5.8 Solving Three-Dimensional Problems by Using Trigonometry

YOU WILL NEED

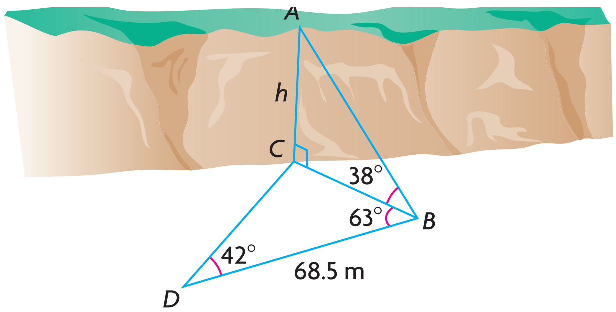
- dynamic geometry software (optional)

GOAL

Solve three-dimensional problems by using trigonometry.

LEARN ABOUT the Math

From point B, Manny uses a clinometer to determine the angle of elevation to the top of a cliff as $38°$. From point D, 68.5 m away from Manny, Joe estimates the angle between the base of the cliff, himself, and Manny to be $42°$, while Manny estimates the angle between the base of the cliff, himself, and his friend Joe to be $63°$.



? What is the height of the cliff to the nearest tenth of a metre?

EXAMPLE 1 Solving a three-dimensional problem by using the sine law

Calculate the height of the cliff to the nearest tenth of a metre.

Matt's Solution

In $\triangle DBC$:

$\angle C = 180° - (63° + 42°)$

$= 75°$

BC is in $\triangle ABC$. In $\triangle ABC$, I don't have enough information to calculate h, but BC is also in $\triangle DBC$.

In $\triangle DBC$, I knew two angles and a side length. Before I could calculate BC, I needed to determine $\angle C$. I used the fact that the sum of all three interior angles is 180°.

$$\frac{BC}{\sin D} = \frac{BD}{\sin C}$$

Using $\triangle DBC$ and the value of $\angle C$, I used the sine law to calculate BC.

$$\frac{BC}{\sin 42°} = \frac{68.5}{\sin 75°}$$

$$\overset{1}{\cancel{\sin 42°}} \times \frac{BC}{\underset{1}{\cancel{\sin 42°}}} = \sin 42° \times \frac{68.5}{\sin 75°}$$

To solve for BC, I multiplied both sides of the equation by sin 42°.

$$BC = \sin 42° \times \frac{68.5}{\sin 75°}$$

I used my calculator to evaluate.

$$BC \doteq 47.45 \text{ m}$$

$$\tan 38° = \frac{h}{BC}$$

$$\tan 38° = \frac{h}{47.45}$$

Then I used $\triangle ABC$ to calculate h. I knew that $\triangle ABC$ is a right triangle and that h is opposite the 38° angle while BC is adjacent to it. So I used tangent.

$$\tan 38° \times 47.45 = \frac{h}{\underset{1}{\cancel{47.45}}} \times \overset{1}{\cancel{47.45}}$$

To evaluate h, I multiplied both sides of the equation by 47.45.

$$37.1 \text{ m} \doteq h$$

The height of the cliff is about 37.1 m.

Reflecting

A. Was the given diagram necessary to help Matt solve the problem? Explain.

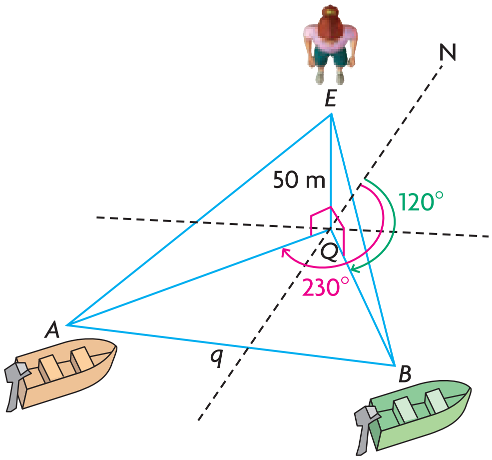
B. Why did Matt begin working with $\triangle DBC$ instead of $\triangle ABC$?

