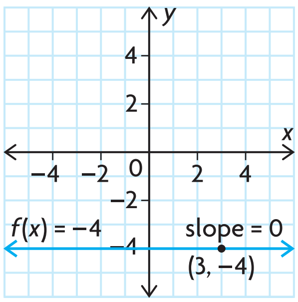
Section 2.2—The Derivatives of Polynomial Functions

We have seen that derivatives of functions are of practical use because they represent instantaneous rates of change.

Computing derivatives from the limit definition, as we did in Section 2.1, is tedious and time-consuming. In this section, we will develop some rules that simplify the process of differentiation.

We will begin developing the rules of differentiation by looking at the constant function, $f(x) = k$. Since the graph of any constant function is a horizontal line with slope zero at each point, the derivative should be zero. For example, if $f(x) = -4$, then $f'(3) = 0$. Alternatively, we can write $\frac{d}{dx}(-4) = 0$.



The Constant Function Rule

If $f(x) = k$, where k is a constant, then $f'(x) = 0$.

In Leibniz notation, $\frac{d}{dx}(k) = 0$.

Proof:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{k-k}{h} \qquad \text{(Since } f(x) = k \text{ and } f(x+h) = k \text{ for all } h\text{)}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

EXAMPLE 1 **Applying the constant function rule**

a. If $f(x) = 5, f'(x) = 0$.

b. If $y = \frac{\pi}{2}, \frac{dy}{dx} = 0$.

A **power function** is a function of the form $f(x) = x^n$, where n is a real number. In the previous section, we observed that for $f(x) = x^2, f'(x) = 2x$; for $g(x) = \sqrt{x} = x^{\frac{1}{2}}, g'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$; and for $h(x) = \frac{1}{x} = x^{-1}, h'(x) = -x^{-2}$. As well, we hypothesized that $\frac{d}{dx}(x^n) = nx^{n-1}$. In fact, this is true and is called the **power rule**.

The Power Rule

If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.
In Leibniz notation, $\frac{d}{dx}(x^n) = nx^{n-1}$.

Proof:
(*Note:* n is a positive integer.)
Using the definition of the derivative,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ where } f(x) = x^n$$

$$= \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \qquad \text{(Factor)}$$

$$= \lim_{h \to 0} \frac{(x + h - x)[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}]}{h}$$

$$= \lim_{h \to 0}[(x+h)^{n-1} + (x+h)^{n-2}x + \ldots + (x+h)x^{n-2} + x^{n-1}] \qquad \text{(Divide out } h\text{)}$$

$$= x^{n-1} + x^{n-2}(x) + \ldots + (x)x^{n-2} + x^{n-1}$$

$$= x^{n-1} + x^{n-1} + \ldots + x^{n-1} + x^{n-1} \qquad \text{(Since there are } n \text{ terms)}$$

$$= nx^{n-1}$$

EXAMPLE 2 Applying the power rule

a. If $f(x) = x^7$, then $f'(x) = 7x^6$.

b. If $g(x) = \frac{1}{x^3} = x^{-3}$, then $g'(x) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$.

c. If $s = t^{\frac{3}{2}}$, $\frac{ds}{dt} = \frac{3}{2}t^{\frac{1}{2}} = \frac{3}{2}\sqrt{t}$.

d. $\frac{d}{dx}(x) = 1x^{1-1} = x^0 = 1$

The Constant Multiple Rule

If $f(x) = kg(x)$, where k is a constant, then $f'(x) = kg'(x)$.
In Leibniz notation, $\frac{d}{dx}(ky) = k\frac{dy}{dx}$.

Proof:
Let $f(x) = kg(x)$. By the definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{kg(x+h)-kg(x)}{h} \qquad \text{(Factor)}$$

$$= \lim_{h\to 0} k\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= k\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right] \qquad \text{(Property of limits)}$$

$$= kg'(x)$$

EXAMPLE 3 Applying the constant multiple rule

Differentiate the following functions:

a. $f(x) = 7x^3$

b. $y = 12x^{\frac{4}{3}}$

Solution

a. $f(x) = 7x^3$

$$f'(x) = 7\frac{d}{dx}(x^3) = 7(3x^2) = 21x^2$$

b. $y = 12x^{\frac{4}{3}}$

$$\frac{dy}{dx} = 12\frac{d}{dx}(x^{\frac{4}{3}}) = 12\left(\frac{4}{3}x^{\left(\frac{4}{3}-1\right)}\right) = 16x^{\frac{1}{3}}$$

We conclude this section with the sum and difference rules.

