Section 1.1

Linear Equations

What is a linear equation? Linear may remind you of lines and that correct

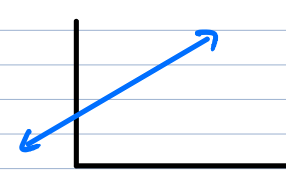
Def: A linear equation is one of the form

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots a_n x_n = c$$

where $a_1, \dots, a_n$ and b are constant coefficients and $x_1, \dots, x_n$ are variables

Think of $\mathbb{R}^2$ and equation of line

$$y = mx + b \quad ☆$$



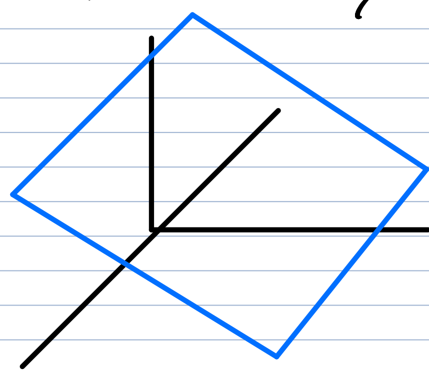
Re-arrange

$$-mx + y = b$$

Rename x as x_1, y as x_2

$$-mx_1 + x_2 = b$$

In $\mathbb{R}^3$, linear equations give us planes.



<u>Note</u>: In $\mathbb{R}^3$

$$a_1x_1 + a_2x_2 = C$$

is a plane, not a line.

Think of it as

$$a_1x_1 + a_2x_2 + 0x_3 = C$$

Should be clear from context

In higher dimensions, things are harder to picture so we must get comfortable with abstract thinking.

Lets consider one linear equation

$$a_1x_1 + \dots a_nx_n = c$$

Will usually be given the a's and c and asked to find the x's that make the equation true.

E1 Consider $\mathbb{R}^2$

$$1x_1 + 2x_2 = 4$$

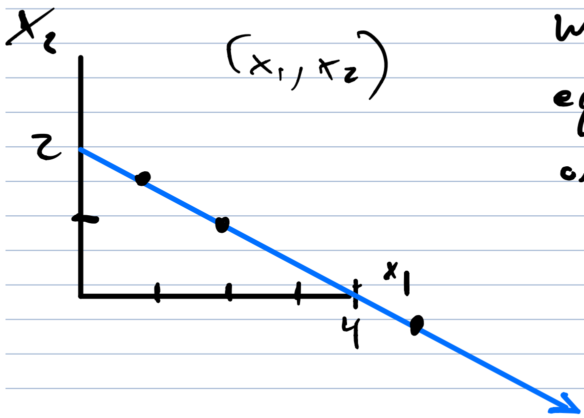
What values of x_1 and x_2 satisfy this?

Clear if we rewrite

$$2x_2 = -x_1 + 4$$

$$x_2 = -\tfrac{1}{2}x_1 + 2$$

This is a line with slope $-\frac{1}{2}$ and y-intercept of 2



What points satisfy our equation? (What points are on the line?)

Lots of solutions! (Infinite). Every point on the line satisfies our single equation.

Get a solution set:

$$x_1 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

More on these later.

Systems of Linear Equations

A single linear equation is not very interesting by itself. Not much to actually solve.

Often work with several linear equations

$$\begin{cases} a_1 x_1 + a_2 x_2 + \dots \; a_n x_n = d_1 \\ b_1 x_1 + b_2 x_2 + \dots \; b_n x_n = d_2 \\ c_1 x_1 + c_2 x_2 + \dots \; c_n x_n = d_3 \end{cases}$$

Maybe not every variable appears in every equation but we should leave space for them

Ex : $2x_1 + x_3 = 5$ and $3x_1 + x_2 = 6$

should be written as a system as:

$$\begin{cases} 2x_1 + + x_3 = 5 \\ 3x_1 + x_2 = 6 \end{cases} \quad \underline{\text{or}} \quad \begin{cases} 2x_1 + 0x_2 + x_3 = 5 \\ 3x_1 + x_2 + 0x_3 = 6 \end{cases}$$

Once again, will usually be given our coefficients (the a's) and the d's and asked to solve for x's

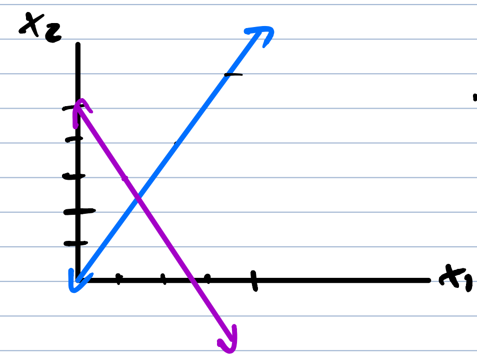
But, now when we are looking for solutions we want x's that make <u>all</u> equations true at same time.

