# 12.1 - 3D Space

Calc 3 is same concepts as Calc 1, 2
but in higher dimensions

## Calc 1, 2

☆ $f(x) = y$

Input Output

Used to having a single number / variable
as input and output

Ex: Price of bracelet as a function
of cost of gold

$x$ = Cost of gold

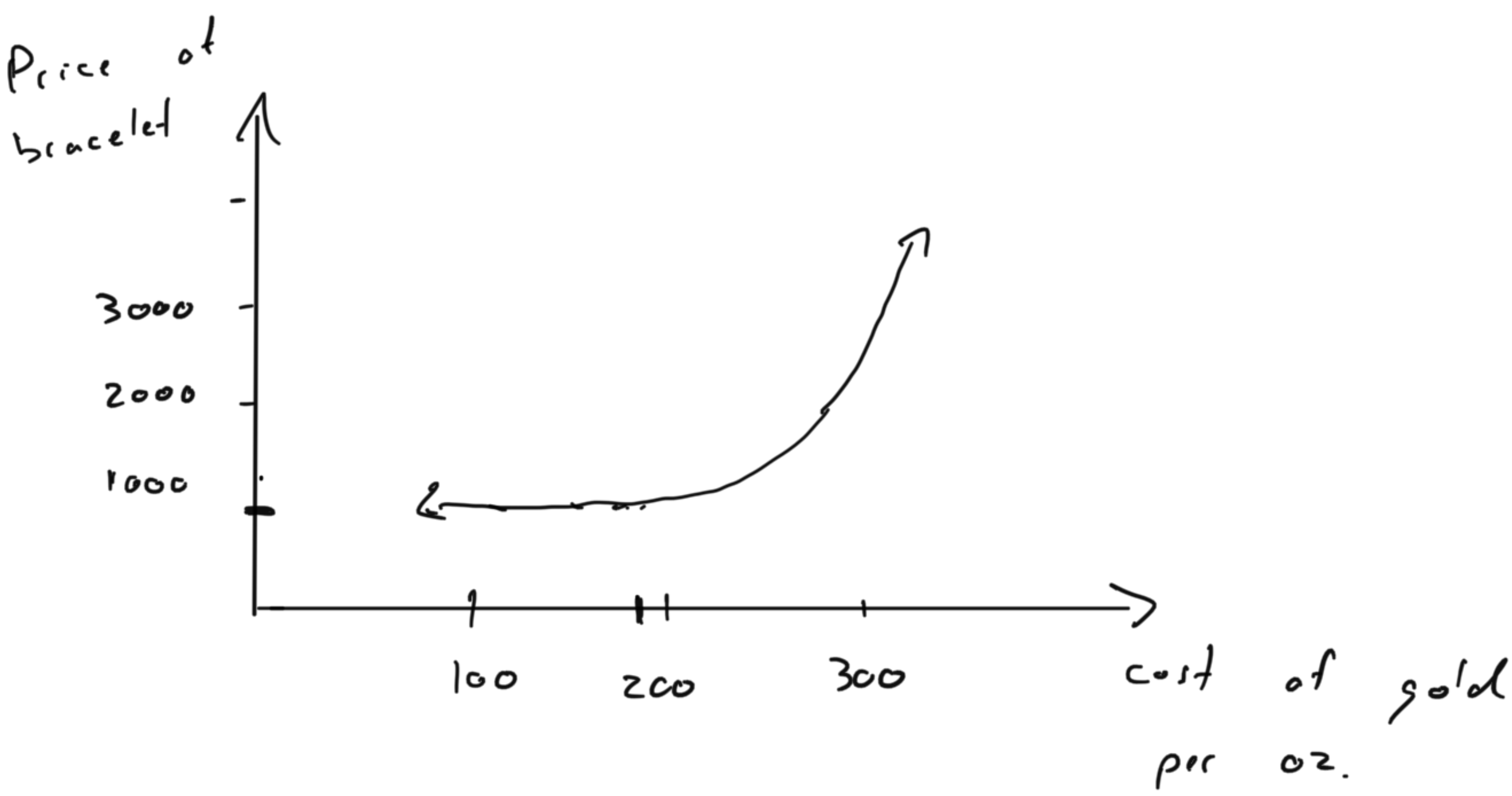
$y$ = price of bracelet

$$y = f(x)$$

Input: $\longleftrightarrow$ $\mathbb{R}$

Output: $\longleftrightarrow$ $\mathbb{R}$

Graph: $\mathbb{R}^2$

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Remember in Calc 1, 2 we were concerned with things like

