6.5 Using Transformations to Sketch the Graphs of Sinusoidal Functions

YOU WILL NEED

- graph paper

GOAL

Sketch the graphs of sinusoidal functions using transformations.

LEARN ABOUT the Math

Glen has been asked to graph the sinusoidal function $f(x) = 3\sin(2(x - 60°)) + 4$ without using technology.

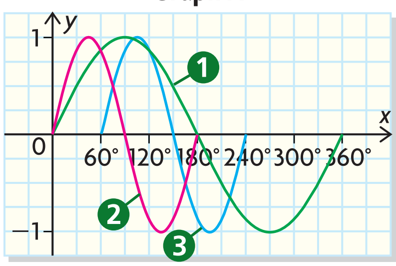
? How can you graph sinusoidal functions using transformations?

EXAMPLE 1 Using transformations to sketch the graph of a sinusoidal function

Sketch the graph of $f(x) = 3\sin(2(x - 60°)) + 4$.

Glen's Solution

Graph A

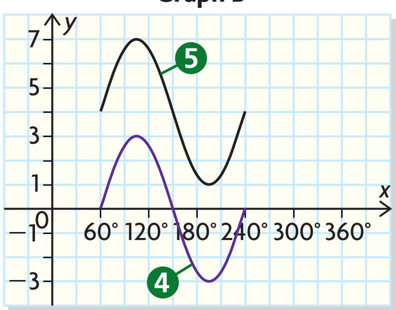


1 I started by graphing $y = \sin x$ (in green).

2 Then I graphed $y = \sin(2x)$ (in red). It has a horizontal compression of $\frac{1}{2}$, so the period is $\frac{360°}{2} = 180°$ instead of 360° because all the x-coordinates of the points on the graph of $y = \sin x$ have been divided by 2.

3 Then I graphed $y = \sin(2(x - 60°))$ (in blue) by applying a horizontal translation of $y = \sin(2x)$ 60° to the right because 60° has been added to all the x-coordinates of the points on the previous graph.

Graph B



4 Next, I graphed $y = 3\sin(2(x - 60°))$ (in purple) by applying a vertical stretch of 3 to $y = \sin(2(x - 60°))$. The amplitude is now 3 because all the y-coordinates of the points on the previous graph have been multiplied by 3.

5 Finally, I graphed $y = 3\sin(2(x - 60°)) + 4$ (in black) by applying a vertical translation of 4 to $y = 3\sin(2(x - 60°))$. This means that the whole graph has slid up 4 units and that the equation of the axis is now $y = 4$ because 4 has been added to all the y-coordinates of the points on the previous graph.

Reflecting

A. In what order were the transformations applied to the function $y = \sin x$?

B. If the equation of the function $y = 3\sin(2(x - 60°)) + 4$ were changed to $y = 3\sin(2(x - 60°)) - 5$, how would the graph of the function change? How would it stay the same?

C. If the equation of the function $y = 3\sin(2(x - 60°)) + 4$ were changed to $y = 3\sin(9(x - 60°)) + 4$, how would the graph of the function change?

D. Which transformations affect the range of the function? How?

E. Which transformations affect the period of the function? How?

F. Could Glen graph this function faster by combining transformations? If so, which ones?

APPLY the Math

EXAMPLE 2 Connecting transformations to the graph of a sinusoidal function

a) Graph $y = -2\cos(3x) - 1$ using transformations.

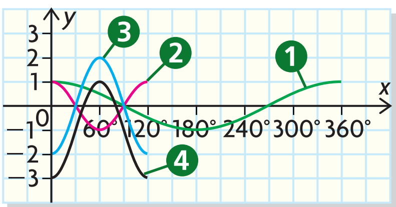
b) State the amplitude, period, equation of the axis, phase shift, and range of this sinusoidal function.

phase shift

the horizontal translation of a sinusoidal function

Steven's Solution

a)



1. I started by graphing $y = \cos x$ (in green).

2. I dealt with the horizontal compression first. I graphed $y = \cos(3x)$ (in red) using a period of $\frac{360°}{3} = 120°$ instead of 360°.

3. I dealt with the vertical stretch and the reflection in the x-axis. I graphed $y = -2\cos(3x)$ (in blue) starting at its lowest value due to the reflection, changing its amplitude to 2 due to the vertical stretch.

4. Finally, I did the vertical translation. I graphed $y = -2\cos(3x) - 1$ (in black) by sliding the previous graph down 1 unit, so the equation of the axis is $y = -1$.

b) The amplitude is 2.

