Section 1.5—Properties of Limits

The statement $\lim_{x \to a} f(x) = L$ says that the values of $f(x)$ become closer and closer to the number L as x gets closer and closer to the number a (from either side of a), such that $x \neq a$. This means that when finding the limit of $f(x)$ as x approaches a, there is no need to consider $x = a$. In fact, $f(a)$ need not even be defined. The only thing that matters is the behaviour of $f(x)$ near $x = a$.

EXAMPLE 1

Reasoning about the limit of a polynomial function

Find $\lim_{x \to 2} (3x^2 + 4x - 1)$.

Solution

It seems clear that when x is close to 2, $3x^2$ is close to 12, and $4x$ is close to 8. Therefore, it appears that $\lim_{x \to 2} (3x^2 + 4x - 1) = 12 + 8 - 1 = 19$.

In Example 1, the limit was arrived at intuitively. It is possible to evaluate limits using the following properties of limits, which can be proved using the formal definition of limits. This is left for more advanced courses.

Properties of Limits

For any real number a, suppose that f and g both have limits that exist at $x = a$.

1. $\lim_{x \to a} k = k$, for any constant k
2. $\lim_{x \to a} x = a$
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a} [cf(x)] = c[\lim_{x \to a} f(x)]$, for any constant c
5. $\lim_{x \to a} [f(x)g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)]$
6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$
7. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$, for any rational number n

EXAMPLE 2 Using the limit properties to evaluate the limit of a polynomial function

Evaluate $\lim_{x\to 2}(3x^2 + 4x - 1)$.

Solution

$$\begin{aligned}\lim_{x\to 2}(3x^2 + 4x - 1) &= \lim_{x\to 2}(3x^2) + \lim_{x\to 2}(4x) - \lim_{x\to 2}(1)\\ &= 3\lim_{x\to 2}(x^2) + 4\lim_{x\to 2}(x) - 1\\ &= 3[\lim_{x\to 2} x]^2 + 4(2) - 1\\ &= 3(2)^2 + 8 - 1\\ &= 19\end{aligned}$$

Note: If f is a polynomial function, then $\lim_{x\to a} f(x) = f(a)$.

EXAMPLE 3 Using the limit properties to evaluate the limit of a rational function

Evaluate $\lim_{x\to -1}\frac{x^2 - 5x + 2}{2x^3 + 3x + 1}$.

Solution

$$\begin{aligned}\lim_{x\to -1}\frac{x^2 - 5x + 2}{2x^3 + 3x + 1} &= \frac{\lim_{x\to -1}(x^2 - 5x + 2)}{\lim_{x\to -1}(2x^3 + 3x + 1)}\\ &= \frac{(-1)^2 - 5(-1) + 2}{2(-1)^3 + 3(-1) + 1}\\ &= \frac{8}{-4}\\ &= -2\end{aligned}$$

EXAMPLE 4 Using the limit properties to evaluate the limit of a root function

Evaluate $\lim_{x\to 5}\sqrt{\frac{x^2}{x - 1}}$.

Solution

$$\begin{aligned}\lim_{x\to 5}\sqrt{\frac{x^2}{x - 1}} &= \sqrt{\lim_{x\to 5}\frac{x^2}{x - 1}}\\ &= \sqrt{\frac{\lim_{x\to 5} x^2}{\lim_{x\to 5}(x - 1)}}\\ &= \sqrt{\frac{25}{4}}\\ &= \frac{5}{2}\end{aligned}$$

Sometimes $\lim_{x \to a} f(x)$ cannot be found by direct substitution. This is particularly interesting when direct substitution results in an **indeterminate form** $\left(\frac{0}{0}\right)$. In such cases, we look for an equivalent function that agrees with f for all values except at $x = a$. Here are some examples.

EXAMPLE 5 Selecting a factoring strategy to evaluate a limit

Evaluate $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}$.

Solution

Substitution produces the indeterminate form $\frac{0}{0}$. The next step is to simplify the function by factoring and reducing to see if the limit of the reduced form can be evaluated.

