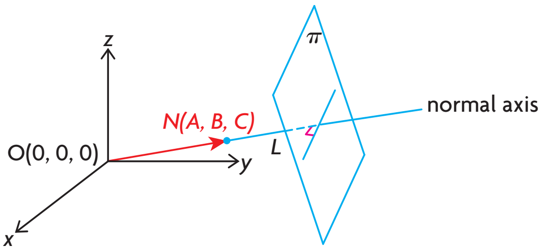
Section 8.5—The Cartesian Equation of a Plane

In the previous section, the vector and parametric equations of a plane were found. In this section, we will show how to derive the Cartesian (or scalar) equation of a plane. The process is very similar to the process used to find the Cartesian equation of a line in R^2.

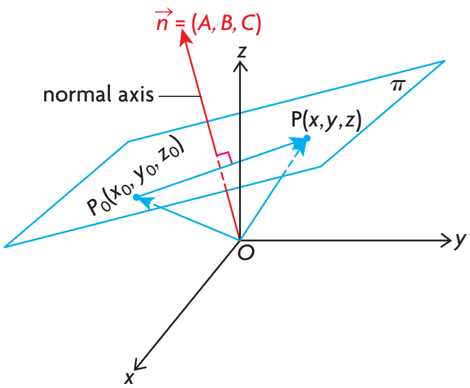


In the diagram above, a plane π is shown, along with a line L drawn from the origin, so that L is perpendicular to the given plane. For any plane in R^3, there is only one possible line that can be drawn through the origin perpendicular to the plane. This line is called the normal axis for the plane. The direction of the normal axis is given by a vector joining the origin to any point on the normal axis. The direction vector is called a normal to the plane. In the diagram, $\overrightarrow{ON}$ is a normal to the plane because it joins the origin to $N(A, B, C)$, a point on the normal axis. Any nonzero vector is a normal to a plane if it lies along the normal axis. This implies that an infinite number of normals exist for all planes.

A plane is completely determined when we know a point $(P_0(x_0, y_0, z_0))$ on the plane and a normal to the plane. This single idea can be used to determine the Cartesian equation of a plane.

Deriving the Cartesian Equation of a Plane

Consider the following diagram:



To derive the equation of this plane, we need two points on the plane, $P_0(x_0, y_0, z_0)$ (its coordinates given) and a general point, $P(x, y, z)$, different from P_0. The vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ represents any vector on the plane. If $\vec{n} = \overrightarrow{ON} = (A, B, C)$ is a known normal to the plane, then the relationship, $\vec{n} \cdot \overrightarrow{P_0P} = 0$ can be used to derive the equation of the plane, since $\vec{n}$ and any vector on the plane are perpendicular.

$\vec{n} \cdot \overrightarrow{P_0P} = 0$

$(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$Ax - Ax_0 + By - By_0 + Cz - Cz_0 = 0$

Or, $Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) = 0$

Since the quantities in the expression $-Ax_0 - By_0 - Cz_0$ are known, we'll replace this with D to make the equation simpler. The Cartesian equation of the plane is, thus, $Ax + By + Cz + D = 0$.

Cartesian Equation of a Plane

The Cartesian (or scalar) equation of a plane in R^3 is of the form $Ax + By + Cz + D = 0$ with normal $\vec{n} = (A, B, C)$. The normal $\vec{n}$ is a nonzero vector perpendicular to all vectors in the plane.

EXAMPLE 1 Representing a plane by its Cartesian equation

The point $A(1, 2, 2)$ is a point on the plane with normal $\vec{n} = (-1, 2, 6)$. Determine the Cartesian equation of this plane.

Solution

Two different methods can be used to determine the Cartesian equation of this plane. Both methods will give the same answer.

Method 1:
Let $P(x, y, z)$ be any point on the plane.

Therefore, $\overrightarrow{AP} = (x - 1, y - 2, z - 2)$ represents any vector on the plane. Since $\vec{n} = (-1, 2, 6)$ and $\vec{n} \cdot \overrightarrow{AP} = 0$,

$$\begin{aligned}
(-1, 2, 6) \cdot (x - 1, y - 2, z - 2) &= 0 \\
-1(x - 1) + 2(y - 2) + 6(z - 2) &= 0 \\
-x + 1 + 2y - 4 + 6z - 12 &= 0 \\
-x + 2y + 6z - 15 &= 0
\end{aligned}$$

Multiplying each side by -1, $x - 2y - 6z + 15 = 0$.

