6.2 Investigating the Properties of Sinusoidal Functions

GOAL

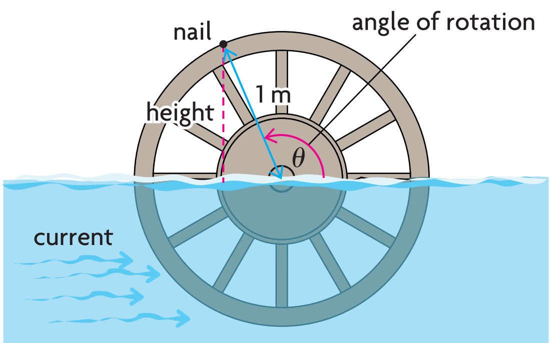
Examine the two functions that are associated with all sinusoidal functions.

YOU WILL NEED

- cardboard
- ruler
- protractor
- metre stick
- thumbtack
- graphing calculator

INVESTIGATE the Math

Paul uses a generator powered by a water wheel to produce electricity. Half the water wheel is submerged below the surface of a river. The wheel has a radius of 1 m. A nail on the circumference of the wheel starts at water level. As the current flows down the river, the wheel rotates counterclockwise to power the generator. The height of the nail changes as the wheel rotates.

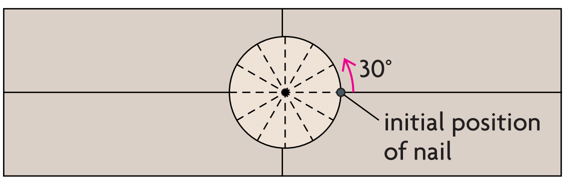


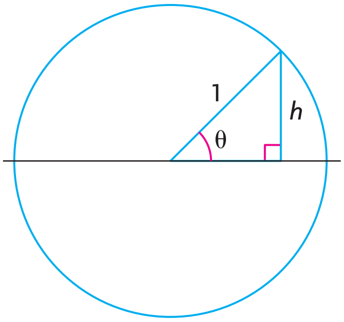
? How can you describe the position of the nail using an equation?

A. Construct a scale model of the water wheel. On a piece of cardboard, cut out a circle with a radius of 10 cm to represent the water wheel's 1 m radius.

B. Locate the centre of the circle. Use a protractor to divide your cardboard wheel into 30° increments through the centre. Draw a dot to represent the nail on the circumference of the circle at one of the lines you drew to divide the wheel.

C. On a rectangular piece of cardboard about 100 cm long and 30 cm wide, draw a horizontal line to represent the water level and a vertical line both through the centre. Attach the cardboard wheel to the centre of the rectangular piece of cardboard with a thumbtack, with the rectangle behind the wheel.



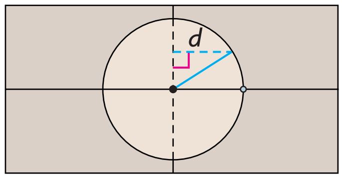


D. Rotate the cardboard wheel 30° counterclockwise. Measure the height, h, of the nail: the perpendicular distance from the nail to the horizontal line. Copy the table, and record the *actual* distance the nail is above the horizontal line at 30° by multiplying the scale height by 10 and converting to metres. Continue to rotate the wheel in 30° increments, measuring h and recording the actual heights. If the nail goes below the horizontal line, record the height as a negative value. Continue until the nail has rotated 720°.

Angle of Rotation, θ (°)	0	30	60	90	120	• • •	690	720
Actual Height of Nail, h (m)	0			1				

E. Use your data to graph height versus angle of rotation.

F. Use your model of the water wheel to examine the horizontal distance, d, the nail is from a vertical line that passes through the centre of the water wheel. Start with the nail initially positioned at water level.



