Measure Theory - Lebesgue Integration

Evan Kim

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Contents

1 References

- Measure, Integration & Real Analysis (MIRA) - Sheldon Axler (2020)
- Measure Theory, Integration, and Hilbert Spaces - Stein, Shakarchi (2005)

2 Measure Theory

2.1 The Exterior Measure

2.1.1 Countable Union of Disjoint Open Intervals

This theorem[1] shows the construction of a set of maximal open intervals, shows that they are disjoint, and shows that these disjoint open intervals are countable. This is a straight forward construction proof with no tricks and the statement is fairly obvious. This theorem is limited because it does not generalize into higher dimensions for *every* type of open subset.

Theorem 1.3 - Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

Proof: For each $x \in \mathcal{O}$, let I_x denote the largest open interval containing x that is contained in $\mathcal{O}$. More precisely, x is contained in some small (non-trivial) interval since $\mathcal{O}$ is open. Therefore if

$$a_x = \inf\{a < x \ : \ (a, x) \subset \mathcal{O}\} \quad \text{and} \quad b_x = \inf\{b > x \ : \ (b, x) \subset \mathcal{O}\}$$

then $a_x < x < b_x$. Let $I_x = (a_x, b_x)$. Then by construction since $x \in I_x$ and $x \in \mathcal{O}$, we have

$$\mathcal{O} = \bigcup_{x \in \mathcal{O} I_x}$$

Next we will show that any two distinct intervals must be disjoint. To see why, suppose that I_x and I_y are distinct and intersect. Since I_x is maximal, then $I_x \cup I_y) \subset I_x$ and $I_x \cup I_y) \subset I_y$. This can only be true when $I_x = I_y$, but this is a contradiction that the intervals intersect.

To show that these intervals are countable, note that every open interval I_x contains a rational number. Since the intervals are disjoint, they must contain distinct rationals. $\square$

2.1.2 Properties of Rectangles

By using sets constructed of almost-disjoint cubes, Theorem 1.3 can be generalized to d dimensions. An **open** rectangle is the product of d open intervals

$$R = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d)$$

and a union of rectangles is said to be **almost disjoint** if the interiors of the rectangles are disjoint. A **cube** Q is a special type of rectangle for which $b_1 - a_1 = b_2 - a_2 = \cdots = b_d - a_d$. If $Q \subset \mathbb{R}^d$ is a cube of common length l, then $|Q| = l^d$. There are two important properties of rectangles that are given as lemmas and proofs can be found in Stein, Shakarchi Lemma 1.1, 1.2 respectively.

Rectangles are **countably sub-additive**. More precisely, if $R, R_1, \ldots, R_N$ are rectangles (not necessarily disjoint) and $R \subset \sum_{k=1}^{N} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R^k|$$

[1] Theorem 1.3 - Stein, Shakarchi

However almost disjoint rectangles are **countably additive**. More precisely, if a rectangle is the almost disjoint union of finitely many other rectangles such that $R = \bigcup_{k=1}^{N} R_k$, then

$$|R| = \sum_{k=1}^{N} |R^k|$$

2.1.3 Countable Union of Disjoint Open Intervals in $\mathbb{R}^d$

Here is the presentation of Theorem 1.3 that is generalized to higher dimensions.

Theorem 1.4 - Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.

Proof: We need to construct a countable collection $\mathcal{Q}$ of closed cubes with disjoint interiors so that

$$\mathcal{O} = \bigcup_{Q \in \mathcal{Q}} Q$$

Consider the grid of lines parallel to the axes, which is the grid generated by the lattice $\mathbb{Z}^d$. This forms a natural collection of closed cubes Q of side lengths 1 with integer vertex coordinates.

Now we make a decision to accept, tentatively accept, or reject each cube Q as part of $\mathcal{Q}$.

(i) If $Q \subset \mathcal{O}$, then accept.

(ii) If $Q \cap \mathcal{O}$ and $Q \cap \mathcal{O}^c$, then tentatively accept.

(iii) If $Q \subset \mathcal{O}^c$, then reject.

Next bisect the tentatively accepted cubes into 2^d cubes with side length $1/2$ and by repeating the decision procedure indefinitely, the collection $\mathcal{Q}$ of all accepted cubes is countable and consists of almost disjoint cubes.

To see why the union of the accepted cubes is $\mathcal{O}$, we note that given $x \in \mathcal{O}$, there exists a cube of side length 2^{-N} that contains x and is entirely contained in $\mathcal{O}$. Either the cube has been accepted or was previously accepted. Then using the same argument from Theorem 1.3, we argue that the union of all cubes in $\mathcal{Q}$ covers $\mathcal{O}$. $\square$

2.1.4 Properties of the Exterior Measure

If E is any subset of $\mathbb{R}^d$, the exterior measure of E is

$$m^*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

and $E \subset \bigcup_{j=1}^{\infty} Q_j$. More precisely, the infimum is taken over all the closed cubes Q_j that cover E. There are five key properties of the exterior measure.

Property 1: Monotonicity - If $E_1 \subset E_2$, then $m^*(E_1) \leq m^*(E_2)$.

Observe that any covering of E_2 by a countable collection of cubes is also a covering of E_1. Monotonicity implies that every bounded subset of $\mathbb{R}^d$ has finite exterior measure.

Property 2: Countable sub-additivity - If $E = \bigcup_{j=1}^{\infty} E_j$, then $m^*(E) \leq \sum_{j=1}^{\infty} m^*(E_j)$.

