Section 6.5—Vectors in R^2 and R^3

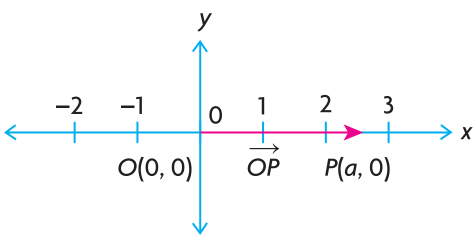
In the introduction to this chapter, we said vectors are important because of their application to a variety of different areas of study. In these areas, the value of using vectors is derived primarily from being able to consider them in coordinate form, or algebraic form, as it is sometimes described. Our experience with coordinate systems in mathematics thus far has been restricted to the *xy*-plane, but we will soon begin to see how ideas in two dimensions can be extended to higher dimensions and how this results in a greater range of applicability.

Introduction to Algebraic Vectors

Mathematicians started using coordinates to analyze physical situations in about the fourteenth century. However, a great deal of the credit for developing the methods used with coordinate systems should be given to the French mathematician Rene Descartes (1596–1650). Descartes was the first to realize that using a coordinate system would allow for the use of algebra in geometry. Since then, this idea has become important in the development of mathematical ideas in many areas. For our purposes, using algebra in this way leads us to the consideration of ideas involving vectors that otherwise would not be possible.

At the beginning of our study of algebraic vectors, there are a number of ideas that must be introduced and that form the foundation for what we are doing. After we start to work with vectors, these ideas are used implicitly without having to be restated each time.

One of the most important ideas that we must consider is that of the **unique** representation of vectors in the *xy*-plane. The unique representation of the vector $\overrightarrow{OP}$ is a matter of showing the unique representation of the point P because $\overrightarrow{OP}$ is determined by this point. The uniqueness of vector representation will be first considered for the **position vector** $\overrightarrow{OP}$, which has its head at the point $P(a, 0)$ and its tail at the origin $O(0, 0)$ shown on the *x*-axis below. The *x*-axis is the set of real numbers, **R**, which is made up of rational and irrational numbers.

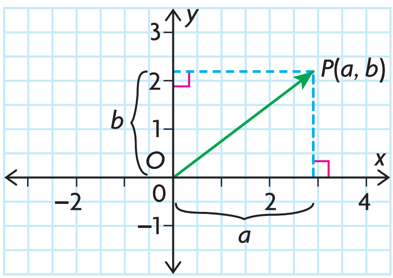


The point P is a distance of a units away from the origin and occupies exactly one position on the *x*-axis. Since each point P has a unique position on this axis, this implies that $\overrightarrow{OP}$ is also unique because this vector is determined by P.

The xy-plane is often referred to as R^2, which means that each of the x- and y-coordinates for any point on the plane is a real number. In technical terms, we would say that $R^2 = \{(x, y)$, where x and y are real numbers$\}$.

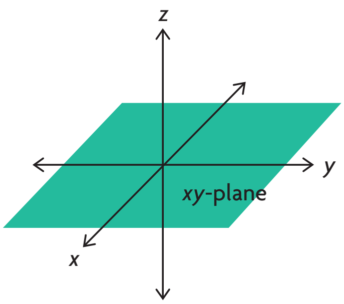
Points and Vectors in R^2

In the following diagram, $\overrightarrow{OP}$ can also be represented in component form by the vector defined as (a, b). This is a vector with its tail at $O(0, 0)$ and its head at $P(a, b)$. Perpendicular lines have been drawn from P to the two axes to help show the meaning of (a, b) in relation to $\overrightarrow{OP}$. a is called the x-component and b is the y-component of $\overrightarrow{OP}$. Again, each of the coordinates of this point is a unique real number and, because of this, the associated vector, $\overrightarrow{OP}$, has a unique location in the xy-plane.