Either $-x + 2y + 6z - 15 = 0$ or $x - 2y - 6z + 15 = 0$ is a correct equation for the plane, but usually we write the equation with integer coefficients and with a positive coefficient for the x-term.

Method 2:
Since the required equation has the form $Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C) = (-1, 2, 6)$, the direction numbers for the normal can be substituted directly into the equation. This gives $-x + 2y + 6z + D = 0$, with D to be determined. Since the point $A(1, 2, 2)$ is on the plane, it satisfies the equation.

Substituting the coordinates of this point into the equation gives $-(1) + 2(2) + 6(2) + D = 0$, and thus $D = -15$.

If $D = -15$ is substituted into $-x + 2y + 6z + D = 0$, the equation will be $-x + 2y + 6z + (-15) = 0$ or $x - 2y - 6z + 15 = 0$.

To find the Cartesian equation of a plane, either Method 1 or Method 2 can be used.

The Cartesian equation of a plane is simpler than either the vector or the parametric form and is used most often.

EXAMPLE 2

Determining the Cartesian equation of a plane from three coplanar points

Determine the Cartesian equation of the plane containing the points $A(-1, 2, 5)$, $B(3, 2, 4)$, and $C(-2, -3, 6)$.

Solution

A normal to this plane is determined by calculating the cross product of the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This results in a vector perpendicular to the plane in which both these vectors lie.

$\overrightarrow{AB} = (3 - (-1), 2 - 2, 4 - 5) = (4, 0, -1)$ and

$\overrightarrow{AC} = (-2 - (-1), -3 - 2, 6 - 5) = (-1, -5, 1)$

$$\text{Thus, } \overrightarrow{AB} \times \overrightarrow{AC} = (0(1) - (-1)(-5), -1(-1) - (4)(1), 4(-5) - (0)(-1))$$
$$= (-5, -3, -20)$$
$$= -1(5, 3, 20)$$

If we let $P(x, y, z)$ be any point on the plane, then $\overrightarrow{AP} = (x + 1, y - 2, z - 5)$, and since a normal to the plane is $(5, 3, 20)$,

$$(5, 3, 20) \cdot (x + 1, y - 2, z - 5) = 0$$

$$5x + 5 + 3y - 6 + 20z - 100 = 0$$

After simplifying, the required equation of the plane is $5x + 3y + 20z - 101 = 0$.

A number of observations can be made about this calculation. If we had used $\overrightarrow{AB}$ and $\overrightarrow{BC}$ as direction vectors, for example, we would have found that $\overrightarrow{BC} = (-5, -5, 2)$ and $\overrightarrow{AB} \times \overrightarrow{BC} = (4, 0, -1) \times (-5, -5, 2) = -1(5, 3, 20)$. When finding the equation of a plane, it is possible to use any pair of direction vectors on the plane to find a normal to the plane. Also, when finding the value for D, if we had used the method of substitution, it would have been possible to substitute any one of the three given points in the equation.

In the next example, we will show how to convert from vector or parametric form to Cartesian form. We will also show how to obtain the vector form of a plane if given its Cartesian form.

EXAMPLE 3 **Connecting the various forms of the equation of a plane**

a. Determine the Cartesian form of the plane whose equation in vector form is $\vec{r} = (1, 2, -1) + s(1, 0, 2) + t(-1, 3, 4), s, t \in \mathbf{R}$.

b. Determine the vector and parametric equations of the plane with Cartesian equation $x - 2y + 5z - 6 = 0$.

Solution

a. To find the Cartesian equation of the plane, two direction vectors are needed so that a normal to the plane can be determined. The two given direction vectors for the plane are $(1, 0, 2)$ and $(-1, 3, 4)$. Their cross product is

$$\begin{aligned}(1, 0, 2) \times (-1, 3, 4) &= (0(4)-2(3), 2(-1)-1(4), 1(3)-0(-1)) \\ &= -3(2, 2, -1)\end{aligned}$$

A normal to the plane is $(2, 2, -1)$, and the Cartesian equation of the plane is of the form $2x + 2y - z + D = 0$. Substituting the point $(1, 2, -1)$ into this equation gives $2(1) + 2(2)-(-1) + D = 0$, or $D = -7$.