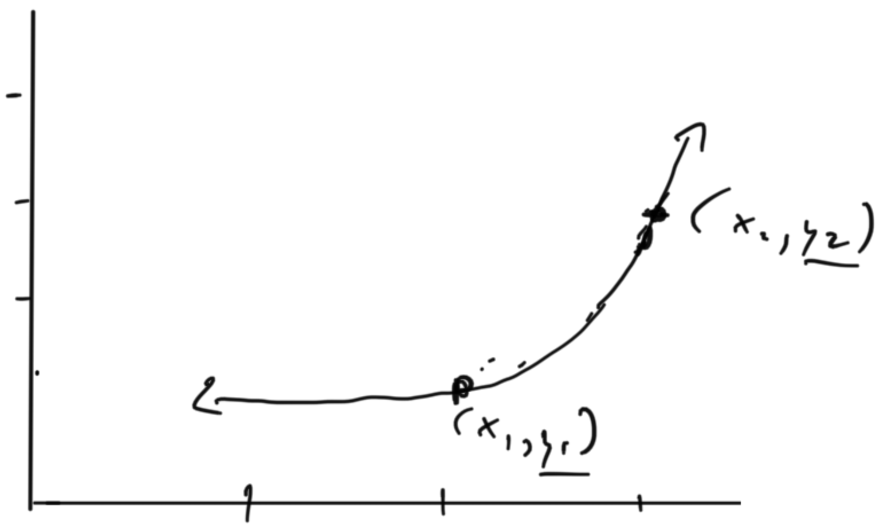
Limits

Continuity

Derivatives

Integrals

Can use graph of function in $\mathbb{R}^2$ to help interpret these things




$$\frac{y_2 - y_1}{x_2 - x_1}$$

So need to be able to work with points in $\mathbb{R}^2$

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## Calc 3

Same concepts in higher dimensions

**Ex.** Price of bracelet a function of cost of gold and cost of diamonds

$x$ = cost gold $\qquad$ $y$ = cost diamonds

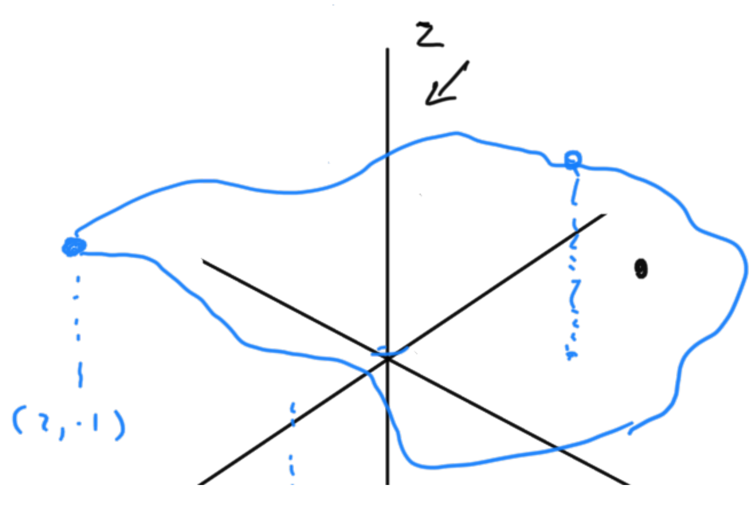
$z$: price of bracelet

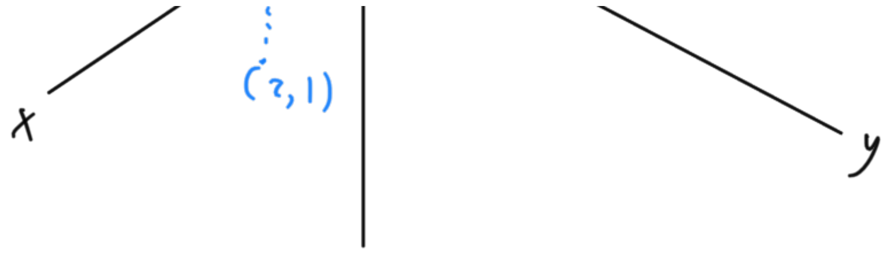
$$z = f(x,y)$$

**Input:** Can think of input as being point $(x,y)$ in $\mathbb{R}^2$

**Output:** A single number in $\mathbb{R}$

**Graph:**




$$\frac{y_2-y_1}{x_2-x_1}$$

Can still talk about derivatives/integrals of this type of function but have slightly more complicated interpretation

For $f:\mathbb{R}\to\mathbb{R}$ had to consider/work with points in $\mathbb{R}^2$ to do these operations

$$\downarrow$$

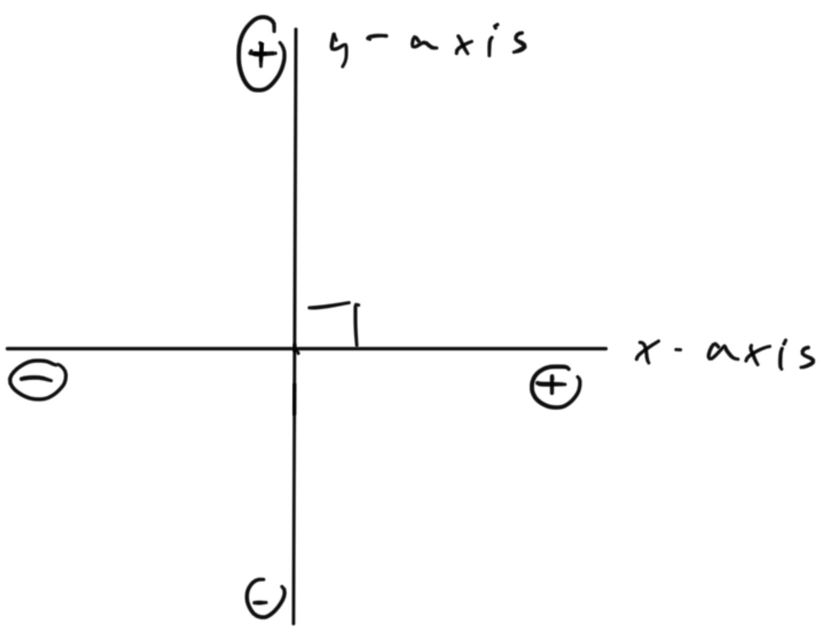
For $f:\mathbb{R}^2\to\mathbb{R}$, need to be able to work with points in $\underline{\mathbb{R}^3}$

Chapter 12 is basics of working in $\mathbb{R}^3$.

