Section 4.3—Vertical and Horizontal Asymptotes

Adding, subtracting, or multiplying two polynomial functions yields another polynomial function. Dividing two polynomial functions results in a function that is not a polynomial. The quotient is a **rational function**. Asymptotes are among the special features of rational functions, and they play a significant role in curve sketching. In this section, we will consider vertical and horizontal asymptotes of rational functions.

INVESTIGATION

The purpose of this investigation is to examine the occurrence of vertical asymptotes for rational functions.

A. Use your graphing calculator to obtain the graph of $f(x) = \frac{1}{x - k}$ and the table of values for each of the following: $k = 3, 1, 0, -2, -4$, and -5.

B. Describe the behaviour of each graph as x approaches k from the right and from the left.

C. Repeat parts A and B for the function $f(x) = \frac{x + 3}{x - k}$ using the same values of k.

D. Repeat parts A and B for the function $f(x) = \frac{1}{x^2 + x - k}$ using the following values: $k = 2, 6$, and 12.

E. Make a general statement about the existence of a vertical asymptote for a rational function of the form $y = \frac{p(x)}{q(x)}$ if there is a value c such that $q(c) = 0$ and $p(c) \neq 0$.

Vertical Asymptotes and Rational Functions

Recall that the notation $x \to c^+$ means that x approaches c from the right. Similarly, $x \to c^-$ means that x approaches c from the left.

You can see from this investigation that as $x \to c$ from either side, the function values get increasingly large and either positive or negative depending on the value of $p(c)$. We say that the function values approach $+\infty$ (positive infinity) or $-\infty$ (negative infinity). These are not numbers. They are symbols that represent the behaviour of a function that increases or decreases without limit.

Because the symbol ∞ is not a number, the limits $\lim_{x \to c^+} \frac{1}{x - c}$ and $\lim_{x \to c^-} \frac{1}{x - c}$ *do not exist*. For convenience, however, we use the notation $\lim_{x \to c^+} \frac{1}{x - c} = +\infty$ and $\lim_{x \to c^-} \frac{1}{x - c} = -\infty$.

These limits form the basis for determining the asymptotes of simple functions.

Vertical Asymptotes of Rational Functions

A rational function of the form $f(x) = \frac{p(x)}{q(x)}$ has a vertical asymptote $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.

EXAMPLE 1 Reasoning about the behaviour of a rational function near its vertical asymptotes

Determine any vertical asymptotes of the function $f(x) = \frac{x}{x^2 + x - 2}$, and describe the behaviour of the graph of the function for values of x near the asymptotes.

Solution

First, determine the values of x for which $f(x)$ is undefined by solving the following:

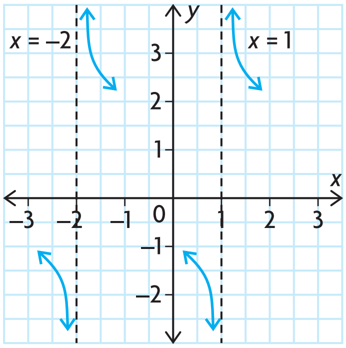
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \text{ or } x = 1$$

Neither of these values of x makes the numerator zero, so both of these values give vertical asymptotes. The equations of the asymptotes are $x = -2$ and $x = 1$.

To determine the behaviour of the graph near the asymptotes, it can be helpful to use a chart.

Values of x	x	$x + 2$	$x - 1$	$f(x) = \frac{x}{(x + 2)(x - 1)}$	$f(x) \to ?$
$x \to -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \to -2^+$	< 0	> 0	< 0	> 0	$+\infty$
$x \to 1^-$	> 0	> 0	< 0	< 0	$-\infty$
$x \to 1^+$	> 0	> 0	> 0	> 0	$+\infty$

The behaviour of the graph can be illustrated as follows:



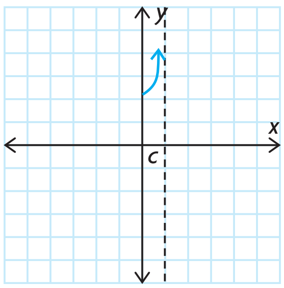
To proceed beyond this point, we require additional information.