Therefore, the Cartesian equation of the plane is $2x + 2y - z - 7 = 0$.

b. *Method 1:*

To find the corresponding vector and parametric equations of a plane, the equation of the plane is first converted to its parametric form. The simplest way to do this is to choose any two of the variables and replace them with a parameter. For example, if we substitute $y = s$ and $z = t$ and solve for x, we obtain $x - 2s + 5t - 6 = 0$ or $x = 2s - 5t + 6$.

This gives us the required parametric equations $x = 2s - 5t + 6$, $y = s$, and $z = t$. The vector form of the plane can be found by rearranging the parametric form.

Therefore, $(x, y, z) = (2s - 5t + 6, s, t)$

$$(x, y, z) = (6, 0, 0) + (2s, s, 0) + (-5t, 0, t)$$
$$\vec{r} = (6, 0, 0) + s(2, 1, 0) + t(-5, 0, 1),\ s, t \in \mathbf{R}$$

The parametric equations of this plane are $x = 2s - 5t + 6$, $y = s$, and $z = t$, and the corresponding vector form is $\vec{r} = (6, 0, 0) + s(2, 1, 0) + t(-5, 0, 1),\ s, t \in \mathbf{R}$.

Check:
This vector equation of the plane can be checked by converting to Cartesian form. A normal to the plane is $(2, 1, 0) \times (-5, 0, 1) = (1, -2, 5)$. The plane has the form $x - 2y + 5z + D = 0$. If $(6, 0, 0)$ is substituted into the equation to find D, we find that $6 - 2(0) + 5(0) + D = 0$, so $D = -6$ and the equation is the given equation $x - 2y + 5z - 6 = 0$.

Method 2:
We rewrite the given equation as $x = 2y - 5z + 6$. We are going to find the coordinates of three points on the plane, and writing the equation in this way allows us to choose integer values for y and z that will give an integer value for x. The values in the table are chosen to make the computation as simple as possible.

The following table shows our choices for y and z, along with the calculation for x.

y	z	$x = 2y - 5z + 6$	**Resulting Point**
0	0	$x = 2(0) - 5(0) + 6 = 6$	$A(6, 0, 0)$
-1	0	$x = 2(-1) - 5(0) + 6 = 4$	$B(4, -1, 0)$
-1	1	$x = 2(-1) - 5(1) + 6 = -1$	$C(-1, -1, 1)$

$\overrightarrow{AB} = (4 - 6, -1 - 0, 0 - 0) = (-2, -1, 0)$ and
$\overrightarrow{AC} = (-1 - 6, -1 - 0, 1 - 0) = (-7, -1, 1)$

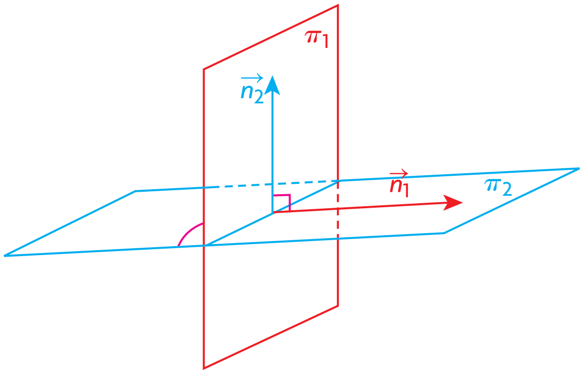
A vector equation is $\vec{r} = (6, 0, 0) + p(-2, -1, 0) + q(-7, -1, 1),\ p, q \in \mathbf{R}$.

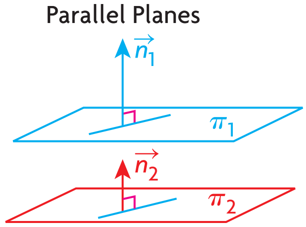
The corresponding parametric form is $x = 6 - 2p - 7q$, $y = -p - q$, and $z = q$.

Check:
To check that these equations are correct, the same procedure shown in Method 1 is used. This gives the identical Cartesian equation, $x - 2y + 5z - 6 = 0$.