## 3-D Space

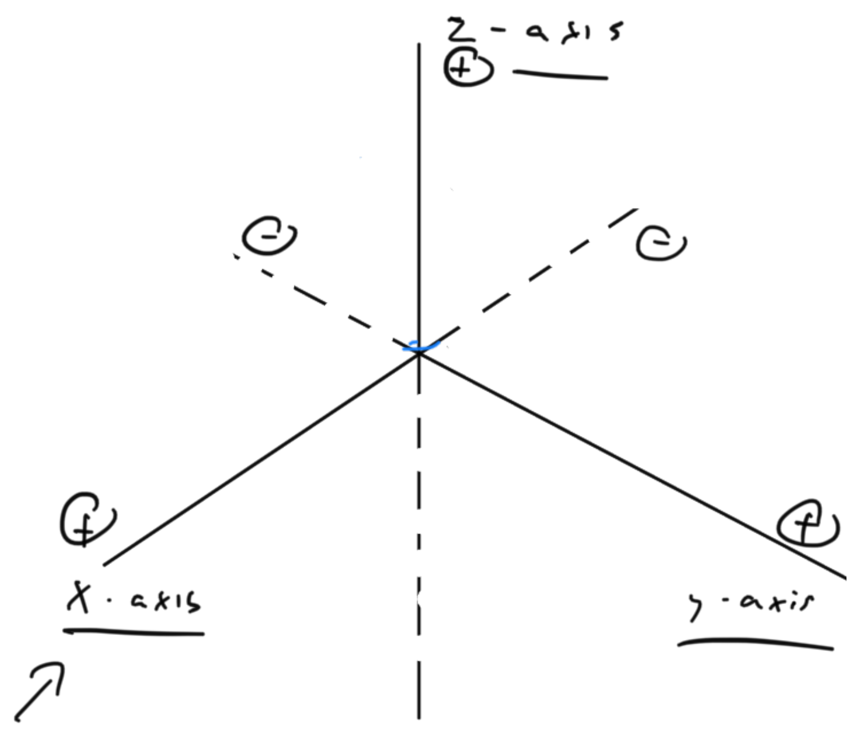
$\mathbb{R}^2$

$\{(x,y) : x, y \in \mathbb{R}\}$



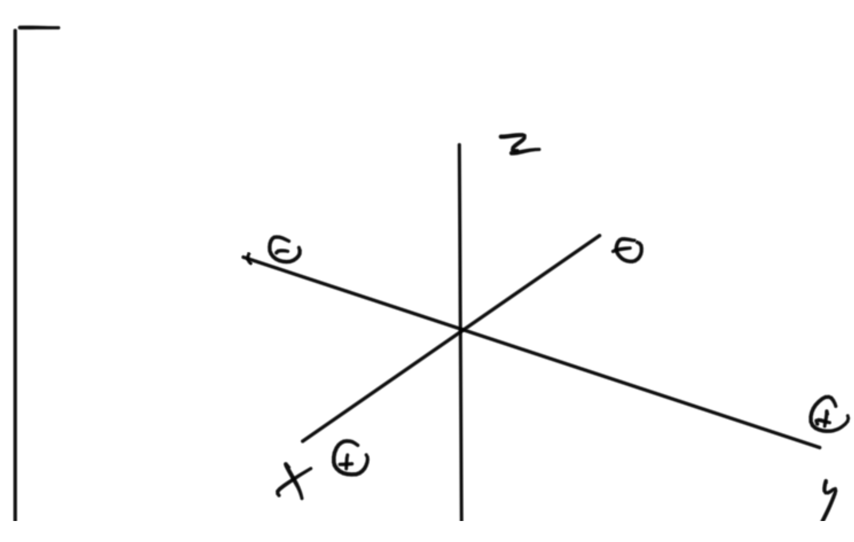

$\mathbb{R}^3$

$\{(x,y,z) : x, y, z \in \mathbb{R}\}$

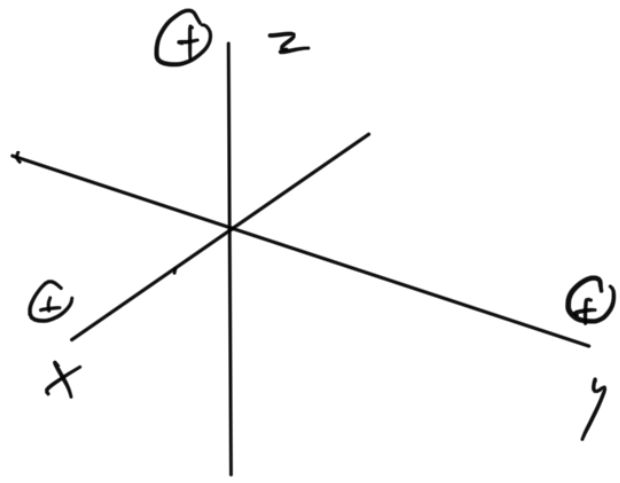



Why do we draw $\mathbb{R}^3$ this way?

