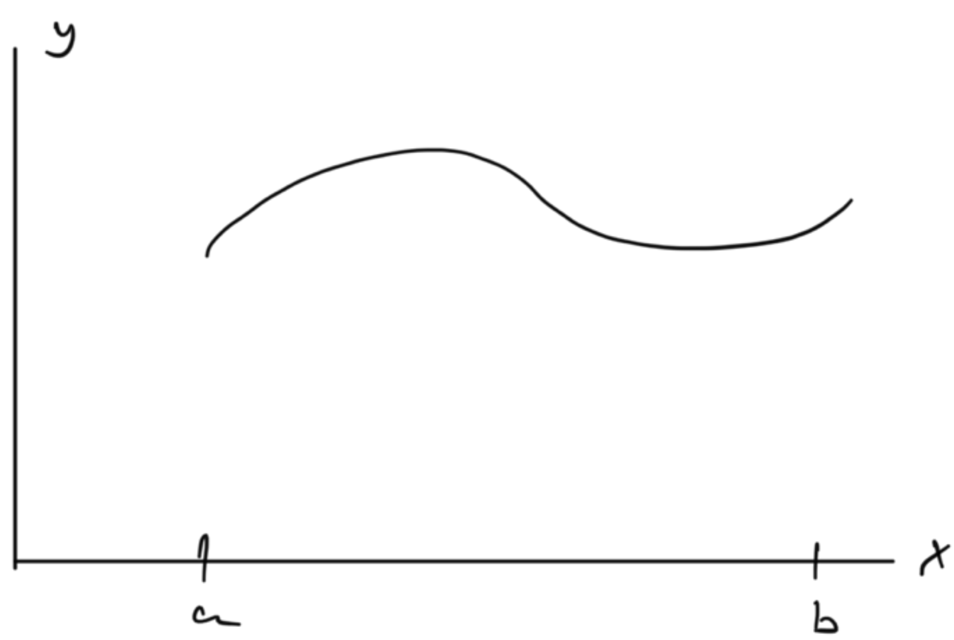
15.2 - Integrals on General Regions

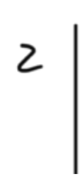
Saw last time how to integrate on rectangles. Great. But integrating over higher dimensions exposes a problem that didn't exist in single variable case