C. What strategies might Matt use to check whether his answer is reasonable?

APPLY the Math

EXAMPLE 2 Solving a three-dimensional problem by using the sine law

Emma is on a 50 m high bridge and sees two boats anchored below. From her position, boat A has a bearing of 230° and boat B has a bearing of 120°. Emma estimates the angles of depression to be 38° for boat A and 35° for boat B. How far apart are the boats to the nearest metre?



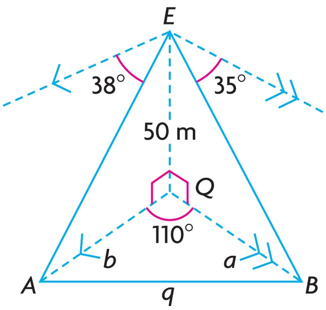
Kelly's Solution

In $\triangle AQB$:

$\angle Q = 230° - 120°$

$= 110°$

In $\triangle AQB$, I knew that the value of $\angle Q$ is equal to the difference of the bearings of boats A and B. So I subtracted 120° from 230°.

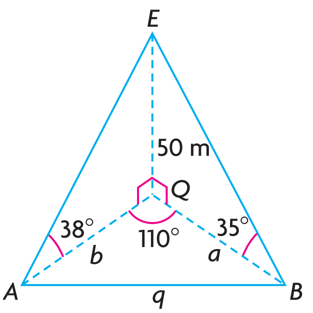


In $\triangle AQB$, I knew only one angle and no side lengths. In order to calculate q, I needed to determine AQ (side b) and BQ (side a) first.

I drew a sketch that included the angles of depression. I used these angles to determine the values of $\angle EAQ$ and $\angle EBQ$.

$\angle EAQ = 38°$ $\angle EBQ = 35°$

The angle of depression to A is measured from a line parallel to AQ. So $\angle EAQ$ is equal to 38°. Using the same reasoning, I determined that $\angle EBQ$ is equal to 35°.



I included the values of $\angle EAQ$ and $\angle EBQ$ in my sketch.

In $\triangle AEQ$:

$$\tan 38° = \frac{50}{b}$$

$$b = \frac{50}{\tan 38°}$$

$$b \doteq 64.0 \text{ m}$$

In $\triangle BEQ$:

$$\tan 35° = \frac{50}{a}$$

$$a = \frac{50}{\tan 35°}$$

$$a \doteq 71.4 \text{ m}$$

Since $\triangle AEQ$ and $\triangle BEQ$ are right triangles, I expressed *AQ* in terms of tan 38° and *BQ* in terms of tan 35°. Then I solved for *b* and *a*.

$$q^2 = b^2 + a^2 - 2ba \cos 110°$$

In $\triangle AQB$, I now knew two side lengths and the angle between those sides. So I used the cosine law to calculate *q*.

$$q^2 = (64.0)^2 + (71.4)^2 - 2(64.0)(71.4)\cos 110°$$
$$q = \sqrt{12\,320.6}$$
$$q \doteq 111 \text{ m}$$

I substituted the values of *b* and *a* into the equation and evaluated *q*.

The boats are about 111 m apart.

In Summary

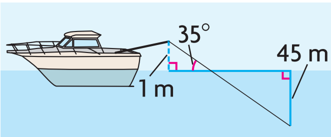
Key Ideas

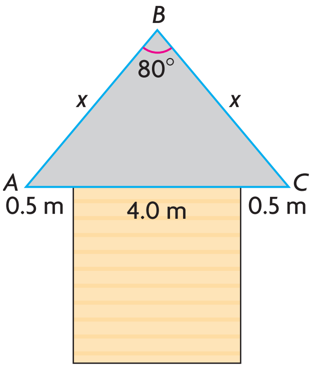
- Three-dimensional problems involving triangles can be solved using some combination of these approaches:
  - trigonometric ratios
  - the Pythagorean theorem
  - the sine law
  - the cosine law
- The approach you use depends on the given information and what you are required to find.

