16.3 - Fundamental Theorem of Line Integrals

For certain types of line integrals we have something very similar to the fundamental theorem of Calculus (which states $\int_a^b F'(x)\,dx = F(b) - F(a)$)

Recall our line integral of a vector field, and assume vector field is a gradient field (i.e. Conservative)

$$F = \nabla f$$

$$\int_C F \cdot dr$$

$$= \int_C \nabla f \cdot d\vec{r} = \int_a^b \left(\nabla f\,(\underbrace{\vec{r}(t)) \cdot \vec{r}\,'(t)}_{\frac{d}{dt}(f(\vec{r}(t))}\right) dt$$

Think about this in terms of chain rule.

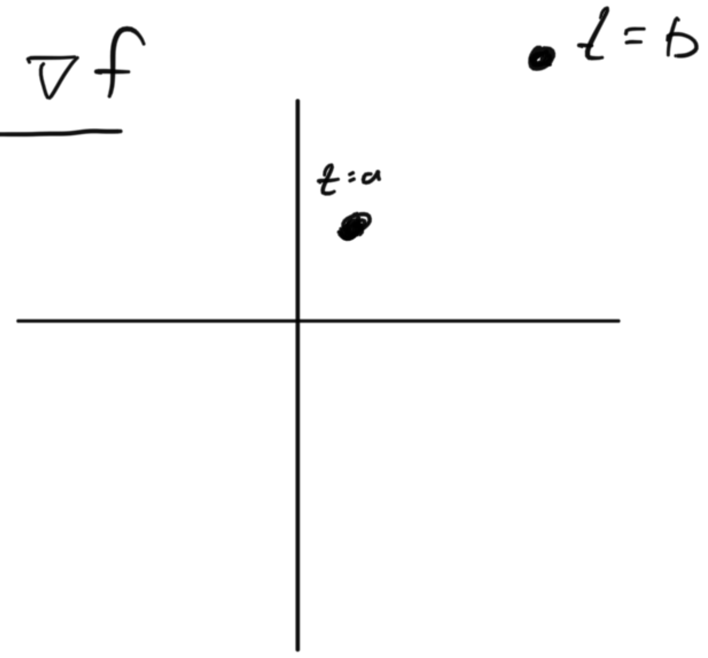
$$= \int_a^b \left[\frac{d}{dt} f(\vec{r}(t)) \right] dt$$

Would expect the integral to be:

$$\boxed{\begin{aligned}\int_C \underline{\overset{F}{\nabla f}\cdot d\vec{r}} &= \int_a^b \overset{\frac{d}{dt}\left(f(\vec{r}(t))\right)dt}{\nabla f(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt} \\ &= f(\vec{r}(b)) - f(\vec{r}(a))\end{aligned}}$$

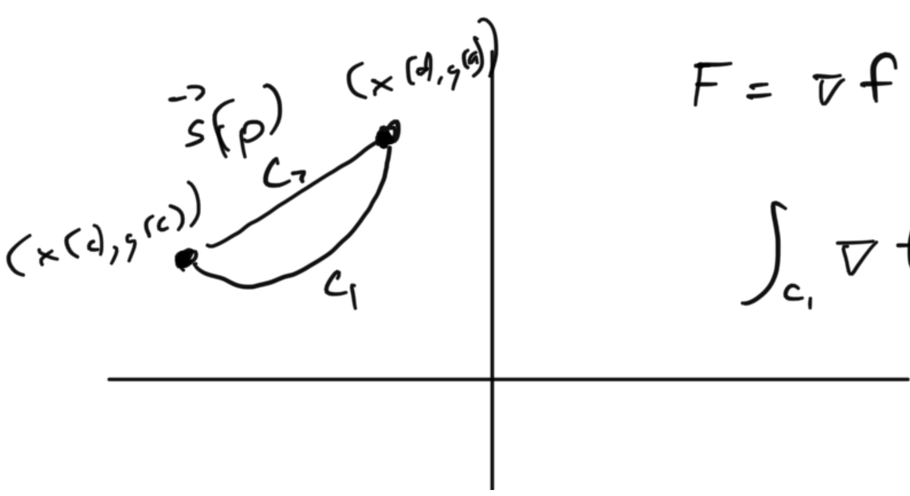
This is "Fundamental Theorem of Line Integrals"

$\underline{F = \nabla f}$



$\underline{f(\vec{r}(b)) - f(\vec{r}(a))}$

<u>Important</u>: Note that when we evaluated, only used the endpoints. What if we integrate ∇f along two different <u>paths</u> with same starting point, endpoint?