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} \frac{(x + 1)(x - 3)}{x - 3} = \lim_{x \to 3} (x + 1)$$

The reduction is valid only if $x \neq 3$. This is not a problem, since $\lim_{x \to 3}$ requires values as x approaches 3, not when $x = 3$. Therefore,

$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3} = \lim_{x \to 3} (x + 1) = 4.$

EXAMPLE 6 Selecting a rationalizing strategy to evaluate a limit

Evaluate $\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$.

Solution

A useful technique for finding a limit is to rationalize either the numerator or the denominator to obtain an algebraic form that is not indeterminate.

Substitution produces the indeterminate form $\frac{0}{0}$, so we will try rationalizing.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} &= \lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x} \times \frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} + 1} && \text{(Rationalize the numerator)} \\
&= \lim_{x \to 0} \frac{x + 1 - 1}{x(\sqrt{x + 1} + 1)} \\
&= \lim_{x \to 0} \frac{x}{x(\sqrt{x + 1} + 1)} && \text{(Simplify)} \\
&= \lim_{x \to 0} \frac{1}{\sqrt{x + 1} + 1} && \text{(Evaluate)} \\
&= \frac{1}{2}
\end{aligned}$$

INVESTIGATION Here is an alternate technique for finding the value of a limit.

A. Find $\lim_{x\to 1} \frac{(x-1)}{\sqrt{x}-1}$ by rationalizing.

B. Let $u = \sqrt{x}$, and rewrite $\lim_{x\to 1} \frac{(x-1)}{\sqrt{x}-1}$ in terms of u. We know $x = u^2$, $\sqrt{x} \geq 0$, and $u \geq 0$. Therefore, as x approaches the value of 1, u approaches the value of 1. Use this substitution to find $\lim_{u\to 1} \frac{(u^2-1)}{u-1}$ by reducing the rational expression.

EXAMPLE 7 Selecting a substitution strategy to evaluate a limit

Evaluate $\lim_{x\to 0} \frac{(x+8)^{\frac{1}{3}}-2}{x}$.

Solution

This quotient is indeterminate $\left(\frac{0}{0}\right)$ when $x = 0$. Rationalizing the numerator $(x+8)^{\frac{1}{3}} - 2$ is not so easy. However, the expression can be simplified by substitution. Let $u = (x+8)^{\frac{1}{3}}$. Then $u^3 = x + 8$ and $x = u^3 - 8$. As x approaches the value 0, u approaches the value 2. The given limit becomes

$$\begin{aligned}
\lim_{x\to 0} \frac{(x+8)^{\frac{1}{3}}-2}{x} &= \lim_{u\to 2} \frac{u-2}{u^3-8} && \text{(Factor)}\\
&= \lim_{u\to 2} \frac{u-2}{(u-2)(u^2+2u+4)} && \text{(Simplify)}\\
&= \lim_{u\to 2} \frac{1}{u^2+2u+4} && \text{(Evaluate)}\\
&= \frac{1}{12}
\end{aligned}$$

EXAMPLE 8 Evaluating a limit that involves absolute value

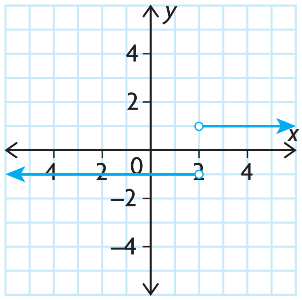
Evaluate $\lim_{x\to 2} \frac{|x-2|}{x-2}$. Illustrate with a graph.

Solution

Consider the following:

$$f(x) = \frac{|x-2|}{x-2} = \begin{cases} \frac{(x-2)}{x-2}, \text{ if } x > 2 \\ \frac{-(x-2)}{x-2}, \text{ if } x < 2 \end{cases}$$

$$= \begin{cases} 1, \text{ if } x > 2 \\ -1, \text{ if } x < 2 \end{cases}$$



Notice that $f(2)$ is not defined. Also note and that we must consider left-hand and right-hand limits.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-1) = -1$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (1) = 1$$

Since the left-hand and right-hand limits are not the same, we conclude that $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$ does not exist.

