Pure Math Notes on Liquidity

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Contents

1 Intro

The concept of financial liquidity is considered using geometric based first principles to improve capital efficiency of decentralized finance primitives. Fundamental properties for these liquidity shapes are examined and used to expand the current understanding of liquidity.

This paper examines liquidity as a surface rather than a curve and establishes fundamental properties using modern concepts from algebraic topology, cohomology, and symplectic geometry. These ideas provide the language to talk about algebraic liquidity invariants, the geometric curvature of the differential manifolds, and the connection between continuous and discrete spaces.

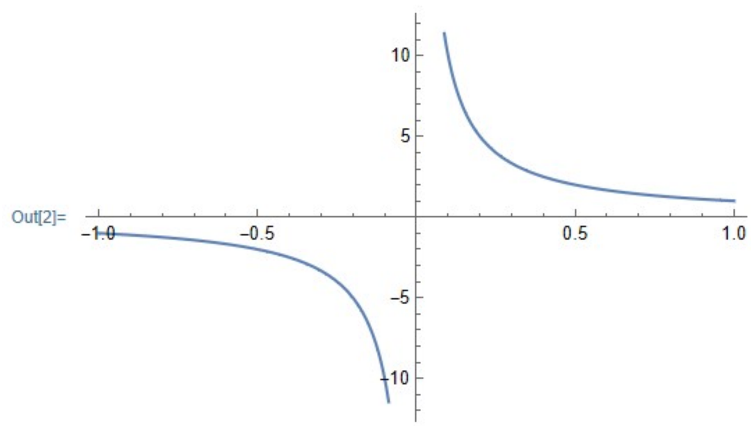
These properties expand on all of the existing DeFi innovations from concentrated liquidity designs Replicating Market Makers (Guillermo, ALex, and Tarun)[1] and Uniswap v3 (Hayden, Noah, River, Dan, Moody)[2] as well as research in perpetual, oracle-free options (Guillame, Jesper)[3].

2 The Shape of Liquidity

2.1 $xy = k$ as a Liquidity Curve in $\mathbb{R}^2$

Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x, y) = xy = k$, which is the unit liquidity curve in DeFi. f is widely used in decentralized exchanges to provide decentralized pools of liquidity for two token assets x and y. AMMs (Automated Market Makers) generate orders for traders by maintaining a scalar k. k represents the amount of reserves in a liquidity pool and only increases when additional reserves are added or when fees are enabled.

Plotting f in $\mathbb{R}^2$ by taking $y = \frac{1}{x}$, we have the following figure, which shows the price curve between swapping x for y.



The top right curve shows the ratio of reserves over time and is what is used in practice. Since liquidity pools are strictly positive, we can disregard the bottom price curve. Note as $x \to 0$, $y \to \infty$. Similarly as $y \to 0$, $x \to \infty$. Informally, the discontinuous points for x and y can be expressed as a 'class' of vectors $\{(x, 0), (0, \frac{1}{x})\}$. In an otherwise smooth space, how do we account for these points of discontinuity at 0 and ∞?

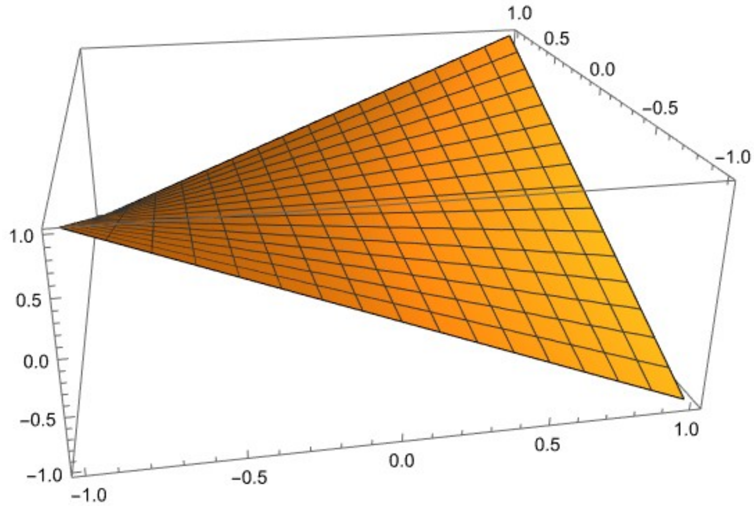
[1]https://arxiv.org/abs/2103.14769

[2]https://uniswap.org/whitepaper-v3.pdf

[3]https://docsend.com/view/ctkizkm97c6sdsaq

2.2 $xy = k$ as a Liquidity Surface in $\mathbb{R}^3$

Let us recast f as a unit liquidity surface by plotting $xy = k$ in $\mathbb{R}^3$. From an algebraic geometry point of view, these kinds of surfaces are called projective varieties.