Rotate the cardboard wheel 30° counterclockwise, and measure the distance the nail is from the vertical line. Copy the table, and record the *actual* distance the nail is from the vertical line at 30°, again adjusting for the scale factor. Continue to rotate the wheel in 30° increments, and record the actual distances. If the nail goes to the left of the vertical line, record the distance as a negative value. Continue until the nail has rotated 720°.

Angle of Rotation, θ (°)	0	30	60	90	120	• • •	690	720
Actual Distance from Vertical Line, d (m)	0			1				

G. Use your data to graph horizontal distance versus angle of rotation.

H. Use your graphing calculator to determine the cosine and sine of each rotation angle. Make sure your calculator is in DEGREE mode and evaluate to the nearest hundredth.

Tech Support

You can generate the tables using the List feature on your graphing calculator. Try putting degrees in L1, replacing L2 with "cos(L1)" and L3 with "sin(L1)."

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos θ													
θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin θ													

I. Based on the tables you created in parts D, F, and H, select the appropriate equation that describes the height, h, of the nail on the water wheel in terms of the rotation. Also, identify another equation that describes the distance, d, the nail is from the vertical line in terms of the rotation.

$$d(\theta) = \sin\theta \qquad d(\theta) = 0.5\,\theta \qquad \theta(d) = \sin d \qquad d(\theta) = \cos\theta$$

$$h(\theta) = \sin\theta \qquad h(\theta) = 0.5\,\theta \qquad \theta(d) = \sin h \qquad h(\theta) = \cos\theta$$

Reflecting

J. Use your graphing calculator to graph $y = \sin x$ and $y = \cos x$, where $0° \leq x \leq 360°$, and compare these graphs to the graphs from parts E and G. Use words such as *amplitude*, *period*, *equation of the axis*, *increasing intervals*, *decreasing intervals*, *domain*, and *range* in your comparison.

K. State the coordinates of five key points that would allow you to draw the sinusoidal function $y = \sin x$ quickly over the interval $0°$ to $360°$.

L. State the coordinates of five key points that would allow you to draw the sinusoidal function $y = \cos x$ quickly over the interval $0°$ to $360°$.

M. What transformation can you apply to the cosine curve that will result in the sine curve?

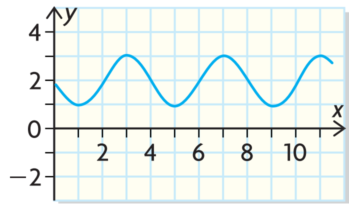
N. What ordered pair could you use to represent the point on the wheel that corresponds to the nail's location in terms of θ, the angle of rotation?

Tech Support

For help graphing trigonometric functions on your graphing calculator, see Technical Appendix, B-14.

sinusoidal function

a periodic function whose graph looks like smooth symmetrical waves, where any portion of the wave can be horizontally translated onto another portion of the curve; graphs of sinusoidal functions can be created by transforming the graph of the function $y = \sin x$ or $y = \cos x$

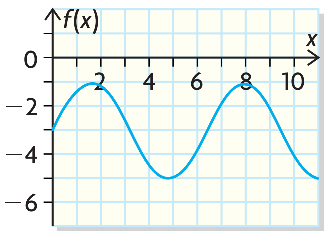


APPLY the Math

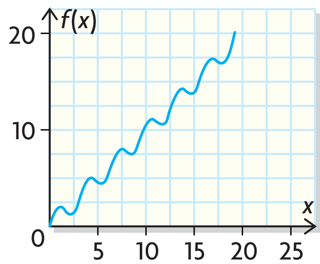
EXAMPLE 1 Identifying the function

Determine whether the graph represents a periodic function. If it does, determine whether it represents a sinusoidal function.