The period is 120°.

The equation of the axis is $y = -1$.

The phase shift is 0.

The range is $\{y \in \mathbf{R} \mid -3 \leq y \leq 1\}$.

You can graph sinusoidal functions more efficiently if you combine and use several transformations at the same time.

EXAMPLE 3 Using a factoring strategy to determine the transformations

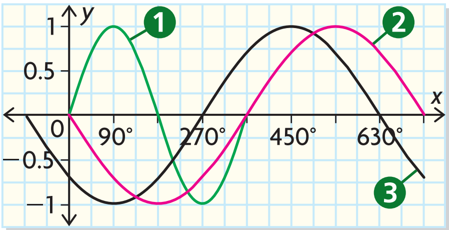
Graph $y = -\sin(0.5x + 45°)$ using transformations.

John's Solution

$y = -\sin(0.5x + 45°)$

$y = -\sin[0.5(x + 90°)]$

I factored the expression inside the brackets so that I could see all the transformations. I divided out the common factor 0.5 from 0.5*x* and 45.



1. I started by graphing $y = \sin x$ (in green).
2. Rather than graph this one transformation at a time, I dealt with all stretches/compressions and reflections at the same time. I graphed $y = -\sin(0.5x)$ (in red) by using a period of $\frac{360°}{0.5} = 720°$ and reflecting this across the *x*-axis.
3. I applied the phase shift and graphed $y = -\sin(0.5(x + 90°))$ (in black) by shifting all the points on the previous graph 90° to the left.

In Summary

Key Idea

- Functions of the form $g(x) = a\sin(k(x - d)) + c$ and $h(x) = a\cos(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the graphs of $f(x) = \sin x$ and $f(x) = \cos x$, respectively, one at a time, following the order of operations (multiplication and division before addition and subtraction) for all vertical transformations and for all horizontal transformations. The horizontal and vertical transformations can be completed in either order.
- As with other functions, you can apply all stretches/compressions and reflections together followed by all translations to graph the transformed function more efficiently.

Need to Know

- To graph $g(x)$, you need to apply the transformations to the key points of $f(x) = \sin x$ or $f(x) = \cos x$ only, not to every point on $f(x)$.
  - Key points for $f(x) = \sin x$
    $(0°, 0)$, $(90°, 1)$, $(180°, 0)$, $(270°, -1)$, $(360°, 0)$
  - Key points for $f(x) = \cos x$
    $(0°, 1)$, $(90°, 0)$, $(180°, -1)$, $(270°, 0)$, $(360°, 1)$

(*continued*)

- By doing so, you end up with a function with
  - an amplitude of $|a|$
  - a period of $\frac{360°}{|k|}$
  - an equation of the axis $y = c$
- Horizontal and vertical translations of sine and cosine functions can be summarized as follows:

  Horizontal
  - Move the graph d units to the right when $d > 0$.
  - Move the graph $|d|$ units to the left when $d < 0$.

  Vertical
  - Move the graph $|c|$ units down when $c < 0$.
  - Move the graph c units up when $c > 0$.
- Horizontal and vertical stretches of sine and cosine functions can be summarized as follows:

  Horizontal
  - Compress the graph by a factor $\left|\frac{1}{k}\right|$ when $|k| > 1$.
  - Stretch the graph by a factor $\left|\frac{1}{k}\right|$ when $0 < |k| < 1$.
  - Reflect the graph in the y-axis if $k < 0$.

  Vertical
  - Stretch the graph by a factor $|a|$ when $|a| > 1$.
  - Compress the graph by a factor $|a|$ when $0 < |a| < 1$.
  - Reflect the graph in the x-axis if $a < 0$.

CHECK Your Understanding

1. State the transformations, in the order you would apply them, for each sinusoidal function.

a) $f(x) = \sin(4x) + 2$

b) $y = 0.25 \cos(x - 20°)$

c) $g(x) = -\sin(0.5x)$

d) $y = 12 \cos(18x) + 3$

e) $f(x) = -20 \sin\left[\frac{1}{3}(x - 40°)\right]$

2. If the function $f(x) = 4 \cos 3x + 6$ starts at $x = 0$ and completes two full cycles, determine the period, amplitude, equation of the axis, domain, and range.

3. Use transformations to predict what the graph of $g(x) = 5 \sin(2(x - 30°)) + 4$ will look like. Verify with a graphing calculator.