$F = \nabla f$

$$\int_{C_1} \nabla f(\vec{r}(t))\cdot r'(t)\,dt$$

$f(x_2, y_2) - f(x_1, y_1)$

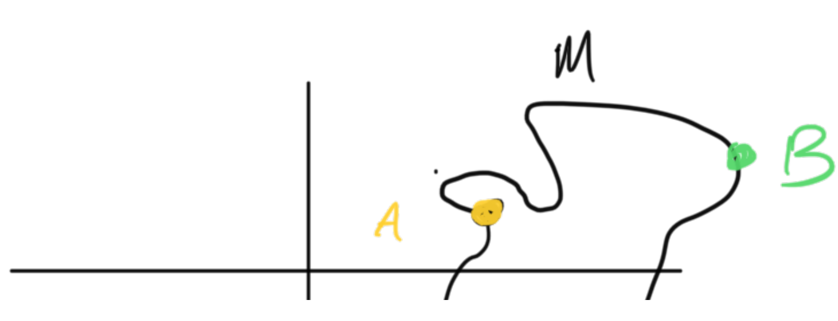
$$\int_{C_1} \nabla f \cdot dr = \int_a^b \nabla f \, \vec{r}\,'(t)\, dt = f(\vec{r}(b)) - f(\vec{r}(a))$$
$$= f(x_2, y_2) - f(x_1, y_1)$$

$$\int_{C_2} \nabla f \cdot d\vec{s} = \int_c^d \nabla f \, \vec{s}\,'(p)\, dp = f(\vec{s}(d)) - f(\vec{s}(c))$$
$$= f(x_2, y_2) - f(x_1, y_1)$$

This concept is called independence of path.

- $F = \nabla f \Rightarrow$ independence of path.
- independence of path $\overset{?}{\Rightarrow}$ $F = \nabla f$
- More generally, when is F independent of path

If we have independence of path, what can we say about line integrals along closed curves?



Thm: $\int_c F\cdot dr$ is independent of path if and only if $\int_c F\cdot dr = 0$ for every closed path C in the domain.

$$\int_{C_1} F\cdot dr = \int_{C_2} F\cdot dr$$

$$\int_{C_1 \cup -C_2} F\cdot dr = 0$$

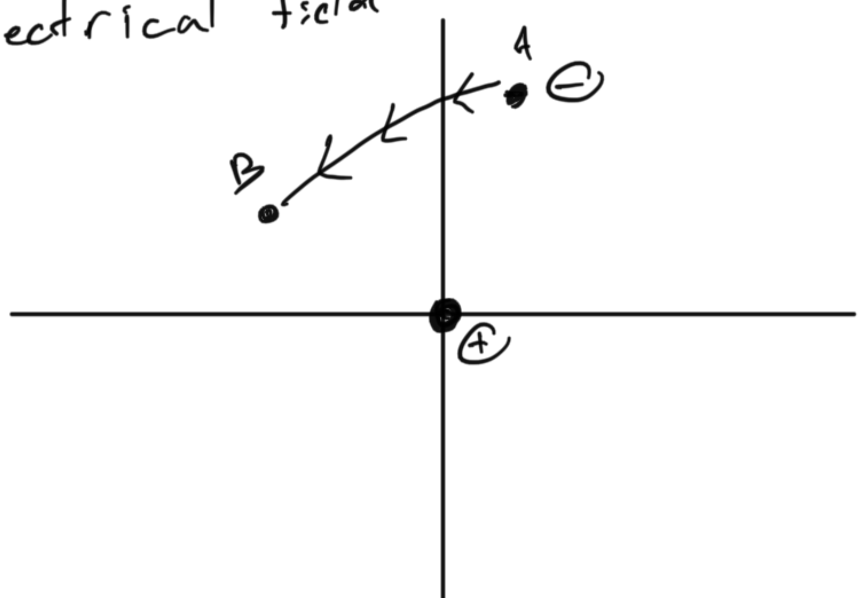
$$= \int_{C_1} F\cdot dr + \int_{-C_2} F\cdot dr$$

