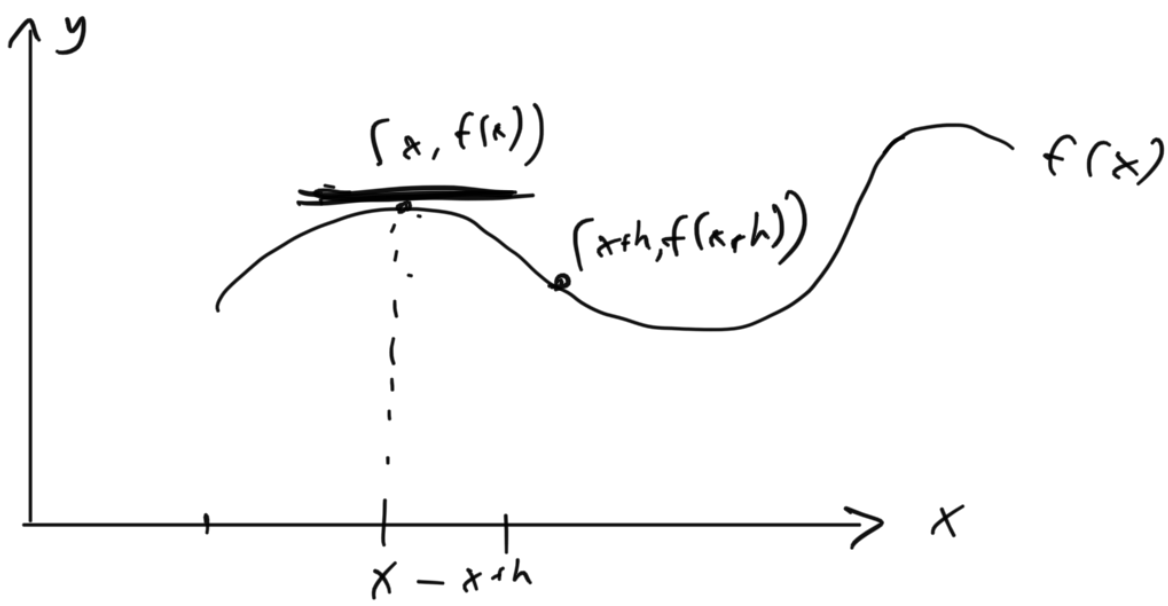
14.3 - Partial Derivatives

Want to develop idea of derivatives for $f: \mathbb{R}^n \to \mathbb{R}$

Recall idea of derivative for $\underline{f: \mathbb{R} \to \mathbb{R}}$



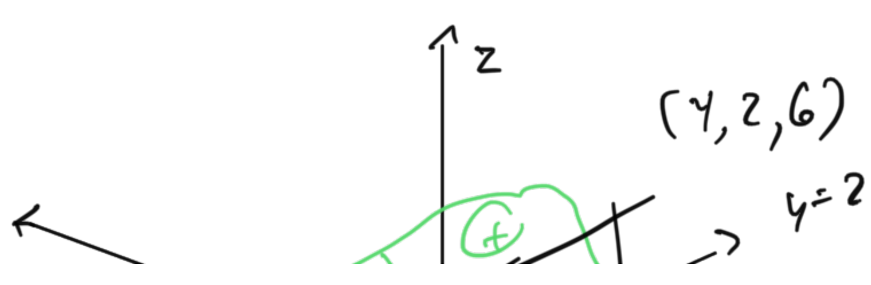
* $f'(x)$ gives slopes of tangent lines

* Analytically,

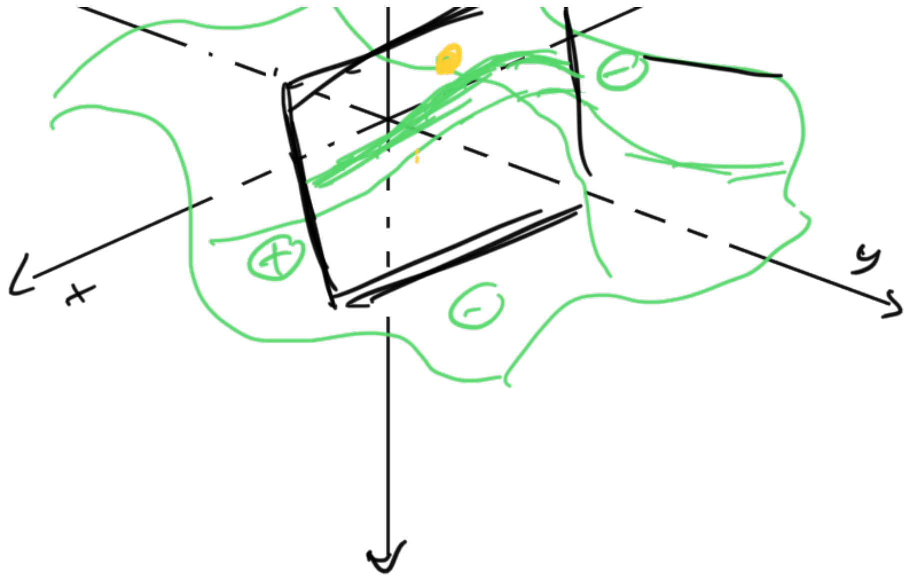
$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