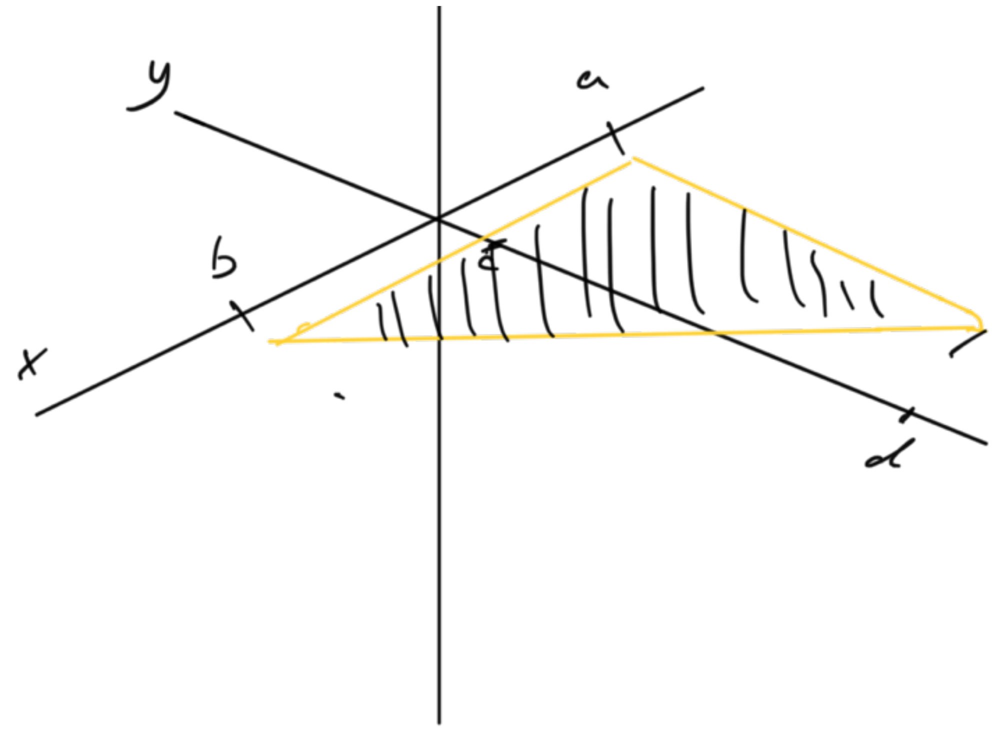


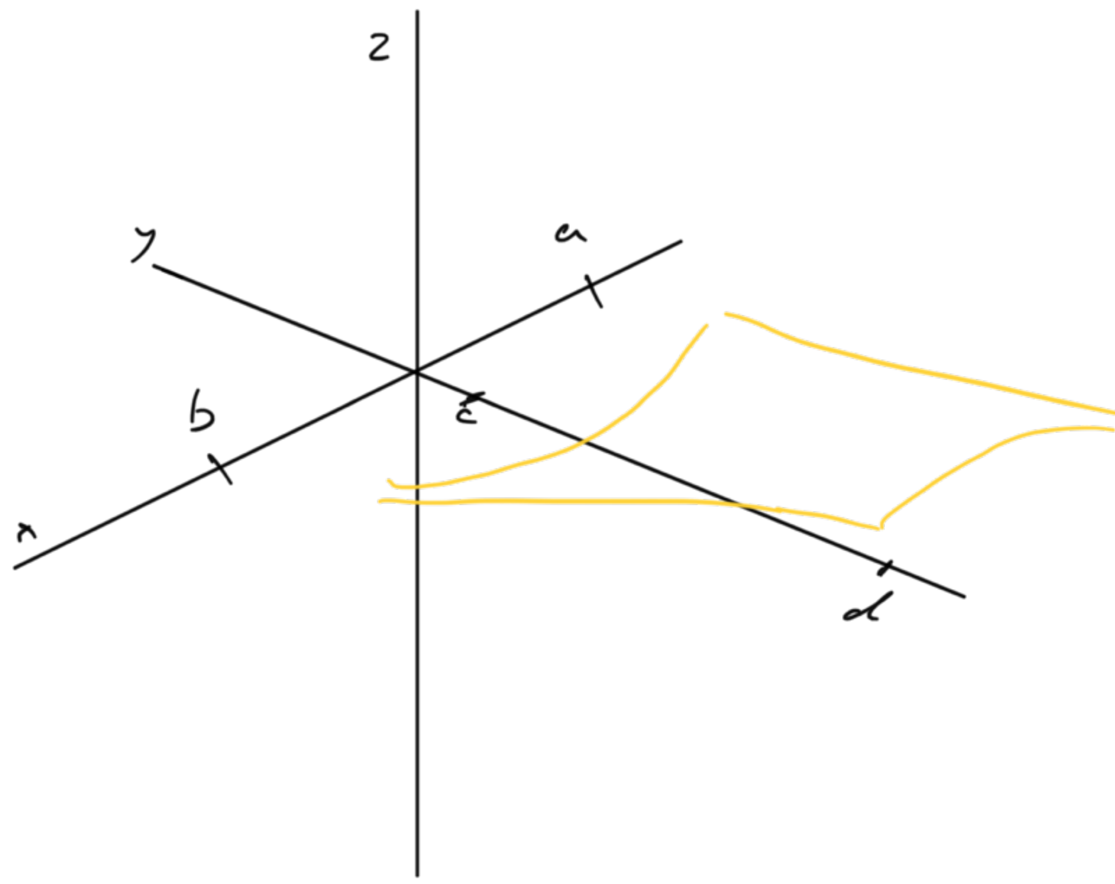
For single variable, not much option except integrating over interval

Boundary is just endpoints. Simple.

In multivariable boundary is more complex



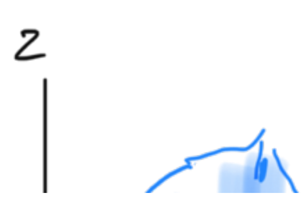


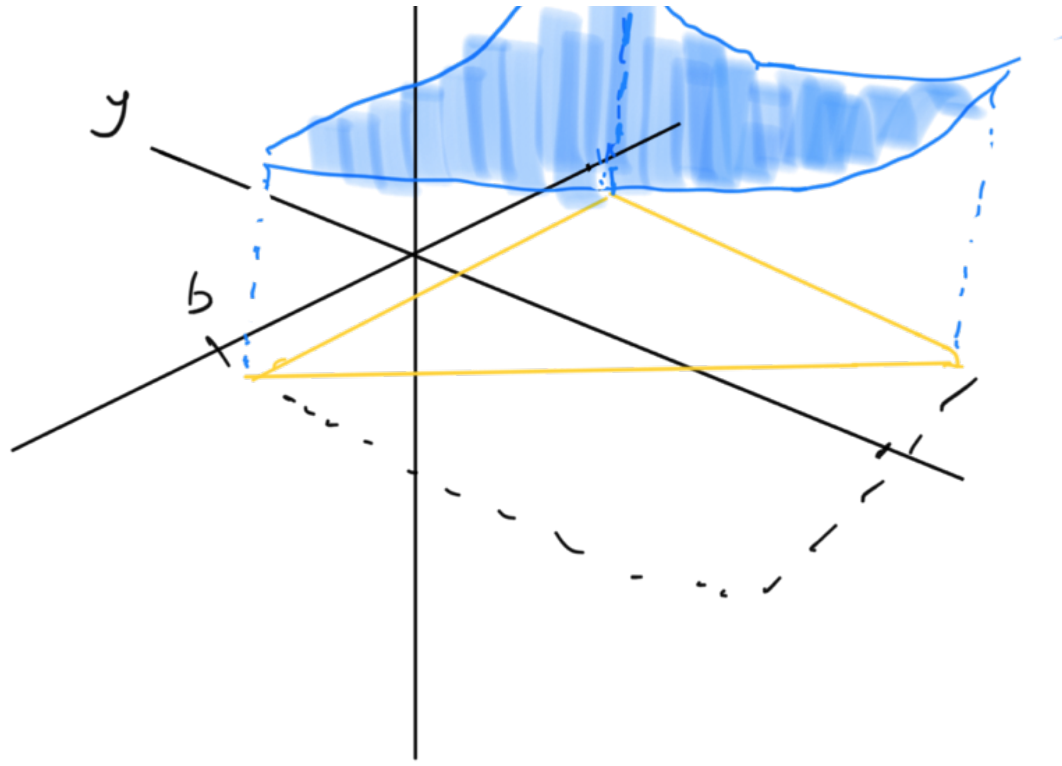


How do we deal with these domains?