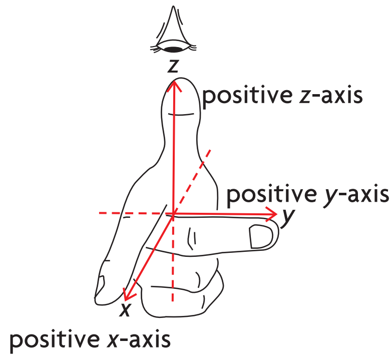
Points and Vectors in R^3

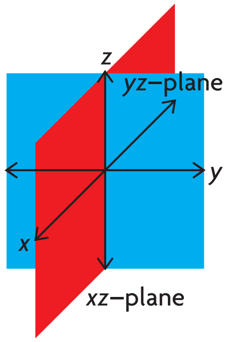
All planes in R^2 are flat surfaces that extend infinitely far in all directions and are said to be two-dimensional because each point is located using an x and a y or two coordinates. It is also useful to be able to represent points and vectors in three dimensions. The designation R^3 is used for three dimensions because each of the coordinates of a point $P(a, b, c)$, and its associated vector $\overrightarrow{OP} = (a, b, c)$, is a real number. Here, $O(0, 0, 0)$ is the origin in three dimensions. As in R^2, each point has a unique location in R^3, which again implies that each position vector $\overrightarrow{OP}$ is unique in R^3.



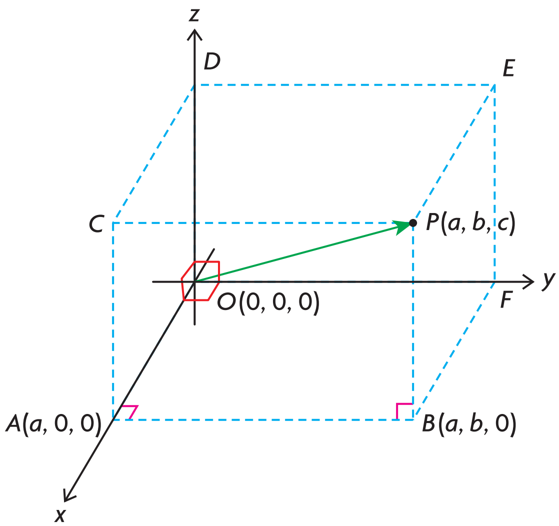
In placing points in R^3, we choose three axes called the x-, y-, and z-axis. Each pair of axes is perpendicular, and each axis is a copy of the real number line. There are several ways to choose the orientation of the positive axes, but we will use what is called a right-handed system. If we imagine ourselves looking down the positive z-axis onto the xy-plane so that, when the positive x-axis is rotated 90° *counterclockwise* it becomes coincident with the positive y-axis, then this is called a right-handed system. A right-handed system is normally what is used to represent R^3, and we will use this convention in this book.

Right-Handed System of Coordinates





Each pair of axes determines a plane. The xz-plane is determined by the x- and z-axes, and the yz-plane is determined by the y- and z-axes. Notice that, when we are discussing, for example, the xy-plane in R^3, this plane extends infinitely far in both the positive and negative directions. One way to visualize a right-handed system is to think of the y- and z-axes as lying in the plane of a book, determining the yz-plane, with the positive x-axis being perpendicular to the plane of the book and pointing directly toward you.



Each point $P(a, b, c)$ in R^3 has its location determined by an ordered triple. In the diagram above, the positive x-, y-, and z-axes are shown such that each pair of axes is perpendicular to the other and each axis represents a real number line. If we wish to locate $P(a, b, c)$, we move along the x-axis to $A(a, 0, 0)$, then in a direction perpendicular to the xz-plane, and parallel to the y-axis, to the point $B(a, b, 0)$. From there, we move in a direction perpendicular to the xy-plane and parallel to the z-axis to the point $P(a, b, c)$. This point is a vertex of a right rectangular prism.

Notice that the coordinates are signed, and so, for example, if we are locating the point $A(-2, 0, 0)$ we would proceed along the *negative* x-axis.

A source of confusion might be the meaning of $P(a, b, c)$ because it may be confused as either being a point or a vector. When referring to a vector, it will be stated explicitly that we are dealing with a vector and will be written as $\overrightarrow{OP} = (a, b, c)$, where a, b, and c are the x-, y-, and z-components respectively of the vector. In the diagram, this position vector is formed by joining the origin $O(0, 0, 0)$ to $P(a, b, c)$. When dealing with points, $P(a, b, c)$ will be named specifically as a point. In most situations, the distinction between the two should be evident from the context.

EXAMPLE 1 Reasoning about the coordinates of points in R^3

In the diagram on the previous page, determine the coordinates of C, D, E, and F.

