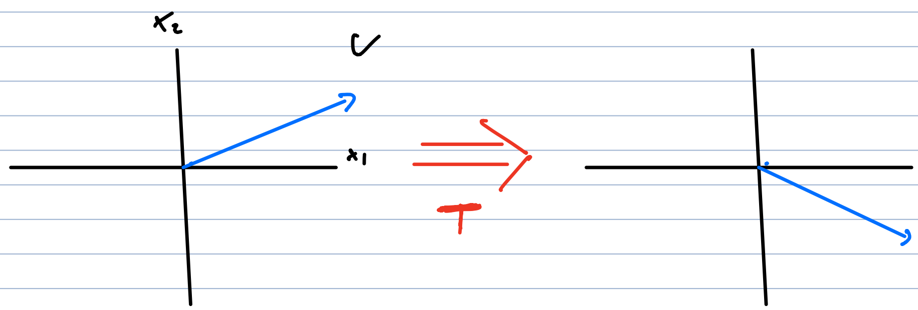
# Section 5.1

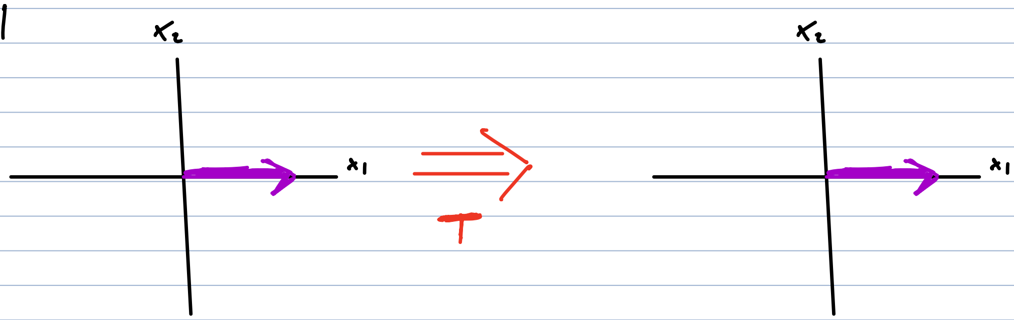
Will focus on linear transformations of type
☆ $T: V \to V$, domain and codomain are same vector space. Usually, use $n \times n$ matrix $A$.
(Remember, $n \times n$ matrix gives us transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$)

Think of the effect transformations have on individual vectors.

~~Ex~~ $A = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$ Flips vector over $x_1$ axis




Some vectors more effected than others.
If vector is on $x_1$ axis, not changed at all



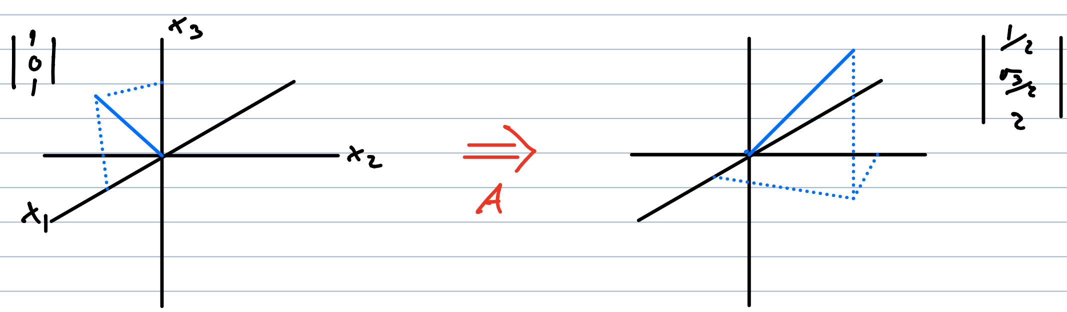
We are interested in vectors that aren't effected very much. Okay for them to be stretched / compressed, but not moved off their original lines.

**Definition:** An eigenvector of an $n \times n$ matrix $A$ is a nonzero vector $\vec{x}$ s.t. $A\vec{x} = \lambda \vec{x}$ for some scalar $\lambda$. This scaler $\lambda$ is called an eigenvalue of $A$. We say $\vec{x}$ is an eigenvector corresponding to $\lambda$.

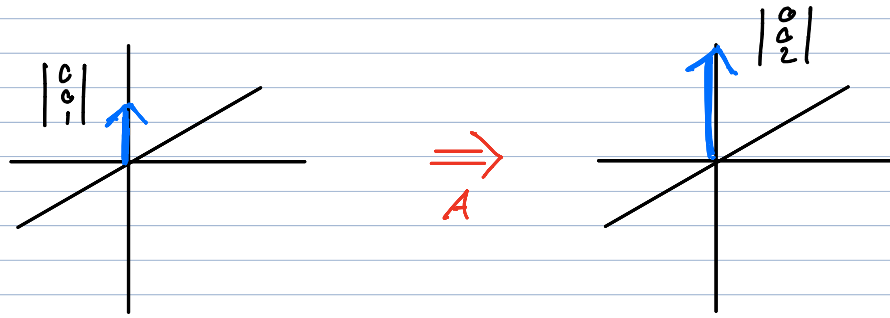
**Ex**

$$A = \begin{bmatrix} \cos \pi/3 & -\sin \pi/3 & 0 \\ \sin \pi/3 & \cos \pi/3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

This matrix rotates vectors $\pi/3$ clockwise around $x_3$-axis, doubles $x_3$ component.



What if vector is on $x_3$-axis?




$\begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$ is an eigenvector of $A$, corresponding to eigenvalue of $\lambda = 2$.

