Section 2.1

Have seen we can represent linear transformations by matrices.

What if we are dealing with a very complicated transformation like:

① ☆ T rotates points $\pi/6$ radians counterclockwise

② ☆ then flips across line $x_1 = x_2$

③ ☆ then rotates clockwise $\pi/3$

Figuring out a single matrix may be difficult but there is another approach.

If we know how to multiply matrices we can find matrix for each transformation then multiply them all together!

Just one of reasons we need to learn how to work with matrices

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Matrix Info

Recall how we number entries of a matrix.

$$a_{ij} \left\langle \begin{array}{l} i = \text{row} \\ j = \text{column} \end{array} \right.$$

Then diagonal entries of a matrix are of the form a_{ii}

The main diagonal of matrix

$$\begin{bmatrix} a_{11} & a_{12} & & \\ a_{21} & a_{22} & & \\ a_{31} & a_{32} & a_{33} & \\ & & & a_{44} \end{bmatrix},$$

A diagonal matrix is square matrix whose nonzero entries are on main diagonal, zero everywhere else.

$$\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ 0 & & a_{33} & \\ & & & a_{44} \end{bmatrix}$$

Ex The identity matrix $I_{n \times n}$ is a diagonal matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Zero matrix is the matrix of all 0's.

Two matrices are equal if they have same dimensions $(m \times n)$ and all entries are equal

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Matrix Arithmetic

Sum: If A, B have same dimensions $(m \times n)$ can add them together by just adding their entries.

$$\begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 5 \end{bmatrix}$$

$$(A+B)_{ij} = \underline{a_{ij}} + \underline{b_{ij}}$$

Scalar multiple: $c \in \mathbb{R}$ $\qquad 3\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & 0 \end{bmatrix}$

$(cA)_{ij} = ca_{ij}$ (just multiple every entry by c)

Theorem: (Properties of matrix arithmetic)

1. $A+B = B+A$ ☆ commutativity of addition
2. $(A+B)+C = A+(B+C)$ ☆ associativity
3. $A + O_{m\times n} = A$
4. $c(A+B) = cA + cB$ for $c \in \mathbb{R}$
5. $(c+d)A = cA + dA$
6. $c(dA) = (cd)A$

Matrix multiplication: Have already given some motivation. Straight to mechanics.

Know how to do (matrix)(vector)

b_i is dot product of i^{th} row of A and $\vec{x}$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \\ \vec{x} \\ \\ \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Recall have requirements.

If A $m \times n$, $\vec{x}$ must be $n \times 1$

$$(m \times n)(n \times 1) = m \times 1$$

Red numbers must match

(Matrix)(matrix) is similar, basically just repeat above process

1^{st} column of (AB) is $(A)(1^{st}$ column of $B)$

$$\underset{m \times n}{\begin{bmatrix} \quad & \quad \end{bmatrix}} \underset{n \times p}{\begin{bmatrix} \quad & \quad \end{bmatrix}} = \underset{m \times p}{\begin{bmatrix} \quad & \quad \end{bmatrix}}$$

In general, $(AB)_{ij}$ =

$(i^{th}$ row $A) \cdot (j^{th}$ column $B)$

Ex $\begin{bmatrix} -1 & 2 \\ 5 & 4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix} =$

Theorem: (Properties of Matrix Multiplication)

(Assume sizes of matrices are all appropriate so products defined)

① $A(BC) = (AB)C$

② $A(B+C) = AB + AC$

③ $(B+C)A = BA + CA$

④ $r(AB) = (rA)B = A(rB)$ for any $r \in \mathbb{R}$

⑤ $IA = AI = A$

Note: Very important!

In general AB does NOT equal BA

$$AB \neq BA$$

Property 5 above is an exception

In fact even if AB is defined, BA may not be.

Other warnings:

① $AB = AC$ does NOT mean $B = C$

Do not have a simple guaranteed way to cancel A from both sides, so can't get this conclusion

② $AB = 0_{n \times p}$ does NOT imply either A or B is the zero matrix

We may have A, B both non-zero, but product is zero matrix.

Powers of matrix: If A is square, can multiply A by itself. Why square?

Let A be $m \times n$

$$\begin{array}{cc} A & A \\ \uparrow & \uparrow \\ m \times n & m \times n \end{array} \qquad n = m$$

So A must be $n \times n$ (or $m \times m$)

Can multiply A by itself arbitrary number of times

$$A^k = \underbrace{A \cdot A \cdot A \dots A}_{k\text{-times}}$$

Transpose of matrix: Essentially flipping a matrix on its side. Turn rows into columns, columns into rows

Ex:

$$A = \begin{bmatrix} 1 & 0 & 5 & 6 \\ 2 & 3 & 1 & 9 \end{bmatrix}$$

$$A^T =$$

In general $(A^T)_{ij} = A_{ji}$

Theorem: A, B matrices of appropriate sizes

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = cA^T$ for any $c \in \mathbb{R}$
4. $(AB)^T = B^T A^T$
5. Generalizing above, $(A_1 A_2 A_3 \dots A_k)^T$

$$= A_k^T A_{k-1}^T \dots A_1^T$$