Solution

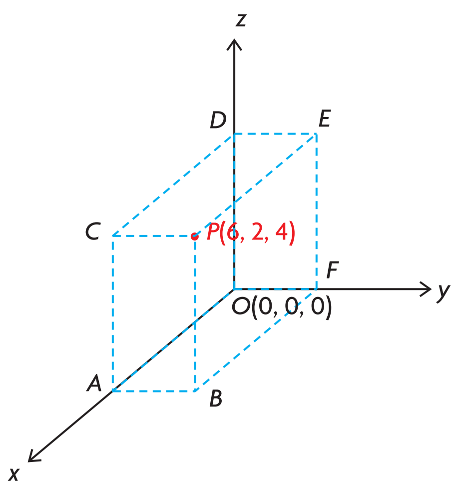
C is on the xz-plane and has coordinates $(a, 0, c)$, D is on the z-axis and has coordinates $(0, 0, c)$, E is on yz-plane and has coordinates $(0, b, c)$, and F is on the y-axis and has coordinates $(0, b, 0)$.

In the following example, we show how to locate points with the use of a rectangular box (prism) and line segments. It is useful, when we first start labelling points in R^3, to draw the box to gain familiarity with the coordinate system.

EXAMPLE 2 Connecting the coordinates of points and vector components in R^3

a. In the following diagram, the point $P(6, 2, 4)$ is located in R^3. What are the coordinates of A, B, C, D, E, and F?

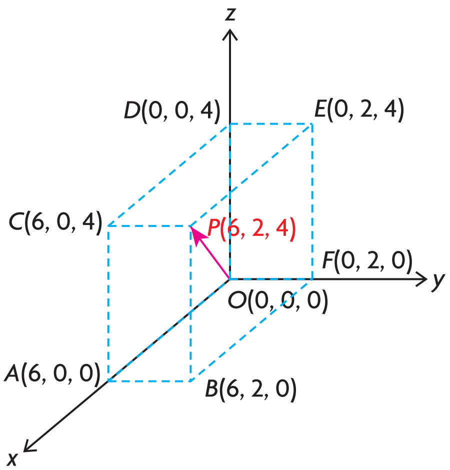
b. Draw the vector $\overrightarrow{OP}$.



Solution

a. $A(6, 0, 0)$ is a point on the positive x-axis, $B(6, 2, 0)$ is a point on the xy-plane, $C(6, 0, 4)$ is a point on the xz-plane, $D(0, 0, 4)$ is a point on the positive z-axis, $E(0, 2, 4)$ is a point on the yz-plane, and $F(0, 2, 0)$ is a point on the positive y-axis.

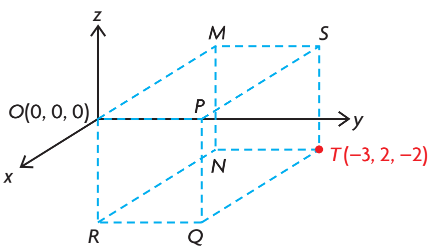
b. The vector $\overrightarrow{OP}$ is the vector associated with the point $P(a, b, c)$. It is the vector with its tail at the origin and its head at $P(6, 2, 4)$ and is named $\overrightarrow{OP} = (6, 2, 4)$.



EXAMPLE 3 Connecting the coordinates of points and vector components in R^3

a. In the following diagram, the point T is located in R^3. What are the coordinates of P, Q, R, M, N, and S?

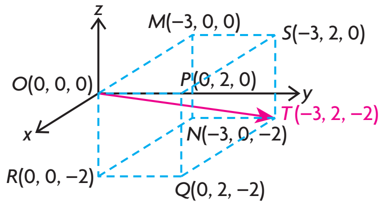
b. Draw the vector $\overrightarrow{OT}$.



Solution

a. The point $P(0, 2, 0)$ is a point on the positive y-axis. The point $Q(0, 2, -2)$ is on the yz-plane. The point $R(0, 0, -2)$ is on the negative z-axis. The point

$M(-3, 0, 0)$ is on the negative x-axis. The point $N(-3, 0, -2)$ is on the xz-plane. The point $S(-3, 2, 0)$ is on the xy-plane.



b. The vector $\overrightarrow{OT}$ is the vector associated with the point $T(-3, 2, -2)$ and is a vector with O as its tail and T as its head and is named $\overrightarrow{OT} = (-3, 2, -2)$.