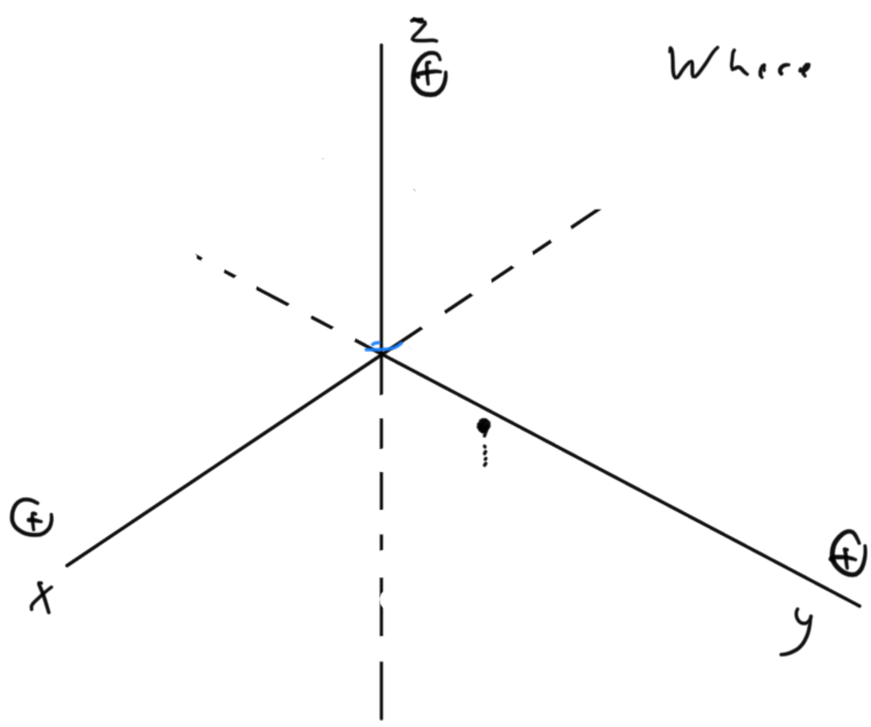
Start with x and y



Now need to add in the z-axis




## Points in $\mathbb{R}^3$



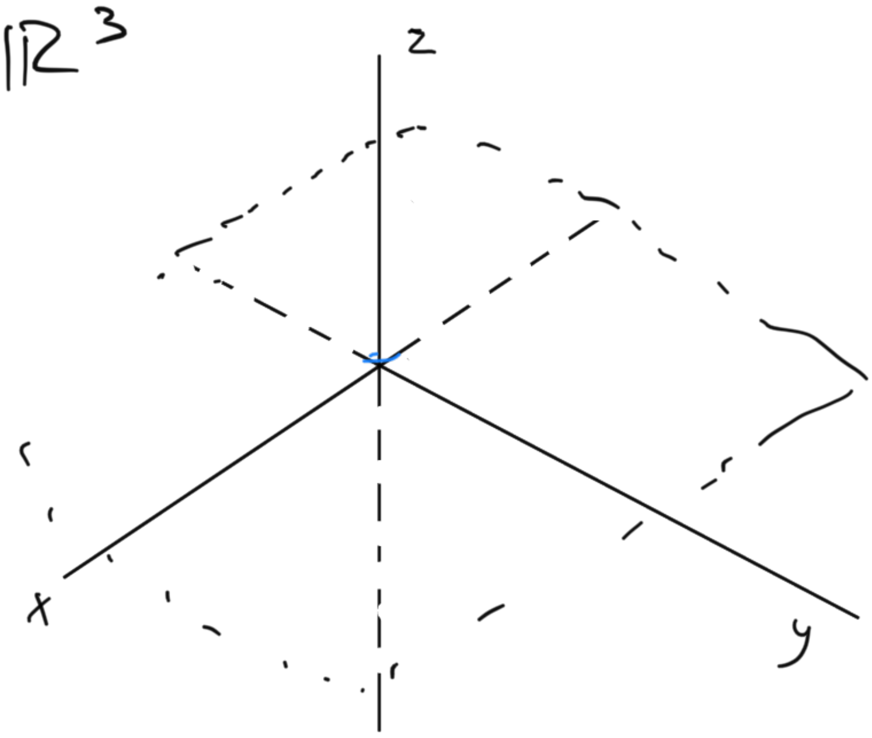

Where is $\underline{P = (1, 2, 1)}$ ?

These numbers are the <u>coordinates</u> of $P$.

Gives us the three-dimensional rectangular coordinate system.

## Coordinate Planes

$\mathbb{R}^3$




Note that the xy-plane still exists inside $\mathbb{R}^3$

Call this a coordinate plane

What does a point in xy-plane look like?

Ex:

$(2, 4, 0)$

$(1, 7, 0)$

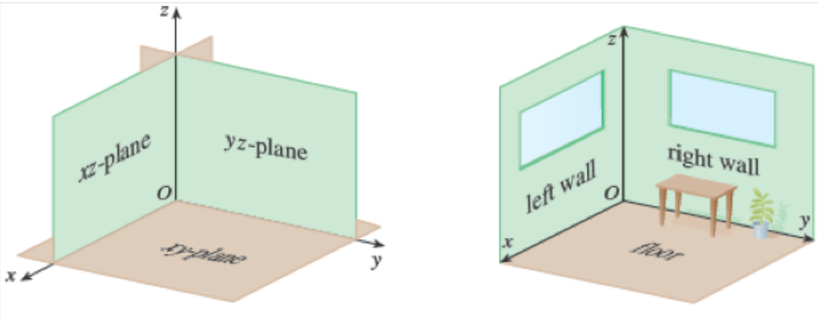
$(100000, -.9, 0)$

z-coordinate is always 0. This is a key feature of xy-plane.