$$0 = \int_{C_1} F\cdot dr + \int_{-C_2} F\cdot dr$$

$$-\int_{-C_2} F\cdot dr = \int_{C_1} F\cdot dr$$

$$\int_{C_2} F\cdot dr = \int_{C_1} F\cdot dr$$

F = electrical field



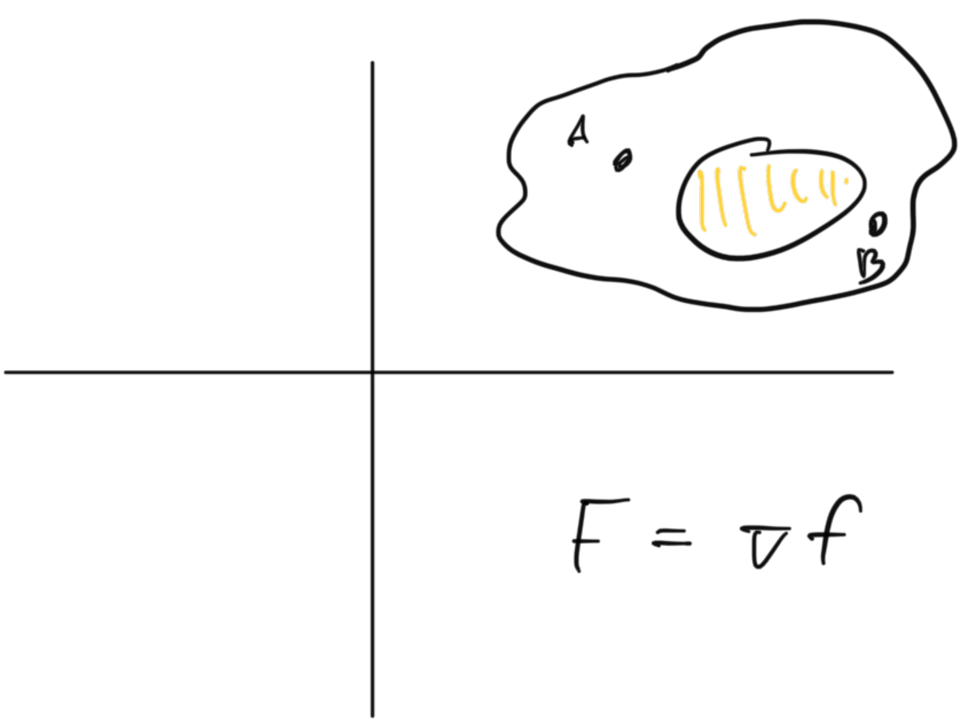
$\int_C F \cdot dr$

So, we see line integrals of conservative vector fields are independent of path.

Converse ?

$F = \langle F_1, F_2, F_3 \rangle$

<u>Thm 4:</u> Suppose F is continuous vector field on open connected region D. If $\int_C F \cdot dr$ is independent of path in D, then <u>F</u> is conservative vector field ($F = \nabla f$ for some potential function f)



$F = \nabla f$

☆ $\int_C F$ ind path

F conservative?

F conservative or not?

~~Could~~ try to check if F is independent of path.

<u>Thm:</u> If $F(x,y) = P(x,\ )\vec{i} + Q(x,y)\vec{j}$ is a conservative vector field where P, Q have continuous 1^{st} order partials on domain

then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$F = \langle P, Q \rangle$ is conservative

$F = \nabla f$

$$\nabla f = \langle P, Q \rangle$$

$$f_x \qquad f_y$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\langle \underline{\underline{P(x,y)}}, \underline{\underline{Q(x,y)}} \rangle = \langle f_x, f_y \rangle$$

$f_{yx} = f_{xy}$

F is conservative

$$\langle P, Q\rangle = F = \nabla f = \langle f_x, f_y \rangle$$

$$\frac{\partial P}{\partial y} = \boxed{f_{xy}} \qquad \frac{\partial Q}{\partial x} = \boxed{f_{yx}}$$

Given $F = \langle P, Q\rangle$ asked if it is conservative.

<u>A test</u>: If <u>P, Q</u> have cont partials <u>but</u> $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ then F <u>not</u> conservative.

F conservative <u>then</u> $\underline{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$

OTOH:

<u>Theorem</u>: Let $\underline{F(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}}$ be

vector field on open ☆ simply connected ☆ region D.

Suppose P, Q ☆ have continuous 1^{st} order partials and

$$☆ \ \partial P/\partial y = \partial Q/\partial x \ ☆ \quad \text{on } D$$

Then F is conservative.

