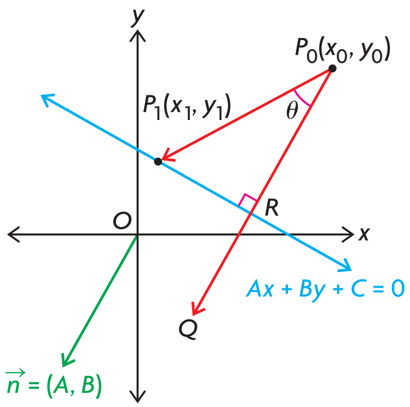
Section 9.5—The Distance from a Point to a Line in R^2 and R^3

In this section, we consider various approaches for determining the distance between a point and a line in R^2 and R^3.

Determining a Formula for the Distance between a Point and a Line in R^2



Consider a line in R^2 that has $Ax + By + C = 0$ as its general equation, as shown in the diagram above. The point $P_0(x_0, y_0)$ is a point not on the line and whose coordinates are known. A line from P_0 is drawn perpendicular to $Ax + By + C = 0$ and meets this line at R. The line from P_0 is extended to point Q. The point $P_1(x_1, y_1)$ represents a second point on the line different from R. We wish to determine a formula for $|\overrightarrow{P_0R}|$, the distance from P_0 to the line. (Note that when we are calculating the distance between a point and either a line or a plane, we are always calculating the perpendicular distance, which is always unique. In simple terms, this means that there is only one shortest distance that can be calculated between P_0 and $Ax + By + C = 0$.)

To determine the formula, we are going to take the scalar projection of $\overrightarrow{P_0P_1}$ on $\overrightarrow{P_0Q}$. Since $\overrightarrow{P_0Q}$ is perpendicular to $Ax + By + C = 0$, what we are doing is equivalent to taking the scalar projection of $\overrightarrow{P_0P_1}$ on the normal to the line, $n = (A, B)$, since n and P_0Q are parallel.

We know that $\overrightarrow{P_0P_1} = (x_1 - x_0, y_1 - y_0)$ and $\vec{n} = (A, B)$. The formula for the dot product is $\overrightarrow{P_0P_1} \cdot \vec{n} = |\overrightarrow{P_0P_1}||\vec{n}|\cos\theta$, where θ is the angle between $\overrightarrow{P_0P_1}$ and $\vec{n}$. Rearranging this formula gives

$$|\overrightarrow{P_0P_1}|\cos\theta = \frac{\overrightarrow{P_0P_1} \cdot \vec{n}}{|\vec{n}|} \qquad \text{(Equation 1)}$$

From triangle P_0RP_1

$$\cos\theta = \frac{|\overrightarrow{P_0R}|}{|\overrightarrow{P_0P_1}|}$$

$$|\overrightarrow{P_0P_1}|\cos\theta = |\overrightarrow{P_0R}|$$

Substituting $|\overrightarrow{P_0P_1}|\cos\theta = |\overrightarrow{P_0R}|$ into the dot product formula (equation 1 above) gives

$$|\overrightarrow{P_0R}| = \frac{\overrightarrow{P_0P_1} \cdot \vec{n}}{|\vec{n}|}$$

Since $\overrightarrow{P_0P_1} \cdot \vec{n} = (x_1 - x_0, y_1 - y_0) \cdot (A, B) = Ax_1 - Ax_0 + By_1 - By_0$ and

$|\vec{n}| = \sqrt{A^2 + B^2}$, (by substitution) we obtain

$$|\overrightarrow{P_0R}| = \frac{Ax_1 + By_1 - Ax_0 - By_0}{\sqrt{A^2 + B^2}}$$

The point $P_1(x_1, y_1)$ is on the line $Ax + By + C = 0$, meaning that $Ax_1 + By_1 + C = 0$ or $Ax_1 + By_1 = -C$. Substituting this into the formula for

$|\overrightarrow{P_0R}|$ gives

$$|\overrightarrow{P_0R}| = \frac{-C - Ax_0 - By_0}{\sqrt{A^2 + B^2}} = \frac{-(C + Ax_0 + By_0)}{\sqrt{A^2 + B^2}}$$

To ensure that this quantity is always positive, it is written as

$$|\overrightarrow{P_0R}| = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

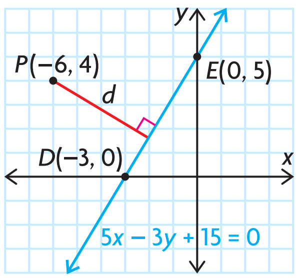
Distance from a Point $P_0(x_0, y_0)$ to the Line with Equation $Ax + By + C = 0$

$d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, where d represents the distance between the point $P_0(x_0, y_0)$, and the line defined by $Ax + By + C = 0$, where the point does not lie on the line. But we don't really need this since the formula gives the correct value of 0 when the point *does* lie on the line.

