

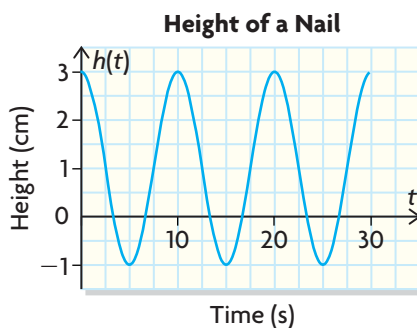
# Investigating Models of Sinusoidal Functions

## GOAL

Determine the equation of a sinusoidal function from a graph or a table of values.

## LEARN ABOUT the Math

A nail located on the circumference of a water wheel is moving as the current pushes on the wheel. The height of the nail in terms of time can be modelled by the graph shown.



- ?** How can you determine the equation of a sinusoidal function from its graph?

### EXAMPLE 1 Representing a sinusoidal graph using the equation of a function

Determine an equation of the given graph.

#### Sasha's Solution

Horizontal compression factor:  $k$

$$\text{period} = \frac{360}{|k|}$$

The period is 10 s.

$k > 0$ , so  $|k| = k$

I calculated the period, equation of the axis, and amplitude. Then I figured out how they are related to different transformations.

The period is 10 s since the peaks are 10 units apart. The horizontal stretch or compression factor  $k$  had to be positive because the graph was not reflected in the  $y$ -axis. I used the formula relating  $k$  to the period.



$$10 = \frac{360}{k}$$

$$k = \frac{360}{10}$$

$$k = 36$$

The graph was compressed by a factor of  $\frac{1}{36}$ .

Vertical translation:  $c$

The axis is halfway between the maximum, 3, and the minimum,  $-1$ . This gave me the vertical translation and the value of  $c$ .

$$\text{equation of the axis} = \frac{\text{max} + \text{min}}{2}$$

$$= \frac{3 + (-1)}{2}$$

$$= 1 \text{ (vertical translation)}$$

$$c = 1$$

I calculated the amplitude by taking the maximum value, 3, and subtracting the axis, 1. Since the amplitude of  $y = \cos x$  is 1, and the amplitude of this graph is 2, the vertical stretch is 2.

Vertical stretch:  $a$

$$a = 2$$

Base graph:  $y = \cos x$

As a cosine curve:

$$y = 2 \cos(36x)^\circ + 1$$

As a sine curve:

$$y = 2 \sin(36(x - 7.5))^\circ + 1$$

For both functions, the domain is restricted to  $x \geq 0$  because it represents the time elapsed.

The cosine curve is easier to use for my equation since the graph has its maximum on the  $y$ -axis, just as this graph does. This means that for a cosine curve, there isn't any horizontal translation, so  $d = 0$ .

I found the equation of the function by substituting the values I calculated into  $f(x) = a \cos(k(x - d)) + c$ .

I could have used the sine function instead.

A sine curve increases from a  $y$ -value of 0 at  $x = 0$ .

On this graph, that happens at 7.5. This means that, for a sine curve, there is a horizontal translation of 7.5, so  $c = 7.5$ .

## Reflecting

- Tanya says that another possible equation of the sinusoidal function created by Sasha is  $y = 2 \cos(36(x - 10))^\circ + 1$ . Is she correct? Why or why not?
- If the period on the original water wheel graph is changed from 10 to 20, what would be the new equation of the sinusoidal function?
- If the maximum value on the original water wheel graph is changed from 3 to 5, what would be the new equation of the sinusoidal function?
- If the speed of the current increases so that the water wheel spins twice as fast, what would be the equation of the resulting function?

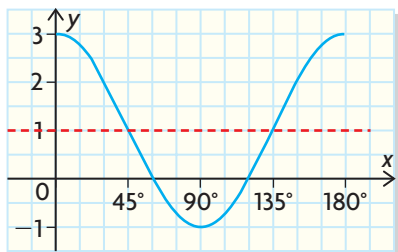
## APPLY the Math

### EXAMPLE 2

Connecting the equation of a sinusoidal function to its features

A sinusoidal function has an amplitude of 2 units, a period of  $180^\circ$ , and a maximum at  $(0, 3)$ . Represent the function with an equation in two different ways.

### Rajiv's Solution



The graph has a maximum at  $(0, 3)$  and a period of  $180^\circ$ , so the next maximum would be at  $(180, 3)$ .

A minimum would be halfway between the two maximums.

Since the amplitude is 2, and  $2 - 3 = -1$ , the minimum would have to be at  $(90^\circ, -1)$ .