The Sum Rule

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$.

Proof:

Let $f(x) = p(x) + q(x)$. By the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0}\frac{[p(x+h) + q(x+h)] - [p(x) + q(x)]}{h} \\ &= \lim_{h\to 0}\left\{\frac{[p(x+h) - p(x)]}{h} + \frac{[q(x+h) - q(x)]}{h}\right\} \\ &= \lim_{h\to 0}\left\{\frac{[p(x+h) - p(x)]}{h}\right\} + \lim_{h\to 0}\left\{\frac{[q(x+h) - q(x)]}{h}\right\} \\ &= p'(x) + q'(x) \end{aligned}$$

The Difference Rule

If functions $p(x)$ and $q(x)$ are differentiable, and $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.

In Leibniz notation, $\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) - \frac{d}{dx}(q(x))$.

The proof for the difference rule is similar to the proof for the sum rule.

EXAMPLE 4 Selecting appropriate rules to determine the derivative

Differentiate the following functions:

a. $f(x) = 3x^2 - 5\sqrt{x}$

b. $y = (3x + 2)^2$

Solution

We apply the constant multiple, power, sum, and difference rules.

a. $f(x) = 3x^2 - 5\sqrt{x}$

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}(3x^2) - \frac{d}{dx}(5x^{\frac{1}{2}}) \\
&= 3\frac{d}{dx}(x^2) - 5\frac{d}{dx}(x^{\frac{1}{2}}) \\
&= 3(2x) - 5\left(\frac{1}{2}x^{-\frac{1}{2}}\right) \\
&= 6x - \frac{5}{2}x^{-\frac{1}{2}}, \text{ or } 6x - \frac{5}{2\sqrt{x}}
\end{aligned}
$$

b. We first expand $y = (3x + 2)^2$.

$$
\begin{aligned}
y &= 9x^2 + 12x + 4 \\
\frac{dy}{dx} &= 9(2x) + 12(1) + 0 \\
&= 18x + 12
\end{aligned}
$$

EXAMPLE 5 **Selecting a strategy to determine the equation of a tangent**

Determine the equation of the tangent to the graph of $f(x) = -x^3 + 3x^2 - 2$ at $x = 1$.

Solution A – Using the derivative

The slope of the tangent to the graph of f at any point is given by the derivative $f'(x)$.

For $\quad f(x) = -x^3 + 3x^2 - 2$

$\qquad f'(x) = -3x^2 + 6x$

Now, $f'(1) = -3(1)^2 + 6(1)$

$\qquad = -3 + 6$

$\qquad = 3$

The slope of the tangent at $x = 1$ is 3 and the point of tangency is $(1, f(1)) = (1, 0)$.

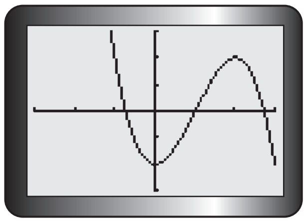
The equation of the tangent is $y - 0 = 3(x - 1)$ or $y = 3x - 3$.

Tech Support

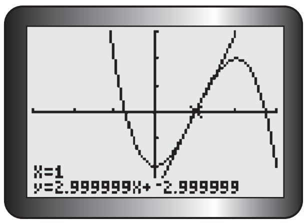
For help using the graphing calculator to graph functions and draw tangent lines See Technical Appendices p. 597 and p. 608.

Solution B – Using the graphing calculator

Draw the graph of the function using the graphing calculator.



Draw the tangent at the point on the function where $x = 1$. The calculator displays the equation of the tangent line.



The equation of the tangent line in this case is $y = 3x - 3$.

EXAMPLE 6 Connecting the derivative to horizontal tangents

Determine points on the graph in Example 5 where the tangents are horizontal.

Solution

Horizontal lines have slope zero. We need to find the values of x that satisfy $f'(x) = 0$.

$-3x^2 + 6x = 0$

$-3x(x - 2) = 0$

$x = 0$ or $x = 2$

The graph of $f(x) = -x^3 + 3x^2 - 2$ has horizontal tangents at (0, –2) and (2, 2).

IN SUMMARY

Key Ideas

The following table summarizes the derivative rules in this section.