EXAMPLE 1 Calculating the distance between a point and a line in R^2

Determine the distance from point $P(-6, 4)$ to the line with equation $5x - 3y + 15 = 0$.

Solution



Since $x_0 = -6$, $y_0 = 4$, $A = 5$, $B = -3$, and $C = 15$,

$$d = \frac{|5(-6) - 3(4) + 15|}{\sqrt{5^2 + (-3)^2}} = \frac{|-27|}{\sqrt{34}} \doteq 4.63$$

The distance from $P(-6, 4)$ to the line with equation $5x - 3y + 15 = 0$ is approximately 4.63.

It is not immediately possible to use the formula for the distance between a point and a line if the line is given in vector form. In the following example, we show how to find the required distance if the line is given in vector form.

EXAMPLE 2 Selecting a strategy to determine the distance between a point and a line in R^2

Determine the distance from point $P(15, -9)$ to the line with equation $\vec{r} = (-2, -1) + s(-4, 3)$, $s \in \mathbf{R}$.

Solution

To use the formula, it is necessary to convert the equation of the line in vector form to its corresponding Cartesian form. The given equation must first be written using parametric form. The parametric equations for this line are $x = -2 - 4s$ and $y = -1 + 3s$. Solving for the parameter s in each equation gives $\frac{x + 2}{-4} = s$ and $\frac{y + 1}{3} = s$. Therefore, $\frac{x + 2}{-4} = \frac{y + 1}{3} = s$. The required equation is $3(x + 2) = -4(y + 1)$ or $3x + 4y + 10 = 0$.

Therefore, $d = \dfrac{|3(15) + 4(-9) + 10|}{\sqrt{3^2 + 4^2}} = \dfrac{19}{5} = 3.80$.

The required distance is 3.80.

EXAMPLE 3

Selecting a strategy to determine the distance between two parallel lines

Calculate the distance between the two parallel lines $5x - 12y + 60 = 0$ and $5x - 12y - 60 = 0$.

Solution

To find the required distance, it is necessary to determine the coordinates of a point on one of the lines and then use the distance formula. For the line with equation $5x - 12y - 60 = 0$, we can determine the coordinates of the point where the line crosses either the x-axis or the y-axis. (This point was chosen because it is easy to calculate and it also makes the resulting computation simpler. In practice, however, any point on the chosen line is satisfactory.) If we let $x = 0$, then $5(0) - 12y - 60 = 0$, or $y = -5$. The line crosses the y-axis at $(0, -5)$. To find the required distance, d, it is necessary to find the distance from $(0, -5)$ to the line with equation $5x - 12y + 60 = 0$.

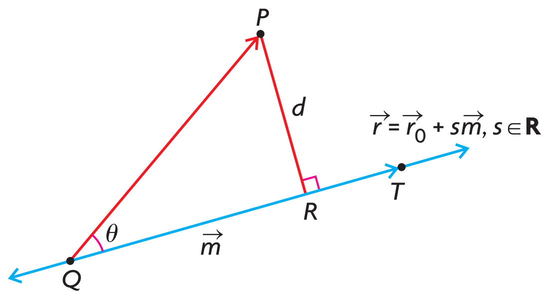
$$d = \frac{|5(0) - 12(-5) + 60|}{\sqrt{5^2 + (-12)^2}} = \frac{|120|}{13} = \frac{120}{13}$$

Therefore, the distance between the two parallel lines is $\frac{120}{13} \doteq 9.23$.

Determining the Distance between a Point and a Line in R^3

It is not possible to use the formula we just developed for finding the distance between a point and a line in R^3 because lines in R^3 are not of the form $Ax + By + C = 0$. We need to use a different approach.

The most efficient way to find the distance between a point and a line in R^3 is to use the cross product. In the following diagram, we would like to find d, which represents the distance between point P, whose coordinates are known, and a line with vector equation $\vec{r} = \vec{r}_0 + s\vec{m}$, $s \in \mathbf{R}$. Point Q is any point on the line whose coordinates are also known. Point T is the point on the line such that $\overrightarrow{QT}$ is a vector representing the direction $\vec{m}$, which is known.