There is also a xz-plane and a yz-plane

For xz-plane, y-coordinate is 0

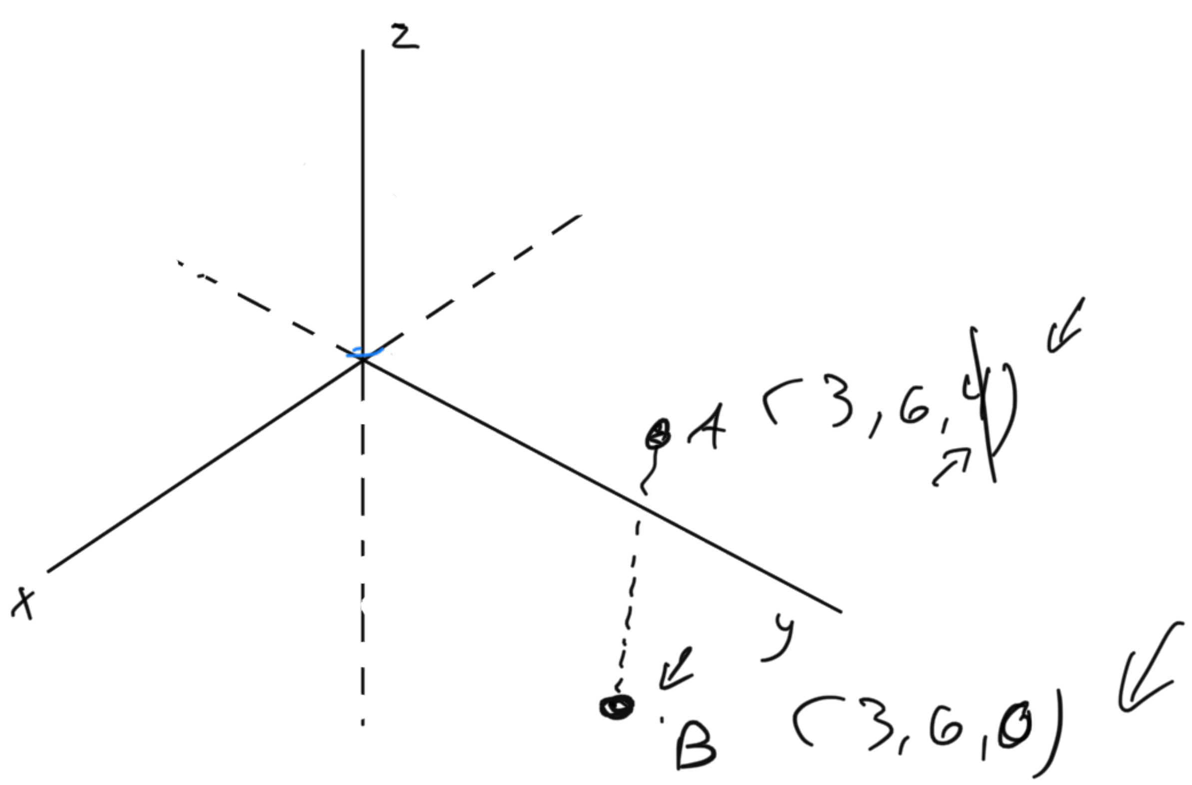
yz-plane , x-coordinate is 0




(a) Coordinate planes (b)

## Projections

Given point in $\mathbb{R}^3$, sometimes want to find <u>version</u> of the point on a coordinate plane (xy-plane for ex.)




