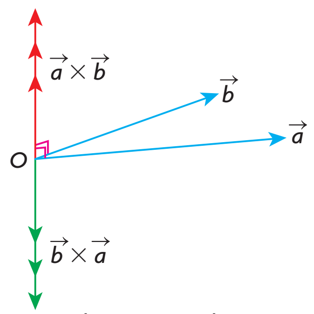
# *Section 7.6*—The Cross Product of Two Vectors




The Cross Product (Vector Product)

In the previous three sections, the dot product along with some of its applications was discussed. In this section, a second product called the **cross product**, denoted as $\vec{a} \times \vec{b}$, is introduced. The cross product is sometimes referred to as a **vector product** because, when it is calculated, the result is a vector and not a scalar. As we shall see, the cross product can be used in physical applications but also in the understanding of the geometry of $R^3$.

If we are given two vectors, $\vec{a}$ and $\vec{b}$, and wish to calculate their cross product, what we are trying to find is a particular vector that is perpendicular to each of the two given vectors. As will be observed, if we consider two nonzero, noncollinear vectors, there is an infinite number of vectors perpendicular to the two vectors. If we want to determine the cross product of these two vectors, we choose just one of these perpendicular vectors as our answer. Finding the cross product of two vectors is shown in the following example.

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**EXAMPLE 1** **Calculating the cross product of two vectors**

Given the vectors $\vec{a} = (1, 1, 0)$ and $\vec{b} = (1, 3, -1)$, determine $\vec{a} \times \vec{b}$.

### Solution

When calculating $\vec{a} \times \vec{b}$, we are determining a vector that is perpendicular to both $\vec{a}$ and $\vec{b}$. We start by letting this vector be $\vec{v} = (x, y, z)$.

Since $\vec{a} \cdot \vec{v} = 0$, $(1, 1, 0) \cdot (x, y, z) = 0$ and $x + y = 0$.

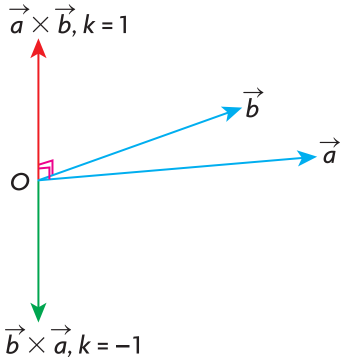
In the same way, since $\vec{b} \cdot \vec{v} = 0$, $(1, 3, -1) \cdot (x, y, z) = 0$ and $x + 3y - z = 0$.

Finding of the cross product of $\vec{a}$ and $\vec{b}$ requires solving a system of two equations in three variables which, under normal circumstances, has an infinite number of solutions. We will eliminate a variable and use substitution to find a solution to this system.

① $x + y = 0$

② $x + 3y - z = 0$

Subtracting eliminates $x$, $-2y + z = 0$, or $z = 2y$. If we substitute $z = 2y$ in equation ②, it will be possible to express $x$ in terms of $y$. Doing so gives $x + 3y - (2y) = 0$ or $x = -y$. Since $x$ and $z$ can both be expressed in terms of $y$, we write the solution as $(-y, y, 2y) = y(-1, 1, 2)$. The solution to this system




is any vector of the form $y(-1, 1, 2)$. It can be left in this form, but is usually written as $k(-1, 1, 2)$, $k \in \mathbf{R}$, where $k$ is a parameter representing any real value. The parameter $k$ indicates that there is an infinite number of solutions and that each of them is a scalar multiple of $(-1, 1, 2)$. In this case, the cross product is defined to be the vector where $k = 1$—that is, $(-1, 1, 2)$. The choosing of $k = 1$ simplifies computation and makes sense mathematically, as we will see in the next section. It should also be noted that the cross product is a vector and, as stated previously, is sometimes called a vector product.

