2.4 Simplifying Rational Functions

GOAL

Define rational functions, and explore methods of simplifying the related rational expression.

LEARN ABOUT the Math

Adonis has designed a game called "2 and 1" to raise money at a charity casino. To start the game, Adonis announces he will draw n numbers from a set that includes all the natural numbers from 1 to $2n$.

The players then pick three numbers.

Adonis draws n numbers and announces them. The players check for matches. Any player who has at least two matches wins.

rational function

any function that is the ratio of two polynomials. A rational function can be expressed as $f(x) = \frac{R(x)}{S(x)}$, where R and S are polynomials and $S \neq 0$; for example, $f(x) = \frac{x^2 - 2x + 3}{4x - 1}, x \neq \frac{1}{4}$

A rational expression is a quotient of polynomials; for example, $\frac{2x - 1}{3x}, x \neq 0$

The probability of a player winning is given by the rational function

$$P(n) = \frac{3n^3 - 3n^2}{8n^3 - 12n^2 + 4n}$$

For example, if Adonis draws 5 numbers from the set 1 to 10, the probability of winning is

$$P(5) = \frac{3(5)^3 - 3(5)^2}{8(5)^3 - 12(5)^2 + 4(5)}$$

$$= \frac{5}{12}$$

The game is played at a rapid pace, and Adonis needs a fast way to determine the range he should use, based on the number of players and their chances of winning.

? What is the simplified expression for the probability of a player winning at "2 and 1"?

EXAMPLE 1 Simplifying rational functions

Write the simplified expression for the function defined by
$P(n) = \dfrac{3n^3 - 3n^2}{8n^3 - 12n^2 + 4n}$.

Faez's Solution

$$P(n) = \frac{3n^2(n-1)}{4n(2n^2 - 3n + 1)}$$

I knew that I could simplify rational numbers by first factoring numerators and denominators and dividing each by the common factor $\left(\text{e.g., } \frac{24}{27} = \frac{3(8)}{3(9)} = \frac{8}{9}\right)$.

So I tried the same idea here.
I factored the numerator and denominator of $P(n)$.

$$= \frac{3n^2\cancel{(n-1)}^{1}}{4n(2n-1)\cancel{(n-1)}_{1}}$$

Then I divided by the common factor, $(n - 1)$.

$$= \frac{3n^2}{4n(2n-1)}$$

Restrictions:

When $4n(2n-1)(n-1) = 0$,

$$4n = 0 \quad (2n-1) = 0 \quad (n-1) = 0$$
$$n = 0 \quad n = \frac{1}{2} \quad n = 1$$

Since I cannot divide by zero, I determined the **restrictions** by calculating the values of n that make the factored denominator zero.
I solved $4n(2n - 1)(n - 1) = 0$ by setting each factor equal to 0.

$$P(n) = \frac{3n^2}{4n(2n-1)};\ n \neq 0, \frac{1}{2}, 1.$$

restrictions

the values of the variable(s) in a rational function or rational expression that cause the function to be undefined. These are the zeros of the denominator or, equivalently, the numbers that are not in the domain of the function.

Reflecting

A. How is working with rational expressions like working with rational numbers? How is it different?

B. How do the restrictions on the rational expression in $P(n)$ relate to the domain of this rational function?

C. How does factoring help to simplify and determine the restrictions on the variable?

APPLY the Math

EXAMPLE 2 **Selecting a strategy for simplifying the quotient of a monomial and a monomial**

Simplify and state any restrictions on the variables.

$$\frac{30x^4y^3}{-6x^7y}$$

Tanya's Solution

$$\frac{30x^4y^3}{-6x^7y} = \frac{\overset{1}{\cancel{-6x^4y}}(-5y^2)}{\underset{1}{\cancel{-6x^4y}}(x^3)}$$

I factored the numerator and denominator by dividing out the GCF $-6x^4y$. Then I divided the numerator and denominator by the GCF.

$$= \frac{-5y^2}{x^3};\ x, y \neq 0$$

I determined the restrictions by finding the zeros of the *original* denominator by solving $-6x^7y = 0$. This gives the restrictions x, $y \neq 0$.

EXAMPLE 3 **Selecting a strategy for simplifying the quotient of a polynomial and a monomial**

Simplify and state any restrictions on the variables.

$$\frac{10x^4 - 8x^2 + 4x}{2x^2}$$

Lee's Solution

$$\frac{10x^4 - 8x^2 + 4x}{2x^2}$$

$$= \frac{\overset{1}{\cancel{2x}}(5x^3 - 4x + 2)}{\underset{1}{\cancel{2x}}(x)}$$

I factored the numerator and denominator by dividing out the GCF $2x$. Then I divided both the numerator and denominator by $2x$.

$$= \frac{5x^3 - 4x + 2}{x};\ x \neq 0$$

I determined the restrictions by solving $2x^2 = 0$ to get the zeros of the original denominator. The only restriction is $x \neq 0$.

