Section 5.3—Optimization Problems Involving Exponential Functions

In earlier chapters, you considered numerous situations in which you were asked to optimize a given situation. As you learned, to optimize means to determine values of variables so that a function representing quantities such as cost, area, number of objects, or distance can be minimized or maximized.

Here we will consider further optimization problems, using exponential function models.

EXAMPLE 1 Solving an optimization problem involving an exponential model

The effectiveness of studying for an exam depends on how many hours a student studies. Some experiments show that if the effectiveness, E, is put on a scale of 0 to 10, then $E(t) = 0.5\left[10 + te^{-\frac{t}{20}}\right]$, where t is the number of hours spent studying for an examination. If a student has up to 30 h for studying, how many hours are needed for maximum effectiveness?

Solution

We wish to find the maximum value of the function $E(t) = 0.5\left[10 + te^{-\frac{t}{20}}\right]$, on the interval $0 \leq t \leq 30$.

First find critical numbers by determining $E'(t)$.

$$E'(t) = 0.5\left(e^{-\frac{t}{20}} + t\left(-\frac{1}{20}e^{-\frac{t}{20}}\right)\right) \quad \text{(Product and chain rules)}$$

$$= 0.5e^{-\frac{t}{20}}\left(1 - \frac{t}{20}\right)$$

E' is defined for $t \in \mathbf{R}$, and $e^{-\frac{t}{20}} > 0$ for all values of t. So, $E'(t) = 0$ when $1 - \frac{t}{20} = 0$.

Therefore, $t = 20$ is the only critical number.

To determine the maximum effectiveness, we use the algorithm for finding extreme values.

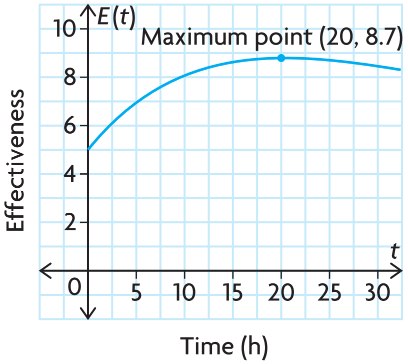
$$E(0) = 0.5(10 + 0e^{0}) = 5$$

$$E(20) = 0.5(10 + 20e^{-1}) \doteq 8.7$$

$$E(30) = 0.5(10 + 30e^{-1.5}) \doteq 8.3$$

Therefore, the maximum effectiveness measure of 8.7 is achieved when a student studies 20 h for the exam.

Examining the graph of the function $E(t) = 0.5[10 + te^{-\frac{t}{20}}]$ confirms our result.



EXAMPLE 2 Using calculus techniques to analyze an exponential business model

A mathematical consultant determines that the proportion of people who will have responded to the advertisement of a new product after it has been marketed for t days is given by $f(t) = 0.7(1 - e^{-0.2t})$. The area covered by the advertisement contains 10 million potential customers, and each response to the advertisement results in revenue to the company of \$0.70 (on average), excluding the cost of advertising. The advertising costs \$30 000 to produce and a further \$5000 per day to run.

a. Determine $\lim_{t\to\infty} f(t)$, and interpret the result.

b. What percent of potential customers have responded after seven days of advertising?

c. Write the function $P(t)$ that represents the average profit after t days of advertising. What is the average profit after seven days?

d. For how many full days should the advertising campaign be run in order to maximize the average profit? Assume an advertising budget of \$200 000.

Solution

a. As $t \to \infty$, $e^{-0.2t} \to 0$, so $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} 0.7(1 - e^{-0.2t}) = 0.7$. This result means that if the advertising is left in place indefinitely (forever), 70% of the population will respond.

b. $f(7) = 0.7(1 - e^{-0.2(7)}) \doteq 0.53$

After seven days of advertising, about 53% of the population has responded.

c. The average profit is the difference between the average revenue received from all customers responding to the ad and the advertising costs. Since the area covered by the ad contains 10 million potential customers, the number of customers responding to the ad after t days is $10^7[0.7(1 - e^{-0.2t})] = 7 \times 10^6(1 - e^{-0.2t})$.