[Remember, for limit to exist it must be the same from both directions]

Now $\underline{f: \mathbb{R}^2 \to \mathbb{R}}$



Can have an <u>infinite</u> number of tangent lines all



through same point

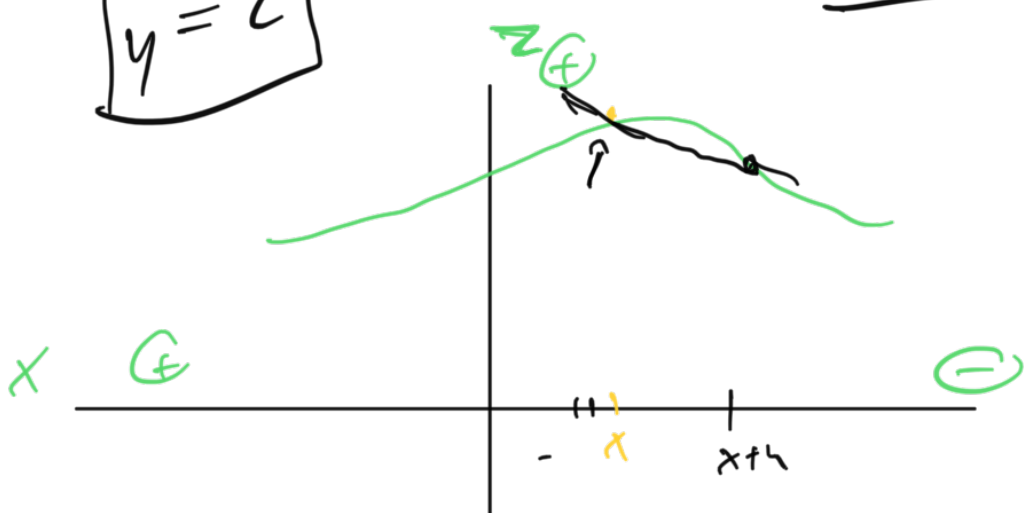
Which ones do we care about?

Too much to consider all possibilities.
Restrict ourselves to derivatives in x, y directions.

Take a slice of function parallel to x-axis

$$f: \mathbb{R}^2 \to \mathbb{R}$$

$$\boxed{y = 2}$$



$$\lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

How do we define them formally (as limits?)

Once we confine ourselves to this slice we can form derivative as normal

$f: \mathbb{R} \to \mathbb{R}$

Hold y constant.

$$\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h}$$

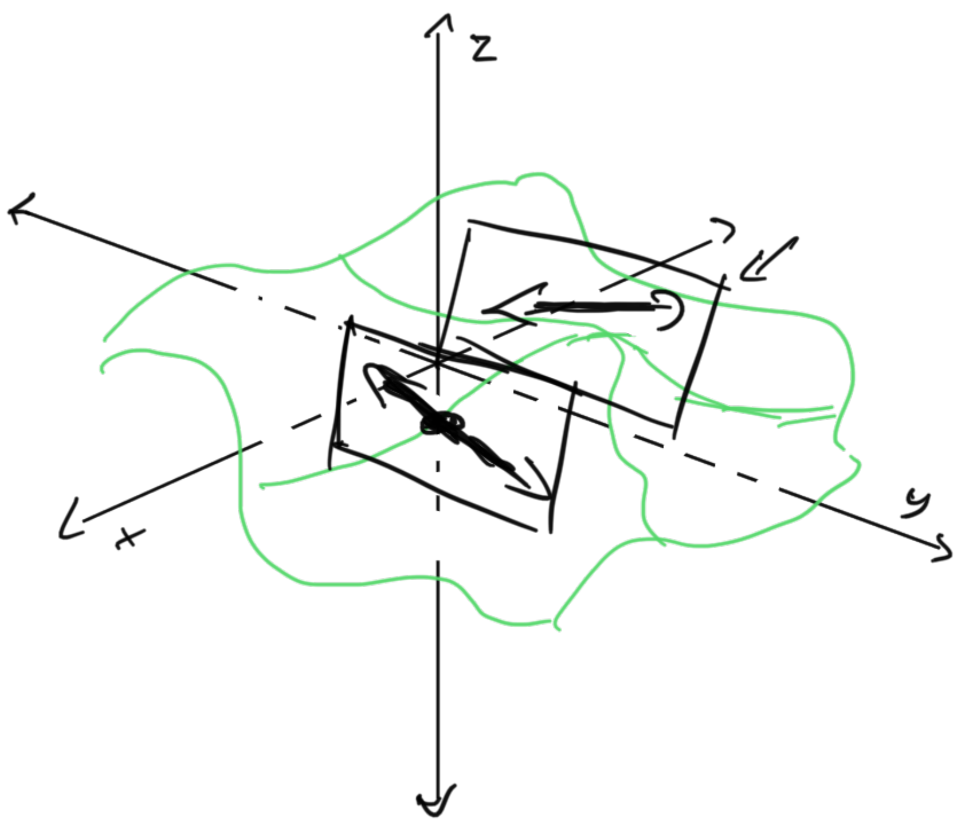
Partial derivative of f with respect to x

