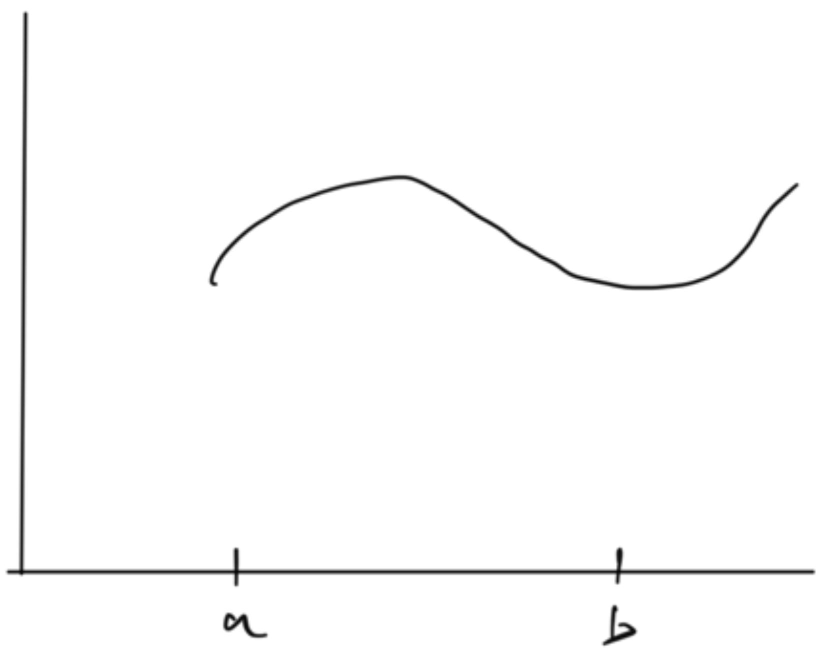
13.3 Arc Length/Curvature

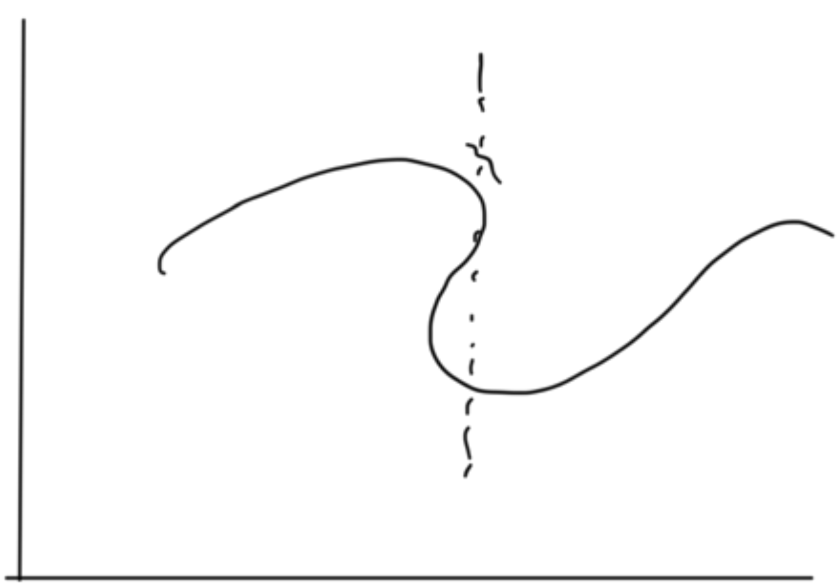
Recall previous concepts of arc length from calc 2