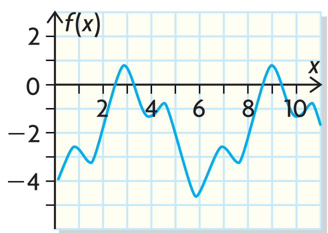
a)



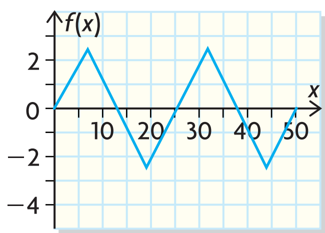
c)



b)



d)



Bridget's Solution

a) periodic and sinusoidal

The function repeats, so it's periodic. It looks like smooth symmetrical waves, where any portion of the wave can be horizontally translated onto another portion of the curve.

b) periodic

The pattern repeats but the waves aren't symmetrical.

c) neither periodic nor sinusoidal

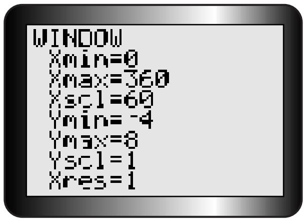
It looks like smooth symmetrical waves; however, I can't horizontally translate any portion of the wave onto another portion of the curve.

d) periodic

The pattern repeats but the waves aren't smooth curves.

EXAMPLE 2 Identifying the properties of a sinusoidal function

Graph the function $f(x) = 4\sin(3x) + 2$ on a graphing calculator using the WINDOW settings shown in DEGREE mode.



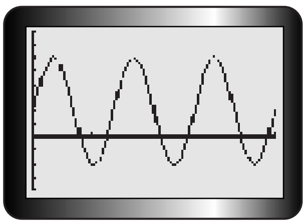
a) Is the function periodic? If it is, is it sinusoidal?

b) From the graph, determine the period, the equation of the axis, the amplitude, and the range.

c) Calculate $f(20°)$.

Beth's Solution

a)



Because it repeats, the graph is periodic.

Since it forms a series of identical, symmetrical smooth waves, it is sinusoidal.

b) period $= 120°$

The graph completes three cycles in 360°, so one cycle, which is the period, must be 120°.

equation of the axis: $y = \dfrac{-2 + 6}{2}$

The axis is halfway between the minimum of -2 and the maximum of 6.

$$y = 2$$

$$6 - 2 = 4$$

$$\text{amplitude} = 4$$

To get the amplitude, I calculated the vertical distance between a maximum and the axis. It's 4.

range: $\{y \in \mathbf{R} \mid -2 \leq y \leq 6\}$

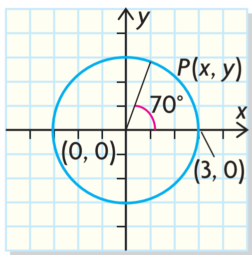
For the range, the greatest *y*-value on the graph (the maximum) is 6, and the least *y*-value (the minimum) is -2.

c)
$$\begin{aligned} f(x) &= 4\sin(3x) + 2 \\ f(20°) &= 4\sin(3(20°)) + 2 \\ &= 4\sin(60°) + 2 \\ &\doteq 4(0.866) + 2 \\ &= 5.464 \end{aligned}$$

$f(20°)$ means find *y* when $x = 20°$. I substituted 20 for *x* and then calculated *y*.

EXAMPLE 3 Determining the coordinates of a point from a rotation angle

Determine the coordinates of the point $P(x, y)$ resulting from a rotation of 70° centred at the origin and starting from the point (3, 0).



Anne's Solution

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}$$

$$\sin \theta = \frac{y}{r}$$

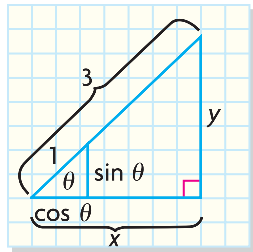
$$\frac{3}{1} = \frac{x}{\cos \theta} \quad \text{and} \quad \frac{3}{1} = \frac{y}{\sin \theta}$$

$$x = 3 \cos \theta \qquad y = 3 \sin \theta$$

$$P(x, y) = (3 \cos \theta, 3 \sin \theta)$$

The water wheel solution was based on a circle of radius 1. The coordinates of the nail after a rotation of θ were $(\cos \theta, \sin \theta)$. But this circle doesn't have a radius of 1. Its radius is 3.

