\sin x$ column.

E. Compare the values in the $\frac{d}{dx}(\cos x)$ column with those in the $-\sin x$ column, and write a concluding equation.

Investigations 1 and 2 lead to the following conclusions:

**Derivatives of Sinusoidal Functions**

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

## EXAMPLE 1 Selecting a strategy to determine the derivative of a sinusoidal function

Determine $\frac{dy}{dx}$ for each function.

a. $y = \cos 3x$

b. $y = x \sin x$

### Solution

a. To differentiate this function, use the chain rule.

$$y = \cos 3x$$

$$\frac{dy}{dx} = \frac{d(\cos 3x)}{d(3x)} \times \frac{d(3x)}{dx} \qquad \text{(Chain rule)}$$

$$= -\sin 3x \times (3)$$

$$= -3 \sin 3x$$

b. To find the derivative, use the product rule.

$$y = x \sin x$$

$$\frac{dy}{dx} = \frac{dx}{dx} \times \sin x + x\frac{d(\sin x)}{dx} \qquad \text{(Product rule)}$$

$$= (1) \times \sin x + x \cos x$$

$$= \sin x + x \cos x$$

---

## EXAMPLE 2 Reasoning about the derivatives of sinusoidal functions

Determine $\frac{dy}{dx}$ for each function.

a. $y = \sin x^2$

b. $y = \sin^2 x$

### Solution

a. To differentiate this composite function, use the chain rule and change of variable.
Here, the inner function is $u = x^2$, and the outer function is $y = \sin u$.

$$\text{Then, } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \text{(Chain rule)}$$

$$= (\cos u)(2x) \qquad \text{(Substitute)}$$

$$= 2x \cos x^2$$

b. Since $y = \sin^2 x = (\sin x)^2$, we use the chain rule with $y = u^2$, where $u = \sin x$.

$$\text{Then, } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \text{(Chain rule)}$$

$$= (2u)(\cos x) \qquad \text{(Substitute)}$$

$$= 2 \sin x \cos x$$

With practice, you will learn how to apply the chain rule without the intermediate step of introducing the variable $u$. For $y = \sin x^2$, for example, you can skip this step and immediately write $\frac{dy}{dx} = (\cos x^2)(2x)$.

**Derivatives of Composite Sinusoidal Functions**

If $y = \sin f(x)$, then $\frac{dy}{dx} = \cos f(x) \times f'(x)$.

In Leibniz notation, $\dfrac{d}{dx}(\sin f(x)) = \dfrac{d(\sin f(x))}{d(f(x))} \times \dfrac{d(f(x))}{dx} = \cos f(x) \times \dfrac{d(f(x))}{dx}$.

If $y = \cos f(x)$, then $\frac{dy}{dx} = -\sin f(x) \times f'(x)$.

In Leibniz notation, $\dfrac{d}{dx}(\cos f(x)) = \dfrac{d(\cos f(x))}{d(f(x))} \times \dfrac{d(f(x))}{dx} = -\sin f(x) \times \dfrac{d(f(x))}{dx}$.

## EXAMPLE 3 Differentiating a composite cosine function

Determine $\frac{dy}{dx}$ for $y = \cos(1 + x^3)$.

### Solution

$$y = \cos(1 + x^3)$$

$$\frac{dy}{dx} = \frac{d[\cos(1 + x^3)]}{d(1 + x^3)} \times \frac{d(1 + x^3)}{dx} \qquad \text{(Chain rule)}$$

$$= -\sin(1 + x^3)(3x^2)$$

$$= -3x^2 \sin(1 + x^3)$$

## EXAMPLE 4 Differentiating a combination of functions

Determine $y'$ for $y = e^{\sin x + \cos x}$.