Rule	Function Notation	Leibniz Notation
Constant Function Rule	If $f(x) = k$, where k is a constant, then $f'(x) = 0$.	$\frac{d}{dx}(k) = 0$
Power Rule	If $f(x) = x^n$, where n is a real number, then $f'(x) = nx^{n-1}$.	$\frac{d}{dx}(x^n) = nx^{n-1}$
Constant Multiple Rule	If $f(x) = kg(x)$, then $f'(x) = kg'(x)$.	$\frac{d}{dx}(ky) = k\frac{dy}{dx}$
Sum Rule	If $f(x) = p(x) + q(x)$, then $f'(x) = p'(x) + q'(x)$.	$\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) + \frac{d}{dx}(q(x))$
Difference Rule	If $f(x) = p(x) - q(x)$, then $f'(x) = p'(x) - q'(x)$.	$\frac{d}{dx}(f(x)) = \frac{d}{dx}(p(x)) - \frac{d}{dx}(q(x))$

Need to Know

- To determine the derivative of a simple rational function, such as $f(x) = \frac{4}{x^6}$, express the function as a power, then use the power rule.
  If $f(x) = 4x^{-6}$, then $f'(x) = 4(-6)x^{(-6-1)} = -24x^{-7}$
- If you have a radical function such as $g(x) = \sqrt[3]{x^5}$, rewrite the function as $g(x) = x^{\frac{5}{3}}$, then use the power rule.
  If $g(x) = x^{\frac{5}{3}}$, then $g'(x) = \frac{5}{3}x^{\frac{2}{3}} = \frac{5}{3}\sqrt[3]{x^2}$

Exercise 2.2

PART A

1. What rules do you know for calculating derivatives? Give examples of each rule.

2. Determine $f'(x)$ for each of the following functions:

a. $f(x) = 4x - 7$
b. $f(x) = x^3 - x^2$
c. $f(x) = -x^2 + 5x + 8$
d. $f(x) = \sqrt[3]{x}$
e. $f(x) = \left(\frac{x}{2}\right)^4$
f. $f(x) = x^{-3}$

K 3. Differentiate each function. Use either Leibniz notation or prime notation, depending on which is appropriate.

a. $h(x) = (2x + 3)(x + 4)$
b. $f(x) = 2x^3 + 5x^2 - 4x - 3.75$
c. $s = t^2(t^2 - 2t)$
d. $y = \frac{1}{5}x^5 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 1$
e. $g(x) = 5(x^2)^4$
f. $s(t) = \frac{t^5 - 3t^2}{2t}, t > 0$

4. Apply the differentiation rules you learned in this section to find the derivatives of the following functions:

a. $y = 3x^{\frac{5}{3}}$
b. $y = 4x^{-\frac{1}{2}} - \frac{6}{x}$
c. $y = \frac{6}{x^3} + \frac{2}{x^2} - 3$
d. $y = 9x^{-2} + 3\sqrt{x}$
e. $y = \sqrt{x} + 6\sqrt{x^3} + \sqrt{2}$
f. $y = \frac{1 + \sqrt{x}}{x}$

PART B

5. Let s represent the position of a moving object at time t. Find the velocity $v = \frac{ds}{dt}$ at time t.

a. $s = -2t^2 + 7t$
b. $s = 18 + 5t - \frac{1}{3}t^3$
c. $s = (t - 3)^2$

6. Determine $f'(a)$ for the given function $f(x)$ at the given value of a.

a. $f(x) = x^3 - \sqrt{x}, a = 4$
b. $f(x) = 7 - 6\sqrt{x} + 5x^{\frac{2}{3}}, a = 64$

7. Determine the slope of the tangent to each of the curves at the given point.

a. $y = 3x^4, (1, 3)$
b. $y = \frac{1}{x^{-5}}, (-1, -1)$
c. $y = \frac{2}{x}, (-2, -1)$
d. $y = \sqrt{16x^3}, (4, 32)$

8. Determine the slope of the tangent to the graph of each function at the point with the given x-coordinate.

a. $y = 2x^3 + 3x, x = 1$

b. $y = 2\sqrt{x} + 5, x = 4$

c. $y = \dfrac{16}{x^2}, x = -2$

d. $y = x^{-3}(x^{-1} + 1), x = 1$

9. Write an equation of the tangent to each curve at the given point.

a. $y = 2x - \dfrac{1}{x}, P(0.5, -1)$

b. $y = \dfrac{3}{x^2} - \dfrac{4}{x^3}, P(-1, 7)$

c. $y = \sqrt{3x^3}, P(3, 9)$

d. $y = \dfrac{1}{x}\left(x^2 + \dfrac{1}{x}\right), P(1, 2)$

e. $y = (\sqrt{x} - 2)(3\sqrt{x} + 8), P(4, 0)$

f. $y = \dfrac{\sqrt{x} - 2}{\sqrt[3]{x}}, P(1, -1)$

C 10. What is a normal to the graph of a function? Determine the equation of the normal to the graph of the function in question 9, part b., at the given point.