Ex: $A = (3, 6, 4)$

Projection on to xy-plane

Projection on to xz-plane

Projection on to yz-plane

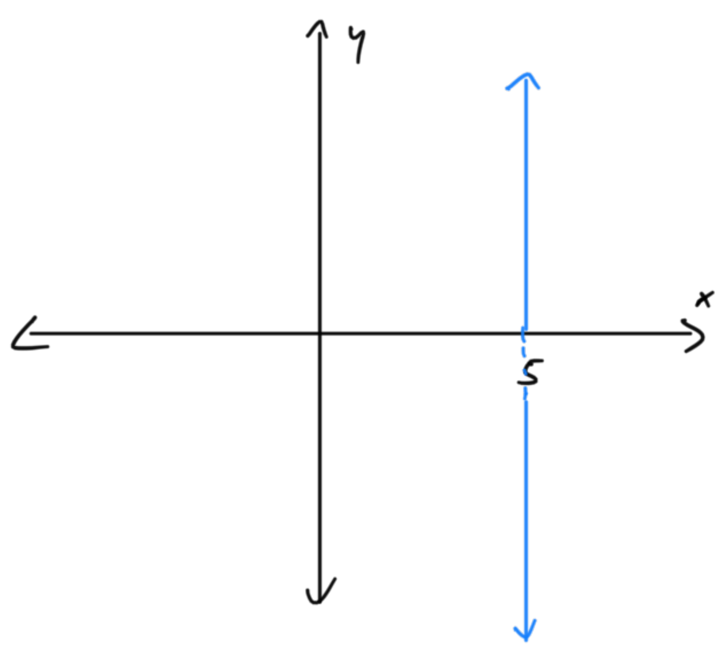
Surfaces

As we saw earlier, function $f(x,y)=z$ gives a surface in $\mathbb{R}^3$

**Note:** May see $f(x,y,z)=c$ instead

## Simple Surfaces

In $\mathbb{R}^2$ equation $x=5$ is a vertical line



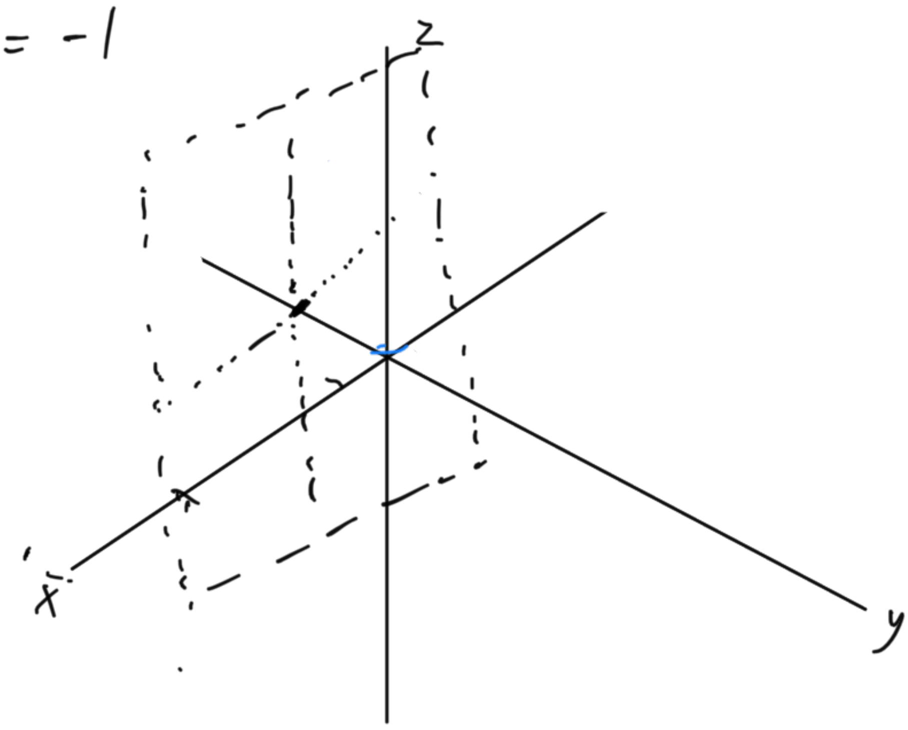

Interpretation: All points in $\mathbb{R}^2$ whose x-coordinates are 5

$$\{(x,y) : x = 5\}$$

What are points in $\mathbb{R}^3$ whose $x$ coor. is $5$?