Vertical Asymptotes and Infinite Limits

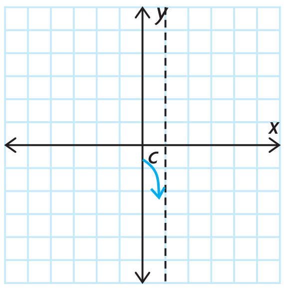
The graph of $f(x)$ has a vertical asymptote, $x = c$, if one of the following infinite limit statements is true:

$$\lim_{x \to c^-} f(x) = +\infty, \quad \lim_{x \to c^-} f(x) = -\infty, \quad \lim_{x \to c^+} f(x) = +\infty \text{ or } \lim_{x \to c^+} f(x) = -\infty$$

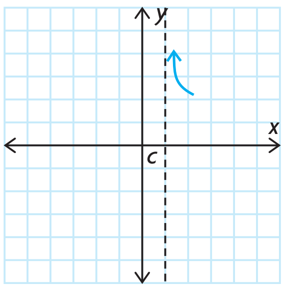
The following graphs correspond to each limit statement above:



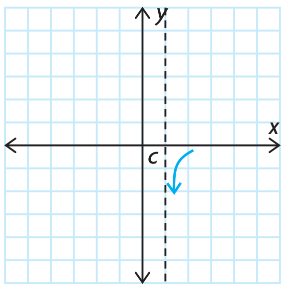
$\lim_{x \to c^-} f(x) = +\infty$



$\lim_{x \to c^-} f(x) = -\infty$



$\lim_{x \to c^+} f(x) = +\infty$



$\lim_{x \to c^+} f(x) = -\infty$

Horizontal Asymptotes and Rational Functions

Consider the behaviour of rational functions $f(x) = \frac{p(x)}{q(x)}$ as x increases without bound in both the positive and negative directions. The following notation is used to describe this behaviour:

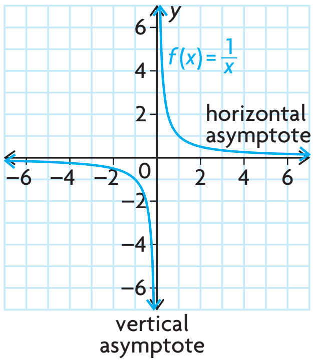
$$\lim_{x \to +\infty} f(x) \text{ and } \lim_{x \to -\infty} f(x)$$

The notation $x \to +\infty$ is read "x tends to positive infinity" and means that the values of x are positive and growing in magnitude without bound. Similarly, the notation $x \to -\infty$ is read "x tends to negative infinity" and means that the values of x are negative and growing in magnitude without bound.

The values of these limits can be determined by making two observations. The first observation is a list of simple limits, similar to those used for determining vertical asymptotes.

The Reciprocal Function and Limits at Infinity

$$\lim_{x \to +\infty} \frac{1}{x} = 0 \text{ and } \lim_{x \to -\infty} \frac{1}{x} = 0$$



The second observation is that a polynomial can always be written so the term of highest degree is a factor.

EXAMPLE 2 Expressing a polynomial function in an equivalent form

Write each function so the term of highest degree is a factor.

a. $p(x) = x^2 + 4x + 1$ b. $q(x) = 3x^2 - 4x + 5$

Solution

a. $p(x) = x^2 + 4x + 1$

$$= x^2\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)$$

b. $q(x) = 3x^2 - 4x + 5$

$$= 3x^2\left(1 - \frac{4}{3x} + \frac{5}{3x^2}\right)$$

The value of writing a polynomial in this form is clear. It is easy to see that as x becomes large (either positive or negative), the value of the second factor always approaches 1.

We can now determine the limit of a rational function in which the degree of $p(x)$ is equal to or less than the degree of $q(x)$.