Need to Know

- When solving problems, always start with a sketch of the given information. Determine any unknown angles by using any geometric facts that apply, such as facts about parallel lines, interior angles in a triangle, and so on. Revise your sketch so that it includes any new information that you determined. Then use trigonometry to solve the original problem.
- In right triangles, use the primary or reciprocal trigonometric ratios.
- In all other triangles, use the sine law and/or the cosine law.

Given Information	Required to Find	Use
SSA	angle	sine law
ASA or AAS	side	sine law
SAS	side	cosine law
SSS	side	cosine law





CHECK Your Understanding

1. Morana is trolling for salmon in Lake Ontario. She sets the fishing rod so that its tip is 1 m above water and the line forms an angle of 35° with the water's surface. She knows that there are fish at a depth of 45 m. Describe the steps you would use to calculate the length of line she must let out.

2. Josh is building a garden shed that is 4.0 m wide. The two sides of the roof are equal in length and must meet at an angle of 80°. There will be a 0.5 m overhang on each side of the shed. Josh wants to determine the length of each side of the roof.

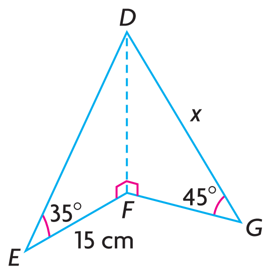
a) Should he use the sine law or the cosine law? Explain.

b) How could Josh use the primary trigonometric ratios to calculate x? Explain.

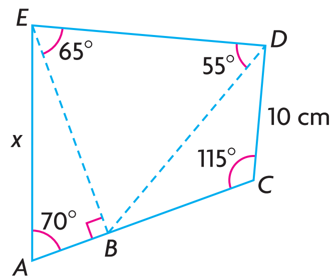
PRACTISING

3. Determine the value of x to the nearest centimetre and θ to the nearest
K degree. Explain your reasoning for each step of your solution.

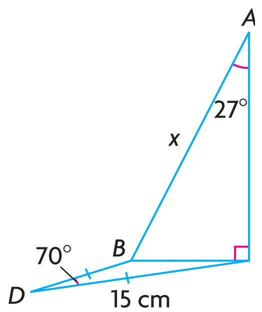
a)



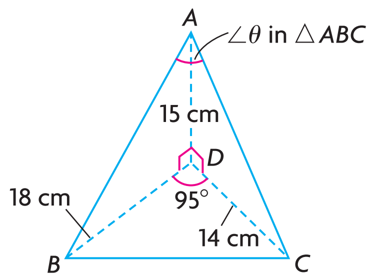
c)

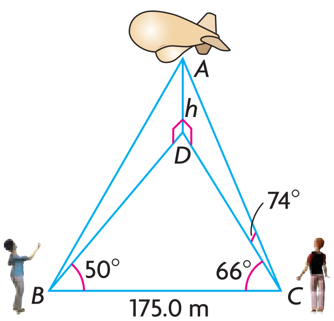


b)



d)





4. As a project, a group of students was asked to determine the altitude, h, of a promotional blimp. The students' measurements are shown in the sketch at the left.

a) Determine h to the nearest tenth of a metre. Explain each of your steps.

b) Is there another way to solve this problem? Explain.

5. While Travis and Bob were flying a hot-air balloon from Beamsville to Vineland in southwestern Ontario, they decided to calculate the straight-line distance, to the nearest metre, between the two towns.

