Section 1.4

Introduce just a bit of new notation.

ij - notation

Let $\underline{A}$ be $m \times n$ matrix. Will write entries of A like so:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

a_{46}

a_{ij}

i is for row

j is for column

So a_{23} is in 2^{nd} row 3^{rd} column

$\vec{a}_1$ is 1^{st} column $\rightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}$

$\vec{a}_5 = \begin{bmatrix} a_{15} \\ a_{25} \\ \vdots \\ a_{m5} \end{bmatrix}$

Use similar concept for system of equations

We have seen how system of equations

$$\begin{array}{l} 1^{st} \text{ eq} \\ 2^{nd} \text{ eq} \end{array} \begin{cases} a_{11} x_1 + \dots \; a_{1n} x_n = d_1 \\ a_{21} x_1 + \dots \; a_{2n} x_n = d_2 \\ a_{31} x_1 + \dots \; a_{3n} x_n = d_3 \end{cases}$$

is same as augmented matrix.

$$\left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & d_1 \\ & \dots & & \\ a_{m1} & \dots & a_{mn} & d_m \end{array}\right]$$

Recently showed that vector equation

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + \dots \; x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

is same as augmented matrix. (So all 3 of the above are same).

Recall we briefly talked about <u>matrix equation</u>

$$\longrightarrow \underbrace{\begin{bmatrix} a_{11} & \dots & a_{1n} \\ & \dots & \\ a_{m1} & \dots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{b} \qquad A\vec{x} = \vec{b}$$

Saw this when we developed augmented matrix.
Will show this is also same as all of the above.

Recall how we multiply matrix by vector.

Let A be $m \times n$ matrix.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Let $\vec{x}$ be vector $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Then $A\vec{x}$ calculated like so:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \\ - \\ - \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} - \\ \\ \\ \end{bmatrix}$$

$$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} - \\ - \\ \\ \end{bmatrix}$$

$= a_{21}x_1 + a_{22}x_2 + \dots a_{2n}x_n$

Note: In $A\vec{x} = \vec{b}$, the i^{th} component of $\vec{b}$ is dot product of i^{th} row of A and $\vec{x}$

$$b_i = (i^{th} \text{ row } A) \cdot \vec{x}$$

Note: For dot products to "line up" # of columns in A must equal # entries in $\vec{x}$

i.e.

If A is $m \times n$

$\vec{x}$ is $n \times 1$

If we have eq. $A\vec{x} = \vec{b}$ where A is $m \times n$, $\vec{x}$ is $n \times 1$ then $\vec{b} = m \times 1$

$$(m \times n)(n \times 1) = (m \times 1)$$

Ex:

$$\underset{(3\times 3)}{\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 4 & 6 & 2 \end{bmatrix}} \underset{(3\times 1)}{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}} = \underset{(3\times 1)}{\begin{bmatrix} 7 \\ 4 \\ 8 \end{bmatrix}}$$

Theorem: For $m \times n$ matrix A, vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and scalar c have

① $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

② $A(c\vec{u}) = cA\vec{u}$

Identity Matrix Consider a square $n \times n$ matrix with 1's on diagonal and all other entries 0.

Then for any vector $\vec{u} \in \mathbb{R}^n$ we have

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & & & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Will use identity matrix a lot. Give it special notation $I_{n \times n}$, or I.

Once we are comfortable with multiplication $A\vec{x}$, can see how $A\vec{x}=\vec{b}$ is equivalent to vector equation

$$x_1 \vec{a}_1 + \ldots x_n \vec{a}_n = \vec{b}$$

Consider $A\vec{x}$

$$\underset{m \times n}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}} \underset{n \times 1}{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m \times 1}{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ \\ a_{m1}x_1 + \ldots \quad a_{mn}x_n \end{bmatrix}}$$

just a vector

Know from algebraic properties of vectors that

$$\begin{bmatrix} a_1 + b_1 + c_1 \\ \vdots \\ a_n + b_n + c_n \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

So this vector:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ \\ a_{m1}x_1 + \ldots \quad a_{mn}x_n \end{bmatrix}$$

can be split into these vectors

$$\begin{bmatrix} a_{11}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \ldots \begin{bmatrix} a_{1n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

Factor x_1 out of first vector

x_2 out of second

and so on

$$\star \quad x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

This is exactly vector equation!

So

$A\vec{x}$ same as $x_1\vec{a} + \ldots + x_n\vec{a_n}$

Recap:

System of linear equations

$$\star \quad \begin{cases} a_{11}x_1 + \ldots a_{1n}x_n = b_1 \\ \\ a_{m1}x_1 + \ldots a_{mn}x_n = b_n \end{cases}$$

How we see data

Same as augmented matrix

$$\star \quad \left[\begin{array}{ccc|c} a_{11} & \ldots & a_{1n} & b_1 \\ & & & \\ a_{m1} & \ldots & a_{mn} & b_n \end{array}\right]$$

Convenient for row reduction and solving

Same as vector equation

Good for geometric intuition

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \cdots \; x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Same as matrix equation

Convenient for matrix algebra

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Thm: System of equations, augmented matrix, vector equation, and matrix equation (as given above) all have same solution sets.

Why do we care about this? Gives us a little mental flexibility in these problems.

This gives geometric intuition

($A\vec{x} = \vec{b}$ has a solution) iff ($\vec{b}$ is linear comb. of columns of A)

$\left(A\vec{x} = \vec{b} \text{ has a solution} \right)$ iff $\left([A \mid b] \text{ has solution} \right)$

So can turn $A\vec{x} = \vec{b}$ into augmented matrix, which is convenient to solve

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Now that we have some different ways to consider systems and their solutions lets consider <u>when</u> these systems have solutions.

In particular when does $A\vec{x} = \vec{b}$ have a solution for <u>all</u> $\vec{b}$?

Consider

Can I get <u>every</u> vector in $\mathbb{R}^m$ as output?

$$\underset{m \times n}{A} \; \underset{\substack{n \times 1 \\ \text{in } \mathbb{R}^n}}{\vec{x}} = \underset{\substack{m \times 1 \\ \text{in } \mathbb{R}^m}}{\vec{b}}$$

Think of in terms of vector equation:

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots x_n\vec{a}_n = \vec{b}$$

Lets say this has solution for all $\vec{b} \in \mathbb{R}^m$. Then it must mean that every $\vec{b} \in \mathbb{R}^m$ is a linear combination of $\vec{a}_1, \dots \vec{a}_n$. So

(all of) $\mathbb{R}^m$ is in $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$

aka

$\{\vec{a}_1, \dots, \vec{a}_n\}$ span $\mathbb{R}^m$

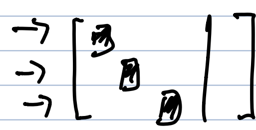
Thm. Let A be $m \times n$ matrix. Then the following statements are equivalent:

→ ⓐ For each $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has solution

→ ⓑ Each $\vec{b} \in \mathbb{R}^m$ is linear comb. of columns of A

→ ⓒ Columns of A span $\mathbb{R}^m$

→ ⓓ A has a pivot in every row (of coefficient matrix)



$A\vec{x} = \vec{b}$