The equation of the axis gave me the vertical translation. Since the equation is  $y = 1$  instead of  $y = 0$ , there was a vertical translation of 1.

The amplitude gives me the vertical stretch.

The period is  $180^\circ$ , so there has been a horizontal compression. Since there was no horizontal reflection,  $k > 0$ . To find  $k$ , I took  $360^\circ$  and divided it by the period.

Cosine curves have a maximum at  $x = 0$ , unless they've been translated horizontally. This curve starts at its maximum, so there would be no horizontal translation with a cosine function as a model.

I found the equation of the function by substituting the values into  $f(x) = a \cos(k(x - d)) + c$ .

The equation of the axis of this cosine curve is  $y = 1$ . On this cosine curve, the point  $(135^\circ, 1)$  corresponds to the start of the cycle of the sine function. The sine curve with the same period and axis as this cosine curve has the equation  $y = 2 \sin(2x) + 1$ , but its starting point is  $(0^\circ, 1)$ . This means the function  $y = 2 \sin(2x) + 1$  must be translated horizontally to the right by  $135^\circ$ , so  $d = 135^\circ$ .

Vertical translation:  $c = 1$

Vertical stretch:  $a$

amplitude  $= 3 - 1 = 2$

$$a = 2$$

Horizontal compression:  $k$

$$\text{period} = \frac{360^\circ}{|k|}$$

$$180^\circ = \frac{360^\circ}{k}$$

$$k = \frac{360^\circ}{180^\circ}$$

$$k = 2$$

Compression factor is  $\frac{1}{2}$ .

For a cosine curve:

No horizontal translation so

$$d = 0$$

Equation:

$$y = 2 \cos(2x) + 1$$

For a sine curve:

horizontal translation  $= 135^\circ$

$$y = 2 \sin(2(x - 135^\circ)) + 1$$

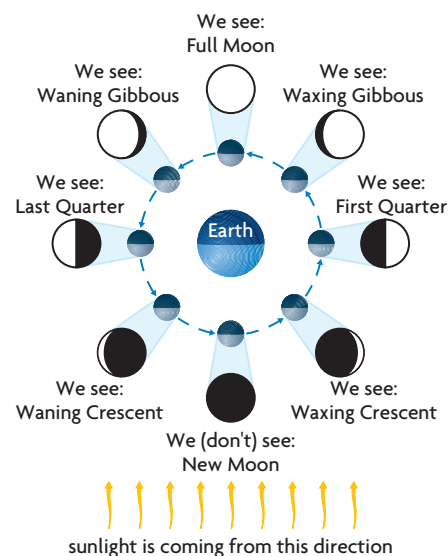
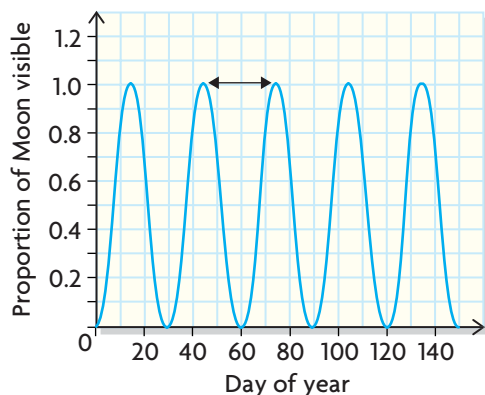
**EXAMPLE 3** Connecting data to the algebraic model of a sinusoidal function

The Moon is always half illuminated by the Sun. How much of the Moon we see depends on where it is in its orbit around Earth. The table shows the proportion of the Moon that was visible from Southern Ontario on days 1 to 74 in the year 2006.

Day of Year	1	4	7	10	14	20	24	29	34
Proportion of Moon Visible	0.02	0.22	0.55	0.83	1.00	0.73	0.34	0.00	0.28

Day of Year	41	44	48	53	56	59	63	70	74
Proportion of Moon Visible	0.92	1.00	0.86	0.41	0.12	0.00	0.23	0.88	1.00

- Determine the equation of the sinusoidal function that models the proportion of visible Moon in terms of time.
- Determine the domain and range of the function.
- Use the equation to determine the proportion of the Moon that is visible on day 110.


**Rosalie's Solution**
**a) Cycle of the Proportion of the Moon Visible**


I plotted the data. When I drew the curve, the graph looked like a sinusoidal function.

The maximum value was 1, and the minimum value 0.

The graph repeats every 30 days, so the period must be 30 days.

