Operator methods

Quantum Mechanics Foundation Course: Problem Sheet 3

1 Operator representation of the harmonic oscillator Hamiltonian

Consider the Hamiltonian for the quantum harmonic oscillator, written in terms of the position and momentum operators:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2,$$

where $[\hat{x}, \hat{p}] = i\hbar$. The **ladder operators** are defined as,

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right).$$

(i) Express the position and momentum operators of the harmonic oscillator in terms of ladder operators

(ii) Using the results of part (i), show that the Hamiltonian can be written in terms of ladder operators as,

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right).$$

(iii) Using the expression for $\hat{H}$ in terms of ladder operators, show that any eigenstate $|n\rangle$ of the Hamiltonian is also an eigenstate of $\hat{a}^\dagger\hat{a}$.

Recall that the energy spectrum of the quantum harmonic oscillator is:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

(iv) Using this result, find the eigenvalue of $\hat{a}^\dagger\hat{a}$ corresponding to a given state $|n\rangle$. What does this suggest about the physical interpretation of $\hat{a}^\dagger\hat{a}$?

2 Commutation relations

Consider the commutator, defined as $[A, B] = AB - BA$.

(i) Using the commutation relations for $\hat{x}$ and $\hat{p}$, show that $[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{a}, \hat{a}] = 0$, $[\hat{a}^\dagger, \hat{a}^\dagger] = 0$ (These are the commutation relations for bosons).

(ii) Evaluate the commutators of the ladder operators with the Hamiltonian, i.e $[\hat{H}, \hat{a}]$ and $[\hat{H}, \hat{a}^\dagger]$.

3 Interpretation of the ladder operators

Consider a general eigenstate of the Hamiltonian $\hat{H}$, $|n\rangle$, having energy E_n, i.e. a solution of $\hat{H}|n\rangle = E_n|n\rangle$.

(i) Using the commutator you found in Part (2.ii), evaluate $\hat{H}\hat{a}^\dagger|n\rangle$ and $\hat{H}\hat{a}|n\rangle$.

(ii) The above results imply that $\hat{H}\hat{a}^\dagger|n\rangle = E_{n+1}\hat{a}^\dagger|n\rangle$ and $\hat{H}\hat{a}|n\rangle = E_{n-1}\hat{a}|n\rangle$. How does this suggest that the action of the ladder operators is $\hat{a}^\dagger|n\rangle \propto |n+1\rangle$ and $\hat{a}|n\rangle \propto |n-1\rangle$?

(iii) The action of the operators can thus be quantified as $\hat{a}^\dagger = \mathcal{C}|n+1\rangle$ and $\hat{a} = \mathcal{D}|n-1\rangle$. Find $\mathcal{C}$ and $\mathcal{D}$ (*Hint: the orthonormality of eigenstates is useful here, i.e* $\langle n|n\rangle = 1$).

(iv) Consider acting the Hamiltonian on the ground state $\hat{H}|0\rangle$, reason that $\hat{a}|0\rangle = 0$.

(v) Use the ladder operators to show that any eigenstate of the harmonic oscillator can be expressed as $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n}}|0\rangle$.

4 Angular momentum

In $3D$, generalizing from classical physics, the angular momentum operator is defined as the cross-product of the position and momentum operators:

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} \qquad \text{where} \qquad \hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z}) \qquad \text{and} \quad \hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)\,. \tag{1}$$

Position and momentum are known as conjugate variables, which means that $[\hat{x}, \hat{p}_x] = i\hbar$ and $[\hat{x}, \hat{p}_{y,z}] = 0$, which similarly holds for the y and z components.

(i) Using the commutation relations for $\hat{x}$ and $\hat{p}_x$ (and similarly for $\hat{y},\ \hat{z}$), show that the angular momentum operators satisfy commutation relations:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z, \quad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x, \quad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y \tag{2}$$

It turns out that elementary quantum particles have a fundamental property known as their "spin", which is directly related to these angular momentum operators: spin operators $\hat{S}_{x,y,z,}$ satisfy analogous commutation relations to $\hat{L}_{x,y,z,}$. The term "spin" is rather unfortunate since elementary particles do not actually spin; the name derives from the fact that the units of this quantum observable are the same as those for angular momentum.

5 Extra: Fluctuations

(i) Using the ladder operators, find the expectation value of $\hat{x}^2$ in the ground state $|0\rangle$ of the quantum harmonic oscillator.

(ii) The uncertainty in position in the ground state is defined as $\Delta\hat{x} = \sqrt{\langle n|\hat{x}^2|n\rangle - \langle n|\hat{x}|n\rangle^2}$. What does this suggest about the meaning of the expectation value computed in part (i)?

6 Extra: Fermions

In contrast to bosons, fermions obey *anti-commutation* relations given by:

$$\{\hat{c}, \hat{c}^\dagger\} = 1 \qquad \{\hat{c}, \hat{c}\} = 0 \qquad \{\hat{c}^\dagger, \hat{c}^\dagger\} = 0,$$

where the *anticommutator* is defined as $\{A, B\} = AB + BA$. Show that $\{\hat{c}^\dagger, \hat{c}^\dagger\} = 0$ implies Pauli's exclusion principle.