Section 4.4

One of main uses of a basis is to act as a coordinate system.

Assume V is a vector space and $\{\vec{v}_1 \dots \vec{v}_n\}$ is a basis for it. By definition of basis:

☆ ① $\{\vec{b}_1, \dots, \vec{b}_n\}$ spans V. So for any $\vec{x} \in V$ can write $\underline{\vec{x}} = \underline{c_1 \vec{b}_1} + \dots + \underline{c_n \vec{b}_n}$ ☆

☆ ② Since $\{\vec{b}_1, \dots \vec{b}_n\}$ lin. ind., there is only <u>one</u> way to write $\vec{x} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n$

Theorem: Let $\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\}$ be a basis for vector space V. Then for each $\vec{x}$ in V there exists a unique set of scalars $c_1 \dots c_n$ such that

$$\vec{x} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n$$

Definition:

The coefficients $c_1 \dots c_n$ are the coordinates of x with respect to basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$. Can collect coordinates in a coordinate vector of $\vec{x}$

$$☆ \quad [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

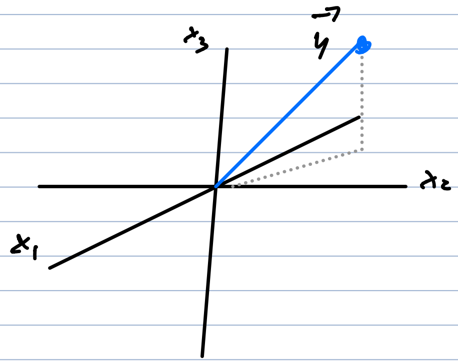
" $\vec{x}$ in terms of basis $\mathcal{B}$ " ..., " is $c_1 \vec{b}_1 + c_2 \vec{b}_2 + \dots c_n \vec{b}_n$ "

$\vec{x} = c_1 \vec{b}_1 + \dots c_n \vec{b}_n$

Ex: Most familiar basis is Standard Basis in $\mathbb{R}^n$

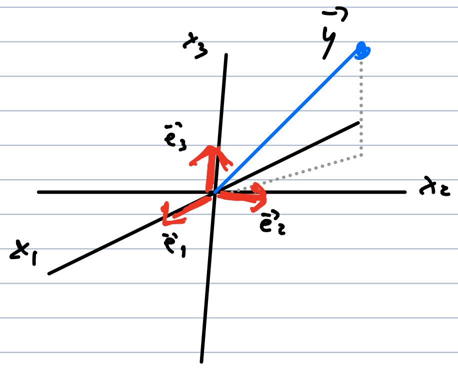
where $\mathcal{B} = \{\vec{e}_1, \dots \vec{e}_n\}$ and $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$, etc.

Consider $\mathbb{R}^3$:



Let's say this is the vector $\vec{y} = \begin{vmatrix} -4 \\ 2 \\ 4 \end{vmatrix}$

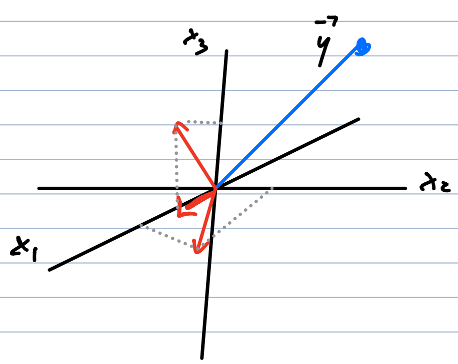
What does that mean?



Means that blue vector is $\vec{y} = -4\vec{e}_1 + 2\vec{e}_2 + 4\vec{e}_3$

What if we had another basis?

$$\mathcal{B} = \left\{ \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}, \begin{vmatrix} 2 \\ 1 \\ 0 \end{vmatrix}, \begin{vmatrix} 1 \\ 0 \\ 2 \end{vmatrix} \right\}$$



$\vec{y} = -10\vec{b}_1 + 2\vec{b}_2 + 2\vec{b}_3$

Thus $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} -10 \\ 2 \\ 2 \end{bmatrix}$

Bases for $\mathbb{R}^n$

If we have a vector $\vec{y}$ in Standard Basis, how can we convert to another basis $\mathcal{B}$?