EXAMPLE 3 Selecting a strategy to evaluate limits at infinity

Determine the value of each of the following:

a. $\lim_{x \to +\infty} \frac{2x - 3}{x + 1}$ b. $\lim_{x \to -\infty} \frac{x}{x^2 + 1}$ c. $\lim_{x \to +\infty} \frac{2x^2 + 3}{3x^2 - x + 4}$

Solution

a. $$f(x) = \frac{2x - 3}{x + 1} = \frac{2x\left(1 - \frac{3}{2x}\right)}{x\left(1 + \frac{1}{x}\right)}$$ (Factor and simplify)

$$= \frac{2\left(1 - \frac{3}{2x}\right)}{1 + \frac{1}{x}}$$

$$\lim_{x \to +\infty} f(x) = \frac{2\left[\lim_{x \to +\infty}\left(1 - \frac{3}{2x}\right)\right]}{\lim_{x \to +\infty}\left(1 + \frac{1}{x}\right)}$$ (Apply limit properties)

$$= \frac{2(1 - 0)}{1 + 0}$$ (Evaluate)

$$= 2$$

b. $$g(x) = \frac{x}{x^2 + 1}$$ (Factor)

$$= \frac{x(1)}{x^2\left(1 + \frac{1}{x^2}\right)}$$ (Simplify)

$$= \frac{1}{x\left(1 + \frac{1}{x^2}\right)}$$

$$\lim_{x \to -\infty} g(x) = \frac{1}{\lim_{x \to -\infty} x \times \lim_{x \to -\infty}\left(1 + \frac{1}{x^2}\right)}$$ (Apply limit properties)

$$= \frac{1}{\lim_{x \to -\infty} x \times (1)}$$

$$= \lim_{x \to -\infty} \frac{1}{x}$$ (Evaluate)

$$= 0$$

c. To evaluate this limit, we can use the technique of dividing the numerator and denominator by the highest power of x in the denominator.

$$p(x) = \frac{2x^2 + 3}{3x^2 - x + 4}$$ (Divide by x^2)

$$= \frac{(2x^2 + 3) \div x^2}{(3x^2 - x + 4) \div x^2}$$ (Simplify)

$$= \frac{2 + \frac{3}{x^2}}{3 - \frac{1}{x} + \frac{4}{x^2}}$$

$$\lim_{x \to +\infty} p(x) = \frac{\lim_{x \to +\infty}\left(2 + \frac{3}{x^2}\right)}{\lim_{x \to +\infty}\left(3 - \frac{1}{x} + \frac{4}{x^2}\right)}$$ (Apply limit properties)

$$= \frac{2 + 0}{3 - 0 + 0}$$ (Evaluate)

$$= \frac{2}{3}$$

When $\lim_{x \to +\infty} f(x) = k$ or $\lim_{x \to -\infty} f(x) = k$, the graph of the function is approaching the line $y = k$. This line is a horizontal asymptote of the function. In Example 3, part a, $y = 2$ is a horizontal asymptote of $f(x) = \frac{2x - 3}{x + 1}$. Therefore, for large positive x-values, the y-values approach 2. This is also the case for large negative x-values.

To sketch the graph of the function, we need to know whether the curve approaches the horizontal asymptote from above or below. To find out, we need to consider $f(x) - k$, where k is the limit we just determined. This is illustrated in the following examples.

EXAMPLE 4 Reasoning about the end behaviours of a rational function

Determine the equations of any horizontal asymptotes of the function $f(x) = \frac{3x + 5}{2x - 1}$. State whether the graph approaches the asymptote from above or below.