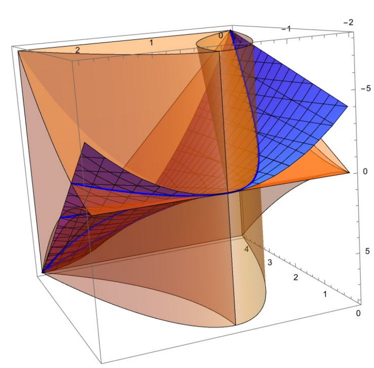


The first observation is that f becomes a smooth surface when projected into a higher dimension $\mathbb{R}^3$. The second observation is that we could informally describe f as a square with curvature.

2.2.1 Projective Varieties

A projective variety over an algebraically closed field k is a subset of a projective n-space $\mathbb{P}^n$ over k that is the zero-locus of some finite family of homogeneous $n + 1$ polynomials that generate a prime ideal.

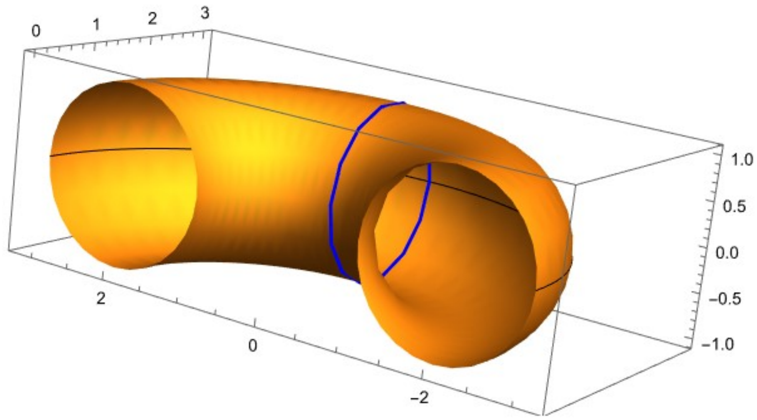
A projective variety is defined by a groebner basis, which is a group of symmetric polynomials. Below is an example of a liquidity surface $f(x, y, z) = (y - x^2) + (xy - z) = 0$ defined by its groebner basis. Conceptually a liquidity surface can be shaped into a space with curvature that maximizes



capital efficiency for any DeFi primitive.

2.3 $xy = k$ as a Liquidity Circle in 2-torus T^2

Using the same liquidity surface f from above, here is f on a 2-torus T^2. Since T^2 is homeomorphic to $\mathbb{R}^2/(\mathbb{Z} \times \mathbb{Z})$, which is almost $\mathbb{R}^2$ but with discontinuous points quotiented out of the space, all topological properties such as homotopy invariance transfer over to f.



3 T^2 Properties

In topology, T^2 is defined as the product of two circles $S^1 \times S^1$. Additionally T^2 is homotopy invariant as a path-connected space and with Euler characteristic zero but not as a compact manifold. T^2 is also a topological space with a single base point which implies that the 0th homotopy set of T^2 has a natural trivial group structure. The fundamental group of T^2 is

$$\pi_1(T^2) = \pi_1(S^1) \times \pi_1(S^1) \equiv \mathbb{Z} \times \mathbb{Z}$$

$\pi_1(T^2)$ is a free abelian group of finite rank 2. In algebraic topology, free abelian groups are used to define groups of chains. In algebraic geometry they are used to define divisors. Free abelian groups have many powerful properties:

1. **Universal Property** - For every function f from basis B to an abelian group A, there exists a unique group homomorphism from the free abelian group F to A which extends f. This implies that the abelian group of base B is unique up to isomorphism.

2. **Torsion-Free** All free abelian groups are torsion free. In differential geometry, the torsion of a curve measures how sharply a curve twists. Since curvature and torsion of a space curve are analogous to the curvature of a plane curve on the Euclidean plane and there is no torsion, we are left with exact measurements of curvature of the space.

3. **Symmetry** - The automorphism group of a free abelian group of finite rank n is the general linear group $GL(n\mathbb{Z})$, which can be described concretely as the set of $n \times n$ invertible integer matrices under matrix multiplication.