EXAMPLE 9 **Reasoning about the existence of a limit**

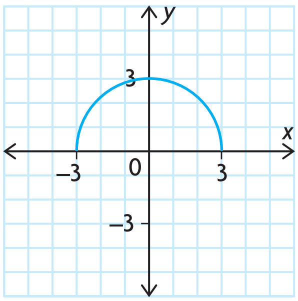
a. Evaluate $\lim_{x \to 3^-} \sqrt{9 - x^2}$

b. Explain why the limit as x approaches 3^+ cannot be determined.

c. What can you conclude about $\lim_{x \to 3} \sqrt{9 - x^2}$?

Solution

a. The graph of $f(x) = \sqrt{9 - x^2}$ is the semicircle illustrated below.



From the graph, the left-hand limit at $x \to 3$ is 0. Therefore, $\lim_{x \to 3^-} \sqrt{9 - x^2} = 0$.

b. The function is not defined for $x > 3$.

c. $\lim_{x \to 3} \sqrt{9 - x^2}$ does not exist because the function is not defined on both sides of 3.

In this section, we learned the properties of limits and developed algebraic methods for evaluating limits. The examples in this section complement the table of values and graphing techniques introduced in previous sections.

IN SUMMARY

Key Ideas

- If f is a polynomial function, then $\lim_{x\to a} f(x) = f(a)$.
- Substituting $x = a$ into $\lim_{x\to a} f(x)$ can yield the indeterminate form $\frac{0}{0}$. If this happens, you may be able to find an equivalent function that is the same as the function f for all values except at $x = a$. Then, substitution can be used to find the limit.

Need to Know

To evaluate a limit algebraically, you can use the following techniques:

- direct substitution
- factoring
- rationalizing
- one-sided limits
- change of variable

For any of these techniques, a graph or table of values can be used to check your result.

Exercise 1.5

PART A

1. Are there different answers for $\lim_{x\to 2}(3 + x)$, $\lim_{x\to 2} 3 + x$, and $\lim_{x\to 2}(x + 3)$?
2. How do you find the limit of a rational function?

C

3. Once you know $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$, do you then know $\lim_{x\to a} f(x)$? Give reasons for your answer.
4. Evaluate each limit.