In fact any other vector on $x_3$ axis is also eigenvector corresponding to 2.
(subspace)

**In general:**

- If $\vec{x}$ is eigenvector of linear transformation $T: V \to V$ then any multiple of $\vec{x}$ is an eigenvector corresponding to same eigenvalue

- If $\vec{x_1}$ and $\vec{x_2}$ are both eigenvectors corresponding to eigenvalue $\lambda$, then sum is also eigenvector for $\lambda$.

$$\begin{aligned} A(x_1 + x_2) &= A\vec{x_1} + A\vec{x_2} \\ &= \lambda \vec{x_1} + \lambda \vec{x_2} \\ &= \lambda(\vec{x_1} + \vec{x_2}) \end{aligned}$$

Note these two conditions are what we need for a set to be a subspace of V

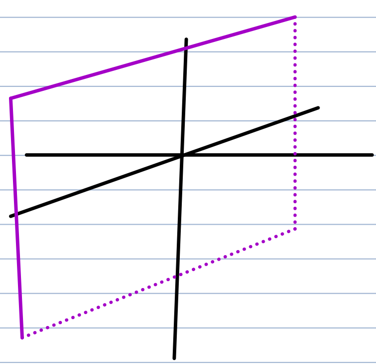
Theorem: Let $\lambda$ be an eigenvalue of $T: V \to V$. The set of all eigenvectors corresponding to $\lambda$ forms a subspace of V, the eigenspace corresponding to $\lambda$.

Ex

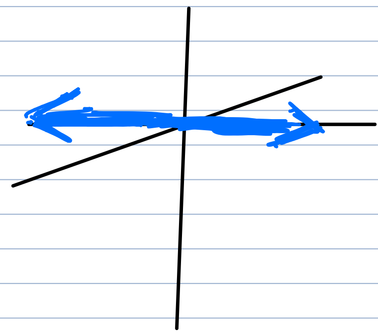
$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} =$$

$\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$ are eigenvectors corresponding to $\lambda = 3$

eigenvector corresponding to $\lambda = 2$



The $x_1 x_3$-plane is eigenspace for $\lambda = 3$



The $x_2$ axis is eigenspace corresponding to $\lambda = 2$.

Find eigenvector for specific eigenvalue:

We would like to be able to examine a matrix and find all its eigenvalues / eigenvectors. This process has two steps:

1. Figure out all eigenvalues of $A$
2. For each eigenvalue, find a basis for corresponding eigenspace.

Lets skip step 1 until next section.
Assume we know that $\lambda$ is eigenvalue of $A$. How to find basis for eigenspace?

1. $A\vec{x} = \lambda \vec{x}$ — Know that eigenvectors satisfy this
2. $A\vec{x} = \lambda I_{n\times n} \vec{x}$ — $\vec{x}$ is same thing as $I_{n\times n}\vec{x}$. Will need that matrix
3. $A\vec{x} - \lambda I_{n\times n}\vec{x} = \vec{0}$ — Move everything to one side
4. $(A - \lambda I_{n\times n})\vec{x} = \vec{0}$ — Factor out $\vec{x}$

Now $(A - \lambda I_{n \times n})\vec{x} = \vec{0}$ is just homogeneous system.
Know how to solve those.
The free variable vectors will give basis of solutions. This basis is exactly basis of our eigenspace.

**Ex:**

$$A = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 3 & 3 \\ 6 & 6 & 2 \end{bmatrix} \qquad \lambda = -4$$

Find basis for eigenspace corresponding $\lambda = -4$

## Checking if vector is an eigenvector:

Easy. If we are given matrix $\underline{A}$ and vector $\underline{\vec{x}}$, may want to know if $\vec{x}$ is an eigenvector.

Multiply $A\vec{x}$. If $\vec{x}$ eigenvector result should be some multiple of $\vec{x}$, $\lambda x$.

**Ex:** Is $\vec{x}$ an eigenvector of $A$? If so, for what eigenvalue?

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underline{\begin{bmatrix} -2 \\ 2 \end{bmatrix}}$$

$\vec{x}$ <u>is</u> eigenvector of $A$ corresponds to eigenvalue $2$

**Theorem:** The eigenvalues of a triangular matrix are the entries on its main diagonal.

□ $A = \begin{bmatrix} a & \cdot & \cdot \\ 0 & b & \cdot \\ 0 & 0 & c \end{bmatrix}$ Easy to check that $a, b, c$ are eigenvalues. Multiply matrix by $\vec{e_1}, \vec{e_2}, \vec{e_3}$

Question is, are there any other eigenvalues "hiding" in $A$. Assume $d$ is some other eigenvalue. Should be vector $\vec{x}$ s.t.

$$A\vec{x} = d\vec{x}$$

or $$(A - dI)\vec{x} = \vec{0}$$

or $$\left[\begin{array}{ccc|c} a-d & \cdot & \cdot & 0 \\ 0 & b-d & \cdot & 0 \\ 0 & 0 & c-d & 0 \end{array}\right] \quad \vec{x} = \vec{0}$$

Writing out solution of this, see

★ $x_3 = 0$. Back substitution gives:

$x_2 = 0$

$x_1 = 0$

So, there is no nonzero $\vec{x}$ that is eigenvector for $d$. Thus, $d$ is <u>not</u> eigenvalue. ∎

**Theorem:** If $\vec{v_1} \ldots \vec{v_r}$ are eigenvectors of $A$ that correspond to <u>distinct</u> eigenvalues, then they are linearly independent.