3.1 De Rham Cohomology of T^2

Cohomology is a general term for a sequence of abelian groups and is a tool in algebraic topology used to assign richer algebraic invariants to a space than homology, which was the primary method of constructing algebraic invariants of topological spaces in the 20th century. De Rham cohomology expresses topological information about smooth manifolds that is favorable for computation and concrete representation of cohomology classes with respect to differential forms. Differential forms provide a unified approach for measuring curves, surfaces, and solids in higher dimensions.

3.2 T^2 is Isomorphic to $\mathbb{R}^2$

It is known that the de Rham cohomology $H^1_{dR}(T^2)$ is isomorphic to $\mathbb{R}^2$. [4] This symplectomorphism between the shapes of circles and surfaces is explained in the next section but implies that the properties of curvature for liquidity surfaces is the same as a circle.

4 Symplectomorphisms

4.1 Symplectic Forms on $\mathbb{R}^2$

Given a 2-form $dx \wedge dy$ and 2 vectors $u, v \in \mathbb{R}^2$, we can compute the wedge product $dx \wedge dy$ as the jacobian determinant of two vectors.

$$dx \wedge dy(u, v) = dx(u)dy(v) - dy(u)dx(v) = u_1 v_2 - u_2 v_1 = \det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

The jacobian determinant is a special diffeomorphism used to determine the stability of equilibrium for systems of differential equations and is also symplectic.[5] Additionally symplectic matrices are nonsingular and skew-symmetric.

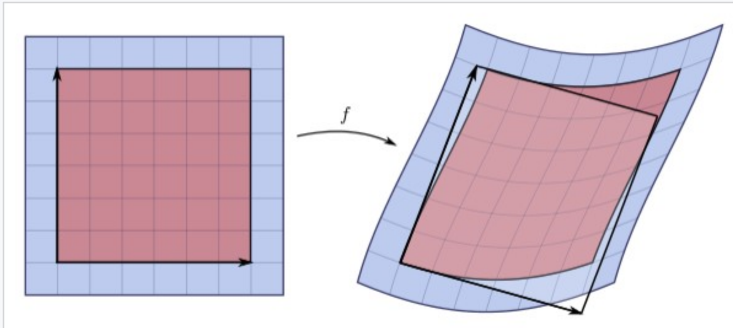
4.2 $dx \wedge dy$ is Symplectic

Symplectic manifolds are smooth manifolds equipped with a symplectic form. A symplectomorphism is a diffeomorphism but with stronger constraints that preserves volume among categories of symplectic manifolds such as $\mathbb{R}^2$ and T^2. On the 2-torus $(T^2, dx \wedge dy)$, the vector fields $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial y}$ are symplectic but not Hamiltonian[6]. In other words the symplectic form $dx \wedge dy$ is closed, but not exact.

4.3 Compute $dx \wedge dy$

Suppose a liquidity pool has $x = 1,000$ reserves and $y = 1000$ reserves where the exchange rate is $x = y = 1$. Let $u = \begin{bmatrix} 1000 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$. Then we have

[4] https://math.stackexchange.com/questions/3003608/first-de-rham-cohomology-group-of-the-2-torus
[5] https://math.stackexchange.com/questions/3283779/prove-a-change-of-coordinates-is-symplectic
[6] https://math.stackexchange.com/questions/3461865/mathbbs1-action-over-t2-is-symplectic-but-not-hamiltonian



A nonlinear map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ sends a small square (left, in red) to a distorted parallelogram (right, in red). The Jacobian at a point gives the best linear approximation of the distorted parallelogram near that point (right, in translucent white), and the Jacobian determinant gives the ratio of the area of the approximating parallelogram to that of the original square.

$$\det \begin{pmatrix} 1000 & 0 \\ 0 & 1000 \end{pmatrix} = u_1 v_2 - u_2 v_1 = 1,000,000$$

Suppose that we swap 50 x for 47.619 y where those amounts are based on the liquidity curve with no fee. Updating u and v, we have $u \begin{bmatrix} 1050 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 952.381 \end{bmatrix}$. Then

$$\det \begin{pmatrix} 1050 & 0 \\ 0 & 952.381 \end{pmatrix} = 1,000,000$$

This implies that k is precisely the jacobian determinant. Financial liquidity measures the volume of a smooth projective variety.