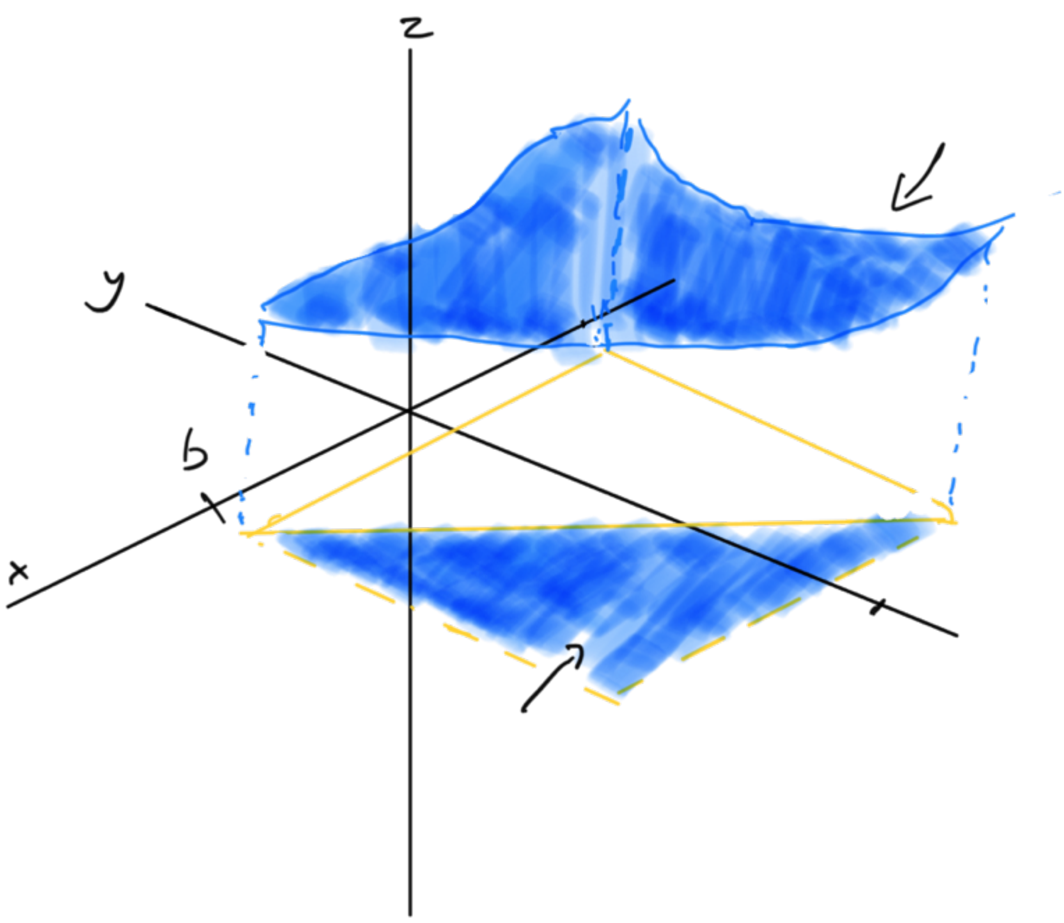
Integral was defined on rectangles. Want to avoid creating whole new method for these special cases

How to proceed?





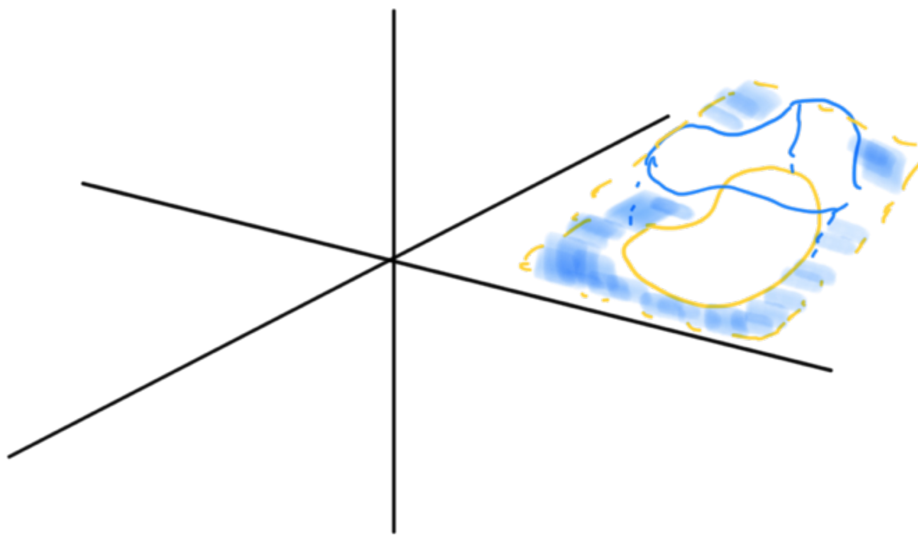
A simple fix is to enlarge our domain D to a rectangle and just call our function 0 on the extra space.



Function may not be continuous now but can still integrate it on the rectangle without changing our theory

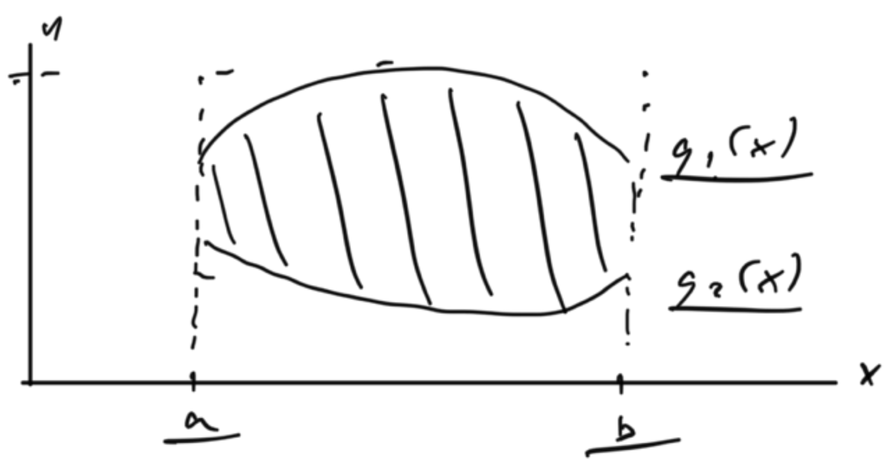
(limit of Riemann sums)

Theory is unchanged but how do we solve these in practice. Classify these irregular regions into 2 types.