$[f : \mathbb{R} \to \mathbb{R}$

For $f(x)$, function of single variable

☆ $\int_a^b \sqrt{1 + [f'(x)]^2}\, dx$



$\{ x = f(t)$

$y = g(t)$

For parametric equations $f(t), g(t)$

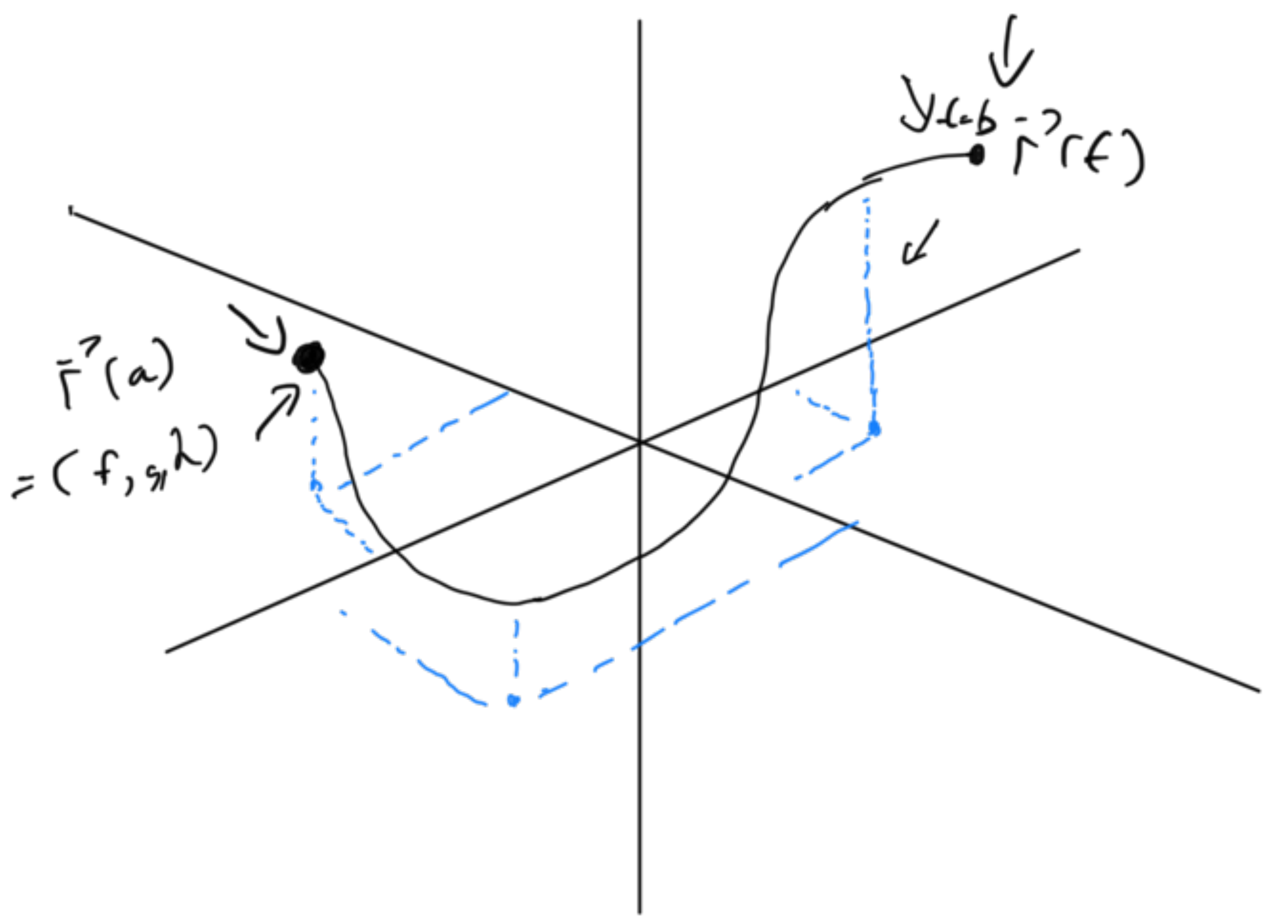
$$\Downarrow$$

$$\int_c^d \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Recall we can think of vector valued function $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ as parametric equations

$$\star\ \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\vec{r} : \mathbb{R} \to \mathbb{R}^3 \quad \begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$$



So, no surprise that length of curve in 3-D is

$$\int_a^b \sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}\,dt$$

$$\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$$

Note that $\sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}$

is exactly $|r'(t)|$.

So can rewrite integral as

$\vec{r}(t)$

arc length

from $t=a$ to $t=b$

$$\int_a^b |r'(t)|\,dt$$

Ex: Find length of curve

$r(t) = \langle t, 3\cos(t), 3\sin(t)\rangle$

for $-5 \le t \le 5$

$$\vec{r}\,'(t) = \langle 1, -3\sin t, 3\cos t\rangle$$

$$\int |r'(t)|$$

$$L = \int_{-5}^{5} \sqrt{1^2 + 9\sin^2 t + 9\cos^2 t}\,dt$$

$$= \int_{-5}^{5} \sqrt{1 + 9}$$

$$\int_{-5}^{5} \sqrt{10}\, dt$$

$$L = 10\sqrt{10}$$

Arc Length Function

$$s(t) = \int_a^t |r'(u)|\, du \quad ☆$$

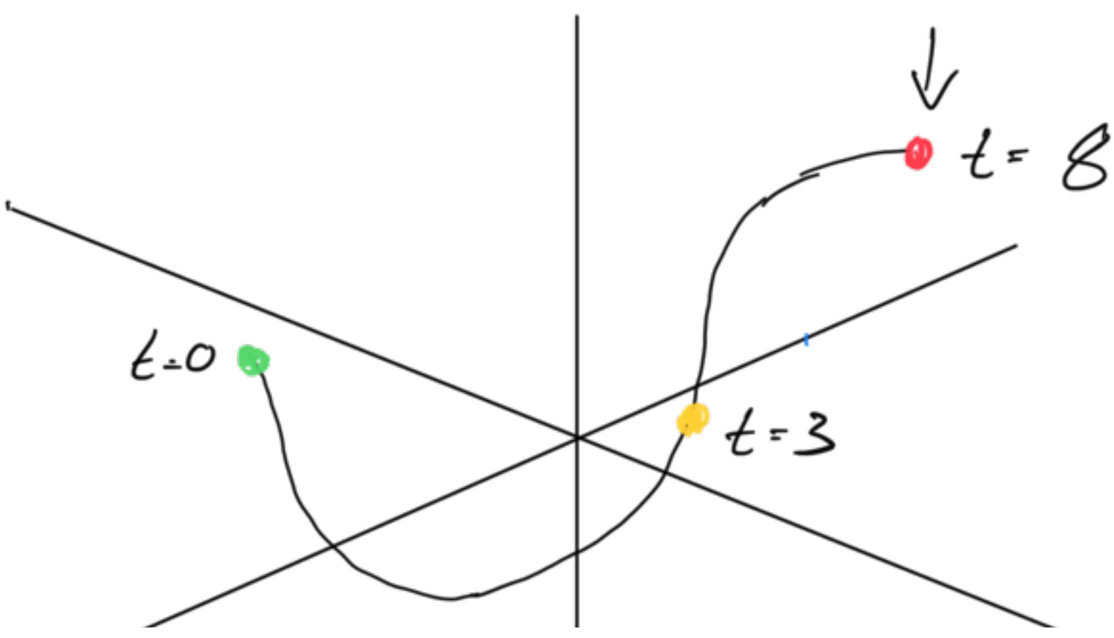
arc length of $\vec{r}(t)$ starting at initial value a, up til value t

Parameters

The parameter in our vector valued function may represent something concrete or something very abstract