I used similar triangles to figure out the coordinates of the larger triangle.



The coordinates for any point $P(x, y)$ on a circle of radius r are $P(x, y) = (r \cos \theta, r \sin \theta)$.

This means that the coordinates of the new point after a rotation of θ from the point $(r, 0)$ about $(0, 0)$ can be determined from $(r \cos \theta, r \sin \theta)$.

$$P(x, y) = (3 \cos 70°, 3 \sin 70°)$$
$$\doteq (1.03, 2.82)$$

I substituted the radius and angle of rotation into the ordered pair $(r \cos \theta, r \sin \theta)$ and got the coordinates of the image point.

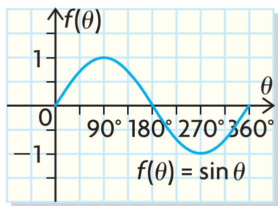
In Summary

Key Idea

- The function $f(\theta) = \sin \theta$ is a periodic function that represents the height (vertical distance) of a point from the x-axis as it rotates $\theta°$ about a circle with radius 1.
- The function $f(\theta) = \cos \theta$ is a periodic function that represents the horizontal distance of a point from the y-axis as it rotates $\theta°$ about a circle with radius 1.

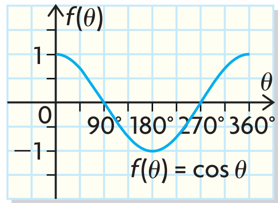
Need to Know

- The graph of $f(\theta) = \sin \theta$ has these characteristics:
  - The period is 360°.
  - The amplitude is 1, the maximum value is 1, and the minimum value is -1.
  - The domain is $\{\theta \in \mathbf{R}\}$, and the range is $-1 \leq f(\theta) \leq 1$.
  - The zeros are located at 0°, 180°, 360°, ...



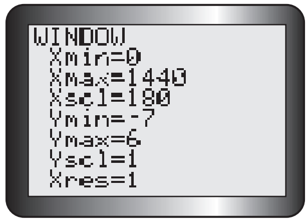
(continued)

- The graph of $f(\theta) = \cos\theta$ has these characteristics:
  - The period is 360°.
  - The amplitude is 1, the maximum value is 1, and the minimum value is -1.
  - The domain is $\{\theta \in \mathbf{R}\}$, and the range is $-1 \le f(\theta) \le 1$.
  - The zeros are located at 90°, 270°, 450°, ...



- The sine function and cosine function are congruent sinusoidal curves; the cosine curve is the sine curve translated 90° to the left.
- Any point $P(x, y)$ on a circle centred at (0, 0) with radius r and rotated through an angle θ can be expressed as an ordered pair $(r\cos\theta, r\sin\theta)$.

CHECK Your Understanding



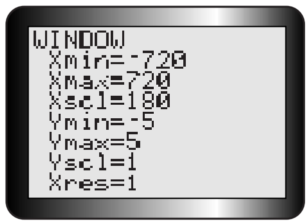
1. Using a graphing calculator in DEGREE mode, graph each sinusoidal function. Use the WINDOW settings shown. From the graph, state the amplitude, period, and equation of the axis for each.
a) $y = 3\sin(2x) + 1$ **b)** $y = 4\cos(0.5x) - 2$

2. a) If $h(x) = \sin(5x) - 1$, calculate $h(25°)$.
b) If $f(x) = \cos x$ and $f(x) = 0$, list the values of x where $0° \le x \le 360°$.

3. A buoy rises and falls as it rides the waves. The equation $h(t) = \cos(36t)°$ models the displacement of the buoy, $h(t)$, in metres at t seconds.
a) Graph the displacement from 0 s to 20 s, in 2.5 s intervals.
b) Determine the period of the function from the graph.
c) What is the displacement at 35 s?
d) At what time, to the nearest second, does the displacement first reach -0.8 m?