I figured out some of the important features of the sinusoidal function.

Vertical translation:  $c$

Equation of the axis is  $y = 0.5$ .

$$c = 0.5$$

The axis is halfway between the maximum of 1 and the minimum of 0.



Vertical stretch:  $a$

$$\begin{aligned}\text{amplitude} &= \frac{(1 - 0)}{2} \\ &= 0.5 \quad \text{or} \quad \frac{1}{2} \\ a &= 0.5\end{aligned}$$

The amplitude is the vertical distance between the maximum and the axis. In this case, it is 0.5, or  $\frac{1}{2}$ .

Horizontal compression:  $k$

$$\begin{aligned}\text{period} &= \frac{360}{|k|} \\ k > 0, \text{ so } |k| &= k \\ 30 &= \frac{360^\circ}{k} \\ k &= \frac{360^\circ}{30} \\ k &= 12\end{aligned}$$

I used the period to get the compression.

Horizontal translation:  $d$

Using a cosine curve:

$$d = 14$$

A sine curve or a cosine curve will work. I used the cosine curve. The horizontal translation is equal to the  $x$ -coordinate of a maximum, since  $y = \cos x$  has a maximum at  $x = 0$ . I chose the  $x$ -coordinate of the maximum closest to the origin,  $x = 14$ .

$$y = \frac{1}{2} \cos(12(x - 14)^\circ) + 0.5$$

I put the information together to get the equation.

b) domain:  $\{x \in \mathbf{R} \mid 0 \leq x \leq 365\}$

range:  $\{y \in \mathbf{R} \mid 0 \leq y \leq 1\}$

The domain is only the non-negative values of  $x$  up to 365, since they are days of the year. The range is 0 to 1.

c)  $y = \frac{1}{2} \cos(12(x - 14)^\circ) + 0.5$

Since  $x$  represents the time in days, I substituted 110 for  $x$  in the equation to calculate the amount of the Moon visible at that time. Then I solved for  $y$ .

$$= \frac{1}{2} \cos(12(110 - 14)^\circ) + 0.5$$

$$= \frac{1}{2} \cos(1152)^\circ + 0.5$$

$$\doteq \frac{1}{2}(0.3090) + 0.5$$

$$= 0.65$$

On day 110, 65% of the Moon is exposed.

## In Summary

### Key Idea

- If you are given a set of data and the corresponding graph is a sinusoidal function, then you can determine the equation by calculating the graph's period, amplitude, and equation of the axis. This information will help you determine the values of  $k$ ,  $a$ , and  $c$ , respectively, in the equations  $g(x) = a \sin(k(x - d)) + c$  and  $h(x) = a \cos(k(x - d)) + c$ . The value of  $d$  is determined by estimating the required horizontal shift (left or right) compared with the graph of the sine or cosine curve.

### Need to Know

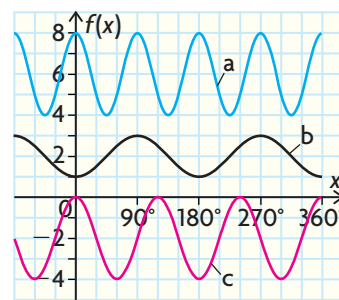
- If the graph begins at a maximum value, it may be easier to use the cosine function as your model.
- The domain and range of a sinusoidal model may need to be restricted for the situation you are dealing with.

## CHECK Your Understanding

- Determine an equation for each sinusoidal function at the right.
- Determine the function that models the data in the table and does not involve a horizontal translation.

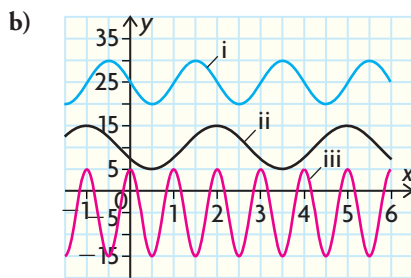
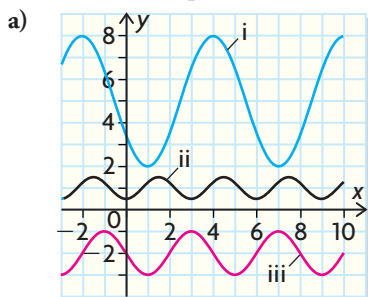
$x$	$0^\circ$	$45^\circ$	$90^\circ$	$135^\circ$	$180^\circ$	$225^\circ$	$270^\circ$
$y$	9	7	5	7	9	7	5

- A sinusoidal function has an amplitude of 4 units, a period of  $120^\circ$ , and a maximum at  $(0, 9)$ . Determine the equation of the function.