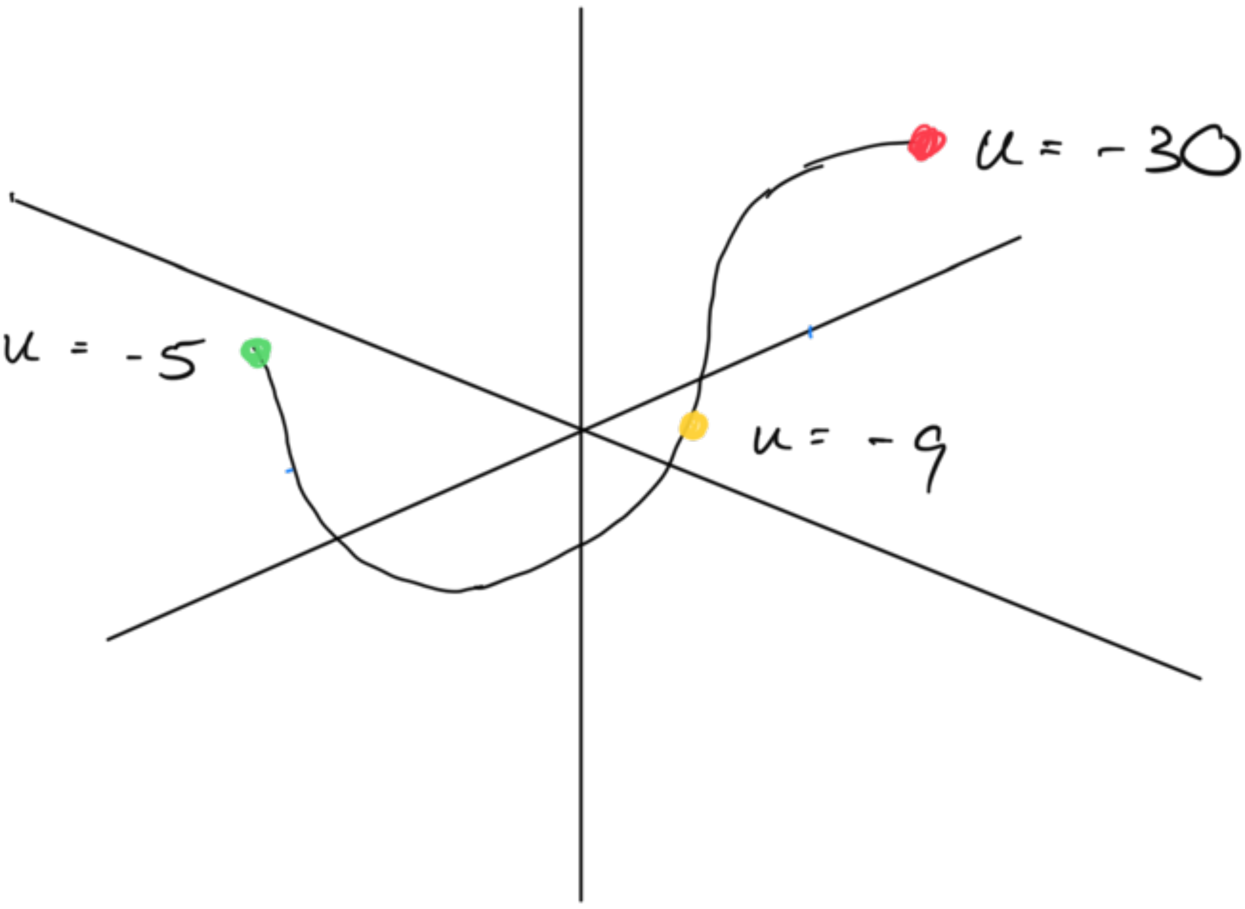
Ex: t may represent "time"

☆ $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$



We could just as easily use some other parameterization

$$\vec{r}(u) = \langle f(u), g(u), h(u) \rangle$$



Can even switch between different parameterizations using essentially u-substitution

Lets say $t = p(u)$

$$dt = p'(u)\,du$$

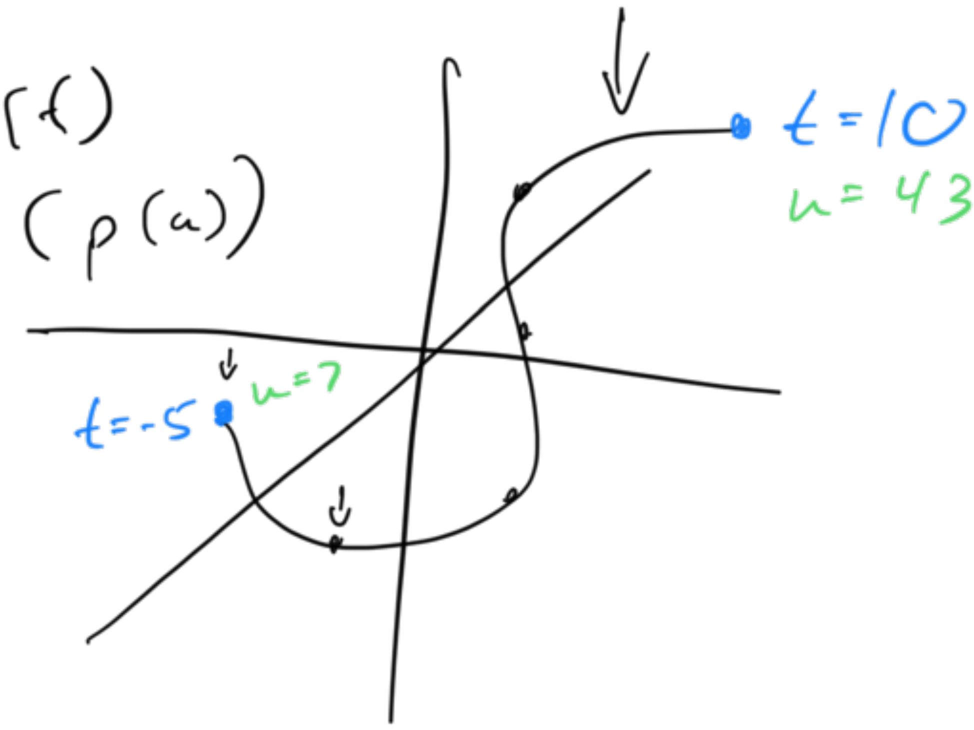
$f(t) = f(p(u))$

$f'(t) = p'(u)\, f'(p(u))$ Chain rule

etc.