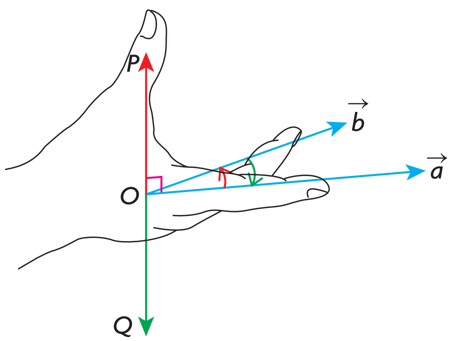
## Deriving a Formula for the Cross Product

What is necessary, to be more efficient in calculating $\vec{a} \times \vec{b}$, is a formula.

The vector $\vec{a} \times \vec{b}$ is a vector that is perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$. An infinite number of vectors satisfy this condition, all of which are scalar multiples of each other, but the cross product is one that is chosen in the simplest possible way, as will be seen when the formula is derived below. Another important point to understand about the cross product is that it exists only in $R^3$. It is not possible to take two noncollinear vectors in $R^2$ and construct a third vector perpendicular to the two vectors, because this vector would be outside the given plane.

It is also difficult, at times, to tell whether we are calculating $\vec{a} \times \vec{b}$ or $\vec{b} \times \vec{a}$. The formula, properly applied, will do the job without difficulty. There are times when it is helpful to be able to identify the cross product without using a formula. From the diagram below, $\vec{a} \times \vec{b}$ is pictured as a vector perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$, and, when looking down the axis from $P$ on $\vec{a} \times \vec{b}$, $\vec{a}$ would have to be rotated *counterclockwise* in order to be collinear with $\vec{b}$. In other words, $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.

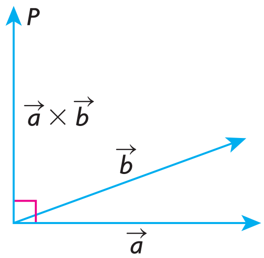
The vector $\vec{b} \times \vec{a}$, the opposite to $\vec{a} \times \vec{b}$, is again perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$, but, when looking from $Q$ down the axis formed by $\vec{b} \times \vec{a}$, $\vec{a}$ would have to be rotated *clockwise* in order to be collinear with $\vec{b}$.



### Definition of a Cross Product

The cross product of two vectors $\vec{a}$ and $\vec{b}$ in $R^3$ (3-space) is the vector that is perpendicular to these vectors such that the vectors $\vec{a}$, $\vec{b}$, and $\vec{a} \times \vec{b}$ form a right-handed system.

The vector $\vec{b} \times \vec{a}$ is the opposite of $\vec{a} \times \vec{b}$ and points in the opposite direction.




To develop a formula for $\vec{a} \times \vec{b}$, we follow a procedure similar to that followed in Example 1. Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, and let $\vec{v} = (x, y, z)$ be the vector that is perpendicular to $\vec{a}$ and $\vec{b}$.

So, ① $\vec{a} \cdot \vec{v} = (a_1, a_2, a_3) \cdot (x, y, z) = a_1x + a_2y + a_3z = 0$

and ② $\vec{b} \cdot \vec{v} = (b_1, b_2, b_3) \cdot (x, y, z) = b_1x + b_2y + b_3z = 0$

As before, we have a system of two equations in three unknowns, which we know from before has an infinite number of solutions. To solve this system of equations, we will multiply the first equation by $b_1$ and the second equation by $a_1$ and then subtract.

①$\times\, b_1 \rightarrow$ ③ $b_1a_1x + b_1a_2y + b_1a_3z = 0$

②$\times\, a_1 \rightarrow$ ④ $a_1b_1x + a_1b_2y + a_1b_3z = 0$

Subtracting ③ and ④ eliminates $x$. Move the $z$-terms to the right.