The average revenue to the company from these respondents is
$R(t) = 0.7[7 \times 10^6(1 - e^{-0.2t})] = 4.9 \times 10^6(1 - e^{-0.2t})$.
The advertising costs for t days are $C(t) = 30\,000 + 5000t$.
Therefore, the average profit earned after t days of advertising is given by

$$\begin{aligned} P(t) &= R(t) - C(t) \\ &= 4.9 \times 10^6(1 - e^{-0.2t}) - 30\,000 - 5000t \end{aligned}$$

After seven days of advertising, the average profit is

$$\begin{aligned} P(7) &= 4.9 \times 10^6(1 - e^{-0.2(7)}) - 30\,000 - 5000(7) \\ &\doteq 3\,627\,000 \end{aligned}$$

d. If the total advertising budget is \$200 000, then we require that

$$\begin{aligned} 30\,000 + 5000t &\le 200\,000 \\ 5000t &\le 170\,000 \\ t &\le 34 \end{aligned}$$

We wish to maximize the average profit function $P(t)$ on the interval $0 \le t \le 34$.
For critical numbers, determine $P'(t)$.

$$\begin{aligned} P'(t) &= 4.9 \times 10^6(0.2e^{-0.2t}) - 5000 \\ &= 9.8 \times 10^5 e^{-0.2t} - 5000 \end{aligned}$$

$P'(t)$ is defined for $t \in \mathbf{R}$. Let $P'(t) = 0$.

$$9.8 \times 10^5 e^{-0.2t} - 5000 = 0$$

$$e^{-0.2t} = \frac{5000}{9.8 \times 10^5} \quad \text{(Isolate } e^{-0.2t}\text{)}$$

$$e^{-0.2t} \doteq 0.005\,102\,04 \quad \text{(Take the ln of both sides)}$$

$$-0.2t = \ln(0.005\,102\,04) \quad \text{(Solve)}$$

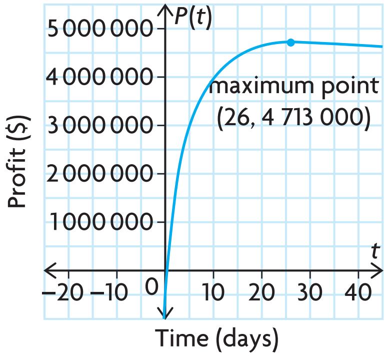
$$t \doteq 26$$

To determine the maximum average profit, we evaluate.

$$\begin{aligned} P(26) &= 4.9 \times 10^6(1 - e^{-0.2(26)}) - 30\,000 - 5000(26) \\ &\doteq 4\,713\,000 \\ P(0) &= 4.9 \times 10^6(1 - e^{0}) - 30\,000 - 0 \\ &= -30\,000 \text{ (They're losing money!)} \\ P(34) &= 4.9 \times 10^6(1 - e^{-0.2(34)}) - 30\,000 - 5000(34) \\ &\doteq 4\,695\,000 \end{aligned}$$

The maximum average profit of \$4 713 000 occurs when the ad campaign runs for 26 days.

Examining the graph of the function $P(t)$ confirms our result.



IN SUMMARY

Key Ideas

- Optimizing means determining the values of the independent variable so that the values of a function that models a situation can be minimized or maximized.
- The techniques used to optimize an exponential function model are the same as those used to optimize polynomial and rational functions.

Need to Know

- Apply the algorithm introduced in Chapter 3 to solve an optimization problem:

1. Understand the problem, and identify quantities that can vary. Determine a function in one variable that represents the quantity to be optimized.
2. Determine the domain of the function to be optimized, using the information given in the problem.
3. Use the algorithm for finding extreme values (from Chapter 3) to find the absolute maximum or minimum value of the function on the domain.
4. Use your result from step 3 to answer the original problem.
5. Graph the original function using technology to confirm your results.

Exercise 5.3

PART A

1. Use graphing technology to graph each of the following functions. From the graph, find the absolute maximum and absolute minimum values of the given functions on the indicated intervals.

   a. $f(x) = e^{-x} - e^{-3x}$ on $0 \leq x \leq 10$

   b. $m(x) = (x + 2)e^{-2x}$ on $x \in [-4, 4]$

2. a. Use the algorithm for finding extreme values to determine the absolute maximum and minimum values of the functions in question 1.

   b. Explain which approach is easier to use for the functions in question 1.

3. The squirrel population in a small self-contained forest was studied by a biologist. The biologist found that the squirrel population, P, measured in hundreds, is a function of time, t, where t is measured in weeks. The function is $P(t) = \frac{20}{1 + 3e^{-0.02t}}$.

   a. Determine the population at the start of the study, when $t = 0$.

   b. The largest population the forest can sustain is represented mathematically by the limit as $t \to \infty$. Determine this limit.

   c. Determine the point of inflection.

   d. Graph the function.

   e. Explain the meaning of the point of inflection in terms of squirrel population growth.

PART B

4. The net monthly profit, in dollars, from the sale of a certain item is given by the formula $P(x) = 10^6[1 + (x - 1)e^{-0.001x}]$, where x is the number of items sold.

   a. Determine the number of items that yield the maximum profit. At full capacity, the factory can produce 2000 items per month.

   b. Repeat part a., assuming that, at most, 500 items can be produced per month.