Solution

$$f(x) = \frac{3x + 5}{2x - 1} = \frac{(3x + 5) \div x}{(2x - 1) \div x} \qquad \text{(Divide by } x\text{)}$$

$$= \frac{3 + \frac{5}{x}}{2 - \frac{1}{x}}$$

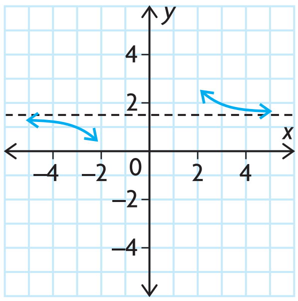
$$\lim_{x \to +\infty} f(x) = \frac{\lim_{x \to +\infty}\left(3 + \frac{5}{x}\right)}{\lim_{x \to +\infty}\left(2 - \frac{1}{x}\right)} \qquad \text{(Evaluate)}$$

$$= \frac{3}{2}$$

Similarly, we can show that $\lim_{x \to -\infty} f(x) = \frac{3}{2}$. So, $y = \frac{3}{2}$ is a horizontal asymptote of the graph of $f(x)$ for both large positive and negative values of x. To determine whether the graph approaches the asymptote from above or below, we consider very large positive and negative values of x.

If x is large and positive (for example, if $x = 1000$), $f(x) = \frac{3005}{1999}$, which is greater than $\frac{3}{2}$. Therefore, the graph approaches the asymptote $y = \frac{3}{2}$ from above.

If x is large and negative (for example, if $x = -1000$), $f(x) = \frac{-2995}{-20\,001}$, which is less than $\frac{3}{2}$. This part of the graph approaches the asymptote $y = \frac{3}{2}$ from below, as illustrated in the diagram.



EXAMPLE 5 Selecting a limit strategy to analyze the behaviour of a rational function near its asymptotes

For the function $f(x) = \frac{3x}{x^2 - x - 6}$, determine the equations of all horizontal or vertical asymptotes. Illustrate the behaviour of the graph as it approaches the asymptotes.

Solution

For vertical asymptotes,

$$x^2 - x - 6 = 0$$
$$(x - 3)(x + 2) = 0$$
$$x = 3 \text{ or } x = -2$$

There are two vertical asymptotes, at $x = 3$ and $x = -2$.

Values of x	x	$x - 3$	$x + 2$	$f(x)$	$f(x) \to ?$
$x \to 3^-$	> 0	< 0	> 0	< 0	$-\infty$
$x \to 3^+$	> 0	> 0	> 0	> 0	$+\infty$
$x \to -2^-$	< 0	< 0	< 0	< 0	$-\infty$
$x \to -2^+$	< 0	< 0	> 0	> 0	$+\infty$

For horizontal asymptotes,

$$f(x) = \frac{3x}{x^2 - x - 6} \qquad \text{(Factor)}$$

$$= \frac{3x}{x^2\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)} \qquad \text{(Simplify)}$$

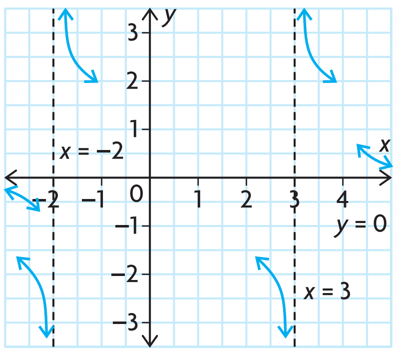
$$= \frac{3}{x\left(1 - \frac{1}{x} - \frac{6}{x^2}\right)}$$

$$\lim_{x \to +\infty} f(x) = \lim_{x \to \infty} \frac{3}{x} = 0$$

Similarly, we can show $\lim_{x \to -\infty} f(x) = 0$. Therefore, $y = 0$ is a horizontal asymptote of the graph of $f(x)$ for both large positive and negative values of x.

As x becomes large positively, $f(x) > 0$, so the graph is above the horizontal asymptote. As x becomes large negatively, $f(x) < 0$, so the graph is below the horizontal asymptote.

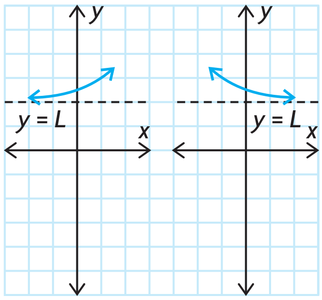
This diagram illustrates the behaviour of the graph as it nears the asymptotes:



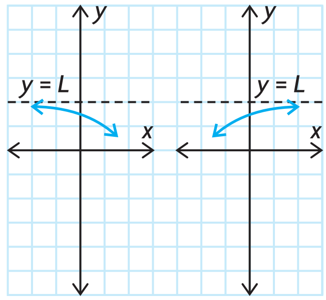
Horizontal Asymptotes and Limits at Infinity

If $\lim_{x \to +\infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, we say that the line $y = L$ is a horizontal asymptote of the graph of $f(x)$.