$\vec{r}(t)$

$\vec{r}(p(u))$



Parameterization does <u>not</u> change arc length

$$s = \int_a^t |\vec{r}\,'(u)|\, du = \cancel{\int |\vec{r}\,'(p(u))|\, du}$$

So no matter what parameter is

$$\frac{ds}{dt} = |r'(t)|$$

$$s(t) = \int_a^t r'(u)\, du$$

$$\boxed{\frac{ds}{dt} = |r'(t)|}$$

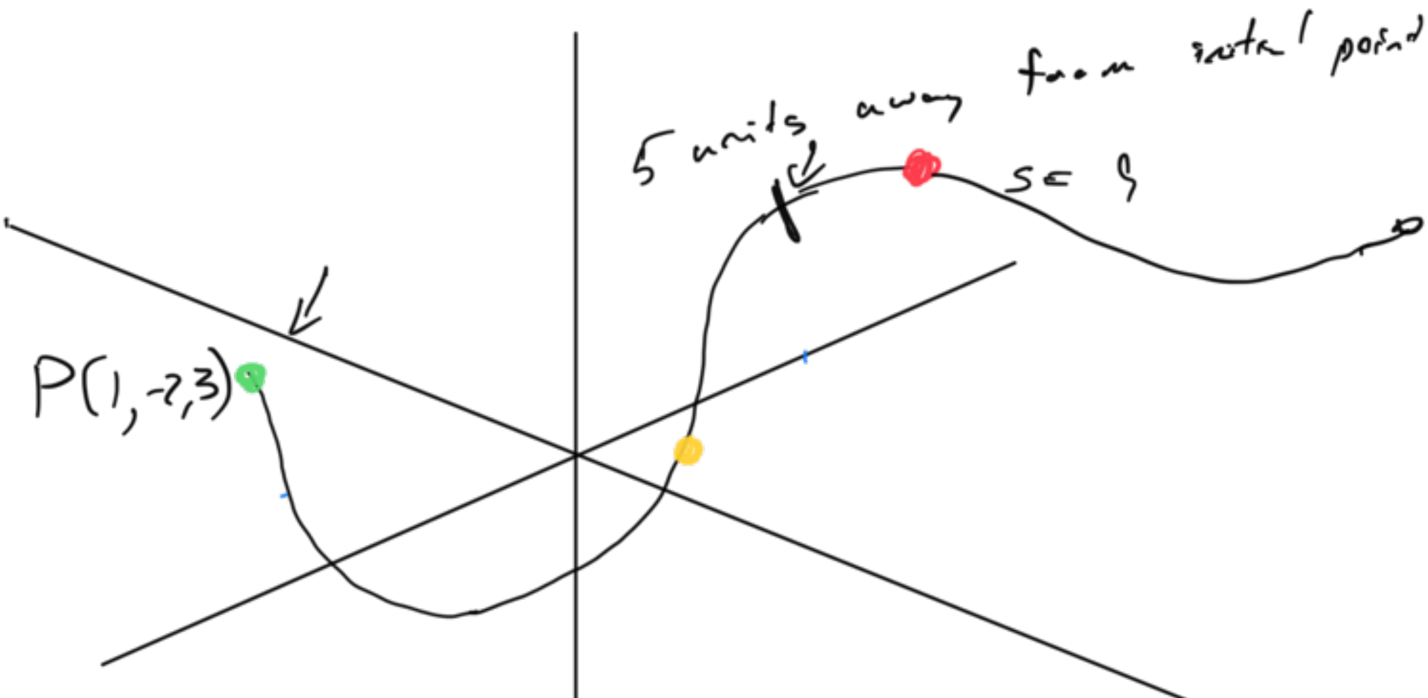
So all parameterizations give same answer for arc length.

Consider other problems. $\boxed{\vec{r}(t)}$

Is there one that is better to use than the others?

Often convenient to work with arc length as parameter.

$$\vec{r}(t) \qquad \vec{r}(p(u))$$



$$r(5) = \langle f(5), g(5), h(5) \rangle$$

$$r(t) = \langle f(t), g(t), h(t) \rangle$$

<u>Don't</u> assume you are given arc

length as parameter.

If we are given $\vec{r}(t)$, is there a way to tell if parameter is arc length or something else?

Yes

Testing

$$\vec{r}(u)$$

Recall $s(t) = \int_a^t |r'(u)| \, du$]

or $\frac{ds}{dt} = \underbrace{|r'(t)|}$ } change in $\vec{r}$ per change in t

change in arc length per change in t }

If $|r'(t)| = 1$ then essentially t changes at same rate as arc length.

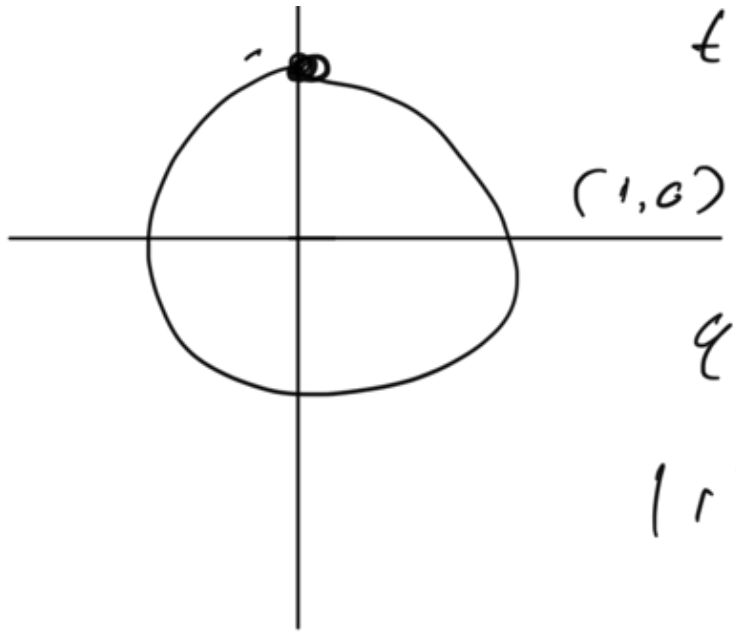
If $r(t)$ = starting point when $t=0$, t will be arc length

Ex: $\vec{r}(t) = \langle \sin(t), \cos(t) \rangle$ ☆

$\vec{r}\,{}^2$

IIC

$t = \pi/4$



s = arc length

$|r'(t)| = 1$

Is this parameterized by arc length measured from $(1,0)$?

$$\vec{r}\,'(t) = \langle \cos(t), -\sin(t) \rangle$$

$$\rightarrow \quad |\vec{r}\,'(t)| = \sqrt{\cos^2(t) + (-\sin(t))^2}$$

$$= 1$$

<u>But</u> $\quad \vec{r}(0) = \langle 0, 1 \rangle$

☆ So if $|r'(t)| = 1$ and $r(0)$ = starting point then parameter we are using is arc length

If we are <u>not</u> given arc length parameter, can we convert to it. Often times yes.