E2

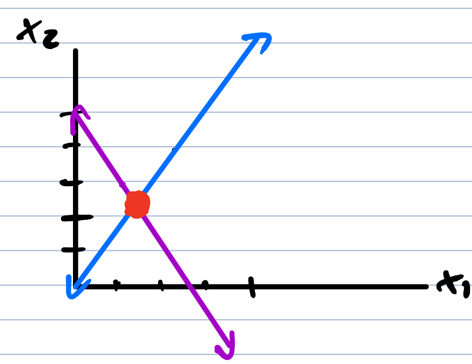
$$\left\{\begin{array}{l} 2x_1 + x_2 = 5 \ast \\ -2x_1 + x_2 = 0 \end{array}\right\} \mathbb{R}^2$$

What is solution for this system?

Lets graph the lines.



What points (x_1, x_2) make both equations true at same time?



Just the single point of intersection of two lines!

$(1.25, \ 2.5)$

- Our systems of equations might have just one solution (a single point like in E2 above)
- System might have infinite number of solutions (like in E1)
- Or a system may have no solutions

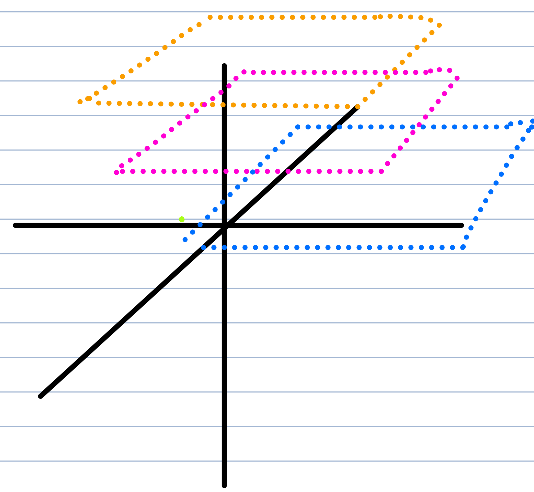
These are only 3 possibilities

Consistent system $\begin{cases} \text{One solution} \\ \text{Infinite solutions} \end{cases}$

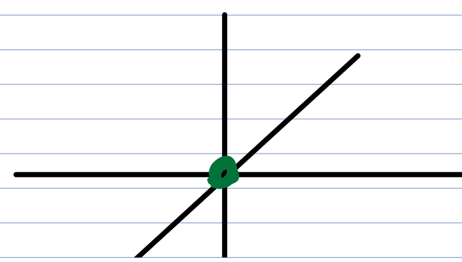
Inconsistent $\{$ No solution

Think of solutions as intersections of the equations.

E3 $\begin{cases} a_1x_1 + a_2x_2 + a_3x_3 = d_1 \\ b_1x_1 + b_2x_2 + b_3x_3 = d_2 \\ c_1x_1 + c_2x_2 + c_3x_3 = d_3 \end{cases}$ In $\mathbb{R}^3$

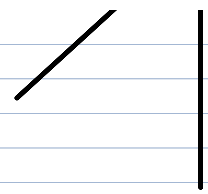


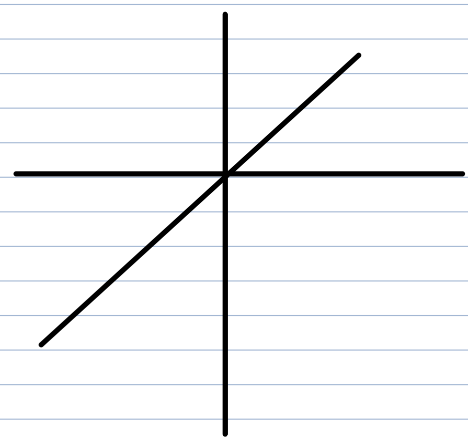
Three parallel planes. No solution



Consider coordinate planes (xy-plane, xz-plane, yz-plane)

Only solution is $(0,0,0)$





Intersection of 2 planes may give us a line (infinite solutions)

Solving Linear Systems

Have probably done some of this in previous courses but we will use new method

We will take our system of equations and turn it into a matrix equation

First, some matrix info!

A matrix is just a box of numbers

A $m \times n$ matrix has m rows, n columns

Ex:

$$\begin{array}{cc} \text{3 rows} & \begin{bmatrix} 1 & 0 & 9 & 5 \\ -3 & 6 & 2 & 0 \\ 8 & 1 & 4 & 6 \end{bmatrix} \\ & \text{4 columns} \end{array}$$

A 3×4 matrix

Each row can be considered a vector and so can each column.