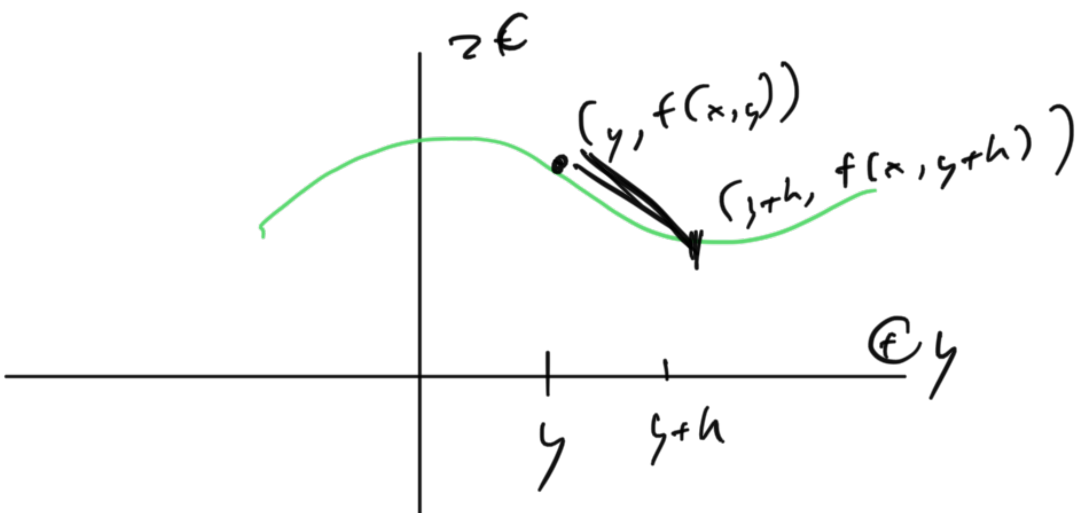
<u>In practice:</u>

$$f(x,y) = x^3 \ln(y) + \frac{e^x}{y} - \sin(y)$$

$$f_x = 3x^2 \ln(y) + \frac{e^x}{y}$$

Works same way for df/dy.





$$\lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Partial derivative of f with respect to y

In practice:

$$f(x,y) = \underbrace{x^3 \ln(y)} + \frac{e^x}{y} - \sin(y)$$

$$f_y = x^3 \frac{1}{y} - \frac{e^x}{y^2} - \cos(y)$$

In fact, can find partial derivatives for function of several variables in same exact way. $f: \mathbb{R}^n \to \mathbb{R}$

Different Notations

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y)$$

$$= \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

Ex. $f(x,y) = \sin\left(\frac{x}{1+y}\right)$

Find f_x, f_y.

$$f_x = \frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)$$

$$f_y = \frac{-x}{(1+y)^2}\cos\left(\frac{x}{1+y}\right)$$

Ex: Assume $z = f(x,y)$. Given that:

$$x^3 + y^3 + z^3 + 6xyz = 1$$

find $\partial z/\partial x$, $\partial z/\partial y$.

$$3x^2 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$3x^2 + 6yz = -3z^2\frac{\partial z}{\partial x} - 6xy\frac{\partial z}{\partial x}$$

$$3x^2 + 6yz = \frac{\partial z}{\partial x}\left(-3z^2 - 6xy\right)$$

$$\frac{3x^2+6yz}{} = \frac{\partial z}{} = -\left(x^2 + 2yz\right)$$

$-3z^2 - 6xy$ $\qquad \partial x \quad \sqrt{z^2 + 2xy}$

Higher Derivatives

$f : \mathbb{R} \to \mathbb{R}$

Can take derivatives of derivatives as well, giving us 2^{nd} derivatives.

Recall for $f : \mathbb{R} \to \mathbb{R}$,

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2 f}{dx^2} = f''$$

Similarly, can take partials of partials.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

But what if we take $\frac{\partial}{\partial y}$ of $\frac{\partial f}{\partial x}$?

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$$

Mixed partials

Does $f_{xy} = f_{yx}$

<u>Not</u> <u>always</u>.

However, under certain (common) conditions this will be the case.

<u>Clairaut's Theorem</u>: Suppose f is defined on a disk D that contains (a,b). If f_{xy} and f_{yx} are both continuous on D then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) \quad \text{Continuous} \quad \checkmark$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) \quad \text{Continuous} \quad \checkmark$$

Differential

Equations

Ordinary
Differential
Eq

Partial
Differential
Equations