5 Ideas

5.1 Dynamic Liquidity Shapes

Let virtual liquidity surface be defined as f' with homotopy invariance to f. If $f' > f$, the liquidity surface becomes larger than the current liquidity curves used in industry today. This allows larger trades to be made with less slippage by imitating the price impact of a larger liquidity pool by adding a larger constant to the jacobian determinant calculation. A $1m$ liquidity pool could have the same slippage levels as a $1b$ liquidity pool. Leverage could also be considered as another use case for virtual liquidity by maintaining a specific shape over a finite duration of time and is left as a direction for future research.

5.2 Discrete Topology of Payoff Functions

1. discrete random variables?

2. how to compute discrete version of CDF?

There is an equivariant mapping between DEX trading curves and respective payoff functions. The trading curve → payoff function direction is well understood and widely used in practice today. The other direction, however, payoff → trading curve is less well understood. The best literature on replicating market makers (RMM Angeris et all March 2021) states that the payoffs that can be represented using the RMM methodology are limited to concave, nonnegative, nondecreasing functions of price.

The most recent implementation that involves constructing trading curves with respect to custom payoff functions to create new financial primitives has been RMM-01 by Primitive. A continuous CDF is discretized by taking the inverse image of the CDF and reversing the process from continuous to discrete. Although the implementation works in process, it is complex to understand and I believe we can derive a simpler differential equation that makes the computation from continuous to discrete space simpler as well as more accurate.

5.2.1 Payoff Function (Game Theory)

In game theory under normal form games, a payoff function for a player is a mapping from the cross-product of players' strategy spaces to the player's set of payoffs (normally assumed to be $\mathbb{R}$ of a player. For example the payoff function of a player takes a strategy profile as input and outputs the resulting yield as the output. Mathematically a payoff function is a function

$$u_i : S_1 \times S_2 \times \cdots \times S_I \to \mathbb{R}$$

for a finite set I of players where each player has a finite k number of available strategies.

5.2.2 Financial Derivatives

A derivative can be seen as a bet based on the behavior of the price movement of the underlying asset. An option is a derivative with a specified payoff function that depends on the price of one or more underlying assets. Options have specific dates that can be exercised and at specific values which is determined by the payoff function. A derivative contract is defined by

- payoff function $f(t, S(t))$
- purchase price f_0

The value of a derivative can be expressed as the present value of the expected future payoff, where the expectation is calculated in a risk neutral world.

One example of derivatives payoff function is the black scholes equation. It assumes the stock price follows a geometric brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

where $|mu$ is constant drift and σ is constant positive volatility and $S_0 \in \mathbb{R}^+$ is the initial stock price. W_t is a one dimensional brownian motion defined on a filtered probability space $(\Omega, P, \mathcal{F})$. Black Scholes equation is integrated by Ito formula.

Misc Remark: Feynman-Kac theorem provides a link between the portfolio replication approach and the risk-neutral valuation approach, according to which we can price a derivative by discounting it's expected future payoff in the risk-neutral world.

5.2.3 Space of Payoff Functions

Problem prompt - Clearly define what a payoff function is and then define the space of payoff functions. Some initial thoughts are

1. What is a payoff function?
2. What is the kernel of a payoff function?
3. what is the domain of an ideal payoff function? $(0, \infty) = \mathbb{R}_+^*$
4. If we consider the function $xy = k$ where xy is the trading curve and k is the payoff function, how do we derive an impermanent loss function as a function of k?

Let $x, y \in \mathbb{R}_+^*$[7] be random variables representing token reserves with $\mathbb{R}_+^* = \{x \in \mathbb{R} : x > 0\}$. Define $\varphi : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}^*$[8] as the payoff function. φ defines a financial derivative with binary operation $(*)$, assumed to be standard multiplication, as

$$\varphi(x, y) = x * y = k.$$

The goal of this exercise is to show that φ is well behaved with respect to the expected payout of the derivative functional. One way we can show that φ is well behaved is to show what functional space φ is contained in. We intuitively assume that φ can be defined in a reproducing kernel Hilbert space (RKHS) and will work towards proving this assumption. The claim is that by showing $\varphi \in H$ and H is RKHS, this will show that the derivative behaves well with respect to 'the greeks', which are a set of differential properties used to define and differentiate financial derivatives in mathematical finance.

A Hilbert space is a specific type of an inner product space. Inner product spaces are important because it satisfies conjugate symmetry, linearity, and positive definiteness[9].