The angle between $\overrightarrow{QP}$ and $\overrightarrow{QT}$ is θ. Note that, for computational purposes, it is possible to determine the coordinates of a position vector equivalent to either $\overrightarrow{QP}$ or $\overrightarrow{PQ}$.

In triangle PQR, $\sin\theta = \dfrac{d}{|\overrightarrow{QP}|}$

equivalently $d = |\overrightarrow{QP}|\sin\theta$

From our earlier discussion on cross products, we know that
$|\vec{m} \times \overrightarrow{QP}| = |\vec{m}||\overrightarrow{QP}|\sin\theta$.

If we substitute $d = |\overrightarrow{QP}|\sin\theta$ into this formula, we find that
$|\vec{m} \times \overrightarrow{QP}| = |\vec{m}|(d)$.

Solving for d gives $d = \dfrac{|\vec{m} \times \overrightarrow{QP}|}{|\vec{m}|}$.

Distance, d, from a Point, P, to the Line $\vec{r} = \vec{r}_0 + s\vec{m}$, $s \in \mathbf{R}$

In R^3, $d = \dfrac{|\vec{m} \times \overrightarrow{QP}|}{|\vec{m}|}$, where Q is a point on the line and P is any other point, both of whose coordinates are known, and $\vec{m}$ is the direction vector of the line.

EXAMPLE 4 Selecting a strategy to calculate the distance between a point and a line in R^3

Determine the distance from point $P(-1, 1, 6)$ to the line with equation
$\vec{r} = (1, 2, -1) + t(0, 1, 1)$, $t \in \mathbf{R}$.

Solution

Method 1: Using the Formula

Since Q is $(1, 2, -1)$ and P is $(-1, 1, 6)$,
$\overrightarrow{QP} = (-1 - 1, 1 - 2, 6 - (-1)) = (-2, -1, 7)$.

From the equation of the line, we note that $\vec{m} = (0, 1, 1)$.

Thus, $d = \dfrac{|(0, 1, 1) \times (-2, -1, 7)|}{|(0, 1, 1)|}$.

Calculating,
$(0, 1, 1) \times (-2, -1, 7) = (7 - (-1), -2 - 0, 0 + 2) = (8, -2, 2)$
$|(8, -2, 2)| = \sqrt{8^2 + (-2)^2 + 2^2} = \sqrt{72} = 6\sqrt{2}$ and
$|(0, 1, 1)| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$

Therefore, the distance from the point to the line is $d = \dfrac{6\sqrt{2}}{\sqrt{2}} = 6$.

This calculation is efficient and gives the required answer quickly. (Note that the vector $(8, -2, 2)$ cannot be reduced by dividing by the common factor 2, or by any other factor.)

Method 2: Using the Dot Product

We start by writing the given equation of the line in parametric form. Doing so gives $x = 1$, $y = 2 + t$, and $z = -1 + t$. We construct a vector from a general point on the line to P and call this vector $\vec{a}$. Thus, $\vec{a} = (-1 - 1, 1 - (2 + t), 6 - (-1 + t)) = (-2, -1 - t, 7 - t)$. What we wish to find is the minimum distance between point $P(-1, 1, 6)$ and the given line. This occurs when $\vec{a}$ is perpendicular to the given line, or when $\vec{m} \cdot \vec{a} = 0$.

Calculating gives
$$\begin{aligned} (0, 1, 1) \cdot (-2, -1 - t, 7 - t) &= 0 \\ 0(-2) + 1(-1 - t) + 1(7 - t) &= 0 \\ -1 - t + 7 - t &= 0 \\ t &= 3 \end{aligned}$$

This means that the minimal distance between $P(-1, 1, 6)$ and the line occurs when $t = 3$, which implies that the point corresponding to $t = 3$ produces the minimal distance between the point and the line. This point has coordinates $x = 1$, $y = 2 + 3 = 5$, and $z = -1 + 3 = 2$. In other words, the minimal distance between the point and the line is the distance between $P(-1, 1, 6)$ and the point, $(1, 5, 2)$. Thus,

$$d = \sqrt{(-1 - 1)^2 + (1 - 5)^2 + (6 - 2)^2} = \sqrt{4 + 16 + 16} = \sqrt{36} = 6$$

This gives the same answer that we found using Method 1. It has the advantage that it also allows us to find the coordinates of the point on the line that produces the minimal distance.

IN SUMMARY

Key Ideas

- In R^2, the distance from point $P_0(x_0, y_0)$ to the line with equation $Ax + By + C = 0$ is $d = \dfrac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, where d represents the distance.
- In R^3, the formula for the distance d from point P to the line $\vec{r} = \vec{r_0} + s\vec{m}$, $s \in \mathbf{R}$, is $d = \dfrac{|\vec{m} \times \vec{QP}|}{|\vec{m}|}$, where Q is a point on the line whose coordinates are known.