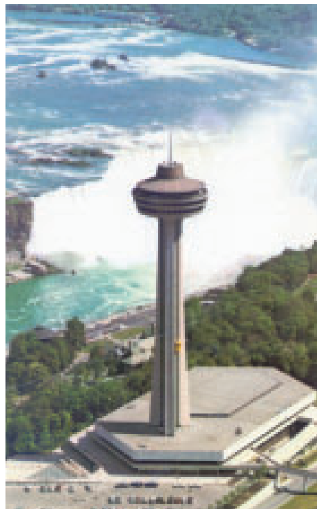
- From an altitude of 226 m, they simultaneously measured the angle of depression to Beamsville as 2° and to Vineland as 3°.
- They measured the angle between the lines of sight to the two towns as 80°.

Is there enough information to calculate the distance between the two towns? Justify your reasoning with calculations.

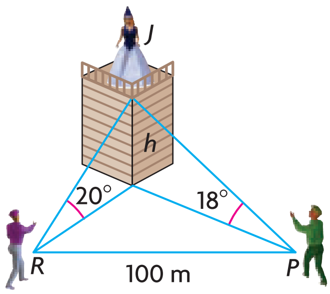
6. A The observation deck of the Skylon Tower in Niagara Falls, Ontario, is 166 m above the Niagara River. A tourist in the observation deck notices two boats on the water. From the tourist's position,

- the bearing of boat A is 180° at an angle of depression of 40°
- the bearing of boat B is 250° at an angle of depression of 34°

Calculate the distance between the two boats to the nearest metre.



7. Suppose Romeo is serenading Juliet while she is on her balcony. Romeo is facing north and sees the balcony at an angle of elevation of 20°. Paris, Juliet's other suitor, is observing the situation and is facing west. Paris sees the balcony at an angle of elevation of 18°. Romeo and Paris are 100 m apart as shown. Determine the height of Juliet's balcony above the ground, to the nearest metre.



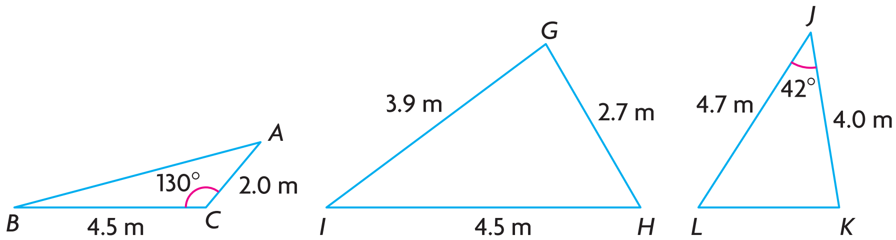
8. A coast guard helicopter hovers between an island and a damaged sailboat.

- From the island, the angle of elevation to the helicopter is 73°.
- From the helicopter, the island and the sailboat are 40° apart.
- A police rescue boat heading toward the sailboat is 800 m away from the scene of the accident. From this position, the angle between the island and the sailboat is 35°.
- At the same moment, an observer on the island notices that the sailboat and police rescue boat are 68° apart.

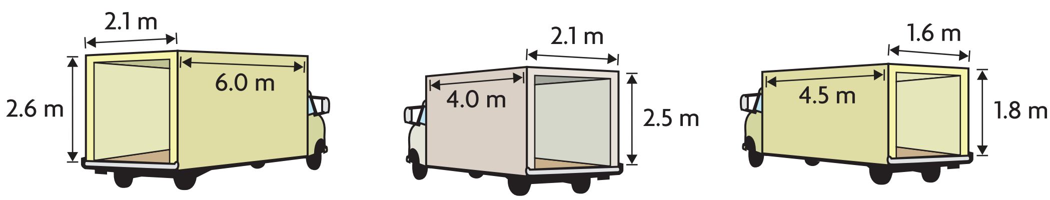
Explain how you would calculate the straight-line distance, to the nearest metre, from the helicopter to the sailboat. Justify your reasoning with calculations.