Converting

Based on same idea that we consider

$$s = s(t) = \int_a^t r'(u)\, du$$

a is value of t that corresponds to starting point

$= 1$ t are length parameter

$$\frac{ds}{dt} = |r'(t)|$$

If $|r'(t)| = b$, a nonzero constant $\neq 1$

will have

$$s = s(t) = \int_a^t |r'(u)|\, du$$

$$= \int_a^t b\, du$$

$$\frac{s}{b} = \frac{\cancel{b} \cdot (t-a)}{\cancel{b}}$$

$$\frac{s}{b} = t - a$$

$\vec{r}(t)$

t value of starting point

$$\frac{s}{|r'(t)|} + a = t \qquad \vec{r}\left(\frac{s}{b} + a\right)$$

Make left side new parameter

Ex: $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ $\mathbb{R}^3$

Reparameterize for arc length measured from $(1, 0, 0)$

D $(1,0,0)^{*}$ "starting point" corresponds to t value of $\boxed{0}$

$$\begin{cases} r'(t) = \langle -\sin(t), \cos(t), 1\rangle \\ |r'(t)| = \sqrt{2} \end{cases}$$

$$\boxed{\frac{s}{\sqrt{2}} + 0} = t$$

$$\boxed{\frac{s}{\sqrt{2}} = t}$$

So $\vec{r}(s) = \langle \cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), \frac{s}{\sqrt{2}} \rangle$ is parameterized by arc length.

$$\frac{s}{|r'(t)|} + a \qquad (1, 0, 0)$$

What if measured from starting point $(0, 1, \pi/2)$? The t value that gives this starting point is $t = \pi/2$. So arc length parameterization is

$$\langle \cos\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \sin\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \frac{s}{\sqrt{2}} + \frac{\pi}{2} \rangle$$

$$\boxed{\frac{s}{\sqrt{2}} + \frac{\pi}{2}} = \boxed{t}$$

Ex: Let $\vec{r}: \mathbb{R} \to \mathbb{R}^3$ be

$$\vec{r}(t) = \langle \sin(4t), \cos(4t), 3t \rangle$$

Determine if $\vec{r}$ is parameterized by arc length. If not, convert to arc length, measured from $(0, 1, 0)$

$$|\vec{r}'(t)| = 1, \quad \text{and} \quad r(0) = (0,1,0) \; ?$$

$$r'(t) = \langle 4\cos(4t), -4\sin(4t), 3 \rangle$$

$$|r'(t)| = \sqrt{16\cos^2(4t) + 16\sin^2(4t) + 9}$$

$$= \sqrt{25}$$

$$= 5$$

$$a = 0$$

$$\frac{s}{|r'(t)|} + a = t$$

$$\frac{s}{5} + 0 = t$$

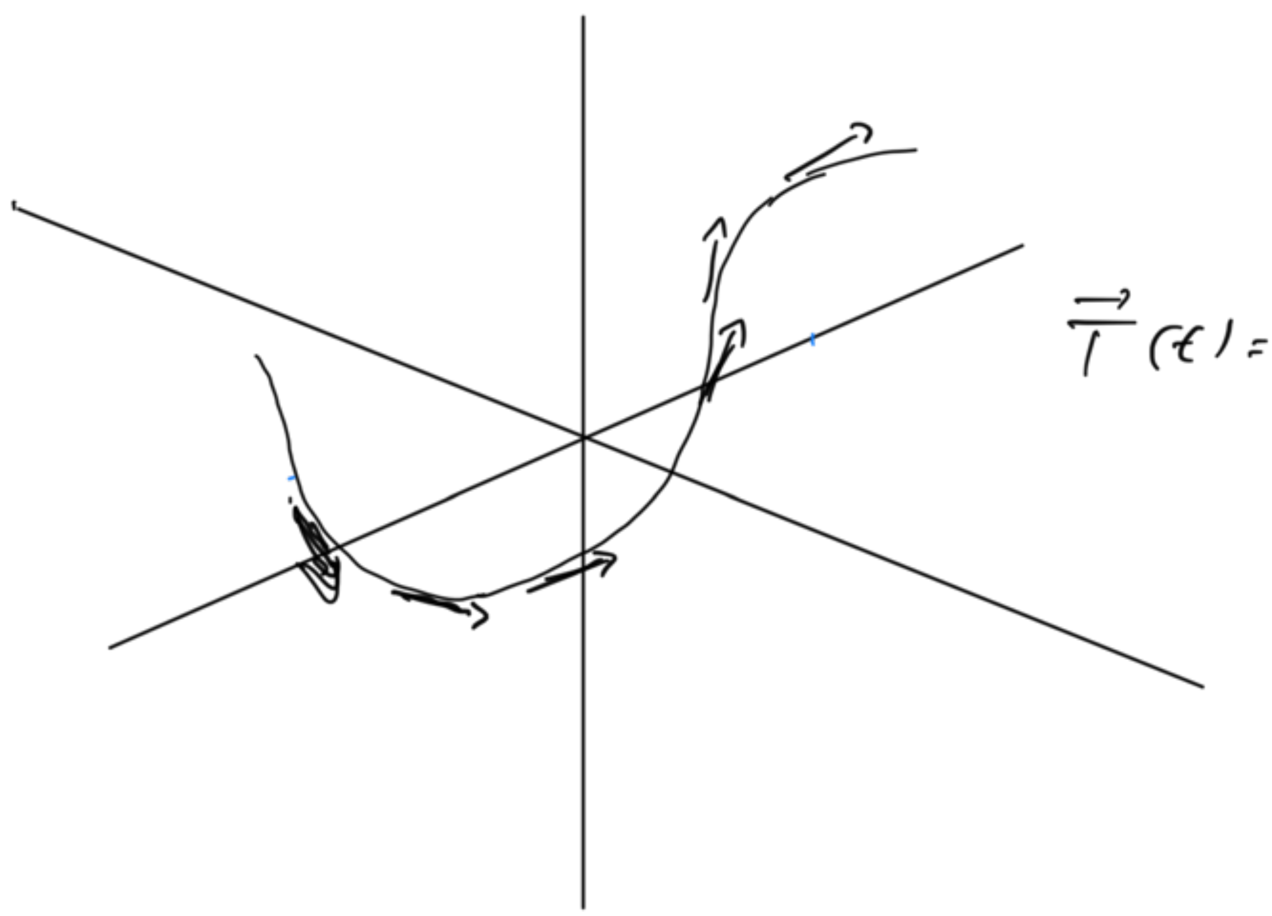
$$\frac{s}{5}$$

$$\vec{r}(s) = \left\langle \sin\left(\frac{4s}{5}\right), \cos\left(\frac{4s}{5}\right), \frac{3s}{5} \right\rangle$$