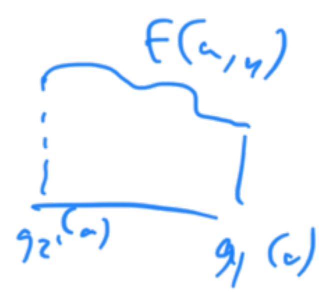


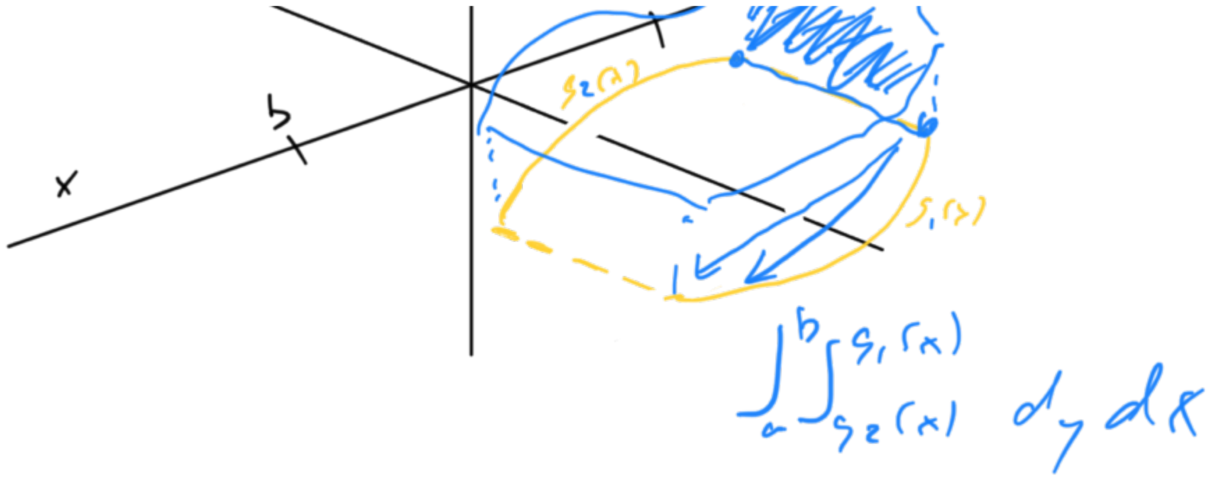
Type 1

Call a region D in xy plane type 1 if it is bounded above and below by functions of x



$\int_a^b \int_{g_1(x)}^{g_1(x)} dy\,dx$





How do we integrate on this region?

Will use the iterated integral again and perhaps it is clear x-bounds are still a to b

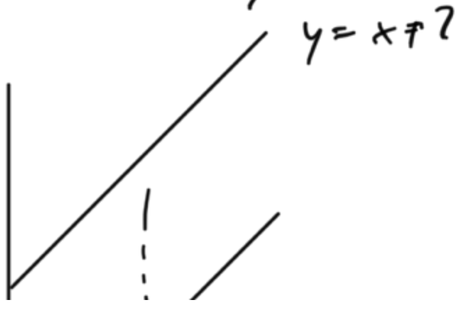
$$\int_a^b \qquad\qquad dx$$

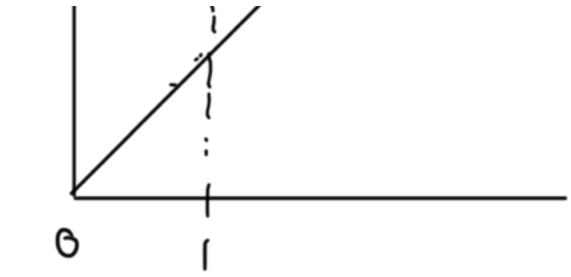
What about y-bounds? Variable y is trapped between my two functions $g_1(x)$ and $g_2(x)$. So lets use those as our bounds.