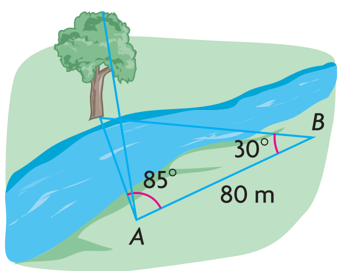
9. Brit and Tara are standing 13.5 m apart on a dock when they observe a sailboat moving parallel to the dock. When the boat is equidistant between both girls, the angle of elevation to the top of its 8.0 m mast is 51° for both observers. Describe how you would calculate the angle, to the nearest degree, between Tara and the boat as viewed from Brit's position. Justify your reasoning with calculations.

10. **T** In setting up for an outdoor concert, a stage platform has been dismantled into three triangular pieces as shown.



There are three vehicles available to transport the pieces. In order to prevent damaging the platform, each piece must fit exactly inside the vehicle. Explain how you would match each piece of the platform to the best-suited vehicle. Justify your reasoning with calculations.





11. Bert wants to calculate the height of a tree on the opposite bank of a river. To do this, he lays out a baseline 80 m long and measures the angles as shown at the left. The angle of elevation from *A* to the top of the tree is 28°. Explain if this information helps Bert to calculate the height of the tree to the nearest metre. Justify your reasoning with calculations.

12. **C** Chandra's homework question reads like this:
Bill and Chris live at different intersections on the same street, which runs north to south. When both of them stand at their front doors, they see a hot-air balloon toward the east at angles of elevation of 41° and 55°, respectively. Calculate the distance between the two friends.

a) Chandra says she doesn't have enough information to answer the question. Evaluate Chandra's statement. Justify your reasoning with calculations.

b) What additional information, if any, would you need to solve the problem? Justify your answer.

Extending

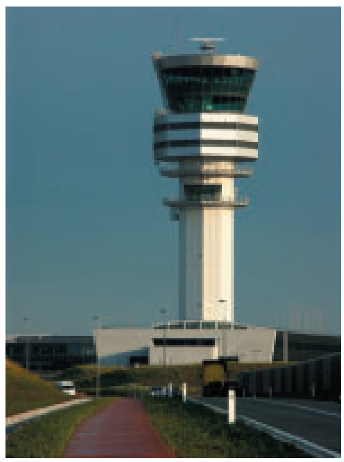
13. Two roads intersect at 34°. Two cars leave the intersection on different roads at speeds of 80 km/h and 100 km/h. After 2 h, a traffic helicopter that is above and between the two cars takes readings on them. The angle of depression to the slower car is 20°, and the straight-line distance from the helicopter to that car is 100 km. Assume that both cars are travelling at constant speed.

a) Calculate the straight-line distance, to the nearest kilometre, from the helicopter to the faster car. Explain your reasoning for each step of your solution.

b) Determine the altitude of the helicopter to the nearest kilometre.

14. Simone is facing north at the entrance of a tunnel through a mountain. She notices that a 1515 m high mountain in the distance has a bearing of 270° and its peak appears at an angle of elevation of 35°. After she exits the tunnel, the same mountain has a bearing of 258° and its peak appears at an angle of elevation of 31°. Assuming that the tunnel is perfectly level and straight, how long is it to the nearest metre?

15. An airport radar operator locates two planes flying toward the airport. The first plane, P, is 120 km from the airport, A, at a bearing of 70° and with an altitude of 2.7 km. The other plane, Q, is 180 km away on a bearing of 125° and with an altitude of 1.8 km. Calculate the distance between the two planes to the nearest tenth of a kilometre.



16. Mario is standing at ground level exactly at the corner where two exterior walls of his apartment building meet. From Mario's position, his apartment window on the north side of the building appears 44.5 m away at an angle of elevation of 55°. Mario notices that his friend Thomas's window on the west side of the building appears 71.0 m away at an angle of elevation of 34°.

a) If a rope were pulled taut from one window to the other, around the outside of the building, how long, to the nearest tenth of a metre, would the rope need to be? Explain your reasoning.

b) What is the straight-line distance through the building between the two windows? Round your answer to the nearest tenth of a metre.