When we considered lines in R^2, we showed how to determine whether lines were parallel or perpendicular. It is possible to use the same formula to determine whether planes are parallel or perpendicular.

Parallel and Perpendicular Planes





1. If π_1 and π_2 are two perpendicular planes, with normals $\vec{n_1}$ and $\vec{n_2}$, respectively, their normals are perpendicular (that is, $\vec{n_1} \cdot \vec{n_2} = 0$).
2. If π_1 and π_2 are two parallel planes, with normals $\vec{n_1}$ and $\vec{n_2}$, respectively, their normals are parallel (that is, $\vec{n_1} = k\vec{n_2}$) for all nonzero real numbers k.

EXAMPLE 4 Reasoning about parallel and perpendicular planes

a. Show that the planes $\pi_1: 2x - 3y + z - 1 = 0$ and $\pi_2: 4x - 3y - 17z = 0$ are perpendicular.

b. Show that the planes $\pi_3: 2x - 3y + 2z - 1 = 0$ and $\pi_4: 2x - 3y + 2z - 3 = 0$ are parallel but not coincident.

Solution

a. For π_1: $\vec{n_1} = (2, -3, 1)$ and for $\pi_2: \vec{n_2} = (4, -3, -17)$.

$$\begin{aligned}\vec{n_1} \cdot \vec{n_2} &= (2, -3, 1) \cdot (4, -3, -17) \\ &= 2(4) - 3(-3) + 1(-17) \\ &= 8 + 9 - 17 \\ &= 0\end{aligned}$$

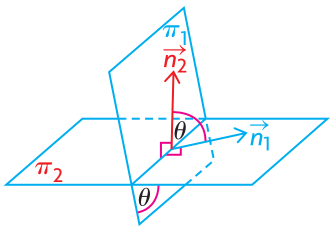
Since $\vec{n_1} \cdot \vec{n_2} = 0$, the two planes are perpendicular to each other.

b. For π_3 and π_4, $\vec{n_3} = \vec{n_4} = (2, -3, 2)$, so the planes are parallel. Because the planes have different constants (that is, -1 and -3), the planes are not coincident.

In general, if planes are coincident, it means that the planes have equations that are scalar multiples of each other. For example, the two planes $2x - y + z - 13 = 0$ and $-6x + 3y - 3z + 39 = 0$ are coincident because $-6x + 3y - 3z + 39 = -3(2x - y + z - 13)$.

It is also possible to find the angle between intersecting planes using their normals and the dot product formula for calculating the angle between two vectors. The angle between two planes is the same as the angle between their normals.

Angle between Intersecting Planes



The angle θ between two planes, π_1 and π_2, with normals of $\vec{n_1}$ and $\vec{n_2}$, respectively, can be calculated using the formula $\cos\theta = \dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$.

EXAMPLE 5 Calculating the angle formed between two intersecting planes

Determine the angle between the two planes $\pi_1: x - y - 2z + 3 = 0$ and $\pi_2: 2x + y - z + 2 = 0$.

Solution

For $\pi_1: \vec{n_1} = (1, -1, -2)$. For $\pi_2: \vec{n_2} = (2, 1, -1)$.

Since $|\vec{n_1}| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$ and

$|\vec{n_2}| = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{6}$,

$$\cos\theta = \frac{(1, -1, -2)\cdot(2, 1, -1)}{\sqrt{6}\sqrt{6}}$$

$$\cos\theta = \frac{2 - 1 + 2}{6}$$

$$\cos\theta = \frac{1}{2}$$

Therefore, the angle between the two planes is 60°. Normally, the angle between planes is given as an acute angle, but it is also correct to express it as 120°.

IN SUMMARY

Key Idea

- The Cartesian equation of a plane in R^3 is $Ax + By + Cz + D = 0$, where $\vec{n} = (A, B, C)$ is a normal to the plane and $\vec{n} = \vec{a} \times \vec{b}$. $\vec{a}$ and $\vec{b}$ are any two noncollinear direction vectors of the plane.

Need to Know

- Two planes whose normals are $\vec{n_1}$ and $\vec{n_2}$
  - are parallel if and only if $\vec{n_1} = k\vec{n_2}$ for any nonzero real number k.
  - are perpendicular if and only if $\vec{n_1} \cdot \vec{n_2} = 0$.
  - have an angle θ between the planes determined by $\theta = \cos^{-1}\left(\frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right)$.