EXAMPLE 4 Selecting a strategy for simplifying a function involving the quotient of a trinomial and a binomial

Simplify $f(x)$ and state the domain, where $f(x) = \dfrac{x^2 + 7x - 8}{2 - 2x}$.

Michel's Solution

$$f(x) = \frac{x^2 + 7x - 8}{2 - 2x}$$

$$= \frac{(x - 1)(x + 8)}{2(1 - x)}$$

I factored the numerator and denominator and noticed that there were two factors that were similar, but with opposite signs.

$$= \frac{-\overset{1}{\cancel{(1 - x)}}(x + 8)}{2\underset{1}{\cancel{(1 - x)}}}$$

I divided out the common factor, -1, from $(x - 1)$ in the numerator, so that it became identical to $(1 - x)$ in the denominator. I divided the numerator and denominator by the GCF $1 - x$.

$$= \frac{-(x + 8)}{2}; x \neq 1$$

The domain is $\{x \in \mathbf{R} \mid x \neq 1\}$.

I determined the restrictions by solving $2(1 - x) = 0$. The only restriction is $x \neq 1$. This means that $f(x)$ is undefined when $x = 1$, so $x = 1$ must be excluded from the domain.

EXAMPLE 5 Selecting a strategy for simplifying the quotient of quadratics in two variables

Simplify and state any restrictions on the variables: $\dfrac{4x^2 - 16y^2}{x^2 + xy - 6y^2}$.

Hermione's Solution

$$\frac{4x^2 - 16y^2}{x^2 + xy - 6y^2}$$

$$= \frac{4\overset{1}{\cancel{(x - 2y)}}(x + 2y)}{(x + 3y)\underset{1}{\cancel{(x - 2y)}}}$$

I factored the numerator and denominator and then divided by the GCF $x - 2y$.

$$= \frac{4(x + 2y)}{x + 3y}; x \neq -3y, 2y$$

I determined the restrictions by finding the zeros of the factored denominator by solving $(x + 3y)(x - 2y) = 0$.

So, $(x + 3y) = 0$ and $(x - 2y) = 0$.

The restrictions are $x \neq -3y, 2y$.

In Summary

Key Ideas

- A rational function can be expressed as the ratio of two polynomial functions. For example,

$$f(x) = \frac{6x + 2}{x - 1}; x \neq 1$$

  A rational expression is the ratio of two polynomials.
  For example,

$$\frac{6x + 2}{x - 1}; x \neq 1$$

- Both rational functions and rational expressions are undefined for numbers that make the denominator zero. These numbers must be excluded or restricted from being possible values for the variables. As a result, for all rational functions, the domain is the set of all real numbers, except those numbers that make the denominator equal zero.

Need to Know

- Rational functions and rational expressions can be simplified by factoring the numerator and denominator and then dividing both by their greatest common factor.
- The restrictions are found by determining all the zeros of the denominator. If the denominator contains two or more terms, the zeros can be determined from its factored form before the function or expression is simplified.

CHECK Your Understanding

1. Simplify. State any restrictions on the variables.

a) $\frac{6 - 4t}{2}$ **b)** $\frac{9x^2}{6x^3}$ **c)** $\frac{7a^2b^3}{21a^4b}$

2. Simplify. State any restrictions on the variables.

a) $\frac{5(x + 3)}{(x + 3)(x - 3)}$ **b)** $\frac{6x - 9}{2x - 3}$ **c)** $\frac{4a^2b - 2ab^2}{(2a - b)^2}$

3. Simplify. State any restrictions on the variables.

a) $\frac{(x - 1)(x - 3)}{(x + 2)(x - 1)}$ **b)** $\frac{5x^2 + x - 4}{25x^2 - 40x + 16}$ **c)** $\frac{x^2 - 7xy + 10y^2}{x^2 + xy - 6y^2}$