$$\int_a^b \int_{g_2(x)}^{g_1(x)} \qquad dy\, dx$$

Ex: $f(x,y) = \underline{x^2 y}$ ☆

☆ $\underline{D} = \{(x,y) : 0 \le x \le 1, \quad x \le y \le x+2\}$





$$\int_0^1\left(\int_x^{x+2} x^2 y \, dy\right)dx$$

$$\int_0^1 \left(\frac{x^2y^2}{2}\Big]_x^{x+2}\right)dx$$

$$\int_0^1\left(\frac{x^2(x+2)^2}{2} - \frac{x^2x^2}{2}\right)dx$$

$$\int_0^1 \frac{x^2(x^2+4x+4)}{2} - \frac{x^4}{2}\,dx$$

$$= \int_0^1 \frac{\cancel{x^4}+4x^3+4x^2-\cancel{x^4}}{2}\,dx$$

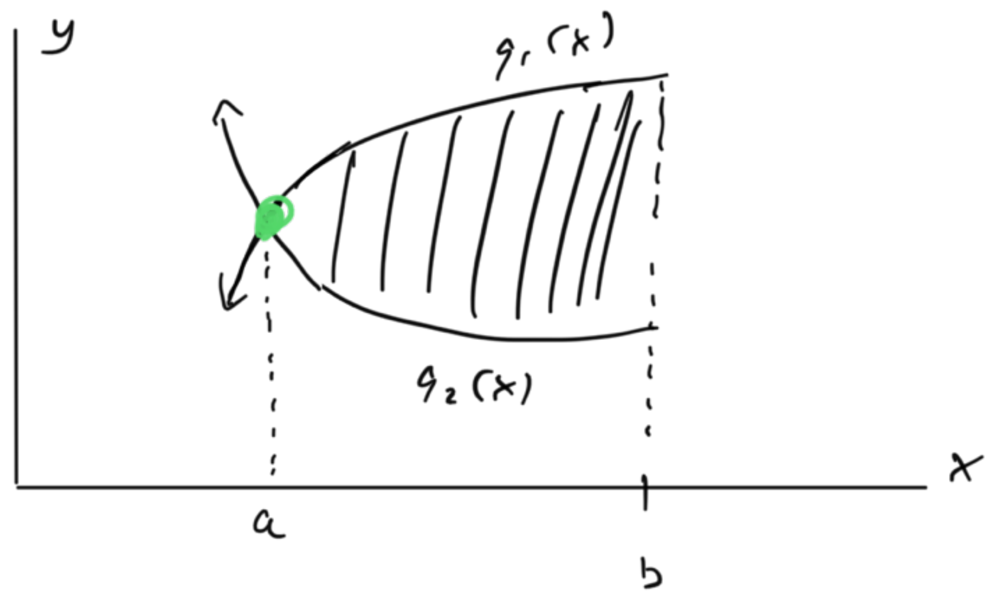
$$= \int_0^1 (2x^3+2x^2)\,dx$$

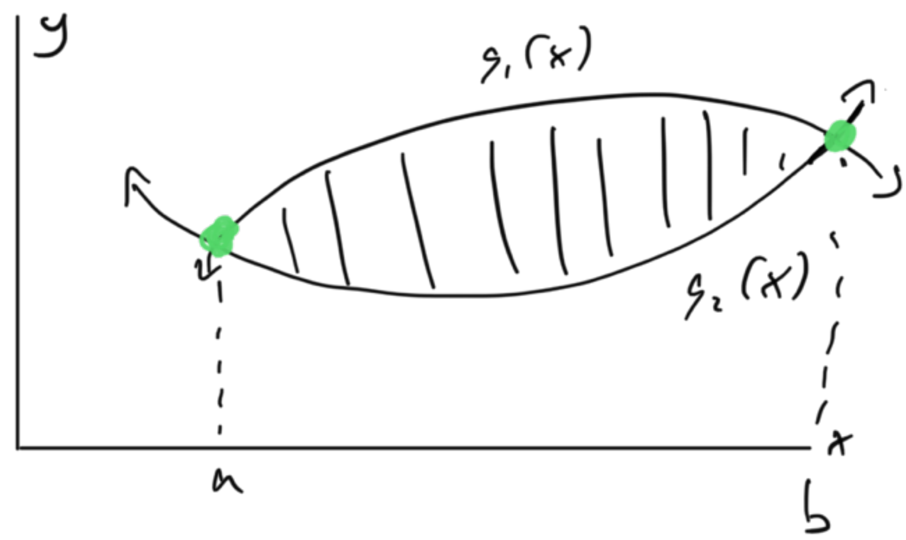
$$= \frac{x^4}{2}+\frac{2}{3}x^3\Big]_0^1$$

$$= \frac{1}{2}+\frac{2}{3}$$

$$= 5$$

More examples of type 1 regions:

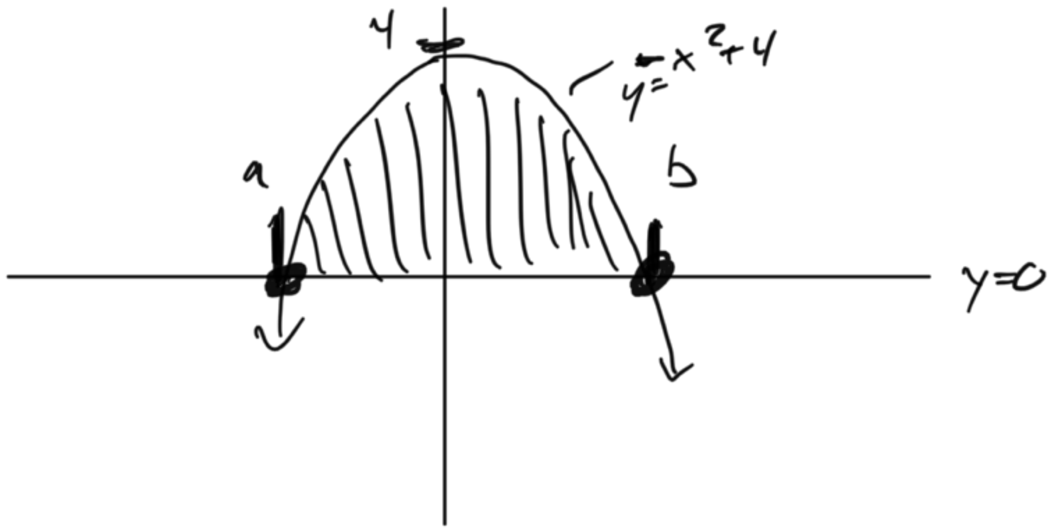




For these, to find our x-bounds we need to find the green points of intersection.