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(3) $F(x,y) = \underbrace{(xy+y^2)}_{P}\vec{i} + \underbrace{(x^2+2xy)}_{Q}\vec{j}$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad ?$$

$$\frac{\partial P}{\partial y} = x + 2y$$

$$\frac{\partial Q}{\partial x} = 2x + 2y$$

$$\partial P/\partial y \neq \partial Q/\partial x$$

F not conservative

(4) $F(x,y) = (y^2 - 2x)\vec{i} + (2xy)\vec{j}$

$$\underbrace{\qquad}_{P} \qquad \underbrace{\qquad}_{Q}$$

$$\partial P/\partial y = \partial Q/\partial x \quad ?$$

$$\partial P/\partial y = 2y$$

$$\partial Q/\partial x = 2y \quad \checkmark$$

$$f \qquad \nabla f = F$$

$$\nabla f = F(x,y) = \underline{(y^2 - 2x)\vec{i} + (2xy)\vec{j}}$$

in $\nabla f = \langle y^2 - 2x,\ 2xy \rangle$

$$\int f_x \, dx = f + g(y)$$

$$\int (y^2 - 2x)\, dx$$

$$\boxed{xy^2 - x^2 + g(y)} \quad \text{I}$$

$$\int f_y \, dy = f + h(x)$$

$$\int 2xy \, dy = \boxed{xy^2 + h(x)} \quad \text{II}$$

$$\boxed{xy^2} - x^2 + \underbrace{\boxed{g(y)}}_{0} = \boxed{xy^2} + \underbrace{\boxed{h(x)}}_{-x^2}$$

$$\boxed{f(x,y) = xy^2 - x^2}$$

5) $F(x,y) = \underbrace{(y^2 e^{xy})}_{P}\vec{i} + \underbrace{(1+xy)e^{xy}}_{Q}\vec{j}$

$$\partial P/\partial y = \partial Q/\partial x \ ?$$

$$\frac{\partial P}{\partial y} = 2ye^{xy} + xy^2e^{xy}$$

$$= \boxed{e^{xy}(2y + xy^2)}$$

$$\partial Q/\partial x = ye^{xy} + (1+xy)ye^{xy}$$

$$= e^{xy}(y + y + xy^2)$$

$$= \boxed{e^{xy}(2y + xy^2)}$$

F conservative

$$F(x,y) = \underbrace{(y^2e^{xy})}_{f_x}\vec{i} + \underbrace{(1+xy)e^{xy}}_{f_y}\vec{j}$$

$$\int y^2e^{xy}\,dx = y^2\frac{e^{xy}}{y} + g(y)$$

$$\boxed{= ye^{xy} + g(y)}$$

$$\int (1+xy)e^{xy}\,dy = \int e^{xy} + xye^{xy}\,dy$$

$$= \int e^{xy}\,dy + \int xye^{xy}\,dy$$

$$\frac{e^{xy}}{x} + x\left(\frac{ye^{xy}}{x} - \frac{e^{xy}}{x^2}\right) + h(x)$$

$$= \cancel{\frac{e^{xy}}{x}} + ye^{xy} - \cancel{\frac{e^{xy}}{x}} + h(x)$$

$$\boxed{= ye^{xy} + h(x)}$$

$$\boxed{ye^{xy}} + K = \boxed{ye^{xy}} + K$$

$$f = ye^{xy} + C$$

(13) $F(x,y) = (x^2y^3)\vec{i} + (x^3y^2)\vec{j}$

$C: \vec{r}(t) = \langle t^3 - 2t, t^3 + 2t \rangle$

$\underline{0} \le t \le \underline{1}$

$$\int x^2y^3\,dx = \frac{x^3}{3}y^3 + g(y)$$

$$\int x^3y^2\,dy = x^3\frac{y^3}{3} + h(x)$$

$f(x,y) = x^3 y^3 + K$

$$\int_C F \cdot dr = f(\vec{r}(1)) - f(\vec{r}(0))$$

$$\vec{r}(1) = \langle -1, 3 \rangle$$
$$\vec{r}(0) = \langle 0, 0 \rangle$$

$$f(-1, 3) - f(0,0)$$
$$= \left(-\frac{1}{3} \cdot 27\right) - (0)$$
$$= -9$$

(23) $F = x^3 \vec{i} + y^3 \vec{j}$

$\underline{P(1,0)}$ $\quad$ $\underline{Q = (2,2)}$

$$r(t) = (1,0) + t(1,2)$$
$$0 \le t \le 1$$
$$\vec{r}(t) = \langle 1+t, 2t \rangle$$
$$r'(t) = \langle 1, 2 \rangle$$
$$\sqrt{5}$$

$$\frac{m}{2} \int_C \frac{d}{dt}\left(r'(t) \cdot r'(t)\right) dt$$
$$\frac{m}{2} \int_C \frac{d}{dt}\left(|r'(t)|^2\right) dt$$

$$\frac{m}{2} \quad |r'(t)|^2 \Big]_a^b$$

$$\frac{m}{2} \left| \sqrt{5} \right|^2 \Big]_0^1$$

$$\frac{m}{2} \cdot 5 \cdot (1)$$