$$\{(x,y,z) : x = 5\}$$

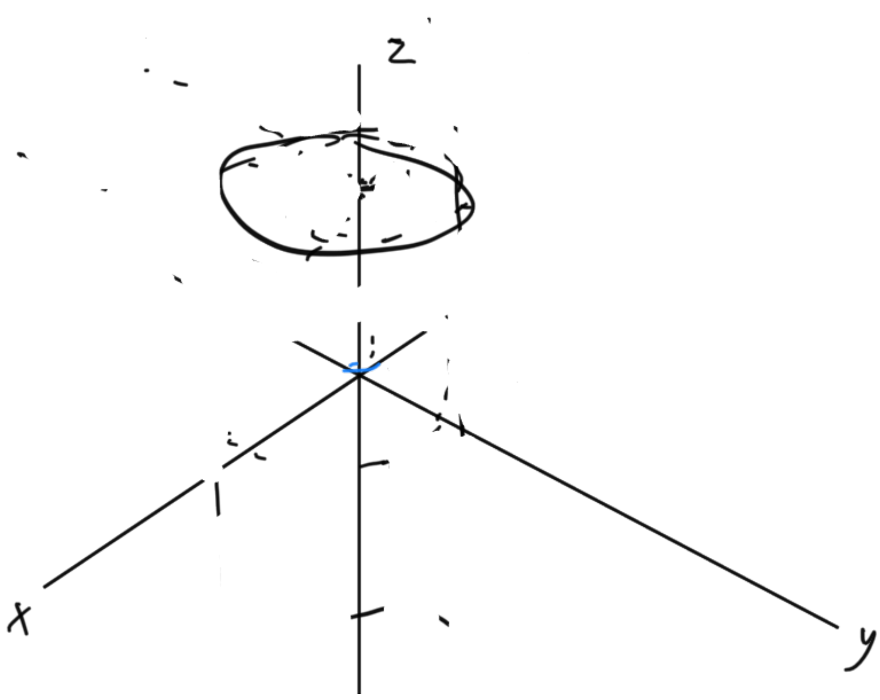
$y = -1$




## Surfaces from multiple equations

**Ex:**

$$\begin{cases} x^2 + y^2 = 1 \\ z = 4 \end{cases}$$

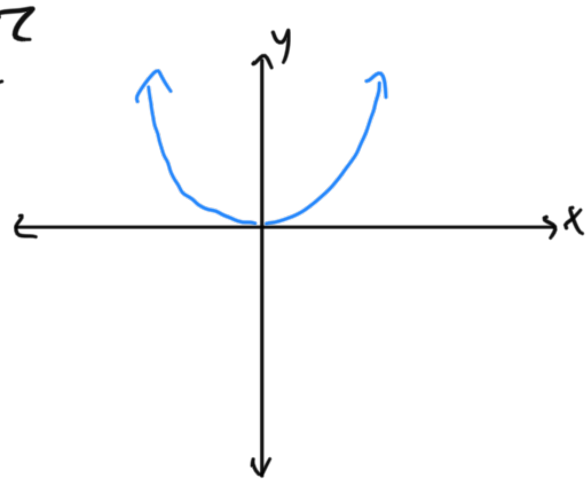


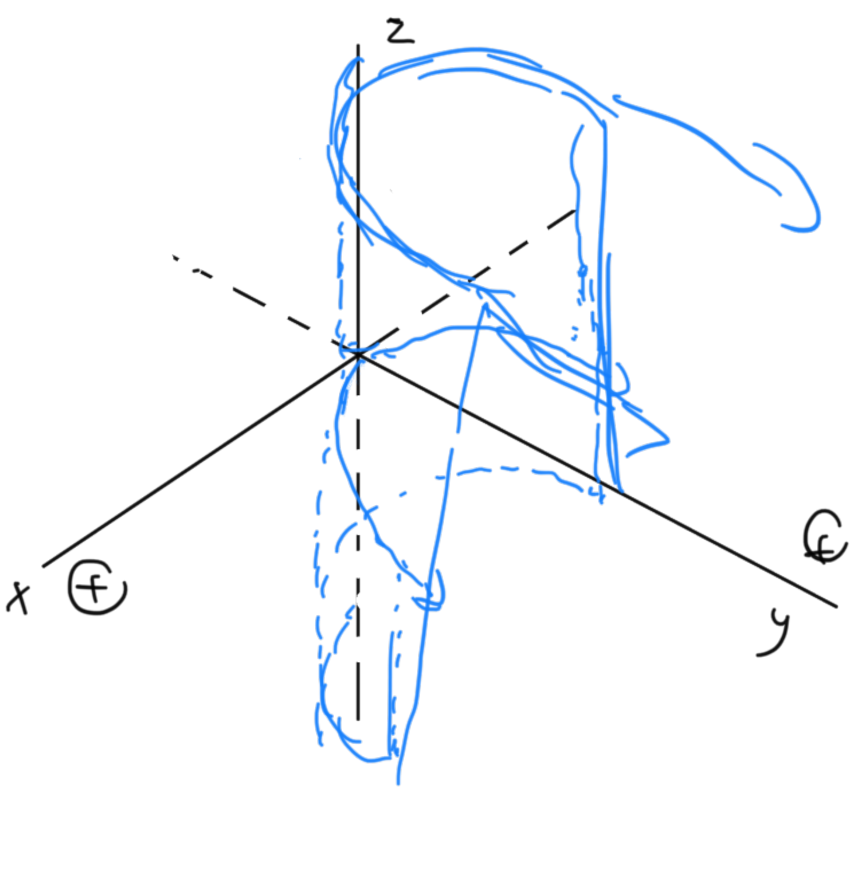
$+$ .

## Other Surfaces

**Ex:** $y = x^2$ $\quad \mathbb{R}^3$

$\mathbb{R}^2$

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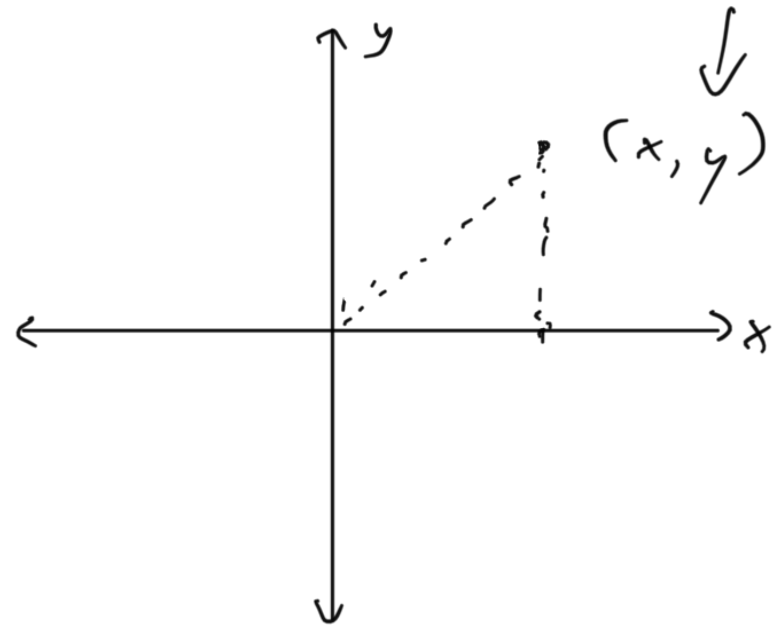
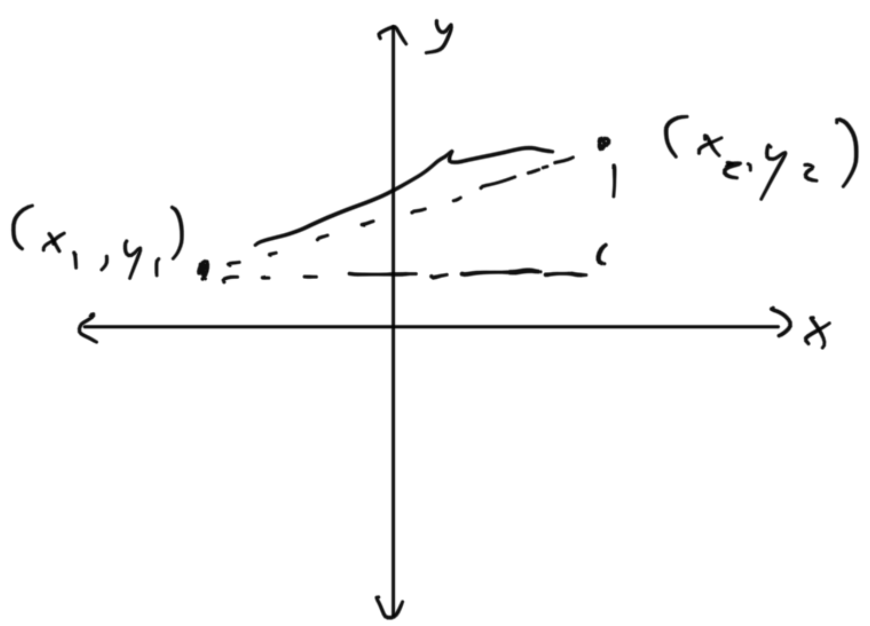
## Distances / Spheres