First we assume that each $m^*(E_j) < \infty$. Otherwise the inequality holds trivially. For any $\epsilon > 0$, the definition of the exterior measure yields, for each j, a covering $E_j \subset \bigcup_{k=1}^{\infty} Q_{k,j}$ by closed cubes with

$$\sum_{k=1}^{\infty} |Q_{k,j}| \leq m^*(E_j) + \frac{\epsilon}{2^j}.$$

Then $E \subset \bigcup_{k=1}^{\infty} Q_{k,j}$ is a covering of E by closed cubes. Therefore we have

$$\begin{aligned} m^*(E) \leq \sum_{j,k} |Q_{j,k}| &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |Q_{k,j}| \\ &\leq \sum_{j=1}^{\infty} \left(m^*(E(j) + \frac{\epsilon}{2^j} \right) \\ &= \sum_{j=1}^{\infty} m^*(E_j) + \epsilon. \end{aligned}$$

Since this holds true for every $\epsilon > 0$, the second observation is shown.

Property 3: Exterior Measure is infimum of all open sets containing E - If $E \subset \mathbb{R}^d$, then $m^*(E) = \inf(m^*(\mathcal{O})$.

By monotonicity, it is clear that the inequality $m^*(E) \leq \inf m^*(\mathcal{O})$ holds. For the reverse inequality $\inf m^*(\mathcal{O} \leq m^*(E)$, let $\epsilon > 0$ and choose cubes Q_j such that $E \subset \bigcup_{j=1}^{\infty} Q_j$ and $\sum_{j=1}^{\infty} |Q_j| \leq m^*(E) + \epsilon/2$. Let Q_j^O denote an open cube containing Q_j such that $|Q_j^O \leq |Q_j| + \epsilon/2^{j+1}$. Then $\mathcal{O} = \bigcup_{j=1}^{\infty} Q_j^o$ is open. By countable sub-additivity have the following inequality

$$\begin{aligned} m^*(\mathcal{O}) \leq \sum_{j=1}^{\infty} m^*(Q_j^O) &= \sum_{j=1}^{\infty} m^*(Q_j^O) \\ &\leq \sum_{j=1}^{\infty} \left(|Q_j| + \frac{\epsilon}{2^{j+1}} \right) \\ &\leq \sum_{j=1}^{\infty} |Q_j| + \frac{\epsilon}{2} \\ &\leq m^*(E) + \epsilon. \end{aligned}$$

Thus $\inf m * (\mathcal{O}) \leq m * (E)$ as desired. □

Property 4: Sum of exterior measure equals its parts - If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$, then $m^*(E) = m^*(E_1) + m^*(E_2)$.

By countable sub-additivity, we know that $m*(E) \leq m^*(E_1) + m^*(E_2)$ so we need to show the reverse inequality $m^*(E_1) + m^*(E_2) \leq m*(E)$. First select δ such that $d(E_1, E_2) > \delta > 0$. Next, choose a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes with

$$\sum_{j=1}^{\infty} |Q_j| \leq m^*(E) + \epsilon.$$

Subdivide the cubes Q_j until each Q_j has a diameter less than δ. In this case, each Q_j can intersect at most one of the two sets E_1 or E_2. If we denote J_1 and J_2 the sets of those indices j for which Q_j intersects E_1 and E_2 respectively, then $J_1 \cap J_2$ is empty and we have

$$E_1 \subset \bigcup_{j\in J_1}^{\infty} Q_j \quad \text{and} \quad E_2 \subset \bigcup_{j\in J_2}^{\infty} Q_j.$$

Therefore we have

$$\begin{aligned} m^*(E_1) + m^*(E_2) &\leq \sum_{j\in J_1} |Q_j| + \sum_{j\in J_2} |Q_j| \\ &\leq \sum_{j=1}^{\infty} |Q_j| \\ &\leq m^*(E) + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we have $m^*(E_1) + m^*(E_2) \leq m*(E)$.

Property 5: Countable additivity of the union of almost disjoint cubes - If a set E is the countable union of almost disjoint cubes $E = \bigcup_{j=1}^{\infty} Q_j$, then $m^*(E) = \sum_{j=1}^{\infty} |Q_j|$.

Let $\widetilde{Q}_j$ denote a cube strictly contained in Q_j such that $|Q_j| \leq |\widetilde{Q}_j + \epsilon/2^j$, where ϵ is arbitrary but fixed. Then, for every N, the cubes $\widetilde{Q}_1, \widetilde{Q}_2, \dots, \widetilde{Q}_N$ are disjoint, hence at a finite distance from one another. With repeated applications of property 4, we have

$$m^*\left(\bigcup_{j=1}^{N} \widetilde{Q}_j\right) = \sum_{j=1}^{N} |\widetilde{Q}_j| \geq \sum_{j=1}^{N}(|Q_j| - \frac{\epsilon}{2^j}).$$

Since $\bigcup_{j=1}^{N} \widetilde{Q}_j \subset E$, we conclude that for every integer N,

$$m^*(E) \geq \sum_{j=1}^{N} |Q_j| - \epsilon.$$

As $N \to \infty$, for every $\epsilon > 0$, we have $\sum_{j=1}^{\infty} |Q_j| \leq m^*(E)$. Combine $\sum_{j=1}^{\infty} |Q_j| \leq m^*(E)$ with property 2 and the equality is obtained.

Despite properties 4 and 5, one cannot conclude in general that if $E_1 \cup E_2$ is a disjoint union of subsets of $\mathbb{R}^d$, then $m^*(E_1 \cup E_2) = m^*(E_1) + m^*(E_2)$.

2.2 Measurable Sets and Lebesgue Measure

A subset $E \in \mathbb{R}^d$ is **Lebesgue measurable** or simply **measurable** if for any $\epsilon > 0$, there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and

$$m^*(\mathcal{O} - E) \leq \epsilon.$$

If E is measurable, we define its **Lebesgue measure** by

$$m(E) = m^*(E).$$

We find that the family of measurable sets is closed under the familiar operations of set theory. Measurable sets are also closed under countable unions and intersections.