$(b_1a_2 - a_1b_2)y = (a_1b_3 - b_1a_3)z$

Multiplying each side by $-1$ and rearranging gives the desired result:

$(a_1b_2 - b_1a_2)y = (b_1a_3 - a_1b_3)z$

Now

$$\frac{y}{a_3b_1 - a_1b_3} = \frac{z}{a_1b_2 - a_2b_1}$$

If we carry out an identical procedure and eliminate $z$ from the system of equations, we have the following:

$$\frac{x}{a_2b_3 - a_3b_2} = \frac{y}{a_3b_1 - a_1b_3}$$

If we combine the two statements and set them equal to a constant $k$, we have

$$\frac{x}{a_2b_3 - a_3b_2} = \frac{y}{a_3b_1 - a_1b_3} = \frac{z}{a_1b_2 - a_2b_1} = k$$

Note that we can make these fractions equal to $k$ because every proportion can be made equal to a constant $k$. (For example, if $\frac{3}{6} = \frac{4}{8} = \frac{5}{10} = k$, then $k$ could be either $\frac{1}{2}$ or $\frac{10}{20}$ or any nonzero multiple of the form $\frac{1n}{2n}$.) This expression gives us a general form for a vector that is perpendicular to $\vec{a}$ and $\vec{b}$. The cross product, $\vec{a} \times \vec{b}$, is defined to occur when $k = 1$, and $\vec{b} \times \vec{a}$ occurs when $k = -1$.

**Formula for Calculating the Cross Product of Algebraic Vectors**

$k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$, $k \in \mathbf{R}$.

If $k = 1$, then $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$

If $k = -1$, then $\vec{b} \times \vec{a} = (a_3b_2 - a_2b_3, a_1b_3 - a_3b_1, a_2b_1 - a_1b_2)$

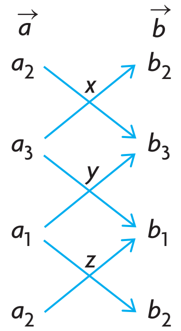
It is not easy to remember this formula for calculating the cross product of two vectors, so we develop a procedure, or a way of writing them, so that the memory work is removed from the calculation.

**Method of Calculating $\vec{a} \times \vec{b}$, where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$**

1. List the components of vector $\vec{a}$ in column form on the left side, starting with $a_2$ and then writing $a_3$, $a_1$, and $a_2$ below each other as shown.
2. Write the components of vector $\vec{b}$ in a column to the right of $\vec{a}$, starting with $b_2$ and then writing $b_3$, $b_1$, and $b_2$ in exactly the same way as the components of $\vec{a}$.
3. The required formula is now a matter of following the arrows and doing the calculation. To find the $x$ component, for example, we take the down product $a_2b_3$ and subtract the up product $a_3b_2$ from it to get $a_2b_3 - a_3b_2$.

(*continued*)

The other components are calculated in exactly the same way, and the formula for each component is listed below.




$x = a_2b_3 - a_3b_2$

$y = a_3b_1 - a_1b_3$

$z = a_1b_2 - a_2b_1$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

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**INVESTIGATION**

A. Given two vectors $\vec{a} = (2, 4, 6)$ and $\vec{b} = (-1, 2, -5)$ calculate $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$. What property does this demonstrate does hold not for the cross product? Explain why the property does not hold.

B. How are the vectors $\vec{a} \times \vec{b}$ and $\vec{b} \times \vec{a}$ related? Write an expression that relates $\vec{a} \times \vec{b}$ with $\vec{b} \times \vec{a}$.

C. Will the expression you wrote in part B be true for any pair of vectors in $R^3$? Explain.

D. Using the two vectors given in part A and a third vector $\vec{c} = (4, 3, -1)$ calculate:

i. $\vec{a} \times (\vec{b} + \vec{c})$ ii. $\vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

E. Compare your results from i and ii in part D. What property does this demonstrate? Write an equivalent expression for $\vec{b} \times (\vec{a} + \vec{c})$ and confirm it using the appropriate calculations.

F. Choose any 3 vectors in $R^3$ and demonstrate that the property you identified in part E holds for your vectors.

G. Using the three vectors given calculate:

i. $(\vec{a} \times \vec{b}) \times \vec{c}$ ii. $\vec{a} \times (\vec{b} \times \vec{c})$

H. Compare your results from i and ii in part G. What property does this demonstrate does hold not for the cross product? Explain why the property does not hold.

