Section 8.3—Vector, Parametric, and Symmetric Equations of a Line in R^3

In Section 8.1, we discussed vector and parametric equations of a line in R^2. In this section, we will continue our discussion, but, instead of R^2, we will examine lines in R^3.

The derivation and form of the vector equation for a line in R^3 is the same as in R^2. If we wish to find a vector equation for a line in R^3, it is necessary that either two points or a point and a direction vector be given. If we are given two points and wish to determine a direction vector for the corresponding line, the coordinates of this vector must first be calculated.

EXAMPLE 1

Determining a direction vector of a line in R^3

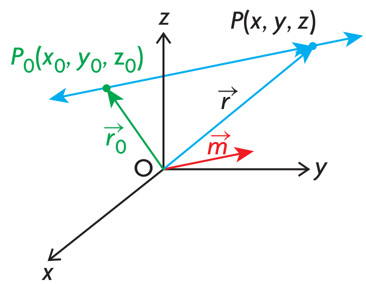
A line passes through the points $A(-1, 3, 5)$ and $B(-3, 3, -4)$. Calculate possible direction vectors for this line.

Solution

A possible direction vector is $\vec{m} = (-1 - (-3), 3 - 3, 5 - (-4)) = (2, 0, 9)$. In general, any vector of the form $t(2, 0, 9)$, $t \in \mathbf{R}$, $t \neq 0$, can be used as a direction vector for this line. As before, the best choice for a direction vector is one in which the direction numbers are integers, with common divisors removed. This implies that either $(2, 0, 9)$ or $(-2, 0, -9)$ are the best choices for a direction vector for this line. Generally speaking, if a line has $\vec{m} = (a, b, c)$ as its direction vector, then any scalar multiple of this vector of the form $t(a, b, c)$, $t \in \mathbf{R}$, $t \neq 0$, can be used as a direction vector.

Vector and Parametric Equations of Lines in R^3

Consider the following diagram.



When determining the vector equation of the line passing through P_0 and P, we know that the point $P_0(x_0, y_0, z_0)$ is a given point on the line, and $\vec{m} = (a, b, c)$

is its direction vector. If $P(x, y, z)$ represents a general point on the line, then $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is a direction vector for this line. This allows us to form the vector equation of the line.

In $\triangle OP_0P$, $\overrightarrow{OP} = \overrightarrow{OP_0} + \overrightarrow{P_0P}$.

Since $\overrightarrow{OP_0} = \vec{r_0}$ and $\overrightarrow{P_0P} = t\vec{m}$, the vector equation of the line is $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$. In component form, this can be written as $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$, $t \in \mathbf{R}$. The parametric equations of the line are found by equating the respective x, y, and z components, giving $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, $t \in \mathbf{R}$.

Vector and Parametric Equations of a Line in R^3

Vector Equation: $\vec{r} = \vec{r_0} + t\vec{m}$, $t \in \mathbf{R}$
Parametric Equations: $x = x_0 + ta$, $y = y_0 + tb$, $z = z_0 + tc$, $t \in \mathbf{R}$
where $\vec{r_0} = (x_0, y_0, z_0)$, the vector from the origin to a point on the line and
$\vec{m} = (a, b, c)$ is a direction vector of the line

EXAMPLE 2 Representing the equation of a line in R^3 in vector and parametric form

Determine the vector and parametric equations of the line passing through $P(-2, 3, 5)$ and $Q(-2, 4, -1)$.

Solution

A direction vector is $\vec{m} = (-2 - (-2), 3 - 4, 5 - (-1)) = (0, -1, 6)$. A vector equation is $\vec{r} = (-2, 3, 5) + s(0, -1, 6)$, $s \in \mathbf{R}$, and its parametric equations are $x = -2$, $y = 3 - s$, $z = 5 + 6s$, $s \in \mathbf{R}$. It would also have been correct to choose any multiple of $(0, -1, 6)$ as a direction vector and any point on the line. For example, the vector equation $\vec{r} = (-2, 4, -1) + t(0, -2, 12)$, $t \in \mathbf{R}$, would also have been correct.

Since a vector equation of a line can be written in many ways, it is useful to be able to tell if different forms are actually equivalent. In the following example, an algebraic approach to this problem is considered.

EXAMPLE 3 Reasoning to establish the equivalence of two lines

a. Show that the following are vector equations for the same line:
$L_1: \vec{r} = (-1, 0, 4) + s(-1, 2, 5), s \in \mathbf{R}$, and
$L_2: \vec{r} = (4, -10, -21) + m(-2, 4, 10), m \in \mathbf{R}$

b. Show that the following are vector equations for different lines:
$L_3: \vec{r} = (1, 6, 1) + l(-1, 1, 2), l \in \mathbf{R}$, and
$L_4: \vec{r} = (-3, 10, 12) + k\left(\frac{1}{2}, -\frac{1}{2}, -1\right), k \in \mathbf{R}$

Solution

a. Since the direction vectors are parallel—that is, $2(-1, 2, 5) = (-2, 4, 10)$—this means that the two lines are parallel. To show that the equations are equivalent, we must show that a point on one of the lines is also on the other line. This is based on the logic that, if the lines are parallel and they share a common point, then the two equations must represent the same line. To check whether $(4, -10, -21)$ is also on L_1 substitute into its vector equation.
$(4, -10, -21) = (-1, 0, 4) + s(-1, 2, 5)$

Using the x component, we find $4 = -1 - s$, or $s = -5$. Substituting $s = -5$ into the above equation, $(-1, 0, 4) + -5(-1, 2, 5) = (4, -10, -21)$. This verifies that the point $(4, -10, -21)$ is also on L_1.