4. Determine the coordinates of the new point after a rotation of 50° about (0, 0) from the point (2, 0).

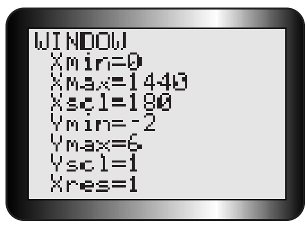
PRACTISING



5. Using a graphing calculator and the WINDOW settings shown, graph each function. Use DEGREE mode. State whether the resulting functions are periodic. If so, state whether they are sinusoidal.
a) $y = 3\sin x + 1$ **c)** $y = \cos(2x) - \sin x$ **e)** $y = 0.5\cos x - 1$
b) $y = (0.004x)\sin x$ **d)** $y = 0.005x + \sin x$ **f)** $y = \sin 90°$

6. Based on your observations in question 5, what can you conclude about any function that possesses sine or cosine in its equation?

7. If $g(x) = \sin x$ and $h(x) = \cos x$, where $0° \le x \le 360°$, calculate each and explain what it means.
a) $g(90°)$ **b)** $h(90°)$



8. **K** Using a graphing calculator in DEGREE mode, graph each sinusoidal function. Use the WINDOW settings shown. From the graph, state the amplitude, period, increasing intervals, decreasing intervals, and equation of the axis for each.
a) $y = 2\sin x + 3$ **c)** $y = \sin(0.5x) + 2$ **e)** $y = 2\sin(0.25x)$
b) $y = 3\sin x + 1$ **d)** $y = \sin(2x) - 1$ **f)** $y = 3\sin(0.5x) + 2$

9. a) If $f(x) = \cos x$, calculate $f(35°)$.
b) If $g(x) = \sin(2x)$, calculate $g(10°)$.
c) If $h(x) = \cos(3x) + 1$, calculate $h(20°)$.
d) If $f(x) = \cos x$ and $f(x) = -1$, calculate x for $0° \leq x \leq 360°$.
e) If $f(x) = \sin x$ and $f(x) = -1$, calculate x for $0° \leq x \leq 360°$.

10. Determine all values where $\sin x = \cos x$ for $-360° \leq x \leq 360°$.

T
11. a) Determine the coordinates of the new point after a rotation of 25° about (0, 0) from the point (1, 0).
b) Determine the coordinates of the new point after a rotation of 80° about (0, 0) from the point (5, 0).
c) Determine the coordinates of the new point after a rotation of 120° about (0, 0) from the point (4, 0).
d) Determine the coordinates of the new point after a rotation of 230° about (0, 0) from the point (3, 0).

12. Sketch the sinusoidal graphs that satisfy the properties in the table.

	Period	Amplitude	Equation of the Axis	Number of Cycles
a)	4	3	$y = 5$	2
b)	20	6	$y = 4$	3
c)	80	5	$y = -2$	2

13. Jim is riding a Ferris wheel, where t is time in seconds. Explain what each of
A the following represents.
a) $h(10)$, where $h(t) = 5\cos(18t)°$
b) $h(10)$, where $h(t) = 5\sin(18t)°$

14. Compare the graphs for $y = \sin x$ and $y = \cos x$, where $0° \leq x \leq 360°$.
C How are they the same, and how are they different?

Extending

15. If the water level in the original water wheel situation was lowered so that three-quarters of the wheel was exposed, determine the equation of the sinusoidal function that describes the height of the nail in terms of the rotation.

16. A spring bounces up and down according to the model $d(t) = 0.5\cos(120t)°$, where $d(t)$ is the displacement in centimetres from the rest position and t is time in seconds. The model does not consider the effects of gravity.
a) Make a table for $0 \leq t \leq 9$. Use 0.5 s intervals.
b) Draw the graph.
c) Explain why the function models periodic behaviour.
d) What is the relationship between the amplitude of the function and the displacement of the spring from its rest position?