Section 5.5—The Derivative of $y = \tan x$

In this section, we will study the derivative of the remaining primary trigonometric function—tangent.

Since this function can be expressed in terms of sine and cosine, we can find its derivative using the product rule.

EXAMPLE 1 Reasoning about the derivative of the tangent function

Determine $\frac{dy}{dx}$ for $y = \tan x$.

Solution

$$\begin{aligned}
y &= \tan x \\
&= \frac{\sin x}{\cos x} \\
&= (\sin x)(\cos x)^{-1} \\
\frac{dy}{dx} &= \frac{d(\sin x)}{dx} \times (\cos x)^{-1} + \sin x \times \frac{d(\cos x)^{-1}}{dx} && \text{(Product rule)} \\
&= (\cos x)(\cos x)^{-1} + \sin x\,(-1)(\cos x)^{-2}(-\sin x) && \text{(Chain rule)} \\
&= 1 + \frac{\sin^2 x}{\cos^2 x} \\
&= 1 + \tan^2 x && \text{(Using the Pythagorean identity)} \\
&= \sec^2 x
\end{aligned}$$

Therefore, $\frac{d(\tan x)}{dx} = \sec^2 x$

EXAMPLE 2 Selecting a strategy to determine the derivative of a composite tangent function

Determine $\frac{dy}{dx}$ for $y = \tan(x^2 + 3x)$.

Solution

$$\begin{aligned}
y &= \tan(x^2 + 3x) \\
\frac{dy}{dx} &= \frac{d\tan(x^2 + 3x)}{d(x^2 + 3x)} \times \frac{d(x^2 + 3x)}{dx} && \text{(Chain rule)} \\
&= \sec^2(x^2 + 3x) \times (2x + 3) \\
&= (2x + 3)\sec^2(x^2 + 3x)
\end{aligned}$$

Derivatives of Composite Functions Involving $y = \tan x$

If $y = \tan f(x)$, then $\frac{dy}{dx} = \sec^2 f(x) \times f'(x)$.

In Leibniz notation, $\frac{d}{dx}(\tan f(x)) = \frac{d(\tan f(x))}{d(f(x))} \times \frac{df(x))}{dx} = \sec^2(f(x)) \times \frac{d(f(x))}{dx}$.

EXAMPLE 3 Determining the derivative of a combination of functions

Determine $\frac{dy}{dx}$ for $y = (\sin x + \tan x)^4$.

Solution

$$y = (\sin x + \tan x)^4$$

$$\frac{dy}{dx} = 4(\sin x + \tan x)^3(\cos x + \sec^2 x)$$ (Chain rule)

EXAMPLE 4 Determining the derivative of a product involving the tangent function

Determine $\frac{dy}{dx}$ for $y = x \tan(2x - 1)$.

Solution

$$y = x\tan(2x - 1)$$

$$\frac{dy}{dx} = (1)\tan(2x - 1) + (x)\sec^2(2x - 1)\frac{d(2x - 1)}{dx}$$ (Product and chain rules)

$$= \tan(2x - 1) + 2x\sec^2(2x - 1)$$

IN SUMMARY

Key Idea

- The derivatives of functions involving the tangent function are found as follows:
  - $\frac{d(\tan x)}{dx} = \sec^2 x$
  - $\frac{d}{dx}(\tan f(x)) = \sec^2 f(x) \times f'(x)$

Need to Know

- Trigonometric identities can be used to write one expression as an equivalent expression and then differentiate. In some cases, the new function will be easier to work with.

Exercise 5.5

PART A

K 1. Determine $\frac{dy}{dx}$ for each of the following:

a. $y = \tan 3x$

b. $y = 2 \tan x - \tan 2x$

c. $y = \tan^2 (x^3)$

d. $y = \dfrac{x^2}{\tan \pi x}$

e. $y = \tan (x^2) - \tan^2 x$

f. $y = 3 \sin 5x \tan 5x$

A 2. Determine an equation for the tangent to each function at the point with the given x-coordinate.

a. $f(x) = \tan x, x = \dfrac{\pi}{4}$

b. $f(x) = 6 \tan x - \tan 2x, x = 0$

PART B

3. Determine y' for each of the following:

a. $y = \tan (\sin x)$

b. $y = [\tan (x^2 - 1)]^{-2}$

c. $y = \tan^2 (\cos x)$

d. $y = (\tan x + \cos x)^2$

e. $y = \sin^3 x \tan x$

f. $y = e^{\tan \sqrt{x}}$

4. Determine $\frac{d^2y}{dx^2}$ for each of the following:

a. $y = \sin x \tan x$

b. $y = \tan^2 x$

5. Determine all the values of x, $0 \le x \le 2\pi$, for which the slope of the tangent to $f(x) = \sin x \tan x$ is zero.

6. Determine the local maximum point on the curve $y = 2x - \tan x$, where $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

T 7. Prove that $y = \sec x + \tan x$ is always increasing on the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

8. Determine the equation of the line that is tangent to $y = 2 \tan x$, where $x = \frac{\pi}{4}$.

C 9. If you forget the expression that results when differentiating the tangent function, explain how you can derive this derivative using an identity.

PART C

10. Determine the derivative of $\cot x$.

11. Determine $f''(x)$, where $f(x) = \cot 4x$.

CHAPTER 5: RATE-OF-CHANGE MODELS IN MICROBIOLOGY

A simplified model for bacterial growth is $P(t) = P_0e^{rt}$, where $P(t)$ is the population of the bacteria colony after t hours, P_0 is the initial population of the colony (the population at $t = 0$), and r determines the growth rate of the colony. The model is simple because it does not account for limited resources, such as space and nutrients. As time increases, so does the population, but there is no bound on the population. While a model like this can describe a population for a short period of time or can be made to describe a population for a longer period of time by adjusting conditions in a laboratory experiment, in general, populations are better described by more complex models.

To determine how the population of a particular type of bacteria will grow over time under controlled conditions, a microbiologist observes the initial population and the population every half hour for 8 h. (The microbiologist also controls the environment in which the colony is growing to make sure that temperature and light conditions remain constant and ensures that the amount of nutrients available to the colony as it grows is sufficient for the increasing population.)

After analyzing the population data, the microbiologist determines that the population of the bacteria colony can be modelled by the equation $P(t) = 500\,e^{0.1t}$.

a. What is the initial population of the bacteria colony?

b. What function describes the instantaneous rate of change in the bacteria population after t hours?

c. What is the instantaneous rate of change in the population after 1 h? What is the instantaneous rate of change after 8 h?

d. How do your answers for part c. help you make a prediction about how long the bacteria colony will take to double in size? Make a prediction for the number of hours the population will take to double, using your answers for part c. and/or other information.

e. Determine the actual doubling time—the time that the colony takes to grow to twice its initial population. (***Hint:*** Solve for t when $P(t) = 1000$.)

f. Compare your prediction for the doubling time with the calculated value. If your prediction was not close to the actual value, what factors do you think might account for the difference?

g. When is the instantaneous rate of change equal to 500 bacteria per hour?