### Solution

$$y = e^{\sin x + \cos x}$$

$$y' = \frac{d(e^{\sin x + \cos x})}{d(\sin x + \cos x)} \times \frac{d(\sin x + \cos x)}{dx} \qquad \text{(Chain rule)}$$

$$= e^{\sin x + \cos x}(\cos x - \sin x)$$

## EXAMPLE 5 Connecting the derivative of a sinusoidal function to the slope of a tangent

Determine the equation of the tangent to the graph of $y = x \cos 2x$ at $x = \frac{\pi}{2}$.

### Solution

When $x = \frac{\pi}{2}$, $y = \frac{\pi}{2} \cos \pi = -\frac{\pi}{2}$.

The point of tangency is $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$.

The slope of the tangent at any point on the graph is given by

$$\frac{dy}{dx} = \frac{dx}{dx} \times \cos 2x + x \times \frac{d(\cos 2x)}{dx} \quad \text{(Product and chain rules)}$$

$$= (1)(\cos 2x) + x(-\sin 2x)(2) \quad \text{(Simplify)}$$

$$= \cos 2x - 2x \sin 2x$$

At $x = \frac{\pi}{2}$, $\frac{dy}{dx} = \cos \pi - \pi(\sin \pi)$ (Evaluate)

$$= -1$$

The equation of the tangent is

$$y + \frac{\pi}{2} = -\left(x - \frac{\pi}{2}\right) \text{ or } y = -x.$$

## EXAMPLE 6 Connecting the derivative of a sinusoidal function to its extreme values

Determine the maximum and minimum values of the function $f(x) = \cos^2 x$ on the interval $x \in [0, 2\pi]$.

### Solution

By the algorithm for finding extreme values, the maximum and minimum values occur at points on the graph where $f'(x) = 0$ or at endpoints of the interval. The derivative of $f(x)$ is

$$f'(x) = 2(\cos x)(-\sin x) \quad \text{(Chain rule)}$$

$$= -2 \sin x \cos x$$

$$= -\sin 2x \quad \text{(Using the double angle identity)}$$

Solving $f'(x) = 0$,

$$-\sin 2x = 0$$

$$\sin 2x = 0$$

$$2x = 0, \pi, 2\pi, 3\pi, \text{ or } 4\pi$$

$$\text{so } x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \text{ or } 2\pi$$

We evaluate $f(x)$ at the critical numbers. (In this example, the endpoints of the interval are included.)

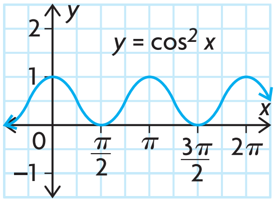
| $x$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| $f(x) = \cos^2 x$ | 1 | 0 | 1 | 0 | 1 |

The maximum value is 1 when $x = 0$, $\pi$, or $2\pi$. The minimum value is 0 when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

The above solution is verified by our knowledge of the cosine function. For the function $y = \cos x$,
- the domain is $x \in \mathbf{R}$
- the range is $-1 \le \cos x \le 1$

For the given function $y = \cos^2 x$,
- the domain is $x \in \mathbf{R}$
- the range is $0 \le \cos^2 x \le 1$

Therefore, the maximum value is 1 and the minimum value is 0.




## IN SUMMARY

### Key Idea

- The derivatives of sinusoidal functions are found as follows:
  - $\frac{d(\sin x)}{dx} = \cos x$ and $\frac{d(\cos x)}{dx} = -\sin x$
  - If $y = \sin f(x)$, then $\frac{dy}{dx} = \cos f(x) \times f'(x)$.
  - If $y = \cos f(x)$, then $\frac{dy}{dx} = -\sin f(x) \times f'(x)$.

### Need to Know

- When you are differentiating a function that involves sinusoidal functions, use the rules given above, along with the sum, difference, product, quotient, and chain rules as required.