## PRACTISING

- Determine the equation for each sinusoidal function.



5. For each table of data, determine the equation of the function that is the simplest model.

a)

<b>x</b>	0°	30°	60°	90°	120°	150°	180°
<b>y</b>	3	2	1	2	3	2	1

b)

<b>x</b>	-180°	0°	180°	360°	540°	720°	900°
<b>y</b>	17	13	17	21	17	13	17

c)

<b>x</b>	-120°	-60°	0°	60°	120°	180°	240°
<b>y</b>	-4	-7	-4	-1	-4	-7	-4

d)

<b>x</b>	-20°	10°	40°	70°	100°	130°	160°
<b>y</b>	2	5	2	-1	2	5	2

6. Determine the equation of the cosine function whose graph has each of the following features.

	Amplitude	Period	Equation of the Axis	Horizontal Translation
a)	3	360°	$y = 11$	0°
b)	4	180°	$y = 15$	30°
c)	2	40°	$y = 0$	7°
d)	0.5	720°	$y = -3$	-56°

7. A sinusoidal function has an amplitude of 6 units, a period of 45°, and a minimum at (0, 1). Determine an equation of the function.

8. The table shows the average monthly high temperature for one year in Kapuskasing, Ontario.

<b>Time (months)</b>	J	F	M	A	M	J	J	A	S	O	N	D
<b>Temperature (°C)</b>	-18.6	-16.3	-9.1	0.4	8.5	13.8	17.0	15.4	10.3	4.4	-4.3	-14.8

- Draw a scatter plot of the data and the curve of best fit. Let January be month 0.
- What type of model describes the graph? Why did you select that model?
- Write an equation for your model. Describe the constants and the variables in the context of this problem.
- What is the average monthly temperature for month 20?

9. The table shows the velocity of air of Nicole's breathing while she is at rest.

Time (s)	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2	2.25	2.5	2.75	3
Velocity (L/s)	0	0.22	0.45	0.61	0.75	0.82	0.85	0.83	0.74	0.61	0.43	0.23	0

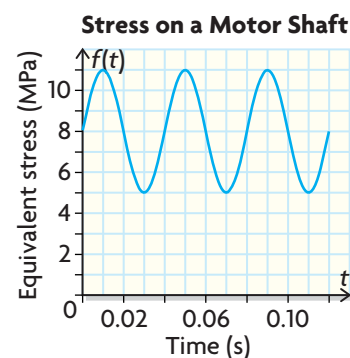
- Explain why breathing is an example of a periodic function.
  - Graph the data, and determine an equation that models the situation.
  - Using a graphing calculator, graph the data as a scatter plot. Enter your equation and graph. Comment on the closeness of fit between the scatter plot and the graph.
  - What is the velocity of Nicole's breathing at 6 s? Justify.
  - How many seconds have passed when the velocity is 0.5 L/s?
10. The table shows the average monthly temperature for three cities: Athens, Lisbon, and Moscow.

Time (month)	J	F	M	A	M	J	J	A	S	O	N	D
Athens (°C)	12	13	15	19	24	30	33	32	28	23	18	14
Lisbon (°C)	13	14	16	18	21	24	26	27	24	21	17	14
Moscow (°C)	-9	-6	0	10	19	21	23	22	16	9	1	-4

- Graph the data to show that temperature is a function of time for each city.
  - Write the equations that model each function.
  - Explain the differences in the amplitude and the vertical translation for each city.
  - What does this tell you about the cities?
11. The relationship between the stress on the shaft of an electric motor and time can be modelled with a sinusoidal function. (The units of stress are megapascals (MPa).)
- Determine an equation of the function that describes the equivalent stress in terms of time.
  - What do the peaks of the function represent in this situation?
  - How much stress was the motor undergoing at 0.143 s?
12. Describe a procedure for writing the equation of a sinusoidal function based on a given graph.

### Tech Support

For help creating a scatter plot using a graphic calculator, see Technical Appendix, B-11.



## Extending

- The diameter of a car's tire is 60 cm. While the car is being driven, the tire picks up a nail. How high above the ground is the nail after the car has travelled 1 km?
- Matthew is riding a Ferris wheel at a constant speed of 10 km/h. The boarding height for the wheel is 1 m, and the wheel has a radius of 7 m. What is the equation of the function that describes Matthew's height in terms of time, assuming Matthew starts at the highest point on the wheel?