PRACTISING

4. State the transformations in the order you would apply them for each sinusoidal function.

a) $y = -2\sin(x + 10°)$

b) $y = \cos(5x) + 7$

c) $y = 9\cos(2(x + 6°)) - 5$

d) $g(x) = \frac{1}{5}\sin(x - 15°) + 1$

e) $h(x) = -\sin\left[\frac{1}{4}(x + 37°)\right] - 2$

f) $d = -6\cos(3t) + 22$

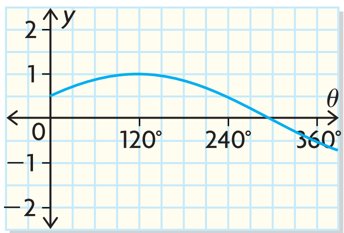
5. Match each function to its corresponding graph.

a) $y = \sin(2\theta - 90°), 0° \leq \theta \leq 360°$

b) $y = \sin(3\theta - 90°), 0° \leq \theta \leq 360°$

c) $y = \sin\left(\frac{\theta}{2} + 30°\right), 0° \leq \theta \leq 360°$

i)



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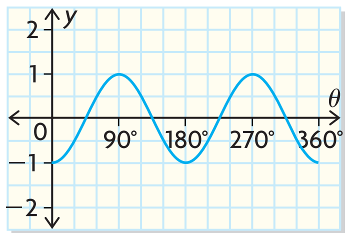
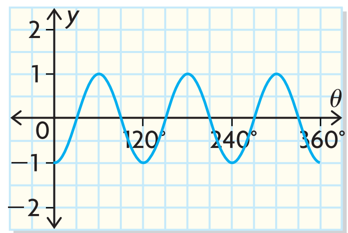


iii)



6. If each function starts at $x = 0$ and finishes after three complete cycles, determine the period, amplitude, equation of the axis, domain, and range of each without graphing.

a) $y = 3\sin x + 2$

b) $g(x) = -4\cos(2x) + 7$

c) $h = -\frac{1}{2}\sin t - 5$

d) $h(x) = \cos(4(x - 12°)) - 9$

e) $d = 10\sin(180(t - 17°)) - 30$

f) $j(x) = 0.5\sin(2x - 30°)$

7. **K** Predict what the graph of each sinusoidal function will look like by describing the transformations of $y = \sin x$ or $y = \cos x$ that would result in the new graph. Sketch the graph, and then verify with a graphing calculator.

a) $y = 2\sin x + 3$

b) $y = -3\cos x + 5$

c) $y = -\sin(6x) + 4$

d) $y = 4\cos(2x) - 3$

e) $y = \frac{1}{2}\cos(3x - 120°)$

f) $y = -8\sin\left[\frac{1}{2}(x + 50°)\right] - 9$

8. Determine the appropriate WINDOW settings on your graphing calculator that enable you to see a complete cycle for each function. There is more than one acceptable answer.

a) $k(x) = -\sin(2x) + 6$

b) $j(x) = -5\sin\left(\frac{1}{2}x\right) + 20$

c) $y = 7\cos(90(x - 1°)) + 82$

d) $f(x) = \frac{1}{2}\sin(360x + 72°) - 27$

9. Each person's blood pressure is different. But there is a range of blood pressure values that is considered healthy. For a person at rest, the function $P(t) = -20\cos(300t)° + 100$ models the blood pressure, $P(t)$, in millimetres of mercury at time t seconds.

A

a) What is the period of the function? What does the period represent for an individual?

b) What is the range of the function? Explain the meaning of the range in terms of a person's blood pressure.

10. a) Determine the equation of a sine function that would have the range $\{y \in \mathbf{R} \mid -1 \leq y \leq 7\}$ and a period of $720°$.

T

b) Determine the equation of the cosine function that results in the same graph as your function in part (a).

11. Explain how you would graph the function $f(x) = -\frac{1}{2}\cos(120x) + 30$ using transformations.

C

Extending

12. If the functions $y = \sin x$ and $y = \cos x$ are subjected to a horizontal compression of 0.5, what transformation would map the resulting sine curve onto the resulting cosine curve?

13. The function $D(t) = 4\sin\left[\frac{360}{365}(t - 80)\right]° + 12$ is a model of the number of hours, $D(t)$, of daylight on a specific day, t, at latitude 50° north.

a) Explain why a trigonometric function is a reasonable model for predicting the number of hours of daylight.

b) How many hours of daylight do March 21 and September 21 have? What is the significance of each of these days?

c) How many hours of daylight do June 21 and December 21 have? What is the significance of each of these days?

d) Explain what the number 12 represents in the model.