Since the lines have the same direction, and a point on one line is also on the second line, the two given equations represent the same line.

b. The first check is to compare the direction vectors of the two lines. Since $-2\left(\frac{1}{2}, -\frac{1}{2}, -1\right) = (-1, 1, 2)$, the lines must be parallel. As in part a, $(-3, 10, 12)$ must be a point on L_3 for the equations to be equivalent. Therefore, $(-3, 10, 12) = (1, 6, 1) + l(-1, 1, 2)$ must give a consistent value of l for each component. If we solve, this gives an inconsistent result since $l = 4$ for the x and y components, and $12 = 1 + 2l$, or $l = \frac{11}{2}$ for the z component. This verifies that the two equations are not equations for the same line.

Symmetric Equations of Lines in R^3

We introduce a new form for a line in R^3, called its **symmetric equation**. The symmetric equation of a line is derived from using its parametric equations and solving for the parameter in each component, as shown below

$$x = x_0 + ta \leftrightarrow t = \frac{x - x_0}{a}, a \neq 0$$

$$y = y_0 + tb \leftrightarrow t = \frac{y - y_0}{b}, b \neq 0$$

$$z = z_0 + tc \leftrightarrow t = \frac{z - z_0}{c}, c \neq 0$$

Combining these statements gives $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, a, b, c \neq 0$.

These equations are called the symmetric equations of a line in R^3.

Symmetric Equations of a Line in R^3

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c},\ a \neq 0, b \neq 0, c \neq 0$$

where (x_0, y_0, z_0) is the vector from the origin to a point on the line, and (a, b, c) is a direction vector of the line.

EXAMPLE 4 Representing the equation of a line in R^3 in symmetric form

a. Write the symmetric equations of the line passing through the points $A(-1, 5, 7)$ and $B(3, -4, 8)$.

b. Write the symmetric equations of the line passing through the points $P(-2, 3, 1)$ and $Q(4, 3, -5)$.

c. Write the symmetric equations of the line passing through the points $X(-1, 2, 5)$ and $Y(-1, 3, 9)$.

Solution

a. A direction vector for this line is $\vec{m} = (-1 - 3, 5 - (-4), 7 - 8) = (-4, 9, -1)$. Using the point $A(-1, 5, 7)$, the parametric equations of the line are $x = -1 - 4t$, $y = 5 + 9t$, and $z = 7 - t$, $t \in \mathbf{R}$. Solving each equation for t gives the required symmetric equations, $\frac{x+1}{-4} = \frac{y-5}{9} = \frac{z-7}{-1}$. It is usually not necessary to find the parametric equations before finding the symmetric equations. The symmetric equations of a line can be written by inspection if the direction vector and a point on the line are known. Using point B and the direction vector found above, the symmetric equations of this line by inspection are $\frac{x-3}{-4} = \frac{y+4}{9} = \frac{z-8}{-1}$.

b. A direction vector for the line is $\vec{m} = (-2 - 4, 3 - 3, 1 - (-5)) = (-6, 0, 6)$. The vector $(1, 0, -1)$ will be used as the direction vector. In a situation like this, where the y direction number is 0, using point P the equation is written as $\frac{x+2}{1} = \frac{z-1}{-1}$, $y = 3$.

c. A direction vector for the line is $\vec{m} = (-1 - (-1), 2 - 3, 5 - 9) = (0, -1, -4)$. Using point X, possible symmetric equations are $\frac{y-2}{-1} = \frac{z-5}{-4}$, $x = -1$.

IN SUMMARY

Key Idea

- In R^3, if $\vec{r_0} = (x_0, y_0, z_0)$ is determined by a point on a line and $\vec{m} = (a, b, c)$ is a direction vector of the same line, then
  - the vector equation of the line is $\vec{r} = \vec{r}_0 + t\vec{m}$, $t \in \mathbf{R}$ or equivalently $(x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$
  - the parametric form of the equation of the line is $x = x_0 + ta$, $y = y_0 + tb$, and $z = z_0 + tc$, $t \in \mathbf{R}$
  - the symmetric form of the equation of the line is $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} (= t)$, $a \neq 0$, $b \neq 0$, $c \neq 0$

Need to Know

- Knowing one of these forms of the equation of a line enables you to find the other two, since all three forms depend on the same information about the line.