Property 1: Every open set in $\mathbb{R}^d$ is measurable

Property 2: If $M * (E) = 0$, then E is measurable. In particular, if F is a subset of a set of exterior measure 0, then F is measurable.

By property 3 of the exterior measure, for every $\epsilon > 0$, there exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $M * (\mathcal{O}) \leq \epsilon$. Since $(\mathcal{O} - E) \subset \mathcal{O}$, monotonicity implies $m^*(\mathcal{O} - E) \leq \epsilon$.

Property 3: A countable union of measurable sets is measurable.

Suppose $E = \bigcup_{j=1}^{\infty} E_j$, where each E_j is measurable. Given $\epsilon > 0$, we may choose, for each j, an open set $\mathcal{O}_j$ such that $E_j \subset \mathcal{O}_j$ and $m^*(\mathcal{O}_j - E_j) \leq \epsilon/2^j$. Then the union $\mathcal{O} = \bigcup_{j=1}^{\infty} \mathcal{O}_j$ is open, $E \subset \mathcal{O}$, and $(\mathcal{O} - E) \subset \bigcup_{j=1}^{\infty}(\mathcal{O}_j - E_j)$. Then monotonicity and sub-additivity properties of the exterior measure imply

$$m^*(\mathcal{O} - E) \leq \sum_{j=1}^{\infty} m^*(\mathcal{O}_j - E_j) \leq \epsilon.$$

Property 4: Closed sets are measurable. First, we observe that it suffices to prove that compact

sets are measurable. More precisely, any closed set F can be written as the union of compact sets $F = \bigcup_{k=1}^{\infty} F \cap B_k$, where B_k denotes the closed ball of radius k centered at the origin so we can apply property 3.

Suppose F is compact. More precisely suppose $m * (F) < \infty$ and let $\epsilon > 0$. By property 3 of the exterior measure we can select an open set $\mathcal{O}$ with $F \subset \mathcal{O}$ and $m * (\mathcal{O}) \leq m * (F) + \epsilon$. Since F is closed, the difference $\mathcal{O} - F$ is open. Thus by Theorem 1.4, we have the equivalency

$$\mathcal{O} - F = \sum_{j=1}^{\infty} Q_j.$$

For a fixed N, the finite union $K = \bigcup_{k=1}^{N} Q_j$ is compact; therefore $d(K, F) > 0$. Since $(K \cup F) \subset \mathcal{O}$, properties 1,4, and 5 of the exterior measure imply

$$\begin{aligned} m^*(\mathcal{O}) &\geq m^*(F) + m^*(K) \\ &= m^*() + \sum_{k=1}^{N} m^*(Q_j) \end{aligned}$$

Thus $\sum_{k=1}^{N} m^*(Q_j) \leq m^*(\mathcal{O}) - m^*(F) \leq \epsilon$ as $N \to \infty$. Then by sub-addivity of the exterior measure, we have

$$m^*(\mathcal{O} - F) \leq \sum_{j=1}^{\infty} m^*(Q_j) \leq \epsilon.$$

Property 5: The complement of a measurable set is measurable.

If E is measurable, then for every positive integer n we may choose an open set $\mathcal{O}_n$ with $E \subset \mathcal{O}_n$ and $m^*(\mathcal{O}_n - E) \leq 1/n$. The complement $\mathcal{O}_n^c$ is closed, hence measurable, which implies that the union $S = \sum_{n=1}^{\infty} \mathcal{O}_n^c$ is also measurable by property 3. Now note that $S \subset E^c$ and $(E^c - S) \subset (\mathcal{O}_n - E)$, such that $m^*(E^c - S) \leq 1/n$ for all n. Then $m^*(E^c - S) = 0$ and $E^c - S$ is measurable by property 2. Thus E^c is measurable since $E^c = S + (E^c - S)$.

Property 6: A countable intersection of measurable sets is measurable.

This follows from properties 3 and 5 since

$$\bigcap_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} E_j \right)^c.$$

2.2.1 Countable Additivity of Disjoint Measurable Sets

Theorem 3.2[2] - If $E_1, E_2, \dots$ are disjoint measurable sets, and $E = \bigcup_{j=1}^{\infty} E_j$, then

$$m(E) = \sum_{j=1}^{\infty} m(E_j)$$

Proof: Since the inequality $\sum_{j=1}^{\infty} m(E_j) \leq m(E)$ holds by countable subadditivity. To show the reverse inequality $m(E) \geq \sum_{j=1}^{\infty} m(E_j)$, first assume that each E_j is bounded. Then, for each j, by the definition of measurability to E_j^c, we can choose a closed subset F_j of E_j with $m^*(E_j - F_j) \leq \epsilon/2^j$. For each fixed N, observe that the sets $F_1, \dots, F_N$ are compact and disjoint so that $m\left(\bigcup_{j=1}^{N} F_j = \sum_{j=1}^{N} m(F)j \right)$. Since $\bigcup_{j=1}^{N} F_j \subset E$, we have

$$m(E) \geq \sum_{j=1}^{N} m(F_j) \geq \sum_{j=1}^{N} m(E_j) - \epsilon.$$

[2] Stein, Shakarchi

As $N \to \infty$ and since ϵ was arbitrary, we have

$$m(E) \geq \sum_{j=1}^{\infty} m(E_j) \quad \square$$

Note that countable additivity can be shown in $\mathbb{R}^d$ by modifying the argument to consider closed cubes. A corollary to Theorem 3.2 are the following. Suppose $E_1, E_2, \dots$ are measurable sets of $\mathbb{R}^d$.

(i) If $E_k \nearrow E$, then $m(E) = \lim_{n\to\infty}(E_N)$

(ii) if $E_k \searrow E$ and $m(E_k) < \infty$ for some k, then $m(E) = \lim_{N\to\infty} m(E_N)$

The next theorem describes the relation of measurable sets to open and closed sets. More precisely, an arbitrary measurable set an be well approximated by the open sets that contain it, and alternatively, by the closed sets it contains.