Ex: $f(x,y) = xy$

$D = \{(x,y): 0 \leq y \leq -x^2+4\}$



$$-x^2 + 4 = 0$$

$$4 = x^2$$

$$x = \pm 2$$

$$\int_{-2}^{2} \left(\int_0^{-x^2+4} xy \, dy \right) dx$$

$$\int_{-2}^{2} \left(\frac{xy^2}{2} \Big]_0^{-x^2+4} \right) dx$$

$$= \int_{-2}^{2} \frac{x(-x^2+4)^2}{2} \, dx$$

$$\int_{-2}^{2} \frac{x(x^4 - 8x^2 + 16)}{2} \, dx$$

$$= \int_{-2}^{2} \frac{x^5 - 8x^3 + 16x}{2} \, dx$$

$$= \frac{x^6}{12} - \frac{8x^4}{8} + \frac{16x^2}{4} \Big]_{-2}^{2}$$

$$= \frac{x^6}{12} - x^4 + 4x^2 \Big]_{-2}^{2}$$

$$= \left(\cancel{\frac{64}{12}} - \cancel{16} + \cancel{16}\right)$$

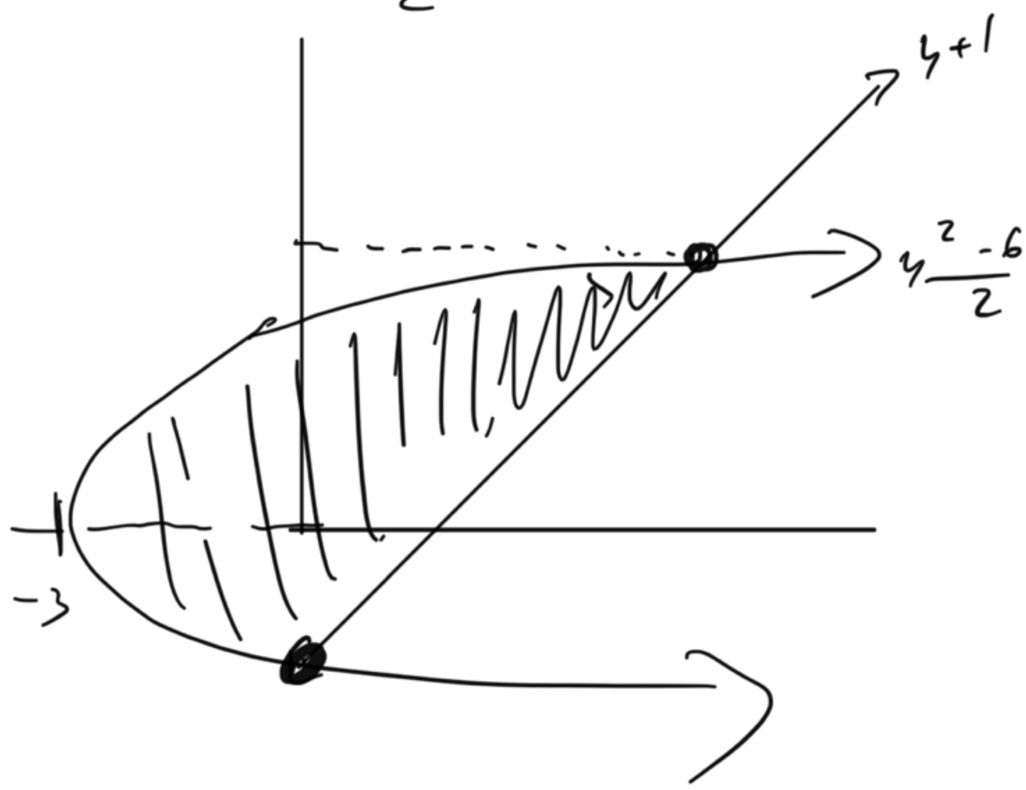
$$- \left(\cancel{\frac{64}{12}} - \cancel{16} + \cancel{16}\right)$$

$$= 0$$

Ex: $\iint_D xy \, dA$ where

D is region bounded by $x = y+1$

and $\frac{y^2-6}{2} = x$



$$\frac{y^2-6}{2} = y+1$$

$$\frac{y^2}{2} - 3 = y+1$$

$$\frac{y^2}{2} - y - 4 = 0$$

$$y^2 - 2y - 8 = 0$$

$$(y-4)(y+2) = 0$$

$y = 4 \quad y = -2$

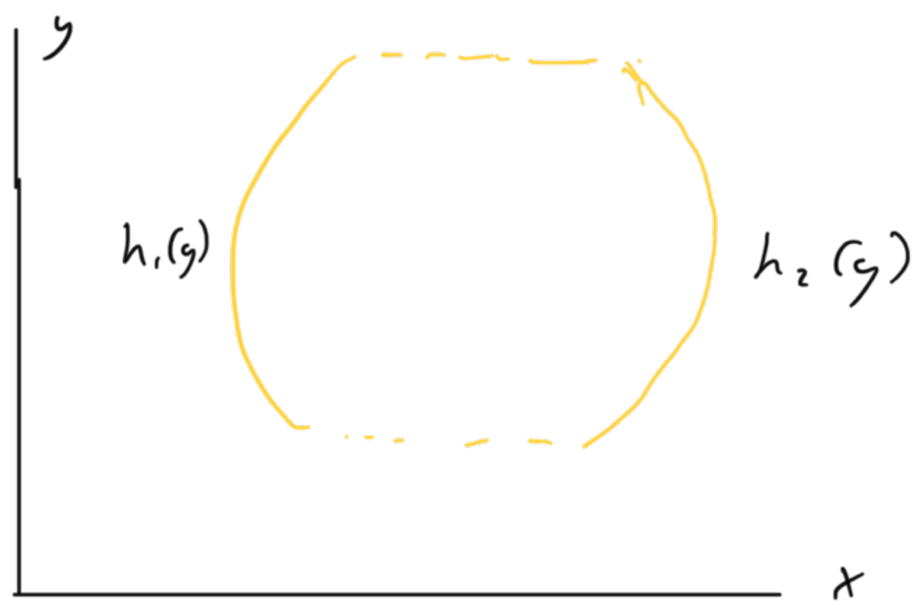
$$\int_{-2}^{4} \left(\int_{\frac{y^2-6}{2}}^{y+1} xy \, dx \right) dy$$

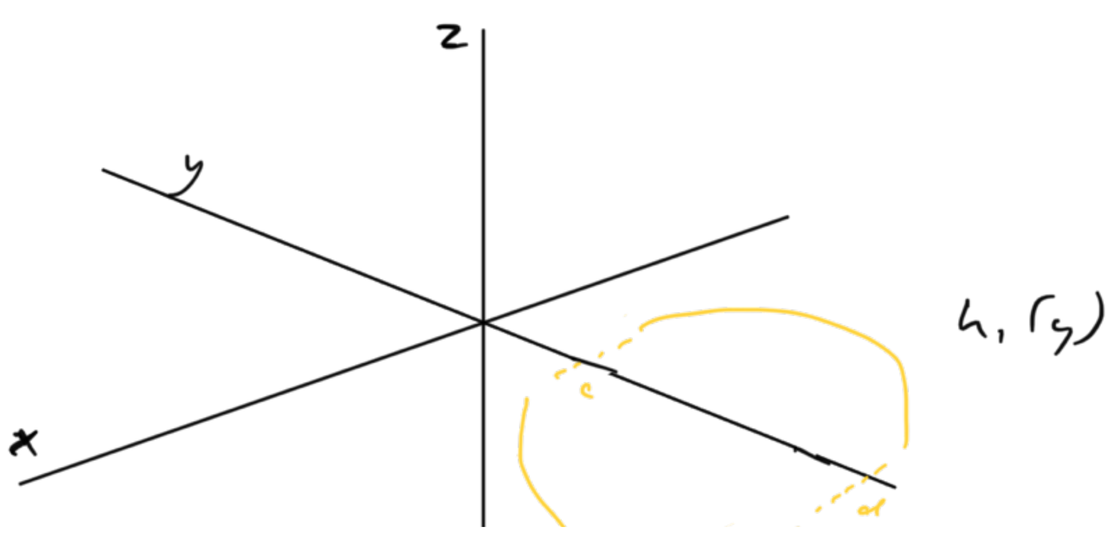
$$= \int_{-2}^{4} \left(\frac{x^2}{2} y \Big]_{\frac{y^2-6}{2}}^{y+1} \right) dy$$