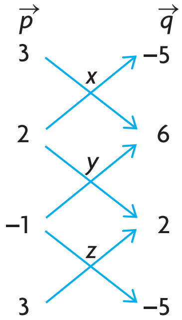
I. Choose any vector that is collinear with $\vec{a}$ (that is, any vector of the form $k\vec{a}$ $k \in \mathbf{R}$). Calculate $\vec{a} \times k\vec{a}$. Repeat using a different value for $k$. What can you conclude?

## EXAMPLE 2 Calculating cross products

If $\vec{p} = (-1, 3, 2)$ and $\vec{q} = (2, -5, 6)$, calculate $\vec{p} \times \vec{q}$ and $\vec{q} \times \vec{p}$.

### Solution

Let $\vec{p} \times \vec{q} = (x, y, z)$. The vectors $\vec{p}$ and $\vec{q}$ are listed in column form, with $\vec{p}$ on the left and $\vec{q}$ on the right, starting from the second component and working down.




$x = 3(6) - (2)(-5) = 28$

$y = 2(2) - (-1)(6) = 10$

$z = -1(-5) - 3(2) = -1$

$\vec{p} \times \vec{q} = (28, 10, -1)$

As already mentioned, $\vec{q} \times \vec{p}$ is the opposite of $\vec{p} \times \vec{q}$, so $\vec{q} \times \vec{p} = -1(28, 10, -1) = (-28, -10, 1)$. It is not actually necessary to calculate $\vec{q} \times \vec{p}$. All that is required is to calculate $\vec{p} \times \vec{q}$ and take the opposite vector to get $\vec{q} \times \vec{p}$.

After completing the calculation of the cross product, the answer should be verified to see if it is perpendicular to the given vectors using the dot product.

*Check:*

$(28, 10, -1) \cdot (-1, 3, 2) = -28 + 30 - 2 = 0$ and

$(28, 10, -1) \cdot (2, -5, 6) = 56 - 50 - 6 = 0$

There are a number of important properties of cross products that are worth noting. Some of these properties will be verified in the exercises.

**Properties of the Cross Product**

Let $\vec{p}$, $\vec{q}$, and $\vec{r}$ be three vectors in $R^3$, and let $k \in \mathbf{R}$.

Vector multiplication is not commutative: $\vec{p} \times \vec{q} = -(\vec{q} \times \vec{p})$,

Distributive law for vector multiplication: $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$,

Scalar law for vector multiplication: $k(\vec{p} \times \vec{q}) = (k\vec{p}) \times \vec{q} = \vec{p} \times (k\vec{q})$, The first property is one that we have seen in this section and is the first instance we have seen where the commutative property for multiplication has failed. Normally, we expect that the order of multiplication does not affect the product. In this case, changing the order of multiplication does change the result. The other two listed results are results that produce exactly what would be expected, and they will be used in this set of exercises and beyond.

**IN SUMMARY**

**Key Idea**

- The cross product $\vec{a} \times \vec{b}$, between two vectors $\vec{a}$ and $\vec{b}$, results in a third vector that is perpendicular to the plane in which the given vectors lie.

**Need to Know**

- $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$
- $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $(k\vec{a}) \times \vec{b} = \vec{a} \times (k\vec{b}) = k(\vec{a} \times \vec{b})$

# Exercise 7.6

## PART A

1. The two vectors $\vec{a}$ and $\vec{b}$ are vectors in $R^3$, and $\vec{a} \times \vec{b}$ is calculated.
   a. Using a diagram, explain why $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ and $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$.
   b. Draw the parallelogram determined by $\vec{a}$ and $\vec{b}$, and then draw the vector $\vec{a} + \vec{b}$. Give a simple explanation of why $(\vec{a} + \vec{b}) \cdot (\vec{a} \times \vec{b}) = 0$.
   c. Why is it true that $(\vec{a} - \vec{b}) \cdot (\vec{a} \times \vec{b}) = 0$? Explain.
2. For vectors in $R^3$, explain why the calculation $(\vec{a} \cdot \vec{b})(\vec{a} \times \vec{b}) = 0$ is meaningless. (Consider whether or not it is possible for the left side to be a scalar.)