PRACTISING

4. Simplify. State any restrictions on the variables.

a) $\dfrac{14x^3 - 7x^2 + 21x}{7x}$

b) $\dfrac{-5x^3y^2}{10xy^3}$

c) $\dfrac{2t(5 - t)}{5t^2(t - 5)}$

d) $\dfrac{5ab}{15a^4b - 10a^2b^2}$

e) $\dfrac{2x^2 + 10x}{-3x - 15}$

f) $\dfrac{2ab - 6a}{9a - 3ab}$

5. Simplify. State any restrictions on the variables.

a) $\dfrac{a + 4}{a^2 + 3a - 4}$

b) $\dfrac{x^2 - 9}{15 - 5x}$

c) $\dfrac{x^2 - 5x + 6}{x^2 + 3x - 10}$

d) $\dfrac{10 + 3p - p^2}{25 - p^2}$

e) $\dfrac{t^2 - 7t + 12}{t^3 - 6t^2 + 9t}$

f) $\dfrac{6t^2 - t - 2}{2t^2 - t - 1}$

6. State the domain of each function. Explain how you found each answer.

a) $f(x) = \dfrac{2 + x}{x}$

b) $g(x) = \dfrac{3}{x(x - 2)}$

c) $h(x) = \dfrac{-3}{(x + 5)(x - 5)}$

d) $f(x) = \dfrac{1}{x^2 - 1}$

e) $g(x) = \dfrac{1}{x^2 + 1}$

f) $h(x) = \dfrac{x - 1}{x^2 - 1}$

7. Determine which pairs of functions are equivalent. Explain your reasoning.

a) $f(x) = 2x^2 + x - 7$ and $g(x) = \dfrac{6x^2 + 3x - 21}{3}$

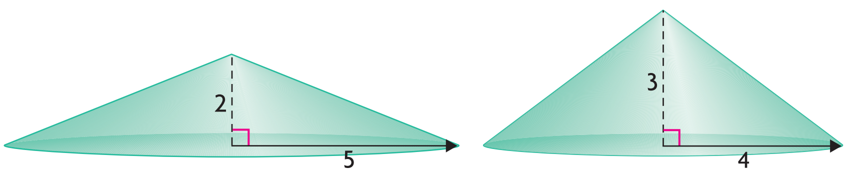
b) $h(x) = 3x^2 + 5x + 1$ and $j(x) = \dfrac{3x^3 + 5x^2 + x}{x}, x \neq 0$

8. **A** An isosceles triangle has two sides of length $9x + 3$. The perimeter of the triangle is $30x + 10$.

a) Determine the ratio of the base to the perimeter, in simplified form. State the restriction on x.

b) Explain why the restriction on x in part (a) is necessary in this situation.

9. Two cones have radii in the ratio 5:4 and heights in the ratio 2:3. Determine the ratio of their volumes, where $V = \frac{1}{3}\pi r^2 h$.



10. **K** Simplify. State any restrictions on the variables.

a) $\dfrac{20t^3 + 15t^2 - 5t}{5t}$

b) $\dfrac{5(4x - 2)}{8(2x - 1)^2}$

c) $\dfrac{x^2 - 9x + 20}{16 - x^2}$

d) $\dfrac{2x^2 - xy - y^2}{x^2 - 2xy + y^2}$

11. A rectangle is six times as long as it is wide. Determine the ratio of its area to its perimeter, in simplest form, if its width is w.

12. The quotient of two polynomials is $3x - 2$. Give two examples of a rational expression equivalent to this polynomial that has the restriction $x \neq 4$.

13. Give an example of a rational function that could have three restrictions that are consecutive numbers.

14. **T** Consider the rational expression $\frac{2x + 1}{x - 4}$.

a) Identify, if possible, a rational expression with integer coefficients that simplifies to $\frac{2x + 1}{x - 4}$, for each set of restrictions.

i) $x \neq -1, 4$ **ii)** $x \neq 0, 4$ **iii)** $x \neq \frac{2}{3}, 4$ **iv)** $x \neq -\frac{1}{2}, 4$

b) Is there a rational expression with denominator of the form $ax^2 + bx + c, a \neq 0$, that simplifies to $\frac{2x + 1}{x - 4}$, and has only the restriction $x \neq 4$? Explain.

15. **C** Can two different rational expressions simplify to the same polynomial? Explain using examples.

Extending

Communication | ***Tip***

"As x approaches positive infinity" is commonly written as $x \to \infty$, and "as x approaches negative infinity" is commonly written as $x \to -\infty$.

The limiting end behaviour of the function $f(x) = \frac{1}{x}$, which approaches zero as x approaches infinity, is written as

$$\lim_{x \to \infty} f(x) = 0$$

16. Mathematicians are often interested in the "end behaviour" of functions, that is, the value of the output as the input, x, gets greater and greater and approaches infinity and as the input, x, gets lesser and lesser and approaches negative infinity. For example, the output of $f(x) = \frac{1}{x}$ as x approaches infinity gets closer and closer to 0. By calculating values of the function, make a conjecture about both end behaviours of each rational function.

a) $f(x) = \dfrac{50x + 73}{x^2 - 10x - 400}$

b) $g(x) = \dfrac{4x^3 - 100}{5x^3 + 87x + 28}$

c) $h(x) = \dfrac{-7x^2 + 3x}{200x + 9999}$

17. Simplify. State any restrictions on the variables.

a) $a(t) = \dfrac{-2(1 + t^2)^2 + 2t(2)(1 + t^2)(2t)}{(1 + t^2)^4}$

b) $f(x) = \dfrac{2(2x + 1)(2)(3x - 2)^3 - (2x + 1)^2(3)(3x - 2)^2}{(3x - 2)^6}$