Exercise 8.3

PART A

1. State the coordinates of a point on each of the given lines.
   a. $\vec{r} = (-3, 1, 8) + s(-1, 1, 9),\ s \in \mathbf{R}$
   b. $\dfrac{x-1}{2} = \dfrac{y+1}{1} = \dfrac{z-3}{-1}$
   c. $x = -2 + 3t,\ y = 1 + (-4t),\ z = 3 - t,\ t \in \mathbf{R}$
   d. $\dfrac{x+2}{-1} = \dfrac{z-1}{2},\ y = -3$
   e. $x = 3,\ y = -2,\ z = -1 + 2k,\ k \in \mathbf{R}$
   f. $\dfrac{x - \frac{1}{3}}{\frac{1}{2}} = \dfrac{y + \frac{3}{4}}{\frac{-1}{4}} = \dfrac{z - \frac{2}{5}}{\frac{1}{2}}$
2. State a direction vector for each line in question 1, making certain that the components for each are integers.

PART B

3. A line passes through the points $A(-1, 2, 4)$ and $B(3, -3, 5)$.
   a. Write two vector equations for this line.
   b. Write the two sets of parametric equations associated with the vector equations you wrote in part a.
4. A line passes through the points $A(-1, 5, -4)$ and $B(2, 5, -4)$.
   a. Write a vector equation for the line containing these points.
   b. Write parametric equations corresponding to the vector equation you wrote in part a.
   c. Explain why there are no symmetric equations for this line.

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5. State where possible vector, parametric, and symmetric equations for each of the following lines.
   a. the line passing through the point $P(-1, 2, 1)$ with direction vector $(3, -2, 1)$
   b. the line passing through the points $A(-1, 1, 0)$ and $B(-1, 2, 1)$
   c. the line passing through the point $B(-2, 3, 0)$ and parallel to the line passing through the points $M(-2, -2, 1)$ and $N(-2, 4, 7)$
   d. the line passing through the points $D(-1, 0, 0)$ and $E(-1, 1, 0)$
   e. the line passing through the points $X(-4, 3, 0)$ and $O(0, 0, 0)$
   f. the line passing through the point $Q(1, 2, 4)$ and parallel to the z-axis

6. a. Determine parametric equations for each of the following lines:
$\frac{x+6}{1} = \frac{y-10}{-1} = \frac{z-7}{1}$ and $\frac{x+7}{1} = \frac{y-11}{-1}, z = 5$

b. Determine the angle between the two lines.

C 7. Show that the following two sets of symmetric equations represent the same straight line: $\frac{x+7}{8} = \frac{y+1}{2} = \frac{z-5}{-2}$ and $\frac{x-1}{-4} = \frac{y-1}{-1} = \frac{z-3}{1}$

8. a. Show that the points $A(6, -2, 15)$ and $B(-15, 5, -27)$ lie on the line that passes through $(0, 0, 3)$ and has the direction vector $(-3, 1, -6)$.

b. Use parametric equations with suitable restrictions on the parameter to describe the line segment from A to B.

A 9. Determine the value of k for which the direction vectors of the lines $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z+1}{k-1}$ and $\frac{x+3}{-2} = \frac{z}{1}, y = -1$ are perpendicular.

10. Determine the coordinates of three different points on each line.

a. $(x, y, z) = (4, -2, 5) + t(-4, -6, 8)$

b. $x = -4 + 5s, y = 2 - s, z = 9 - 6s$

c. $\frac{x+1}{3} = \frac{y-2}{-1} = \frac{z}{4}$

d. $x = -4, \frac{y-2}{3} = \frac{z-3}{5}$

11. Express each equation in question 10 in two other equivalent forms. (i.e. vector, parametric or symmetric form)

PART C

12. Determine the parametric equations of the line whose direction vector is perpendicular to the direction vectors of the two lines $\frac{x}{-4} = \frac{y+10}{-7} = \frac{z+2}{3}$ and $\frac{x-5}{3} = \frac{y-5}{2} = \frac{z+5}{4}$ and passes through the point $(2, -5, 0)$.

T 13. A line with parametric equations $x = 10 + 2s, y = 5 + s, z = 2, s \in \mathbf{R}$, intersects a sphere with the equation $x^2 + y^2 + z^2 = 9$ at the points A and B. Determine the coordinates of these points.

14. You are given the two lines $L_1: x = 4 + 2t, y = 4 + t, z = -3 - t, t \in \mathbf{R}$, and $L_2: x = -2 + 3s, y = -7 + 2s, z = 2 - 3s, s \in \mathbf{R}$. If the point P_1 lies on L_1 and the point P_2 lies on L_2, determine the coordinates of these two points if $\overrightarrow{P_1P_2}$ is perpendicular to each of the two lines. (*Hint*: The vector $\overrightarrow{P_1P_2}$ is perpendicular to the direction vector of each of the two lines.)

15. Determine the angle formed by the intersection of the lines defined by $\frac{x-1}{2} = \frac{y+3}{1}, z = -3$ and $\frac{x-2}{3} = \frac{y+1}{2} = \frac{z}{1}$.