When working with coordinate systems in R^3, it is possible to label planes using equations, which is demonstrated in the following example.

EXAMPLE 4 Representing planes in R^3 with equations

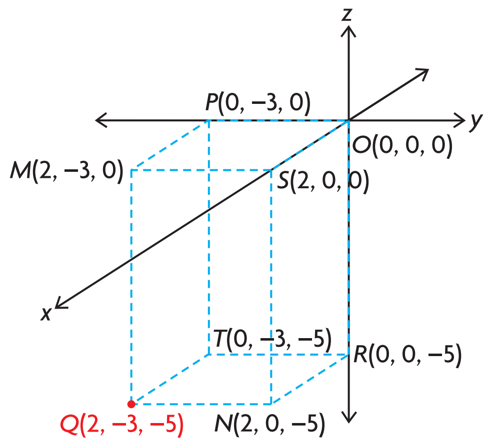
The point $Q(2, -3, -5)$ is shown in R^3.

a. Write an equation for the xy-plane.

b. Write an equation for the plane containing the points P, M, Q, and T.

c. Write a mathematical description of the set of points in rectangle $PMQT$.

d. What is the equation of the plane parallel to the xy-plane passing through $R(0, 0, -5)$?



Solution

a. Every point on the xy-plane has a z-component of 0, with every point on the plane having the form $(x, y, 0)$, where x and y are real numbers. The equation is $z = 0$.

b. Every point on this plane has a y-component equal to -3, with every point on the plane having the form $(x, -3, z)$, where x and z are real numbers. The equation is $y = -3$.

c. Every point in the rectangle has a y-component equal to -3, with every point in the rectangle having the form $(x, -3, z)$, where x and z are real numbers such that $0 \leq x \leq 2$ and $-5 \leq z \leq 0$.

d. Every point on this plane has a z-component equal to -5, with every point on the plane having the form $(x, y, -5)$, where x and y are real numbers. The equation is $z = -5$.

There is one further observation that should be made about placing points on coordinate axes. When using R^2 to describe the plane, which is two-dimensional, the exponent, n, in R^n is 2. Similarly, in three dimensions, the exponent is 3. The exponent in R^n corresponds to the number of dimensions of the coordinate system.

IN SUMMARY

Key Idea

- In R^2 or R^3, the location of every point is unique. As a result, every vector drawn with its tail at the origin and its head at a point is also unique. This type of vector is called a position vector.

Need to Know

- In R^2, $P(a, b)$ is a point that is a units from $O(0, 0)$ along the x-axis and b units parallel to the y-axis.
- The position vector $\overrightarrow{OP}$ has its tail located at $O(0, 0)$ and its head at $P(a, b)$. $\overrightarrow{OP} = (a, b)$
- In R^3, $P(a, b, c)$ is a point that is a units from $O(0, 0, 0)$ along the x-axis, b units parallel to the y-axis, and c units parallel to the z-axis. The position vector $\overrightarrow{OP}$ has its tail located at $O(0, 0, 0)$ and its head at $P(a, b, c)$. $\overrightarrow{OP} = (a, b, c)$
- In R^3, the three mutually perpendicular axes form a *right-handed* system.

Exercise 6.5

PART A

1. In R^3, is it possible to locate the point $P\left(\frac{1}{2}, \sqrt{-1}, 3\right)$? Explain.

2. a. Describe in your own words what it means for a point and its associated vector to be uniquely represented in R^3.

   b. Suppose that $\overrightarrow{OP} = (a, -3, c)$ and $\overrightarrow{OP} = (-4, b, -8)$. What are the corresponding values for a, b, and c? Why are we able to be certain that the determined values are correct?

3. a. The points $A(5, b, c)$ and $B(a, -3, 8)$ are located at the same point in R^3. What are the values of a, b, and c?

   b. Write the vector corresponding to $\overrightarrow{OA}$.

4. In R^3, each of the components for each point or vector is a real number. If we use the notation I^3, where I represents the set of integers, explain why $\overrightarrow{OP} = (-2, 4, -\sqrt{3})$ would not be an acceptable vector in I^3. Why is $\overrightarrow{OP}$ an acceptable vector in R^3?