Exercise 9.5

PART A

1. Determine the distance from $P(-4, 5)$ to each of the following lines:
   a. $3x + 4y - 5 = 0$
   b. $5x - 12y + 24 = 0$
   c. $9x - 40y = 0$

2. Determine the distance between the following parallel lines:
   a. $2x - y + 1 = 0, 2x - y + 6 = 0$
   b. $7x - 24y + 168 = 0, 7x - 24y - 336 = 0$

3. Determine the distance from $R(-2, 3)$ to each of the following lines:
   a. $\vec{r} = (-1, 2) + s(3, 4), s \in \mathbf{R}$
   b. $\vec{r} = (1, 0) + t(5, 12), t \in \mathbf{R}$
   c. $\vec{r} = (1, 3) + p(7, -24), p \in \mathbf{R}$

PART B

C 4. a. The formula for the distance from a point to a line is $d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$. Show that this formula can be modified so the distance from the origin, $O(0, 0)$, to the line $Ax + By + C = 0$ is given by the formula $d = \frac{|C|}{\sqrt{A^2 + B^2}}$.
   b. Determine the distance between $L_1: 3x - 4y - 12 = 0$ and $L_2: 3x - 4y + 12 = 0$ by first finding the distance from the origin to L_1 and then finding the distance from the origin to L_2.
   c. Find the distance between the two lines directly by first determining a point on one of the lines and then using the distance formula. How does this answer compare with the answer you found in part b.?

K 5. Calculate the distance between the following lines:
   a. $\vec{r} = (-2, 1) + s(3, 4); s \in \mathbf{R}; \vec{r} = (1, 0) + t(3, 4), t \in \mathbf{R}$
   b. $\frac{x - 1}{4} = \frac{y}{-3}, \frac{x}{4} = \frac{y + 1}{-3}$
   c. $2x - 3y + 1 = 0, 2x - 3y - 3 = 0$
   d. $5x + 12y = 120, 5x + 12y + 120 = 0$

6. Calculate the distance between point P and the given line.

a. $P(1, 2, -1)$; $\vec{r} = (1, 0, 0) + s(2, -1, 2)$, $s \in \mathbf{R}$

b. $P(0, -1, 0)$; $\vec{r} = (2, 1, 0) + t(-4, 5, 20)$, $t \in \mathbf{R}$

c. $P(2, 3, 1)$; $\vec{r} = p(12, -3, 4)$, $p \in \mathbf{R}$

7. Calculate the distance between the following parallel lines.

a. $\vec{r} = (1, 1, 0) + s(2, 1, 2)$, $s \in \mathbf{R}$; $\vec{r} = (-1, 1, 2) + t(2, 1, 2)$, $t \in \mathbf{R}$

b. $\vec{r} = (3, 1, -2) + m(1, 1, 3)$, $m \in \mathbf{R}$; $\vec{r} = (1, 0, 1) + n(1, 1, 3)$, $n \in \mathbf{R}$

A 8. a. Determine the coordinates of the point on the line $\vec{r} = (1, -1, 2) + s(1, 3, -1)$, $s \in \mathbf{R}$, that produces the shortest distance between the line and a point with coordinates $(2, 1, 3)$.

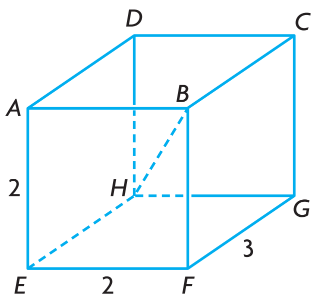
b. What is the distance between the given point and the line?

PART C

9. Two planes with equations $x - y + 2z = 2$ and $x + y - z = -2$ intersect along line L. Determine the distance from $P(-1, 2, -1)$ to L, and determine the coordinates of the point on L that gives this minimal distance.

10. The point $A(2, 4, -5)$ is reflected in the line with equation $\vec{r} = (0, 0, 1) + s(4, 2, 1)$, $s \in \mathbf{R}$, to give the point A'. Determine the coordinates of A'.

T 11. A rectangular box with an open top, measuring 2 by 2 by 3, is constructed. Its vertices are labelled as shown.



a. Determine the distance from A to the line segment HB.

b. What other vertices on the box will give the same distance to HB as the distance you found in part a.?

c. Determine the area of the $\triangle AHB$.