The following graphs illustrate some typical situations:



$f(x) > L$, so the graph approaches from above.



$f(x) < L$, so the graph approaches from below.

In addition to vertical and horizontal asymptotes, it is possible for a graph to have **oblique asymptotes**. These are straight lines that are slanted and to which the curve becomes increasingly close. They occur with rational functions in which the degree of the numerator exceeds the degree of the denominator by exactly one. This is illustrated in the following example.

EXAMPLE 6

Reasoning about oblique asymptotes

Determine the equations of all asymptotes of the graph of $f(x) = \dfrac{2x^2 + 3x - 1}{x + 1}$.

Solution

Since $x + 1 = 0$ for $x = -1$, and $2x^2 + 3x - 1 \neq 0$ for $x = -1$, $x = -1$ is a vertical asymptote.

$$\text{Now } \lim_{x\to\infty} \frac{2x^2 + 3x - 1}{x + 1} = \lim_{x\to\infty} \frac{2x^2\left(1 + \frac{3}{2x} - \frac{1}{2x^2}\right)}{x\left(1 + \frac{1}{x}\right)}$$
$$= \lim_{x\to\infty} 2x$$

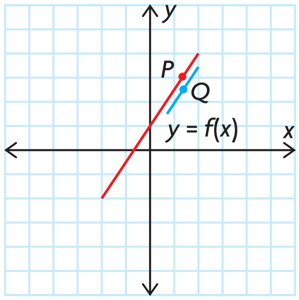
This limit does not exist, and, by a similar calculation, $\lim_{x\to-\infty} f(x)$ does not exist, so there is no horizontal asymptote.

Dividing the numerator by the denominator,

$$\begin{array}{r@{}l}
 & 2x + 1 \\
x + 1 \big) & \overline{2x^2 + 3x - 1} \\
 & \underline{2x^2 + 2x} \\
 & \qquad\;\; x - 1 \\
 & \qquad\;\; \underline{x + 1} \\
 & \qquad\quad\; -2
\end{array}$$

Thus, we can write $f(x)$ in the form $f(x) = 2x + 1 - \dfrac{2}{x + 1}$.

Now let's consider the straight line $y = 2x + 1$ and the graph of $y = f(x)$. For any value of x, we can determine point $P(x, 2x + 1)$ on the line and point $Q\left(x, 2x + 1 - \frac{2}{x + 1}\right)$ on the curve.



Then the vertical distance QP from the curve to the line is

$$QP = 2x + 1 - \left(2x + 1 - \frac{2}{x + 1}\right)$$
$$= \frac{2}{x + 1}$$

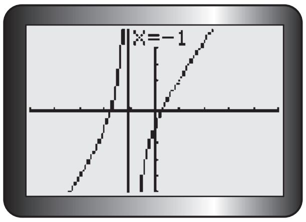
$$\lim_{x\to\infty} QP = \lim_{x\to\infty} \frac{2}{x + 1}$$
$$= 0$$

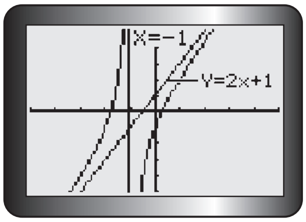
That is, as x gets very large, the curve approaches the line but never touches it. Therefore, the line $y = 2x + 1$ is an asymptote of the curve.

Since $\lim_{x\to-\infty} \frac{2}{x + 1} = 0$, the line is also an asymptote for large negative values of x.

In conclusion, there are two asymptotes of the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$. They are $y = 2x + 1$ and $x = -1$.

Use a graphing calculator to obtain the graph of $f(x) = \frac{2x^2 + 3x - 1}{x + 1}$.