5. Locate the points $A(4, -4, -2)$, $B(-4, 4, 2)$, and $C(4, 4, -2)$ using coordinate axes that you construct yourself. Draw the corresponding rectangular box (prism) for each, and label the coordinates of its vertices.

6. a. On what axis is $A(0, -1, 0)$ located? Name three other points on this axis.
   b. Name the vector $\overrightarrow{OA}$ associated with point A.

7. a. Name three vectors with their tails at the origin and their heads on the z-axis.
   b. Are the vectors you named in part a. collinear? Explain.
   c. How would you represent a general vector with its head on the z-axis and its tail at the origin?

K 8. Draw a set of x-, y-, and z-axes and plot the following points:

a. $A(1, 0, 0)$	c. $C(0, 0, -3)$	e. $E(2, 0, 3)$
b. $B(0, -2, 0)$	d. $D(2, 3, 0)$	f. $F(0, 2, 3)$

PART B

9. a. Draw a set of x-, y-, and z-axes and plot the following points: $A(3, 2, -4)$, $B(1, 1, -4)$, and $C(0, 1, -4)$.
   b. Determine the equation of the plane containing the points A, B, and C.

10. Plot the following points in R^3, using a rectangular prism to illustrate each coordinate.

a. $A(1, 2, 3)$	c. $C(1, -2, 1)$	e. $E(1, -1, 1)$
b. $B(-2, 1, 1)$	d. $D(1, 1, 1)$	f. $F(1, -1, -1)$

11. Name the vector associated with each point in question 10, express it in component form, and show the vectors associated with each of the points in the diagrams.

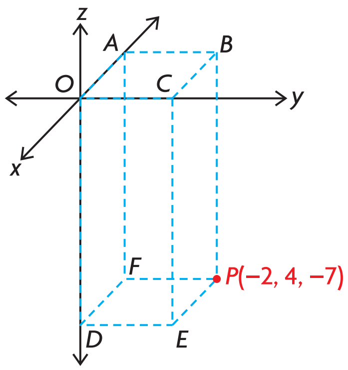
12. $P(2, a - c, a)$ and $Q(2, 6, 11)$ represent the same point in R^3.
    a. What are the values of a and c?
    b. Does $|\overrightarrow{OP}| = |\overrightarrow{OQ}|$? Explain.

13. Each of the points $P(x, y, 0)$, $Q(x, 0, z)$, and $R(0, y, z)$ represent general points on three different planes. Name the three planes to which each corresponds.

C 14. a. What is the equation of the plane that contains the points $M(1, 0, 3)$, $N(4, 0, 6)$, and $P(7, 0, 9)$? Explain your answer.

b. Explain why the plane that contains the points M, N, and P also contains the vectors $\overrightarrow{OM}$, $\overrightarrow{ON}$, and $\overrightarrow{OP}$.

A 15. The point $P(-2, 4, -7)$ is located in R^3 as shown on the coordinate axes below.



a. Determine the coordinates of points A, B, C, D, E, and F.

b. What are the vectors associated with each of the points in part a.?

c. How far below the xy-plane is the rectangle $DEPF$?

d. What is the equation of the plane containing the points B, C, E, and P?

e. Describe mathematically the set of points contained in rectangle $BCEP$.

16. Draw a diagram on the appropriate coordinate system for each of the following vectors:

a. $\overrightarrow{OP} = (4, -2)$
b. $\overrightarrow{OD} = (-3, 4)$
c. $\overrightarrow{OC} = (2, 4, 5)$
d. $\overrightarrow{OM} = (-1, 3, -2)$
e. $\overrightarrow{OF} = (0, 0, 5)$
f. $\overrightarrow{OJ} = (-2, -2, 0)$

PART C

T 17. Draw a diagram illustrating the set of points $\{(x, y, z) \in R^3 \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$.

18. Show that if $\overrightarrow{OP} = (5, -10, -10)$, then $|\overrightarrow{OP}| = 15$.

19. If $\overrightarrow{OP} = (-2, 3, 6)$ and $B(4, -2, 8)$, determine the coordinates of point A such that $\overrightarrow{OP} = \overrightarrow{AB}$.