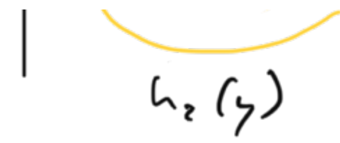
$$= \int_{-2}^{4} \left[\frac{(y+1)^2}{2} y - \frac{\left(\frac{y^2-6}{2}\right)^2}{2} y \right] dy$$

Type 2

Type 2 regions are similar but instead have region bounded by functions of y



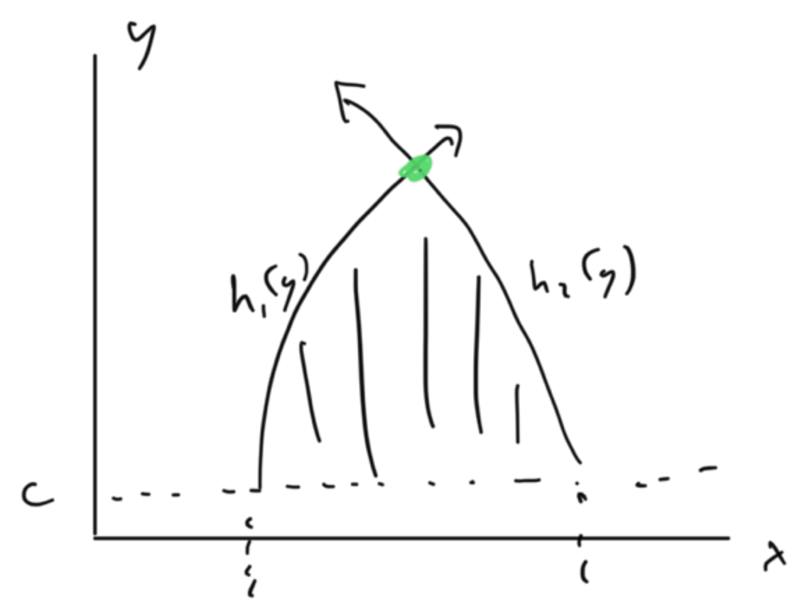


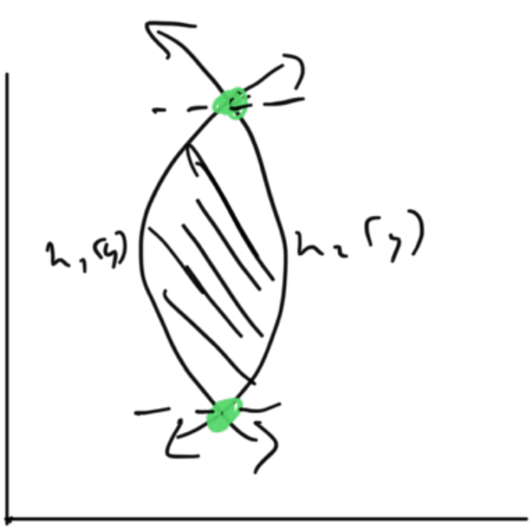


In these cases, y-bounds will be constants and x-bounds will be functions of y

$$\int_c^d \int_{h_1(y)}^{h_2(y)} dx\,dy$$

Again, type 2 regions might involve solving for y-bounds





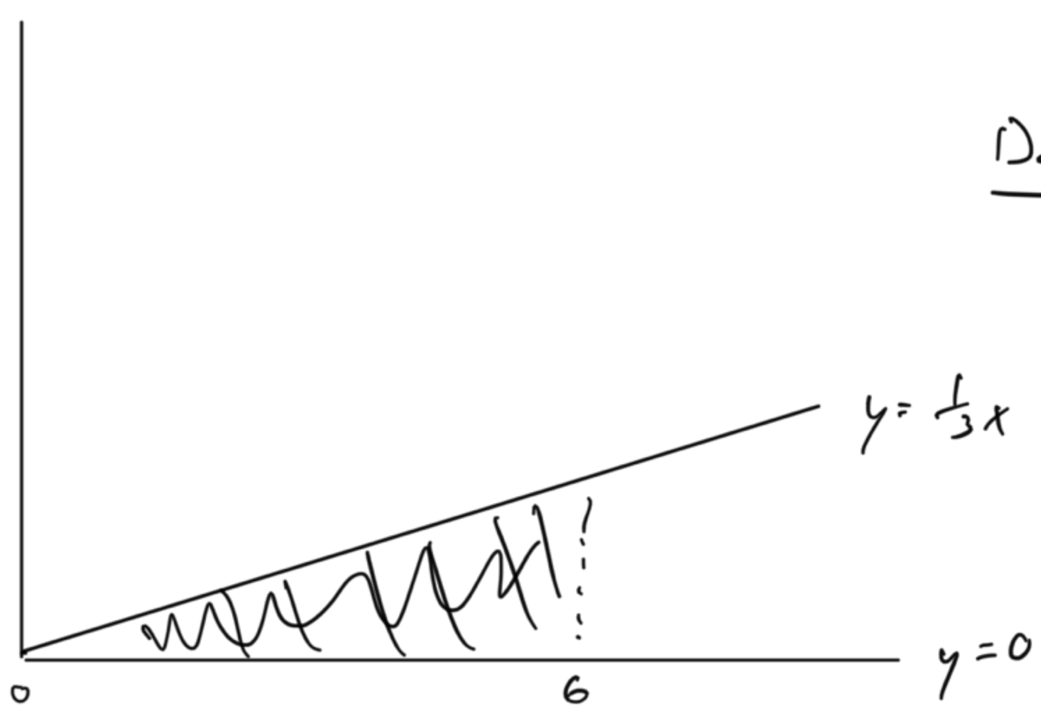
Ex: $\iint_D (x+2y)\,dA$

$D = \{(x,y) : 2x^2 \le y \le 1+x^2\}$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Switching order of integration when involving type 1, type 2:



Seems we have type 1 integral

$$\int_0^6 \int_0^{\frac{1}{3}x} \qquad dy\,dx$$

But what if it is more convenient to integrate in terms of y for some reason? Can we switch order of integration?