2.2.2 Approximation of Measurable Sets by Open/Closed Sets

Theorem 3.4[3] - Suppose E is a measurable subset of $\mathbb{R}^d$. Then for every $\epsilon > 0$,

(i) There exists an open set $\mathcal{O}$ with $E \subset \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$.

(ii) There exists a closed set F with $F \subset E$ and $m(E - F) \leq \epsilon$.

(iii) If $m(E)$ is finite, there exists a compact set K with $K \subset E$ and $m(E - K) \leq \epsilon$.

(iv) If $m(E)$ is finite, then there exists a finite union $F = \bigcup_{j=1}^{N} Q_j$ of closed cubes such that $m(E \bigtriangleup F) \leq \epsilon$.

The notation $E \bigtriangleup F$ is the **symmetric difference** between sets E and F, which consists of points that belong to only one of the two sets E or F and is defined as

$$E \bigtriangleup F = (E - F) \cup (F - E)$$

Proof: (i) is the definition of measurability. For (ii), we know that E^c is measurable, so there exists an open set $\mathcal{O}$ with $E^c \subset \mathcal{O}$ and $m(\mathcal{O} - E^c) \leq \epsilon$. Then let $F = \mathcal{O}^c$. For (iii), pick a closed set F so that $F \subset E$ and $m(E - F) \leq \epsilon/2$. For each n, let B_n denote the ball centered at the origin of radius n and define compact sets $K_n = F \cap B_n$. Then $E - K_n$ is a sequence of measurable sets that decreases to $E - F$ since $m(E) < \infty$. Thus for all large n we have $m(E - K_n) \leq \epsilon$.

For (iv), choose a family of closed cubes $\{Q_j\}_{j=1}^{\infty}$ so that

$$E \subset \bigcup_{j=1}^{\infty} Q_j \quad \text{and} \quad \sum_{j=1}^{\infty} |Q_j| \leq m(E) + \epsilon/2$$

[3] Stein, Shakarchi

Since $m(E) < \infty$, the series converges and there exists $N > 0$ such that

$$\sum_{j=N+1}^{\infty} |Q_j| < \epsilon/2.$$

If $F = \bigcup_{j=1}^{N} Q_j$, then we have

$$\begin{aligned}
m(E \triangle F) &= m(E - F) + m(F - E) \\
&\leq m\left(\bigcup_{j=N+1}^{\infty} Q_j\right) + m\left(\bigcup_{j=1}^{\infty} Q_j - E\right) \\
&\leq \sum_{N+1}^{\infty} |Q_j + \sum_{j=1}^{\infty} |Q_j| - m(E) \\
&\leq \epsilon \quad \square
\end{aligned}$$

2.2.3 Invariance Properties

The **translation-invariance** property of Lebesgue measure in $\mathbb{R}^d$ is stated as follows: if E is a measurable set and $h \in \mathbb{R}^d$, then the set $E_h = E + h = \{x + h \ : \ x \in E\}$ is also measurable, and $m(E + h) = m(E)$. Observe that this holds for the special case when E is a cube and then observe that this should hold for the exterior measure of arbitrary sets E. To prove the measurability of E_h, under the assumption that E is measurable, note that if $\mathcal{O}$ is open, $E \subset \mathcal{O}$, and $m^*(\mathcal{O} - E) < \epsilon$, then $\mathcal{O}_h$ is open, $E_h \subset \mathcal{O}_h$, and $m^*(\mathcal{O}_h - E_h) < \epsilon$.

We can prove the **dilation-invariance** of Lebesgue measure in the same way. Suppose $\delta > 0$, and denote $\delta E = \{\delta x \ : \ x \in E\}$. We can then assert that δE is measurable whenever E is and $m(\delta E) = \delta^d m(E)$. It is also easy to see that Lebesgue measure is **reflection-invariant**. More precisely, whenever E is measurable, so is $-E = \{-x \ : \ x \in E\}$ and $m(-E) = m(E)$.

2.2.4 σ-algebras and Borel Sets

A **σ-algebra** of sets, denoted Ω, is a collection of subsets of $\mathbb{R}^d$ that is closed under countable unions, countable intersections, and complements. A **σ-field** of measurable subsets of a set Ω is a collection $\mathcal{F}$ of subsets of Ω that has $\varnothing$ as a member and is also closed under complementation and countable unions.

Borel σ-algebra is the smallest σ-algebra that contains all open sets. Elements of this σ-algebra are called **Borel Sets**.

2.3 Exercises

2.3.1 Exercise 1 - Cantor sets are perfect sets

Discussion: This exercise shows that Cantor sets are perfect sets. A **perfect set** is a closed set such that every single point in the set is a limit point of the set. In $\mathbb{R}^n$, any perfect set has an uncountable number of points. This can be shown by taking increasingly smaller neighborhoods about a point in the set.

Exercise 1: Prove that the Cantor set C constructed in the text is totally disconnected and perfect. In other words, given two distinct points $x, y \in C$, there is a point $z \notin C$ that lies between x and y, and yet C has no isolated points. *Hint:* If $x, y \in C$ and $|x-y| > 1/3^k$, then x and y belong to two different intervals in C_k. Also, given any $x \in C$, there is an end point y_k of some interval C_k that satisfies $x \neq y_k$ and $|x - y_k| \leq 1/3^k$.

Solution: If $x, y \in C$ and $|x-y| > 1/3^k$, then x and y belong to two different intervals in C_k because by construction, C_k is a disjoint union of 2^k intervals of length $1/3^k$. We see that any two points in C are disconnected because as $k \to \infty$, $|x-y| > 0$. Given any $x \in C$, there is an endpoint y_k of some interval in C_k such that $x \neq y_k$ and $|x-y_k| \leq 1/3^k$ because $C_0 \supset C_1 \supset \cdots \supset C_k$. Thus for every point $x \in C$, there exists a corresponding limit point $y_k \in C$. Thus we conclude that the Cantor set is a totally disconnected, perfect set.