Curvature

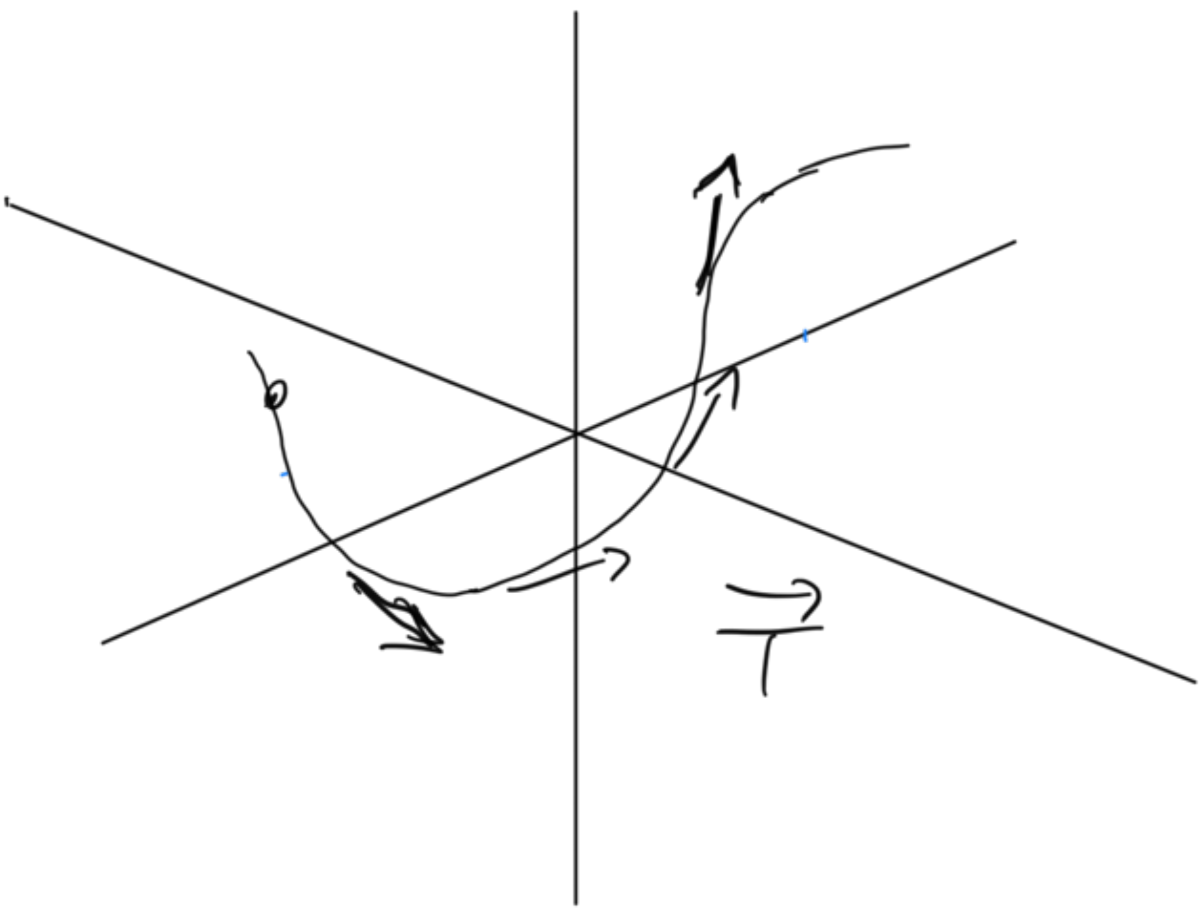
Given a curve, we would like some

way to evaluate how quickly it changes direction.



$\vec{r}(t)$

Know that <u>tangent vector</u> $\vec{r}\,'(t)$ corresponds to change in direction. But magnitude of tangent may be misleading

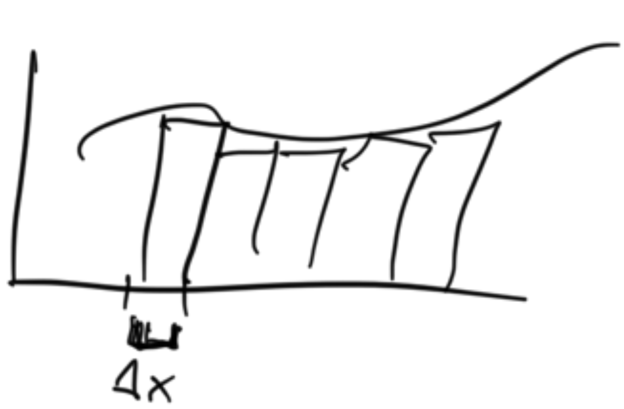


That is why we developed unit tangent vector, $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

Magnitude always 1 so change in $\vec{T}$, aka $d\vec{T}$, is just change in direction

$\Delta \vec{T}$ change is "discrete," <u>not</u> continuous

ΔT



$d T$

$d\vec{T}$ "change in $\vec{T}$"

$$\vec{T}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\frac{d\vec{T}}{dt} = \vec{T}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

Thus, how "quickly" $\vec{T}$ changes measures how quickly direction is changing

What should we measure $\vec{T}$ against?

Arc length.

"How quickly unit tangent changes in response to change in arc length"

Thus $\left|\frac{d\vec{T}}{ds}\right|$ is a measure of how quickly a curve, parameterized by arc length, changes direction

Call this quantity **curvature**, κ

$$\kappa = \left|\frac{dT}{ds}\right| \qquad \vec{r}'(t)$$

$$\kappa$$

If we are given $\underline{\vec{r}(t)}$ and want to find $\underline{\kappa}$, dont want to convert to arc length everytime.

<u>But</u> by chain rule

$$\kappa = \frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds}$$

$$\Rightarrow \kappa = \frac{d\vec{T}/dt}{ds/dt}$$

Recall $ds/dt = |r'(t)|$, so

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{\vec{T}'(t)}{\vec{r}'(t)}\right|$$

$$= |\vec{T}'(t)|$$

$$|\vec{r}\,'(t)|$$

Ex:

