$X_1, \ldots, X_n \mid \theta \overset{iid}{\sim} \text{Exp}(\theta)$ $\qquad$ θ is unknown;

$\theta \sim \text{Gamma}(a, b)$ $\qquad$ a, b are known.

$$p(\theta \mid x_{1:n}) = \text{Gamma}\left(\theta \mid \alpha = a + n, \beta = b + \sum x_i\right). \quad (1)$$

$x' = x_{n+1}$

i.) new data point x_{n+1}. derive $p(x_{n+1} \mid x_n)$

posterior → updated gamma

$$\boxed{p(x_{n+1} \mid x_{1:n})} \overset{\text{defn}}{=} \int \underbrace{p(x_{n+1} \mid \theta)}_{\text{likelihood}} \overbrace{p(\theta \mid x_{1:n})}^{\text{posterior}} \, d\theta$$

no concept of proportionality!

likelihood eval at new data pt

exponential distⁿ

$$= \int_0^\infty \theta e^{-\theta x_{n+1}} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha - 1} e^{-\beta\theta} \, d\theta$$

common mistake: drop constants

DON'T DO THIS

$$= \underbrace{\frac{\beta^\alpha}{\Gamma(\alpha)}}_{\substack{\text{take}\\\text{constants}\\\text{out}}} \int_0^\infty \underbrace{\theta^{\alpha+1-1} e^{-\theta(x_{n+1} + \beta)}}_{} \, d\theta \times \frac{\beta_n^{\alpha_n}}{\Gamma(\alpha_n)} \times \frac{\Gamma(\alpha_n)}{\beta_n^{\alpha_n}}$$

combined like terms

common mistake

Kernel Gamma$(\alpha + 1, x_{n+1} + \beta)$

$\qquad\qquad\quad$ α_n $\qquad\quad$ β_n

(its a Gamma w/out its normalizing constants).

The integral is not a Gamma!

$$\frac{\beta_n^{\alpha_n}}{\Gamma(\alpha_n)} \qquad \text{Gamma}(\theta \mid \alpha_n, \beta_n)$$

α_n $\qquad$ β_n $\qquad$ → 1

$$\frac{\beta^\alpha}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha_n)}{\beta_n^{\alpha_n}} \cancel{\int_0^\infty \theta^{\overbrace{\alpha+1}-1} e^{-\theta(\overbrace{x_{n+1}+\beta})}} \times \frac{\beta_n}{\Gamma(\alpha_n)} d\theta$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{(x_{n+1}+\beta)^{\alpha+1}}$$

ii.) The marginal likelihood is defined as

$$p(x_{1:n}) \overset{\text{defn}}{=} \int \underbrace{p(x_{1:n}|\theta)}_{\text{likelihood}} \underbrace{p(\theta)}_{\text{prior}} d\theta$$

common mistake is plugging in just one value of x

$$= \int_0^\infty \theta^n e^{-\theta \sum_{i=1}^n x_i} \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} d\theta$$

$$= \frac{b^a}{\Gamma(a)} \int_0^\infty \underbrace{\theta^{n+a-1} e^{-\theta\left(\sum_{i=1}^n x_i + b\right)}}_{\text{kernel of Gamma}(a+n, \sum x_i + b)} d\theta$$

common mistake: dropping constants!

$$= \frac{b^a}{\Gamma(a)} \frac{\Gamma(a+n)}{(\sum x_i + b)^{a+n}} \underbrace{\int_0^\infty \theta^{n+a-1} e^{-\theta(\sum x_i + b)} \frac{(\sum x_i + b)^{a+n}}{\Gamma(a+n)} d\theta}_{\text{Gamma}(\theta | a+n, \sum x_i + b)}$$

1

$$= \frac{b^{a}}{\Gamma(a)} \frac{\Gamma(a+n)}{\left(\sum x_i + b\right)^{a+n}} .$$