Exercise 8.5

PART A

1. A plane is defined by the equation $x - 7y - 18z = 0$.
   a. What is a normal vector to this plane?
   b. Explain how you know that this plane passes through the origin.
   c. Write the coordinates of three points on this plane.
2. A plane is defined by the equation $2x - 5y = 0$.
   a. What is a normal vector to this plane?
   b. Explain how you know that this plane passes through the origin.
   c. Write the coordinates of three points on this plane.
3. A plane is defined by the equation $x = 0$.
   a. What is a normal vector to this plane?
   b. Explain how you know that this plane passes through the origin.
   c. Write the coordinates of three points on this plane.
4. a. A plane is determined by a normal, $\vec{n} = (15, 75, -105)$, and passes through the origin. Write the Cartesian equation of this plane, where the normal is in reduced form.
   b. A plane has a normal of $\vec{n} = \left(-\frac{1}{2}, \frac{3}{4}, \frac{7}{16}\right)$ and passes through the origin. Determine the Cartesian equation of this plane.

PART B

5. A plane is determined by a normal, $\vec{n} = (1, 7, 5)$, and contains the point $P(-3, 3, 5)$. Determine a Cartesian equation for this plane using the *two* methods shown in Example 1.

K 6. The three noncollinear points $P(-1, 2, 1)$, $Q(3, 1, 4)$, and $R(-2, 3, 5)$ lie on a plane.

a. Using $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ as direction vectors and the point $R(-2, 3, 5)$, determine the Cartesian equation of this plane.

b. Using $\overrightarrow{QP}$ and $\overrightarrow{PR}$ as direction vectors and the point $P(-1, 2, 1)$, determine the Cartesian equation of this plane.

c. Explain why the two equations must be the same.

7. Determine the Cartesian equation of the plane that contains the points $A(-2, 3, 1)$, $B(3, 4, 5)$, and $C(1, 1, 0)$.

8. The line with vector equation $\vec{r} = (2, 0, 1) + s(-4, 5, 5)$, $s \in \mathbf{R}$, lies on the plane π, as does the point $P(1, 3, 0)$. Determine the Cartesian equation of π.

9. Determine unit vectors that are normal to each of the following planes:

a. $2x + 2y - z - 1 = 0$

b. $4x - 3y + z - 3 = 0$

c. $3x - 4y + 12z - 1 = 0$

10. A plane contains the point $A(2, 2, -1)$ and the line $\vec{r} = (1, 1, 5) + s(2, 1, 3)$, $s \in \mathbf{R}$. Determine the Cartesian equation of this plane.

A 11. Determine the Cartesian equation of the plane containing the point $(-1, 1, 0)$ and perpendicular to the line joining the points $(1, 2, 1)$ and $(3, -2, 0)$.

C 12. a. Explain the process you would use to determine the angle formed between two intersecting planes.

b. Determine the angle between the planes $x - z + 7 = 0$ and $2x + y - z + 8 = 0$.

13. a. Determine the angle between the planes $x + 2y - 3z - 4 = 0$ and $x + 2y - 1 = 0$.

b. Determine the Cartesian equation of the plane that passes through the point $P(1, 2, 1)$ and is perpendicular to the line $\frac{x-3}{-2} = \frac{y+1}{3} = \frac{z+4}{1}$.

14. a. What is the value of k that makes the planes $4x + ky - 2z + 1 = 0$ and $2x + 4y - z + 4 = 0$ parallel?

b. What is the value of k that makes these two planes perpendicular?

c. Can these two planes ever be coincident? Explain.

T 15. Determine the Cartesian equation of the plane that passes through the points $(1, 4, 5)$ and $(3, 2, 1)$ and is perpendicular to the plane $2x - y + z - 1 = 0$.

PART C

16. Determine an equation of the plane that is perpendicular to the plane $x + 2y + 4 = 0$, contains the origin, and has a normal that makes an angle of $30°$ with the z-axis.

17. Determine the equation of the plane that lies between the points $(-1, 2, 4)$ and $(3, 1, -4)$ and is equidistant from them.