Find curvature of $\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$

$$\vec{T}(t) = \frac{r'(t)}{|r'(t)|} \qquad K = \frac{|T'(t)|}{|r'(t)|}$$

$$\begin{cases} r'(t) = \langle \cos(t), -\sin(t), 1 \rangle \\ |r'(t)| = \sqrt{\cos^2(t) + \sin^2(t) + 1} \end{cases}$$

$$= \sqrt{2}$$

$$\vec{T}(t) = \frac{1}{\sqrt{2}} \langle \cos(t), -\sin(t), 1 \rangle$$

$$\rightarrow \vec{T}\,'(t) = \frac{1}{\sqrt{2}} \langle -\sin(t), -\cos(t), 0 \rangle$$

$$|T'(t)| = \left| \frac{1}{\sqrt{2}} \langle -\sin(t), -\cos(t), 0 \rangle \right|$$

$$= \left|\frac{1}{\sqrt{2}}\right| \, |\langle -\sin(t), -\cos(t), 0 \rangle|$$

$$= \left|\frac{1}{\sqrt{2}}\right| \sqrt{\sin^2 t + \cos^2(t)}$$

$$|\vec{T}\,'(t)| = \frac{1}{\sqrt{2}}$$

$$K = \frac{|T'(t)|}{|r'(t)|} = \frac{1/\sqrt{2}}{\sqrt{2}}$$

$$\star \; K(t) = \frac{1}{2}$$

Another formula for κ:

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

$$\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$$

$$\begin{cases} \vec{r}\,' = \langle \cos, -\sin, 1 \rangle = \\ |r'(t)| = \sqrt{2} \\ \vec{r}\,'' = \langle -\sin, -\cos, 0 \rangle \end{cases}$$

$$r' \times r'' = \begin{array}{ccc} i & j & k \\ \cos & -\sin & 1 \\ -\sin & -\cos & 0 \end{array}$$

$$= (0 + \cos(t))i - (0 + \sin(t))j + (-\cos^2 - \sin^2(t))k$$

$$\star\ \langle \cos(t), -\sin(t), -1 \rangle$$

$$\star\ \frac{|\vec{r}\,' \times r''|}{|r'|^3} = \frac{\sqrt{2}}{\sqrt{2}^3} = \frac{1}{(\sqrt{2})^2}$$

$$= \frac{1}{2}$$

Recapping $K(t)$

Mostly theory $\quad K = \left| \dfrac{d\vec{T}}{ds} \right|$

$\vec{r}(t)$

Practical

$$K(t) = \left| \frac{\vec{T}'(t)}{\vec{r}'(t)} \right|$$

$$K(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Normal vectors

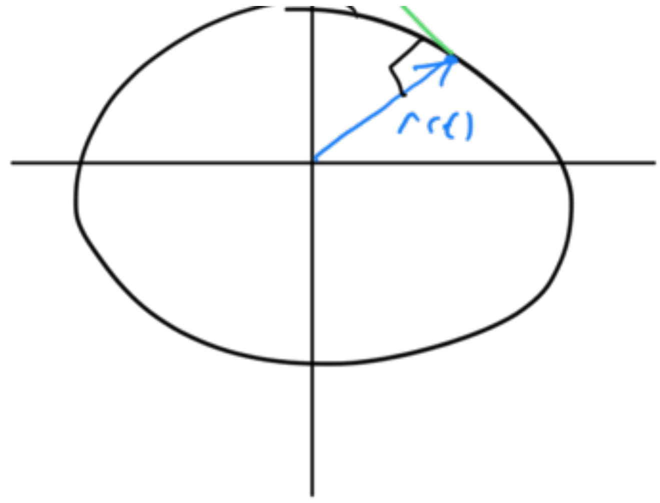
First, a factoid.

If $\vec{r}(t)$ a vector valued function

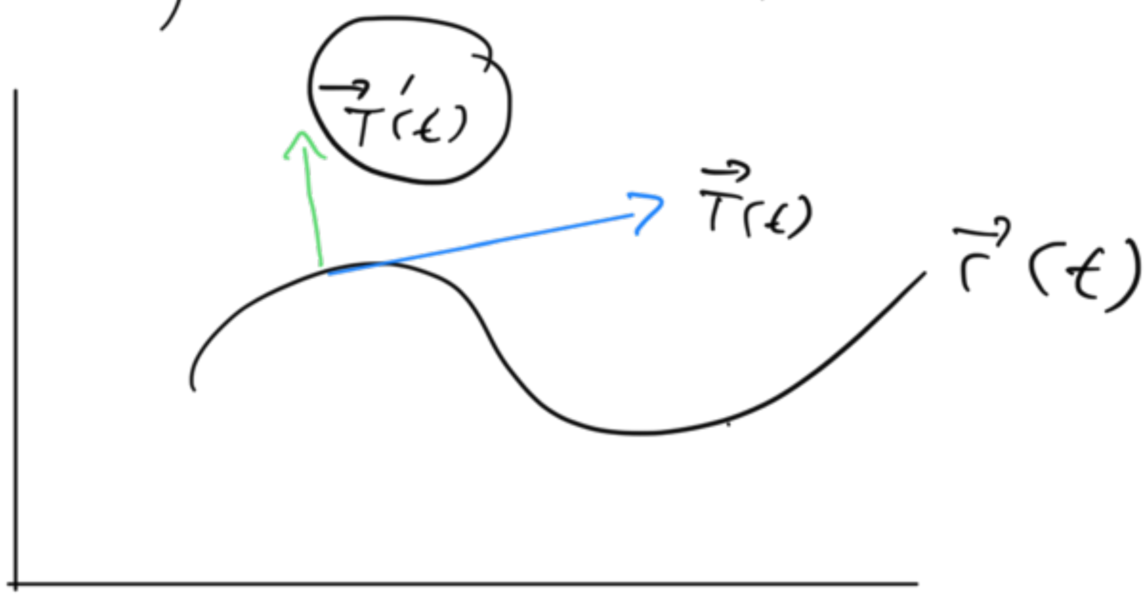
and $|\vec{r}(t)| = c$, constant, then

$$\vec{r}(t) \cdot \underline{\vec{r}'(t)} = 0$$

So $\vec{r}$ orthogonal to its tangent $r'(t)$



Applying this fact to $\vec{T}(t)$, since $|\vec{T}(t)|=1$, $\vec{T}'(t)$ orthogonal to $\vec{T}$