## PART B

3. For each of the following calculations, say which are possible for vectors in $R^3$ and which are meaningless. Give a brief explanation for each.

| | | |
|---|---|---|
| a. $\vec{a} \cdot (\vec{b} \times \vec{c})$ | c. $(\vec{a} \times \vec{b}) \cdot (\vec{c} + \vec{d})$ | e. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ |
| b. $(\vec{a} \cdot \vec{b}) \times \vec{c}$ | d. $(\vec{a} \cdot \vec{b})(\vec{c} \times \vec{d})$ | f. $\vec{a} \times \vec{b} + \vec{c}$ |

K

4. Calculate the cross product for each of the following pairs of vectors, and verify your answer by using the dot product.
   a. $(2, -3, 5)$ and $(0, -1, 4)$
   b. $(2, -1, 3)$ and $(3, -1, 2)$
   c. $(5, -1, 1)$ and $(2, 4, 7)$
   d. $(1, 2, 9)$ and $(-2, 3, 4)$
   e. $(-2, 3, 3)$ and $(1, -1, 0)$
   f. $(5, 1, 6)$ and $(-1, 2, 4)$
5. If $(-1, 3, 5) \times (0, a, 1) = (-2, 1, -1)$, determine $a$.
6. a. Calculate the vector product for $\vec{a} = (0, 1, 1)$ and $\vec{b} = (0, 5, 1)$.
   b. Explain geometrically why it makes sense for vectors of the form $(0, b, c)$ and $(0, d, e)$ to have a cross product of the form $(a, 0, 0)$.
7. a. For the vectors $(1, 2, 1)$ and $(2, 4, 2)$, show that their vector product is $\vec{0}$.
   b. In general, show that the vector product of two collinear vectors, $(a, b, c)$ and $(ka, kb, kc)$, is always $\vec{0}$.
8. In the discussion, it was stated that $\vec{p} \times (\vec{q} + \vec{r}) = \vec{p} \times \vec{q} + \vec{p} \times \vec{r}$ for vectors in $R^3$. Verify that this rule is true for the following vectors.
   a. $\vec{p} = (1, -2, 4)$, $\vec{q} = (1, 2, 7)$, and $\vec{r} = (-1, 1, 0)$
   b. $\vec{p} = (4, 1, 2)$, $\vec{q} = (3, 1, -1)$, and $\vec{r} = (0, 1, 2)$

A

9. Verify each of the following:
   a. $\vec{i} \times \vec{j} = \vec{k} = -\vec{j} \times \vec{i}$
   b. $\vec{j} \times \vec{k} = \vec{i} = -\vec{k} \times \vec{j}$
   c. $\vec{k} \times \vec{i} = \vec{j} = -\vec{i} \times \vec{k}$

C

10. Show algebraically that $k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot \vec{a} = 0$. What is the meaning of this result?
11. You are given the vectors $\vec{a} = (2, 0, 0)$, $\vec{b} = (0, 3, 0)$, $\vec{c} = (2, 3, 0)$, and $\vec{d} = (4, 3, 0)$.
    a. Calculate $\vec{a} \times \vec{b}$ and $\vec{c} \times \vec{d}$.
    b. Calculate $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.
    c. Without doing any calculations (that is, by visualizing the four vectors and using properties of cross products), say why $(\vec{a} \times \vec{c}) \times (\vec{b} \times \vec{d}) = \vec{0}$.

## PART C

12. Show that the cross product is not associative by finding vectors $\vec{x}$, $\vec{y}$, and $\vec{z}$ such that $(\vec{x} \times \vec{y}) \times \vec{z} \neq \vec{x} \times (\vec{y} \times \vec{z})$.

T

13. Prove that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$ is true.