Section 4.5

Consider V be a vector space with a basis $\mathcal{B}$. Roughly, the number of vectors in $\mathcal{B}$ is called the dimension of V. Will formalize it soon.

Ex: $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_7\}$. Then the dimension of V is 7.

We will focus of finite-dimensional vector spaces but it is possible to have infinite dimensional ones as well (countable and uncountable)

☆ **Ex:** Space of all polynomials is infinite dimensional (countable)

☆ **Ex:** Space of all continuous functions is infinite dimensional (uncountable)

Theorem: Let V be a vector space of dimension n. Then it is isomorphic to $\mathbb{R}^n$.

$\square$ Since V has dimension n there is a basis for V with n vectors, $\mathcal{B} = \{\vec{v}_1 \ldots \vec{v}_n\}$.

Define $T: V \to \mathbb{R}^n$ to be the linear transformation such that $T(\vec{v}_i) = \vec{e}_i$ (stand. basis vector in $\mathbb{R}^n$).

Then T is one-to-one (check) } ①
and onto (check) } ②

Then T is an isomorphism and V isomorphic to $\mathbb{R}^n$.

■

The next theorem connects number of vectors in a basis to our idea of dimension.

Theorem: If a vector space V has a basis with n vectors then <u>every</u> basis must have n vectors.

□ Proof book gives is not very good formally but may give some intuition. Here is another approach.

Remember from prev. section can think of basis as ① largest # of lin. ind vectors or ② smallest set of spanning vectors.

☆ Assume $\mathcal{B}$ basis with n vectors and $\mathcal{C}$ set with k vectors.

If $k < n$, too small to span V

If $k > n$ too big to be lin. ind.

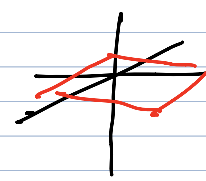
So if $\mathcal{C}$ is a basis, must also have n vectors. ■

Formal Definitions: If vector space V is spanned by finite set, say V is <u>finite dimensional</u>, and <u>dimension</u> of V, <u>$\dim V$</u>, is number of vectors in $\mathcal{B}$, basis of V. Dimension of trivial vector space $\{\vec{0}\}$ defined to be zero. If V not spanned by finite set, say it is <u>infinite dimensional</u>.
In that case say $\dim V = \infty$

Subspaces of $\mathbb{R}^n$ can be classified by dimension. Recall a subspace must contain $\vec{0}$ so all these pass through origin

- 0-dimensional: Just origin itself
- 1-dimension: Line
- 2-dimensional: Plane

Etc.



H V

If we think about dimension of one of these subspaces in relation to vector spaces that contains it, seems clear subspace should have smaller dimension.

Theorem: Let H be a subspace of finite dimensional vector space V. Any linearly independent set in H can be expanded to basis for H (and then to V). H must be finite dimensional as well and

$\dim H \le \dim V$

$\square$ Argument: ($\dim H \le \dim V$)

Assume V has dimension n.

Know H has a basis. Assume it has k vectors, $\vec{v}_1 \dots \vec{v}_k$.

These are lin ind vectors in H (and thus in V).

Since basis V is maximal linear ind set in V,

must be that $k \le n$, i.e. $\dim H \le \dim V$.

$\dim V = p$ $\boxtimes$

V $\quad \{\vec{v}_1, \dots \vec{v}_p\}$

Basis Theorem: Let V be p-dimensional vector space, $p \ge 1$. Any linearly independent set with exactly p elements must be basis for V. Any set with exactly p elements that spans V must be basis.

$\square$ Agrees with concepts of basis as maximal linear independent set / minimal spanning set.

- From prev. know all bases have at most p elements.

So if lin ind set has p elements, its maximal. So basis.

- Also know all bases have at least p elements

So if spanning set has p elements, it is minimal. Thus basis.

$\boxtimes$

Recall for every matrix A we have two special subspaces

A is $m \times n$

$$\text{Col } A \quad (\text{subspace } \mathbb{R}^m)$$
$$\text{Nul } A \quad (\text{subspace } \mathbb{R}^n)$$

In previous section we saw how to find bases for these.

Maybe I don't need to know basis of them, just their dimensions. Is there easy way to figure out or do I need to find basis then manually count the vectors?

There <u>is</u> a shortcut.

Definitions: For $m \times n$ matrix A

- **Rank of A** is dimension of Col A
- **Nullity of A** is dimension of Nul A

Recall basis vectors for Col A were pivot vectors of A. So dimension of Col A (rank) should be number pivot columns/rows

$$A \quad \begin{bmatrix} \blacksquare & \bullet & \bullet & \bullet & \bullet \\ 0 & \blacksquare & \bullet & \bullet & \bullet \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

Recall to find basis Nul A we solved homogeneous system. Free variable vectors became basis Nul A.

So # free variables = dimension Nul A (nullity)

Finally, note that # pivot columns + # free variables equals total # columns. Thus:

Rank Theorem: Dimensions of column space and null space of $m \times n$ matrix A satisfy equation:

$$\text{rank } A + \text{nullity } A = n \quad (\text{\# columns in } A)$$

Ex $A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$ $\underline{\text{Rank } A}$, Nullity A ?

$$\underbrace{\text{Rank } A}_{2} + \underbrace{\text{Nullity } A}_{2} = 4$$

Recall Invertible Matrix Theorem with its many equivalent statements. Can rephrase many of them in terms of Rank A, Nullity A.

Invertible Matrix Theorem (Rewritten):

Let A be $n \times n$ matrix. Then tfae:

⓪ A is invertible

① Columns of A form basis $\mathbb{R}^n$

② Col $A = \mathbb{R}^n$

③ rank $A = n$ $\qquad \underbrace{\text{Rank } A}_{n} + \underbrace{\text{Nullity}}_{0} = \underline{n \text{ columns}}$

④ nullity $A = 0$

⑤ Nul $A = \{\vec{0}\}$

HW Problems

\# 1-6

\# 9

\# 11-25

27+ Challenge Problems

orthogonal polynomials { Hermite Polynomials ☆

Laguerre Polynomials

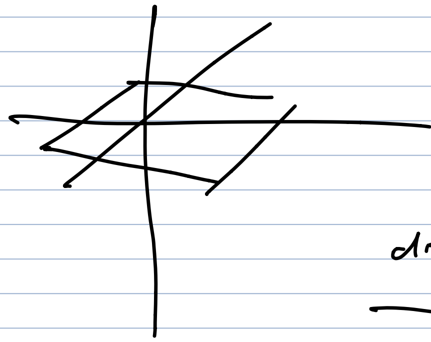
$H \quad V$

$\dim H \leq \dim V$

Finite dimensional vector spaces

Also true for infinite dimensional vector spaces

subspace ✓



$\dim H < \dim V$

$\dim \mathbb{P} = \infty$

$\mathbb{P}_{2k}$ $a_0 + a_2 x^2 + a_4 x^4$

$\mathbb{P}_{2k}$ proper subspace $\mathbb{P}$

$\dim \mathbb{P}_{2k} = \infty$ $\dim \mathbb{P} = \infty$