2.3.2 Exercise 3 - Cantor sets of constant dissection

Discussion: This exercises focuses on Cantor sets with a different fixed dissection length. I think the key observation is that the Cantor set can be constructed with any geometric progression with $0 < \xi < 1$.

Exercise 3: Consider the unit interval $[0, 1]$ and let ξ be a fixed real number with $0 < \xi < 1$ (the case $\xi = 1/3$ corresponds to the Cantor set C in the text). In stage 1 of the construction, remove the centrally situated open interval in $[0, 1]$ of length ξ. In stage 2, remove two central intervals each of relative length ξ, one in each of the remaining intervals after stage 1, and so on.

Solution: Let C_ξ denote the set which remains after applying the procedure indefinitely.

(a) Prove that the complement of C_ξ in $[0, 1]$ is the union of open intervals of total length equal to 1.

(b) Show directly that $m^*(C_\xi) = 0$.

(a) The Cantor set can be generalized by removing an interval of relative length ξ. After the first step, we remove the open interval $\Big((1-\xi)/2, (1+\xi)/2\Big)$, leaving a length of $1-\xi$ in C_ξ. At each k step, an interval of relative length ξ is $\xi \cdot (1-\xi)/2$. Thus over k steps, we remove $2^k \xi \cdot (1-\xi)^k/2^k = \xi(1-\xi)^k$. As $k \to \infty$ and by the geometric progression, we have

$$\xi + (\xi(1-\xi) + \xi(1-\xi)^2 + \cdots + \xi(1-\xi)^k = \xi \cdot \frac{1-(1-\xi)^{k+1}}{1-(1-\xi)} = 1 - (1-\xi)^{k+1} = 1.$$

Since every set we removed was an open set, the complement of C_ξ is a union of open sets with length 1.

(b) Since C_ξ are disjoint, by property 4 of the exterior measures, we have

$$\begin{aligned} m([0,1]) &= m(C_\xi) + m(C_\xi^c) \\ 1 &= m(C_\xi) + 1 \\ 0 &= m(C_\xi). \end{aligned}$$

2.3.3 Exercise 5 - Topology of measurable sets

Discussion: This is a topology based exercise and how it influences measurability.

Exercise 5: Suppose E is a given set, and $\mathcal{O}_n$ is the open set:

$$\mathcal{O}_n = \{x : d(x, E) < 1/n\}.$$

Show

(a) If E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n$.

(b) However, the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

Solution:

(a) Note that $E \subset \mathcal{O}_N \subset \mathcal{O}_n$. As $n \to \infty$, there exists $N \geq n$ such that $m(E - \mathcal{O}_N) < \epsilon$. Thus $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$ as desired because ϵ is arbitrary.

(b) Suppose E is the closed and unbounded set of all integers $\mathbb{Z}$. Then $m(E) = \infty$, but $\lim_{n\to\infty} m(\mathcal{O}_n = 0$.

Let E be $\mathbb{Z}^+$ such that $E = \{x \in \mathbb{Z}^+ \ : \ x < \infty\}$. Thus E is open and bounded by 0 and ∞. Clearly $m(E)$ equals some positive number while $\lim_{n\to\infty} m(\mathcal{O}_n) = 0$.

2.3.4 Exercise 13: G_δ and F_δ sets

Discussion: I don't really know anything about G_δ and F_δ except that maybe they are used more in topology?

Exercise 13:

(a) Show that a closed set is a G_δ and an open set an F_δ. *Hint: If F is closed, consider* $\mathcal{O}_n = \{x : d(x, F) < 1/n\}$.

(b) Given an example of an F_δ that is not a G_δ.

(c) Give an example of a Borel set which is not a G_δ nor a F_δ

Solution:

(a) Let $A \subseteq X$ be closed. For all $n \in \mathbb{N}$, define

$$U_n = \bigcup_{a \in A} B(a, 1/n)$$

where U_n is open as a union of open balls. We will now show that $A = \bigcup_{n\in\mathbb{N}} U_n$. In one direction, clearly $A \subseteq \bigcup_{n\in\mathbb{N}} U_n$. To prove $A \supseteq \bigcap_{n\in\mathbb{N}} U_n$, we take $x \in A^c$ and show that $x \notin \bigcap U_n$. Since A is closed, A^c is open. Therefore there exists $n \in \mathbb{N}$ such that $B(x, 1/n) \bigcap A = \emptyset$. More precisely, for all $a \in A \ : \ a \notin B(x, 1/n)$, we have $x \notin B(a, 1/n) \implies x \notin \bigcup_{a\in A} B(a, 1/n) \implies x \notin U_n \implies x \notin \bigcap U_n$ as desired. **(Why do we conclude this? Not entirely clear to me)**

(b) ??? 6.23.20

(c) ??? 6.23.20

2.3.5 Exercise 16 - Borel-Cantelli Lemma

Discussion: The Borel-Cantelli lemma says that under a suitable decay condition on the probability of E_k (namely convergence in infinite series), the probability of the event E, which is the event that an infinite amount of events E_k occur simultaneously, is 0.

Exercise 16: Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$E = \{x \in \mathbb{R}^d \ : \ x \in E_k, \text{ for infinitely many k} = \lim_{k\to\infty} \sup(E_k).$$

(a) Show that E is measurable.

(b) Prove that $m(E) = 0$.