Note that the vertical asymptote $x = -1$ appears on the graph on the left, but the oblique asymptote $y = 2x + 1$ does not. Use the Y2 function to graph the oblique asymptote $y = 2x + 1$.

IN SUMMARY

Key Ideas

- The graph of $f(x)$ has a **vertical asymptote** $x = c$ if any of the following is true:
  $\lim_{x \to c^-} f(x) = +\infty$ $\lim_{x \to c^-} f(x) = -\infty$
  $\lim_{x \to c^+} f(x) = +\infty$ $\lim_{x \to c^+} f(x) = -\infty$
- The line $y = L$ is a **horizontal asymptote** of the graph of $f(x)$ if $\lim_{x \to +\infty} f(x) = L$ *or* $\lim_{x \to -\infty} f(x) = L$.
- In a rational function, an **oblique asymptote** occurs when the degree of the numerator is exactly one greater than the degree of the denominator**.**

Need to Know

The techniques for curve sketching developed to this point are described in the following algorithm. As we develop new ideas, the algorithm will be extended.

Algorithm for Curve Sketching (so far)

To sketch a curve, apply these steps in the order given.

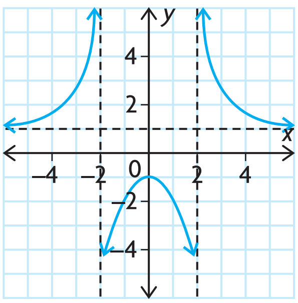
1. Check for any discontinuities in the domain. Determine if there are vertical asymptotes at these discontinuities, and determine the direction from which the curve approaches these asymptotes.
2. Find **both intercepts**.
3. Find any critical points.
4. Use the first derivative test to determine the type of critical points that may be present.
5. **Test end behaviour** by determining $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.
6. Construct an interval of increase/decrease table and identify all local or absolute extrema.
7. Sketch the curve.

Exercise 4.3

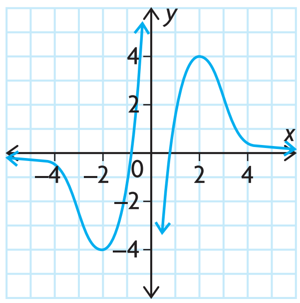
PART A

1. State the equations of the vertical and horizontal asymptotes of the curves shown.

a.



b.



C

2. Under what conditions does a rational function have vertical, horizontal, and oblique asymptotes?

3. Evaluate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, using the symbol "∞" when appropriate.

a. $f(x) = \dfrac{2x + 3}{x - 1}$

b. $f(x) = \dfrac{5x^2 - 3}{x^2 + 2}$

c. $f(x) = \dfrac{-5x^2 + 3x}{2x^2 - 5}$

d. $f(x) = \dfrac{2x^5 - 3x^2 + 5}{3x^4 + 5x - 4}$

4. For each of the following, check for discontinuities and state the equation of any vertical asymptotes. Conduct a limit test to determine the behaviour of the curve on either side of the asymptote.

a. $y = \dfrac{x}{x + 5}$

b. $f(x) = \dfrac{x + 2}{x - 2}$

c. $s = \dfrac{1}{(t - 3)^2}$

d. $y = \dfrac{x^2 - x - 6}{x - 3}$

e. $f(x) = \dfrac{6}{(x + 3)(x - 1)}$

f. $y = \dfrac{x^2}{x^2 - 1}$

5. For each of the following, determine the equations of any horizontal asymptotes. Then state whether the curve approaches the asymptote from above or below.

a. $y = \dfrac{x}{x + 4}$

b. $f(x) = \dfrac{2x}{x^2 - 1}$

c. $g(t) = \dfrac{3t^2 + 4}{t^2 - 1}$

d. $y = \dfrac{3x^2 - 8x - 7}{x - 4}$

PART B

K

6. For each of the following, check for discontinuities and then use at least two other tests to make a rough sketch of the curve. Verify using a calculator.