Could just try to eyeball it but there is a more methodical way.

Standard basis

$$\star \quad \underline{\mathcal{B}} = \{\vec{b}_1, \ldots, \vec{b}_n\} \qquad \vec{y}$$

Create augmented matrix, columns the basis vectors, $\vec{y}$

$$\left[\begin{array}{ccc|c} | & & | & \\ \vec{b}_1 & \cdots & \vec{b}_n & \vec{y} \\ | & & | & \end{array}\right]$$

Since $\mathcal{B}$ basis, there <u>will</u> be unique solution

The unique solution $\vec{c}$ will be coordinates of $\vec{y}$ in terms of $\mathcal{B}$

$$\star \quad \vec{c} = [\vec{y}]_{\mathcal{B}}$$

Ex: $B = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\} \qquad \vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\star$ Will skip "change-of-basis matrix" for now but see it again in 4.6 $\star$

Taking $\vec{x}$ written in terms of one basis (standard) and rewriting it in terms of another ($\mathcal{B}$) is called a coordinate mapping.

Theorem: Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for vector space $\vec{V}$. Then the coordinate mapping

$$\vec{x} \longmapsto [\vec{x}]_{\mathcal{B}}$$

is a one-to-one and onto linear transformation from V to $\mathbb{R}^n$.

□ Why? Since $\mathcal{B}$ spans V, every vector in V can be written as linear comb. of $\vec{b}_i$'s. (So onto). Since $\vec{b}_i$'s linear ind., this process is unique (so one-to-one). ■

Linear transformations that are one-to-one and onto are given a special name, isomorphism.

iso - same

morph - form/structure

If there is an isomorphism between two vector spaces ($T: V \to W$) we say the spaces are isomorphic (to each other)

Spaces that are isomorphic to each other are essentially the same (they have same "dimensions" and their vectors behave the same way). Can usually think of isomorphic spaces as interchangeable.

Ex: $\mathbb{R}^2$ and the xy-plane in $\mathbb{R}^3$

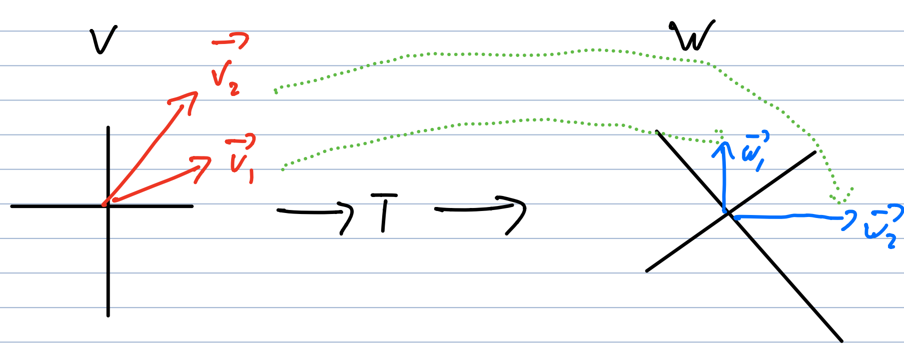
Isomorphisms "preserve" properties such as linear independence.

Meaning:

If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent in V and isomorphism T transforms V to W such that

$$T(\vec{v}_1) = \vec{w}_1 \quad T(\vec{v}_2) = \vec{w}_2 \quad T(\vec{v}_3) = \vec{w}_3$$

then $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ linearly independent in W.

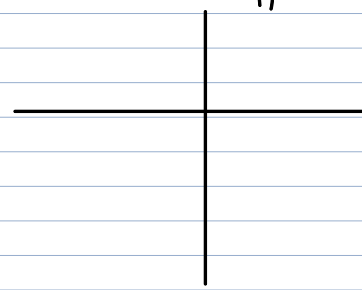


$\vec{w}_1, \vec{w}_2$ lin. ind.

$\vec{w}_1, \vec{w}_2$ span W

$\mathbb{R}^3$

$\mathbb{R}^2$



$T : \underline{V} \to \underline{W}$