Solution:

(a) We prove (a) by showing that

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k.$$

Then we can make the conclusion that E is measurable because for every fixed n, the countable union of E_k sets is measurable. Then E would be the countable intersection of measurable sets. Suppose $x \in E$. Then $x \in E_k$ for infinitely many k. Thus x is in the intersection of the countable union of E_k and we have $x \in \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$. Suppose $x \in \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$. Note that $\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k \subset \bigcup_{k\geq n} E_k$. Thus $x \in \bigcup_{k\geq n} E_k$ for infinitely many k and $x \in E$ as desired and we see the equality holds.

(b) Since E is measurable, we have $m(E) \leq \sum_{k=1}^{\infty} m(E_k) < \infty$. By the monotonicity property, there exists $n \geq k$ such that $| \sum_{k=n}^{\infty} m(E_k)| \leq \epsilon$. Thus $m(E) = 0$.

2.3.6 Exercise 19 - Set operation properties of $A + B$

Discussion: The following exercises demonstrate how measurable properties may differ from basic set operation properties.

Exercise 19: Here are some observations regarding the set operation $A + B$.

(a) Show that if either A and B is open, then $A + B$ is open.

(b) Show that if A and B are closed, then $A + B$ is measurable.

(c) Show, however, that $A + B$ might not be closed even though A and B are closed.

Solution:

(a) Suppose A and B are open. Then note that $A + B$ is the union of translates of an open set, which is a union of open sets so we have

$$A + B = \bigcup_{a \in A}(a + B) = \bigcup_{a \in A}(a + \bigcup_{b \in B} b)$$

(b) Suppose A and B are closed. Note that $A + B$ is also the union of translates of a closed set, which is a union of closed sets, which by definition is an F_σ-set. Then recall that Corollary 3.5 states that a subset E of $\mathbb{R}^d$ is measurable if and only if E differs from an F_σ by a set of measure zero. Let $E = A + B$ and we are done.

(c) in $\mathbb{R}$, let $A = \mathbb{Z}$ and $B = a\mathbb{Z}$, where a is an irrational number. Both of the sets are closed, but their sum is not closed because it does the sum does not contain $\mathbb{Q}$. Another example is to let A be the set of negative integers. Let B be the set of all $n + 1/2^n$, where n ranges over the positive integers. Then both A and B are closed, but $A + B$ is not closed because it does not contain 0.

2.3.7 Exercise 20 - The sum of measurable sets is not always equal to its parts.

Discussion: The preliminary question we pose is whether one can establish any general estimate for the measure of $A + B$ in terms of the measures of A and B. This exercise shows that it is not possible to obtain an upper bound for $m(A + B)$ in terms of $m(A)$ and $m(B)$.

Exercise 20: Show that there exist closed set A and B with $m(A) = 0 = m(B)$, but $m(A + B) > 0$.

Solution: In general, lines in $\mathbb{R}^2$ can be represented as rectangles in $\mathbb{R}^2$ with side length of 0. Thus $m(A) = 0$ and $m(B) = 0$ for two lines in $\mathbb{R}^2$. For $I = [0, 1]$, if $A = I \times \{0\}$ and $B = \{0\} \times I$, then $A + B = \{(x, y) \in \mathbb{R}^2 \ : \ x \in A, y \in B\}$ for $x, y \in [0, 1]$. Thus $m(A + B)$ is the closed cube with sides of length 1. Clearly $[0, 1] \subset A + B \in \mathbb{R}^2$.

2.3.8 Exercise 34 - The Cantor Function

Discussion: This exercise focuses on the function F between different bisections of a cantor set. (i) shows that F is a measurable, bijective function (all continuous functions are measurable, but the converse is not true). (ii) is a property of measurable functions and (iii) is a property of Cantor sets.

Exercise 34: Let C_1 and C_2 be any two Cantor sets of constant dissection (constructed in Exercise 3). Show that there exists a function $F : [0,1] \to [0,1]$ with the following properties:

(i) F is continuous and bijective

(ii) F is monotonically increasing

(iii) F maps C_1 surjectively onto C_2.

Solution:

(i) C is made up of closed, bounded intervals. By the Heine-Borel Theorem, C is compact. C^c is also compact because every open cover of C^c has a finite subcover. Thus we have a surjective, continuous function $F : [0,1] \to [0,1]$ that maps each element $x \in [0,1]$ to C or C^c. Since C and C^c are disjoint, F is injective so we conclude that F is also bijective.

(ii) To see why F is monotonically increasing, note that each interval in C and C^c are non-negative.

(iii) Recall that the cantor set is defined as $C = \bigcup_{k=0}^{\infty} C_k$ and $C_1 \supset C_2 \supset \cdots \supset C_k$, which implies that F maps C_1 surjectively to C_2. Note however that this does not imply that C^c is surjective.

2.3.9 Exercise 35 - Completing the σ-algebra of Borel Sets

Discussion: The Borel σ-algebra is contained in the σ-algebra of measurable sets. However the inclusion is not strict because there exists Lebesgue measurable sets which are not Borel sets. The observation here is that null sets have Lebesgue measure 0, but are not Borel sets because they are not open. The Lebesgue measurable sets completes the σ-algebra of Borel sets.

Exercise 35: Give an example of a measurable function f and a continuous function ϕ so that $f \circ \phi$ is non-measurable. *Hint:* Let $\phi : C_1 \to C_2$, as in Exercise 34, with $m(C_1) > 0$ and $m(C_2) = 0$. Let $N \subset C_1$ be non-measurable and take $f = \chi_{\phi(N)}$.
Solution: Let ϕ be the continuous function F from exercise 34. Let $m(C_1) > 0$ and $m(C_2) = 0$ after a large number of k bissection steps. Since $m(C_1) > 0$, there exists non-measurable null sets $N \subset C_1$. Let f be the measurable characteristic function on n. Then $f \circ \phi$ is non-measurable because ϕ maps the null sets in C_1 to null sets in C_2, of which there are uncountably many of.