a. $\lim_{x\to 2} \dfrac{3x}{x^2 + 2}$

b. $\lim_{x\to -1}(x^4 + x^3 + x^2)$

c. $\lim_{x\to 9}\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$

d. $\lim_{x\to 2\pi}(x^3 + \pi^2 x - 5\pi^3)$

e. $\lim_{x\to 0}\left(\sqrt{3 + \sqrt{1 + x}}\right)$

f. $\lim_{x\to -3}\sqrt{\dfrac{x - 3}{2x + 4}}$

PART B

5. Use a graphing calculator to graph each function and estimate the limit. Then find the limit by substitution.

a. $\lim\limits_{x \to -2} \dfrac{x^3}{x - 2}$

b. $\lim\limits_{x \to 1} \dfrac{2x}{\sqrt{x^2 + 1}}$

6. Show that $\lim\limits_{t \to 1} \dfrac{t^3 - t^2 - 5t}{6 - t^2} = -1$.

K

7. Evaluate the limit of each indeterminate quotient.

a. $\lim\limits_{x \to 2} \dfrac{4 - x^2}{2 - x}$

b. $\lim\limits_{x \to -1} \dfrac{2x^2 + 5x + 3}{x + 1}$

c. $\lim\limits_{x \to 3} \dfrac{x^3 - 27}{x - 3}$

d. $\lim\limits_{x \to 0} \dfrac{2 - \sqrt{4 + x}}{x}$

e. $\lim\limits_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$

f. $\lim\limits_{x \to 0} \dfrac{\sqrt{7 - x} - \sqrt{7 + x}}{x}$

8. Evaluate the limit by using a change of variable.

a. $\lim\limits_{x \to 8} \dfrac{\sqrt[3]{x} - 2}{x - 8}$

b. $\lim\limits_{x \to 27} \dfrac{27 - x}{x^{\frac{1}{3}} - 3}$

c. $\lim\limits_{x \to 1} \dfrac{x^{\frac{1}{6}} - 1}{x - 1}$

d. $\lim\limits_{x \to 1} \dfrac{x^{\frac{1}{6}} - 1}{x^{\frac{1}{3}} - 1}$

e. $\lim\limits_{x \to 4} \dfrac{\sqrt{x} - 2}{\sqrt{x^3} - 8}$

f. $\lim\limits_{x \to 0} \dfrac{(x + 8)^{\frac{1}{3}} - 2}{x}$

9. Evaluate each limit, if it exists, using any appropriate technique.

a. $\lim\limits_{x \to 4} \dfrac{16 - x^2}{x^3 + 64}$

b. $\lim\limits_{x \to 4} \dfrac{x^2 - 16}{x^2 - 5x + 6}$

c. $\lim\limits_{x \to -1} \dfrac{x^2 + x}{x + 1}$

d. $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x}$

e. $\lim\limits_{h \to 0} \dfrac{(x + h)^2 - x^2}{h}$

f. $\lim\limits_{x \to 1} \left[\left(\dfrac{1}{x - 1} \right) \left(\dfrac{1}{x + 3} - \dfrac{2}{3x + 5} \right) \right]$

10. By using one-sided limits, determine whether each limit exists. Illustrate your results geometrically by sketching the graph of the function.

a. $\lim\limits_{x \to 5} \dfrac{|x - 5|}{x - 5}$

b. $\lim\limits_{x \to \frac{5}{2}} \dfrac{|2x - 5|(x + 1)}{2x - 5}$

c. $\lim\limits_{x \to 2} \dfrac{x^2 - x - 2}{|x - 2|}$

d. $\lim\limits_{x \to -2} \dfrac{(x + 2)^3}{|x + 2|}$

A 11. Jacques Charles (1746–1823) discovered that the volume of a gas at a constant pressure increases linearly with the temperature of the gas. To obtain the data in the following table, one mole of hydrogen was held at a constant pressure of one atmosphere. The volume V was measured in litres, and the temperature T was measured in degrees Celsius.

T (°C)	−40	−20	0	20	40	60	80
V (L)	19.1482	20.7908	22.4334	24.0760	25.7186	27.3612	29.0038

a. Calculate first differences, and show that T and V are related by a linear relation.

b. Find the linear equation for V in terms of T.

c. Solve for T in terms of V for the equation in part b.

d. Show that $\lim_{V \to 0^+} T$ is approximately -273.15. *Note:* This represents the approximate number of degrees on the Celsius scale for absolute zero on the Kelvin scale (0 K).

e. Using the information you found in parts b and d, draw a graph of V versus T.

T 12. Show, using the properties of limits, that if $\lim_{x \to 5} f(x) = 3$, then $\lim_{x \to 5} \frac{x^2 - 4}{f(x)} = 7$.

13. If $\lim_{x \to 4} f(x) = 3$, use the properties of limits to evaluate each limit.

a. $\lim_{x \to 4} [f(x)]^3$ b. $\lim_{x \to 4} \frac{[f(x)]^2 - x^2}{f(x) + x}$ c. $\lim_{x \to 4} \sqrt{3f(x) - 2x}$

PART C

14. If $\lim_{x \to 0} \frac{f(x)}{x} = 1$ and $\lim_{x \to 0} g(x)$ exists and is nonzero, then evaluate each limit.

a. $\lim_{x \to 0} f(x)$ b. $\lim_{x \to 0} \frac{f(x)}{g(x)}$

15. If $\lim_{x \to 0} \frac{f(x)}{x} = 1$ and $\lim_{x \to 0} \frac{g(x)}{x} = 2$, then evaluate each limit.

a. $\lim_{x \to 0} g(x)$ b. $\lim_{x \to 0} \frac{f(x)}{g(x)}$

16. Evaluate $\lim_{x \to 0} \frac{\sqrt{x + 1} - \sqrt{2x + 1}}{\sqrt{3x + 4} - \sqrt{2x + 4}}$.

17. Does $\lim_{x \to 1} \frac{x^2 + |x - 1| - 1}{|x - 1|}$ exist? Illustrate your answer by sketching a graph of the function.