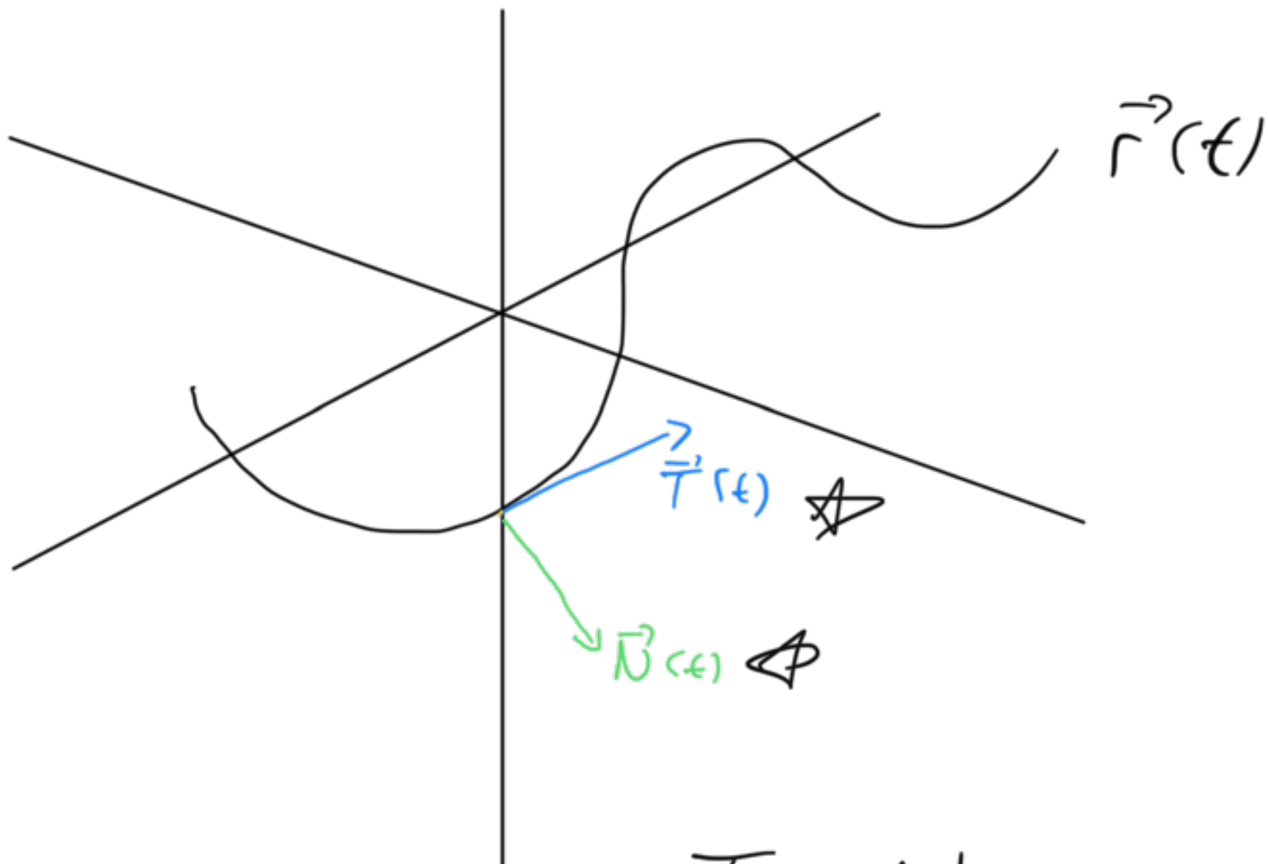


This vector will thus be hitting original curve $\vec{r}(t)$ at $90°$ angle

We say $\vec{T}'(t)$ is **normal** to $\vec{r}(t)$.

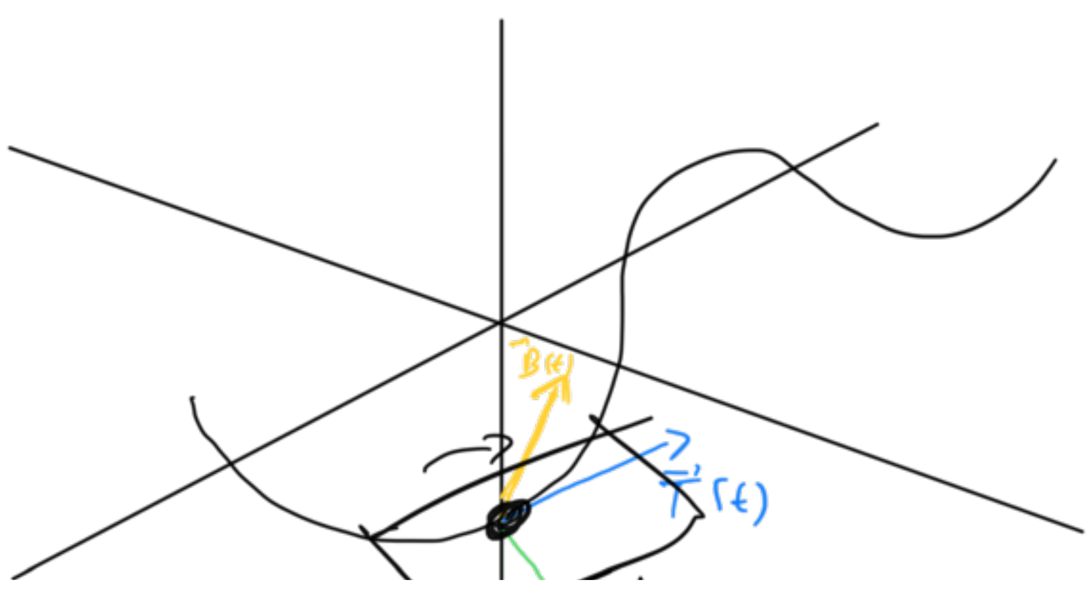
Once again, to avoid confusion, usually make length of this vector 1, giving us **principal unit normal vector**, aka the unit normal, $N(t)$

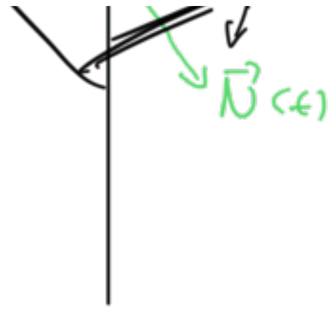
$$N(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$ unit vector Normal to $\vec{r}(t)$



$T \times N$

Can take cross product of $\vec{T}$ and $\vec{N}$ to get 3rd vector orthogonal to both, called <u>Binormal vector</u>, $\vec{B}(t)$





$$\left|\vec{B}(t)\right| = \left|\vec{T} \times \vec{N}\right|$$

$$= \underbrace{\left|\vec{T}\right|}_{1} \cdot \underbrace{\left|\vec{N}\right|}_{1} \cdot \underbrace{\sin(\theta)}_{1}$$

$$= 1$$

Normal plane vs Osculating Plane