2.3.10 Exercises

Prove that if $m^*(A) = 0$, then for each B, $m^*(A \cup B) = m^*(B)$.[4] Well since $m^*(A) = 0$, then $\sum_{k=1}^{\infty} l(a_k) = 0$. Then we have

$$m^*(A \cup B) = \inf\left\{\sum_{k=1}^{\infty}[l(I_k^A) + l(I_k^B)]\right\} = \inf\left\{\sum_{k=1}^{\infty} l(I_k^B)\right\} = m^*(B) \quad \square$$

Prove that if $m^*(A \triangle B) = 0$, then $m^*(A) = m^*(B)$.[5]

$$\begin{aligned} &m^*(A \triangle B) = m^*(A) - m^*(A \cap B) + m^*(B) - m^*(A \cap B) = 0 \\ &\Leftrightarrow m^*(A + B) = 2m^*(A \cap B) \\ &\Leftrightarrow m^*(A) = m^*(A \cap B) = m^*(B) \quad \square \end{aligned}$$

Show that the outer measure is translation invariant, that is $m^*(A) = m^*(A + t)$ for any $A, t \in \mathbb{R}$.[6] First note that intervals are translation invariant so

$$\sum_{k=1}^{\infty} l[a, b] = \sum_{k=1}^{\infty} l[a + t, b + t]$$

By the definition of outer measure, it follows that $m^*(A) = m^*(A + t)$.

Suppose $m(A) = m(B)$. Then does $m(A \triangle B) = 0$?[7].

Suppose that A and B are disjoint and non-empty. Then $A \cap B = \varnothing$ and $m(A \cap B) = 0$. Then we have

$$m(A \triangle B) = \Big[m(A) - m(A \cap B)\Big] + \Big[m(B) - m(A \cap B)\Big] = m(A) + m(B)$$

Thus the converse does not hold.

Suppose A and B are not disjoint, not empty, and $A \cap B \neq \varnothing$. Then we have

$$\begin{aligned} m(A \triangle B) &= m(A) + m(B) - 2m(A \cap B) \\ &= 2m(A) - 2m(A \cap B) \\ \frac{1}{2}m(A \triangle B) &= m(A) - m(A \cap B) \geq 0 \end{aligned}$$

Since A is not empty and $A \cap B \neq \varnothing$, the right hand side is greater than 0 unless $A \cap B = A$. Thus we get the condition that if $A = B$, A and B are not disjoint and not empty, then $m(A \triangle B) = 0$.

[4] Exercise 2.4, Capinski
[5] Capinski, Exercise 2.5
[6] Proposition 2.6, Capinski
[7] Converse of Proposition 2.11, Capinski

3 Integration Theory

3.1 Properties and Convergence Theorems of Lebesgue Integral

3.2 Exercises

3.2.1 Exercise 1 - Every non-disjoint union of sets has a unique decomposition

Discussion: This is an elementary set theory exercise that shows that non-disjoint unions of sets can be decomposed into a mutually disjoint set (is this unique?).

Problem: Given a collection of sets $F_1, F_2, \dots, F_n$, construct another collection $F_1^*, F_2^*, \dots, F_N^*$ with $N = 2^n - 1$ so that

$$\bigcup_{k=1}^{n} F_k = \bigcup_{j=1}^{N} F_j^*,$$

the collection $\{F_j^*\}$ is disjoint, and

$$F_k = \bigcup_{F_j^* \subset F_k} F_j^*.$$

Solution: The first observation is that each F_k in $F_1, \dots, F_n$ can be decomposed into two mutually disjoint sets, F_k and the complement, F_k^c. Consider the intersection of these 2^n sets. Note that $F_1' \cap F_2' \cap \cdots \cap F_{2^n}' = \emptyset$. This implies that the union of these sets is also mutually disjoint. Take the union of these sets minus the null set to see there will be $2^n - 1$ sets remaining.

3.2.2 Exercise 10 - a.e. finite, non-negative, and disjoint functions have non-infinite measures.

Discussion: If we make some assumptions about functions such as being a.e. finite, being non-negative, and disjoint, then these functions should not have infinite measure.

Problem: Suppose $f \geq 0$, and let $E_{2^k} = \{x : f(x) > 2^k\}$ and $F_k = \{x : 2^k < f(x) \leq 2^{k+1}\}$. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\},$$

and the sets F_k are disjoint. Prove that f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \text{ if and only if } \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

Use this result to verify the following assertions. Let

$$f(x) = \begin{cases} |x|^{-a} \text{ if } |x| \leq 1, \\ 0 \text{ otherwise,} \end{cases} \quad \text{and } g(x) = \begin{cases} |x|^{-b} \text{ if } |x| > 1, \\ 0 \text{ otherwise.} \end{cases}$$

Then f is integrable on $\mathbb{R}^d$ if and only if $a < d$; also g is integrable on $\mathbb{R}^d$ if and only if $b > d$.
Solution: Observe each F_k is bounded (E^{2^k} is not bounded) and since f is a.e. finite, then $m(F_k) = 0$ because the measure of sets with finite values is 0. Thus

$$\sum_{k=-\infty}^{\infty} 2^k \cdot m(F_k) = 0 < \infty.$$

Since f is a.e. finite, then each E_{2^k} has a finite number of elements. Using the same reasoning as above, we get

$$\sum_{k=-\infty}^{\infty} 2^k \cdot m(E_{2^k}) = 0 < \infty.$$

The converse of the statement is proved in Proposition 1.6 (v).

If $a \geq d$, then $1/|x|^d$ blows up to infinity, but that contradicts what we just proved for functions that are a.e. finite. A similar verification works for g being integrable when $b > d$.

3.2.3 Exercise 6 - Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$ (uniform convergence).

Discussion: Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$ (uniform convergence). Intuitively, this makes sense because not all integral functions are uniform convergent or uniform continuous.