To show that $\varphi \in H$, we will show that the properties of the dot product between x, y is satisfied, mainly

1. symmetric, satisfied because x, y are random variables and thus independent
2. linear in first argument, I believe this is verified because a payoff function is 1-homogeneous?
3. positive definite, I think this is most difficult property to show and requires careful reasoning.

By construction symmetry and linearity are satisfied because x, y are both random variables which implies linear independence. Given two linearly independent vectors, observe that the dot product is equivalent to taking the determinant of the matrix M:

$$\det(M) = \det \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = xy = k$$

for $k \in \mathbb{R}^+$.

How to show that φ is positive definite? We must show that the following conditions are satisfied:

[7] Note that $*$ means that the zero element is excluded.
[8] unsure if it should be $\mathbb{R}_+^*$ or $\mathbb{R}^*$
[9] Some work could be done here to weaken positive definiteness condition to positive semi-definiteness

1. M is congruent with a diagonal matrix with positive real entries.
2. M is symmetric or Hermitian, and all its eigenvalues are real and positive .
3. M is symmetric or Hermitian, and all its leading principal minors are positive.
4. There exists an invertible matrix B with conjugate transpose B^* such that $M = B^* B$

The first three conditions are immediate. For the last condition, we must reason carefully around when the dot product of M equals 0. In a sense this is a contradiction to the fact that x, y are random variables. However a payoff function will clearly have non-trivial 0 values such that $\varphi(x, y) = 0$ when $x \neq 0$ and $y \neq 0$ in practice. In order to satisfy this conundrum, we have to reason about the possible 0 value combinations before the dot product is computed. If these values are "quotiented" out, then the remaining range of φ will be positive definite.

The **set of positive real numbers** $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$ is the subset of real numbers strictly greater than 0. $\mathbb{R}_{>0}$ is closed under addition, multiplication, and division. Topologically $\mathbb{R}_{>0}$ is a subset of $\mathbb{R}$ and inherits the structure of a multiplicative topological group or of an additive topological semigroup.

From a group point of view, φ can be thought of as a map $GL_2(\mathbb{R}) \to \mathbb{R}_{>0}$.

5.2.4 Idea - Payoff Function as a determinant operator

Suppose we define a payoff function as a determinant operator[10]. Then we would start with a strong assumption that the space of payoff functions is defined by an inner product norm. Suppose we have a determinant operator $\varphi : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that

$$\varphi(x, y) = \det(M) = \det \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = k.$$

Observe that applying conjugate symmetry property gives

$$\det(M) = \frac{1}{\det(M^{-1})} = -\frac{1}{\det(M)}.$$

Applying other linear algebraic properties (invertibility, linearity, homogeneity) to derive an explicit inverse formula with respect to φ we have

$$\begin{aligned} M * M^{-1} &= I \\ \det(M) * \det(M^{-1}) &= 1 \\ \det(M) * \det(M^{-1}) &= k * \frac{1}{k} = 1 \end{aligned}$$

Conclusion: It appears that by classifying the payoff function φ as a determinant operator, showing that $\varphi \in H$ becomes immediate. We can explicitly derive discrete inverse functions with respect to φ.

Misc Remark: $\mathbb{R}$ is one of the simplest examples of an inner product space. The real numbers $\mathbb{R}$ are a vector space over the field $\mathbb{R}$ and becomes an inner product space when endowed with standard multiplication of the real inner product

$$\langle x, y \rangle := xy, \qquad x, y \in \mathbb{R}.$$

[10] Do we need to use an absolute value here to restrict all values to being positive? Is this even desirable?

6 Terminology

1. Replicating Portfolio - In mathematical finance, a replicating portfolio for a given asset or series of cash flows is a portfolio of assets with the same properties. This is meant in two distinct senses: **static replication**, where the portfolio has same cash flows as the reference asset and **dynamic replication**, where the portfolio does not have the same cash flows but has the same "greeks".

7 MEV

7.1 MEV via arbitrage and LPing

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8 Notes

8.1 Research Questions

1. Provide a general definition of a payoff function. How does this apply to derivatives?

8.2 DeFi Resources

1. RMM Angeris et al March 2021

2. RMM01 Estelle et al October 2021

8.3 Math Resources

1. Payoff Function Approximations Vorobeychik et al September 2005

2. Financial Derivatives Giulia Iori

3. Exotic Greeks (2019)

4. On Completeness of Groups of Diffeomorphisms