a. $y = \dfrac{x - 3}{x + 5}$

b. $f(x) = \dfrac{5}{(x + 2)^2}$

c. $g(t) = \dfrac{t^2 - 2t - 15}{t - 5}$

d. $y = \dfrac{(2 + x)(3 - 2x)}{(x^2 - 3x)}$

7. Determine the equation of the oblique asymptote for each of the following:

a. $f(x) = \dfrac{3x^2 - 2x - 17}{x - 3}$

b. $f(x) = \dfrac{2x^2 + 9x + 2}{2x + 3}$

c. $f(x) = \dfrac{x^3 - 1}{x^2 + 2x}$

d. $f(x) = \dfrac{x^3 - x^2 - 9x + 15}{x^2 - 4x + 3}$

8. a. For question 7, part a., determine whether the curve approaches the asymptote from above or below.

b. For question 7, part b., determine the direction from which the curve approaches the asymptote.

9. For each function, determine any vertical or horizontal asymptotes and describe its behaviour on each side of any vertical asymptote.

a. $f(x) = \dfrac{3x - 1}{x + 5}$

b. $g(x) = \dfrac{x^2 + 3x - 2}{(x - 1)^2}$

c. $h(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$

d. $m(x) = \dfrac{5x^2 - 3x + 2}{x - 2}$

A

10. Use the algorithm for curve sketching to sketch the graph of each function.

a. $f(x) = \dfrac{3 - x}{2x + 5}$

b. $h(t) = 2t^3 - 15t^2 + 36t - 10$

c. $y = \dfrac{20}{x^2 + 4}$

d. $s(t) = t + \dfrac{1}{t}$

e. $g(x) = \dfrac{2x^2 + 5x + 2}{x + 3}$

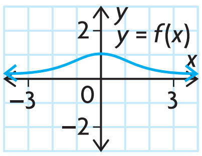
f. $s(t) = \dfrac{t^2 + 4t - 21}{t - 3}, t \geq -7$

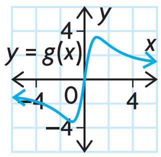
11. Consider the function $y = \frac{ax + b}{cx + d}$, where a, b, c, and d are constants, $a \neq 0, c \neq 0$.

a. Determine the horizontal asymptote of the graph.

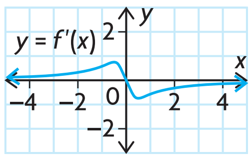
b. Determine the vertical asymptote of the graph.

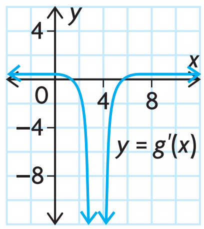
12. Use the features of each function's graph to sketch the graph of its first derivative.

a. 

b. 

13. A function’s derivative is shown in each graph. Use the graph to sketch a possible graph for the original function.

a. 

b. 

14. Let $f(x) = \frac{-x - 3}{x^2 - 5x - 14}$, $g(x) = \frac{x - x^3}{x - 3}$, $h(x) = \frac{x^3 - 1}{x^2 + 4}$, and $r(x) = \frac{x^2 + x - 6}{x^2 - 16}$. How can you can tell from its equation which of these functions has

a. a horizontal asymptote?

b. an oblique asymptote?

c. no vertical asymptote?

Explain. Determine the equations of all asymptote(s) for each function. Describe the behaviour of each function close to its asymptotes.

PART C

T 15. Find constants a and b such that the graph of the function defined by $f(x) = \frac{ax + 5}{3 - bx}$ will have a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = -3$.

16. To understand why we cannot work with the symbol ∞ as though it were a real number, consider the functions $f(x) = \frac{x^2 + 1}{x + 1}$ and $g(x) = \frac{x^2 + 2x + 1}{x + 1}$.

a. Show that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} g(x) = +\infty$.

b. Evaluate $\lim_{x \to +\infty} [f(x) - g(x)]$, and show that the limit is not zero.

17. Use the algorithm for curve sketching to sketch the graph of the function $f(x) = \frac{2x^2 - 2x}{x^2 - 9}$.