Problem: Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.

(a) There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but yet $\limsup_{x\to\infty} f(x) = \infty$.

(b) However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then $\lim_{|x|\to\infty} f(x) = 0$.

Hint: For (a), construct a continuous version of the function equal to n on the segment $[n, n + 1/n^3), n \geq 1$.
Solution: TBD

3.2.4 Exercise 9 - Tchebychev inequality

Discussion: No discussion yet

Problem: Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, prove that

$$m(E_\alpha) \leq \frac{1}{\alpha} \int f.$$

Solution: TBD

3.2.5 Exercise 12 - Needs a name

Discussion: No discussion yet

Problem: Show that there are $f \in \check{\ }^1(\mathbb{R}^d)$ and a sequence $\{f_n\}$ with $f_n \in \check{\ }^1(\mathbb{R}^d)$ such that

$$||f - f_n||_{L^1} \to 0,$$

but $f_n(x) \to f(x)$ for no x.

Hint: In $\mathbb{R}$, let $f_n = \chi_{I_n}$, where I_n is an appropriately chosen sequence of intervals with $m(I_n) \to 0$.

3.2.6 Problem 3 - Cauchy in Measure

Discussion: No discussion yet

Problem: A sequence $\{f_k\}$ of measurable functions on $\mathbb{R}^d$ is **Cauchy in measure** if for every $\epsilon > 0$,

$$m(\{x : |f_k(x) - f_l(x)| > \epsilon\}) \to 0, \text{ as } k, l \to \infty.$$

Solution: TBD

4 Lebesgue Differentiation Theorem

4.1 Vitali Covering Argument

This is a common argument in the theory of differentiation. Informally, the lemma states that we may always find a disjoint sub-collection of balls that covers a fraction of the region covered by the original collection of balls. Stated formally as Lemma 1.2 from Stein Chapter 3 Section 1:

Suppose $\mathcal{B} = \{B_1, B_2, \ldots, B_N\}$ is a finite collection of open balls in $\mathbb{R}^d$. Then there exists a disjoint sub-collection $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$ of $\mathcal{B}$ that satisfies

$$m\left(\bigcup_{l=1}^{N} B_l\right) \leq 3^d \sum_{j=1}^{k} m(B_{i_j}).$$

The argument given is constructive and relies on the following simple observation: Suppose B and B' are a pair of balls that intersect, with the radius of B' being not greater than that of B. Then B' is contained in the ball $\widetilde{B}$ that is concentric with B but with 3 times its radius.

As a first step, we pick a ball B_{i_1} in $\mathcal{B}$ with maximal (that is, largest) radius, and then delete from $\mathcal{B}$ the ball B_{i_1}, as well as any balls that intersect B_{i_1}. Thus all the balls that are deleted are contained in the ball $\widetilde{B_{i_1}}$, concentric with B_{i_1}, but with 3 times its radius.

The remaining balls yield a new collection $\mathcal{B}'$, for which we repeat the procedure. We pick B_{i_2} with largest radius in $\mathcal{B}'$, and then delete from $\mathcal{B}'$ the ball B_{i_2}

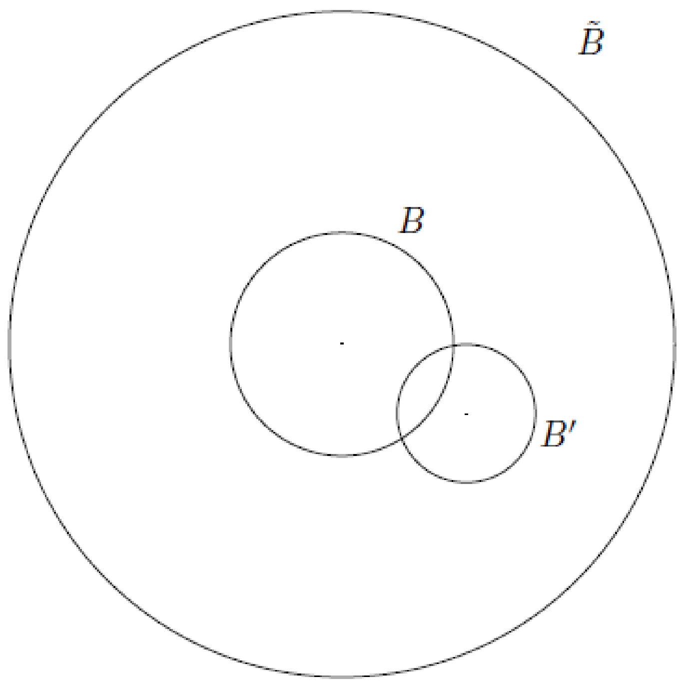

Figure 1. The balls B and $\tilde{B}$

Figure 1: Two concentric circles, figure made in Geogebra

5 Metric Spaces

5.0.1 K-Balls are Convex Sets

Given $x \in \mathbb{R}^k$ and $r > 0$, the **open ball** B is the set defined as

$$B := \{y \in \mathbb{R}^k \ : \ |y - x| < r\}$$

If B is a **closed ball**, then the inequality is no longer strict. A set $E \subset \mathbb{R}^k$ is said to be **convex** if for $x, y \in E$ and $0 < \lambda < 1$, then

$$\lambda x + (1 - \lambda)y \in E$$

Suppose B is an open ball, $x, y, z \in B$, and $0 < \lambda < 1$. Then B is a convex set.

$$\begin{aligned} |[\lambda y + (1 - \lambda)z] - x| &= |\lambda(y - x) + (1 - \lambda)(z - x)| \\ &\leq \lambda|y - x| + (1 - \lambda)|z - x| \\ &< \lambda r + (1 - \lambda)r = r \end{aligned}$$

To see why the first equality holds, factor out the left hand side and rearrange terms.