Yes

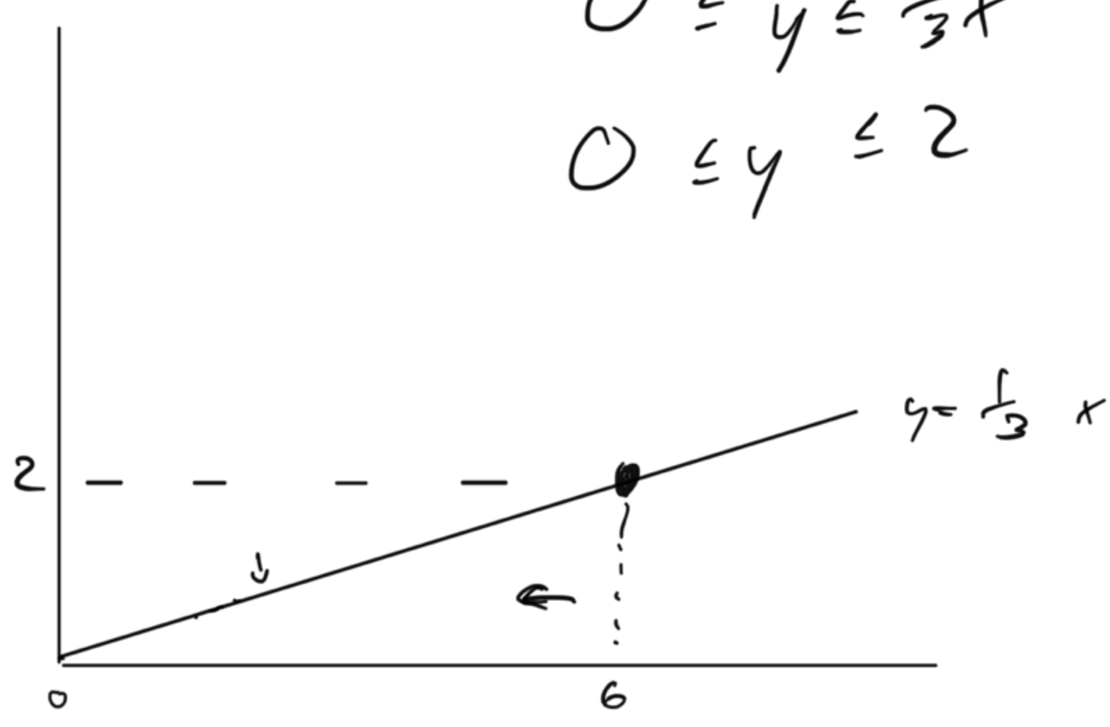
To do this, want to write y-bounds as constants and x bounds as one or two functions

y-bounds

What is largest value y-coordinate reaches?

$$0 \le y \le \tfrac{1}{3}x \qquad \int_0^{\frac{1}{3}x}$$

$$0 \le y \le 2 \qquad \int_0^2$$



What is

- lowest point y- reaches?

0

What function(s) is x-bounded by?

On left:

Greater (to right of) the line

How to write this as inequality?

$\leq x$

Take $y = \frac{1}{3}x$, solve for x

$3y = x$

Turn into inequality as appropriate

$3y \leq x$ $\int_{3y}^{6}$

Right bound:

$x \leq 6$

☆

$$\int_0^6 \int_0^{\frac{1}{3}x} dy\,dx \Rightarrow \int_0^2 \int_{3y}^{6} \underline{dx}\,dy$$

Properties of Integrals (Double)

① $$\iint_D \big(\overset{f(x,y)}{f(\vec{x})} + g(\vec{x})\big)\, dA = \iint_D f(\vec{x})\, dA + \iint_D g(\vec{x})\, dA$$

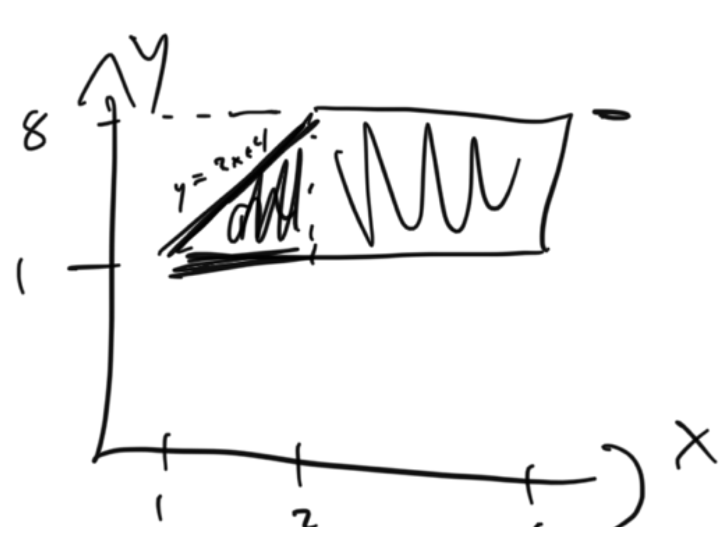
② $$\iint_D c\, f(\vec{x})\, dA = c \iint_D f(\vec{x})\, dA$$

③ If $f(\vec{x}) \le g(\vec{x})$ for all $\vec{x}$ in D then

$$\iint_D f(\vec{x})\, dA \le \iint_D g(\vec{x})\, dA$$

④ If D_1 and D_2 dont intercept except possibly on a boundardy then

$$\iint_{D_1 \cup D_2} f(\vec{x})\, dA = \iint_{D_1} f(\vec{x})\, dA + \iint_{D_2} f(\vec{x})\, dA$$



$$\int_1^2 \left(\int_1^{2x+4} dy \right) dx + \int_2^6 \left(\int_1^8 dy \right) dx$$

(5) If $m \le f(\vec{x}) \le M$ on D

then

$$\underline{\iint_D m \, dA} \le \iint_D f(\vec{x}) \, dA \le \underline{\iint M \, dA}$$

$$m \, A(D) \le \iint_D f(\vec{x}) \, dA \le M \, A(D)$$

(18)

$$\iint_D (x^2 + 2y) \, dA$$

D bounded by