T 11. Determine the values of x so that the tangent to the function $y = \frac{3}{\sqrt[3]{x}}$ is parallel to the line $x + 16y + 3 = 0$.

12. Do the functions $y = \frac{1}{x}$ and $y = x^3$ ever have the same slope? If so, where?

13. Tangents are drawn to the parabola $y = x^2$ at $(2, 4)$ and $\left(-\frac{1}{8}, \frac{1}{64}\right)$. Prove that these lines are perpendicular. Illustrate with a sketch.

14. Determine the point on the parabola $y = -x^2 + 3x + 4$ where the slope of the tangent is 5. Illustrate your answer with a sketch.

15. Determine the coordinates of the points on the graph of $y = x^3 + 2$ at which the slope of the tangent is 12.

16. Show that there are two tangents to the curve $y = \frac{1}{5}x^5 - 10x$ that have a slope of 6.

17. Determine the equations of the tangents to the curve $y = 2x^2 + 3$ that pass through the following points:

a. point $(2, 3)$

b. point $(2, -7)$

18. Determine the value of a, given that the line $ax - 4y + 21 = 0$ is tangent to the graph of $y = \frac{a}{x^2}$ at $x = -2$.

A 19. It can be shown that, from a height of h metres, a person can see a distance of d kilometres to the horizon, where $d = 3.53\sqrt{h}$.

a. When the elevator of the CN Tower passes the 200 m height, how far can the passengers in the elevator see across Lake Ontario?

b. Find the rate of change of this distance with respect to height when the height of the elevator is 200 m.

20. An object drops from a cliff that is 150 m high. The distance, d, in metres, that the object has dropped at t seconds in modelled by $d(t) = 4.9t^2$.

a. Find the average rate of change of distance with respect to time from 2 s to 5 s.

b. Find the instantaneous rate of change of distance with respect to time at 4 s.

c. Find the rate at which the object hits the ground to the nearest tenth.

21. A subway train travels from one station to the next in 2 min. Its distance, in kilometres, from the first station after t minutes is $s(t) = t^2 - \frac{1}{3}t^3$. At what times will the train have a velocity of 0.5 km/min?

22. While working on a high-rise building, a construction worker drops a bolt from 320 m above the ground. After t seconds, the bolt has fallen a distance of s metres, where $s(t) = 5t^2, 0 \le t \le 8$. The function that gives the height of the bolt above ground at time t is $R(t) = 320 - 5t^2$. Use this function to determine the velocity of the bolt at $t = 2$.

23. Tangents are drawn from the point $(0, 3)$ to the parabola $y = -3x^2$. Find the coordinates of the points at which these tangents touch the curve. Illustrate your answer with a sketch.

24. The tangent to the cubic function that is defined by $y = x^3 - 6x^2 + 8x$ at point $A(3, -3)$ intersects the curve at another point, B. Find the coordinates of point B. Illustrate with a sketch.

25. a. Find the coordinates of the points, if any, where each function has a horizontal tangent line.

i. $f(x) = 2x - 5x^2$

ii. $f(x) = 4x^2 + 2x - 3$

iii. $f(x) = x^3 - 8x^2 + 5x + 3$

b. Suggest a graphical interpretation for each of these points.

PART C

26. Let $P(a, b)$ be a point on the curve $\sqrt{x} + \sqrt{y} = 1$. Show that the slope of the tangent at P is $-\sqrt{\frac{b}{a}}$.

27. For the power function $f(x) = x^n$, find the x-intercept of the tangent to its graph at point $(1, 1)$. What happens to the x-intercept as n increases without bound $(n \to +\infty)$? Explain the result geometrically.

28. For each function, sketch the graph of $y = f(x)$ and find an expression for $f'(x)$. Indicate any points at which $f'(x)$ does not exist.

a. $f(x) = \begin{cases} x^2, x < 3 \\ x + 6, x \ge 3 \end{cases}$ b. $f(x) = |3x^2 - 6|$ c. $f(x) = ||x| - 1|$