K 5. Suppose that the monthly revenue in thousands of dollars, for the sale of x hundred units of an electronic item is given by the function $R(x) = 40x^2e^{-0.4x} + 30$, where the maximum capacity of the plant is 800 units. Determine the number of units to produce in order to maximize revenue.

6. A rumour spreads through a population in such a way that t hours after the rumour starts, the percent of people involved in passing it on is given by $P(t) = 100\left(e^{-t} - e^{-4t}\right)$. What is the highest percent of people involved in spreading the rumour within the first 3 h? When does this occur?

7. Small countries trying to develop an industrial economy rapidly often try to achieve their objectives by importing foreign capital and technology. Statistics Canada data show that when Canada attempted this strategy from 1867 to 1967, the amount of U.S. investment in Canada increased from about $\$15 \times 10^6$ to $\$280\,305 \times 10^6$. This increase in foreign investment can be represented by the simple mathematical model $C(t) = 0.015 \times 10^9 e^{0.075\,33t}$, where t represents the number of years (starting with 1867 as zero) and C represents the total capital investment from U.S. sources in dollars.

   a. Graph the curve for the 100-year period.

   b. Compare the growth rate of U.S. investment in 1947 with the rate in 1967.

   c. Determine the growth rate of investment in 1967 as a percent of the amount invested.

   d. If this model is used up to 1977, calculate the total U.S. investment and the growth rate in this year.

   e. Use the Internet to determine the actual total U.S. investment in 1977, and calculate the error in the model.

   f. If the model is used up to 2007, calculate the expected U.S. investment and the expected growth rate.

A

8. A colony of bacteria in a culture grows at a rate given by $N(t) = 2^{\frac{t}{5}}$, where N is the number of bacteria t minutes from the beginning. The colony is allowed to grow for 60 min, at which time a drug is introduced to kill the bacteria. The number of bacteria killed is given by $K(t) = e^{\frac{t}{3}}$, where K bacteria are killed at time t minutes.

   a. Determine the maximum number of bacteria present and the time at which this occurs.

   b. Determine the time at which the bacteria colony is obliterated.

9. Lorianne is studying for two different exams. Because of the nature of the courses, the measure of study effectiveness on a scale from 0 to 10 for the first course is $E_1 = 0.6(9 + te^{-\frac{t}{20}})$, while the measure for the second course is $E_2 = 0.5(10 + te^{-\frac{t}{10}})$. Lorianne is prepared to spend up to 30 h, in total, studying for the exams. The total effectiveness is given by $f(t) = E_1 + E_2$. How should this time be allocated to maximize total effectiveness?

C

10. Explain the steps you would use to determine the absolute extrema of $f(x) = x - e^{2x}$ on the interval $x \in [-2, 2]$.

T

11. a. For $f(x) = x^2 e^x$, determine the intervals of increase and decrease.

    b. Determine the absolute minimum value of $f(x)$.

12. Find the maximum and minimum values of each function. Graph each function.

a. $y = e^x + 2$

b. $y = xe^x + 3$

c. $y = 2xe^{2x}$

d. $y = 3xe^{-x} + x$

13. The profit function of a commodity is $P(x) = xe^{-0.5x^2}$, where $x > 0$. Find the maximum value of the function if x is measured in hundreds of units and P is measured in thousands of dollars.

14. You have just walked out the front door of your home. You notice that it closes quickly at first and then closes more slowly. In fact, a model of the movement of the door is given by $d(t) = 200\,t(2)^{-t}$, where d is the number of degrees between the door frame and the door at t seconds.

a. Graph this relation.

b. Determine when the speed of the moving door is increasing and decreasing.

c. Determine the maximum speed of the moving door.

d. At what point would you consider the door closed?

PART C

15. Suppose that, in question 9, Lorianne has only 25 h to study for the two exams. Is it possible to determine the time to be allocated to each exam? If so, how?

16. Although it is true that many animal populations grow exponentially for a period of time, it must be remembered that the food available to sustain the population is limited and the population will level off because of this. Over a period of time, the population will level out to the maximum attainable value, L. One mathematical model to describe a population that grows exponentially at the beginning and then levels off to a limiting value, L, is the **logistic model**. The equation for this model is $P = \frac{aL}{a + (L - a)e^{-kLt}}$, where the independent variable t represents the time and P represents the size of the population. The constant a is the size of the population at $t = 0$, L is the limiting value of the population, and k is a mathematical constant.

a. Suppose that a biologist starts a cell colony with 100 cells and finds that the limiting size of the colony is 10 000 cells. If the constant $k = 0.0001$, draw a graph to illustrate this population, where t is in days.

b. At what point in time does the cell colony stop growing exponentially? How large is the colony at this point?

c. Compare the growth rate of the colony at the end of day 3 with the growth rate at the end of day 8. Explain what is happening.