Recall distance from origin to point in $\mathbb{R}^2$




$$d = \sqrt{x^2 + y^2}$$

And distance between two points

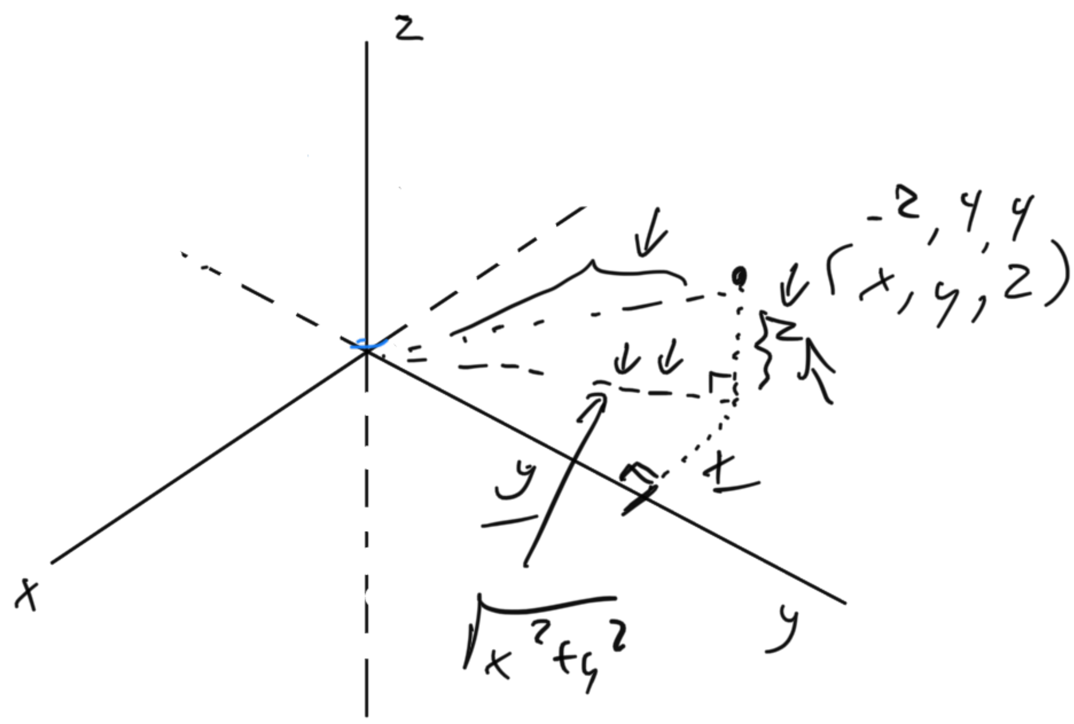



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Similar formulas exist for points in $\mathbb{R}^3$

Again, based on pythagorean theorem

$\mathbb{R}^3$




$$\sqrt{\left(\sqrt{x^2+y^2}\right)^2 + z^2}$$
$$= \sqrt{x^2+y^2+z^2}$$

$$d = \sqrt{x^2+y^2+z^2}$$

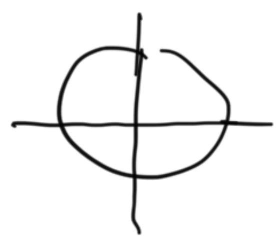
Ex: Distance between points

$P = (1, 0, 4)$ and $Q = (2, -1, 3)$

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

Now recall circles in $\mathbb{R}^2$ are essentially defined in terms of distance

$$x^2 + y^2 = 4$$



$$\Rightarrow \sqrt{x^2+y^2} = 2$$

"All points $(x,y)$ that are distance 2 from origin"

Similarly

$(2,3)$ $r=3$

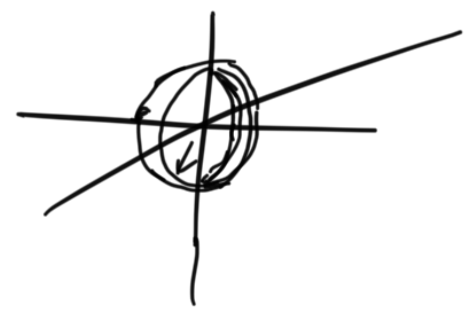
$$(x-2)^2 + (y-3)^2 = 9$$

$$\Rightarrow \sqrt{(x-2)^2 + (y-3)^2} = 3$$

"All points 3 units away from the point $(2,3)$"

Can define spheres in $\mathbb{R}^3$ in a similar way.

$$x^2 + y^2 + z^2 = r^2$$



$$\sqrt{x^2 + y^2 + z^2} = r$$

Eq. of spere centered at origin, radius $r$

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Eq of spere centered at $(a, b, c)$ with radius $r$

Find standard form of this equation for the sphere.

**Ex:** $x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$

$(x-a)^2$

$(x^2 + 4x + 4) - 4 + (y^2 - 2y + 1) - 1 + (z^2 + 4z + 4) - 4$

$= -5$

$$(x+2)^2 - 4 + (y-1)^2 - 1 + (z+2)^2 - 4 = -5$$

$$(x+2)^2 + (y-1)^2 + (z+2)^2 = 4$$

Ex:

$$(x-2)^2 + (y-1)^2 + (z-1)^2 \leq 4$$