# Exercise 5.4

## PART A

K 1. Determine $\frac{dy}{dx}$ for each of the following:

a. $y = \sin 2x$
b. $y = 2 \cos 3x$
c. $y = \sin (x^3 - 2x + 4)$
d. $y = 2 \cos (-4x)$
e. $y = \sin 3x - \cos 4x$
f. $y = 2^x + 2 \sin x - 2 \cos x$
g. $y = \sin (e^x)$
h. $y = 3 \sin (3x + 2\pi)$
i. $y = x^2 + \cos x + \sin \frac{\pi}{4}$
j. $y = \sin \frac{1}{x}$

2. Differentiate the following functions:

a. $y = 2 \sin x \cos x$
b. $y = \frac{\cos 2x}{x}$
c. $y = \cos (\sin 2x)$
d. $y = \frac{\sin x}{1 + \cos x}$
e. $y = e^x(\cos x + \sin x)$
f. $y = 2x^3 \sin x - 3x \cos x$

## PART B

3. Determine an equation for the tangent at the point with the given $x$-coordinate for each of the following functions:

a. $f(x) = \sin x, x = \frac{\pi}{3}$
b. $f(x) = x + \sin x, x = 0$
c. $f(x) = \cos (4x), x = \frac{\pi}{4}$
d. $f(x) = \sin 2x + \cos x, x = \frac{\pi}{2}$
e. $f(x) = \cos \left(2x + \frac{\pi}{3}\right), x = \frac{\pi}{4}$
f. $f(x) = 2 \sin x \cos x, x = \frac{\pi}{2}$

C 4. a. If $f(x) = \sin^2 x$ and $g(x) = 1 - \cos^2 x$, explain why $f'(x) = g'(x)$.
b. If $f(x) = \sin^2 x$ and $g(x) = 1 + \cos^2 x$, how are $f'(x)$ and $g'(x)$ related?

5. Differentiate each function.

a. $v(t) = \sin^2 (\sqrt{t})$
b. $v(t) = \sqrt{1 + \cos t + \sin^2 t}$
c. $h(x) = \sin x \sin 2x \sin 3x$
d. $m(x) = (x^2 + \cos^2 x)^3$

6. Determine the absolute extreme values of each function on the given interval. (Verify your results with graphing technology.)

   a. $y = \cos x + \sin x, 0 \leq x \leq 2\pi$
   b. $y = x + 2\cos x, -\pi \leq x \leq \pi$
   c. $y = \sin x - \cos x, x \in [0, 2\pi]$
   d. $y = 3\sin x + 4\cos x, x \in [0, 2\pi]$

A 7. A particle moves along a line so that, at time $t$, its position is $s(t) = 8\sin 2t$.

   a. For what values of $t$ does the particle change direction?
   b. What is the particle's maximum velocity?

8. a. Graph the function $f(x) = \cos x + \sin x$.
   b. Determine the coordinates of the point where the tangent to the curve of $f(x)$ is horizontal, on the interval $0 \leq x \leq \pi$.

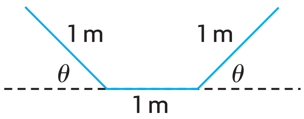
9. Determine expressions for the derivatives of $\csc x$ and $\sec x$.

10. Determine the slope of the tangent to the curve $y = \cos 2x$ at point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

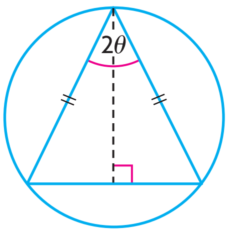
11. A particle moves along a line so that at time $t$, its position is $s = 4\sin 4t$.

   a. When does the particle change direction?
   b. What is the particle's maximum velocity?
   c. What is the particle's minimum distance from the origin? What is its maximum distance from the origin?

T 12. An irrigation channel is constructed by bending a sheet of metal that is 3 m wide, as shown in the diagram. What angle $\theta$ will maximize the cross-sectional area (and thus the capacity) of the channel?




13. An isosceles triangle is inscribed in a circle of radius **R**. Find the value of $\theta$ that maximizes the area of the triangle.




## PART C

14. If $y = A\cos kt + B\sin kt$, where $A$, $B$, and $k$ are constants, show that $y'' + k^2y = 0$.