So above matrix made up of row vectors

$[1 \quad 0 \quad 9 \quad 5]$ $\quad [-3, 6, 2, 0]$

$[8 \quad 1 \quad 4 \quad 6]$

Or could say it is made up of column vectors

$$\begin{bmatrix} 1 \\ -3 \\ 8 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} \begin{bmatrix} 9 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 6 \end{bmatrix}$$

We can multiply a matrix by a vector to get another vector

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \_ \\ \_ \end{bmatrix}$$

To multiply, take dot product of row vector with the column vector

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 + 2\cdot 0 + 3\cdot 1 \\ 4\cdot 1 + 1\cdot 0 + 2\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Why do we care? Consider a system of equations

$$\begin{cases} \underline{a_1} x_1 + \underline{a_2} x_2 + \underline{a_3} x_3 = d_1 \\ \underline{b_1} x_1 + \underline{b_2} x_2 + \underline{b_3} x_3 = d_2 \end{cases}$$

Take all coefficients and put them in a matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$ Coefficient matrix

Multiply by vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \end{bmatrix}$$

What is the vector we get?

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 x_1 + a_2 x_2 + a_3 x_3 \\ b_1 x_1 + b_2 x_2 + b_3 x_3 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Get:

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

This is all the same info as our system of equations

$$A\vec{x} = \vec{d}$$

matrix

vector

vector

The $\vec{x}$ vector is somewhat redundant so we can omit it, make things compact

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \end{array}\right]$$

Augmented matrix

Ex:

$$\begin{cases} 3x_1 + 2x_2 = 6 \\ 7x_1 + 1x_2 + 9x_3 = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 0 & 6 \\ 7 & 1 & 9 & 0 \end{array}\right]$$

Back to solving:

Lets work backwards. Consider:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

Notice the triangular shape that the nonzero entries make

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

Lets interpret this matrix from bottom up

Last line translates to

$$2x_3 = 4$$

So we must have that $\boxed{x_3 = 2}$

Now second line!

$$3x_2 = 9$$

So $\boxed{x_2 = 3}$

What about first line?

$$1x_1 + 2x_3 = 5$$

But we know $x_3 = 2$. So

$$1x_1 + 4 = 5$$

$$\boxed{x_1 = 1}$$

So if matrix is "upper triangular" we can see our solution

So to solve a linear system we should:

- ☆ 1 Turn system into augmented matrix
- ☆ 2 Turn that matrix into upper triangular
- ☆ 3 Interpret results

How do we do step 2?

Row Reduction

The process of turning an arbitrary matrix into one with nice triangular form is called row reduction.

Elementary Row Operations

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right]$$

① Exchange two rows

$$\left[\begin{array}{ccc|c} b_1 & b_2 & b_3 & d_2 \\ a_1 & a_2 & a_3 & d_1 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right]$$

② Multiply a row by non-zero constant

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ m b_1 & m b_2 & m b_3 & m d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right]$$

③ Add multiple of one row to another

multiply by m → adding to row 3

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ m b_1 + c_1 & m b_2 + c_2 & m b_3 + c_3 & m d_2 + d_3 \end{array}\right]$$

These three operations are all we need to make a matrix upper triangular

E4

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 9 \\ 1 & 8 & 2 & 7 \\ 0 & 4 & 5 & 0 \\ 2 & 16 & 4 & 14 \end{array}\right]$$

Get some non-zero number as far up and to left as possible using row switches

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 9 \\ 1 & 8 & 2 & 7 \\ 0 & 4 & 5 & 0 \\ 2 & 16 & 4 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 8 & 2 & 7 \\ 0 & 1 & -1 & 9 \\ 0 & 4 & 5 & 0 \\ 2 & 16 & 4 & 14 \end{array}\right]$$

Once you have done this, try to cancel everything below your upper left term by adding multiples of top row to rows below

$$\left[\begin{array}{ccc|c} 1 & 8 & 2 & 7 \\ 0 & 1 & -1 & 9 \\ 0 & 4 & 5 & 0 \\ 2 & 16 & 4 & 14 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 8 & 2 & 7 \\ 0 & 1 & -1 & 9 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now repeat the process, leaving top row alone

$$\left[\begin{array}{ccc|c} 1 & 8 & 2 & 7 \\ 0 & 1 & -1 & 9 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This row is already well placed. Now eliminate terms below the 1

$$\left[\begin{array}{ccc|c} 1 & 8 & 2 & 7 \\ 0 & 1 & -1 & 9 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 8 & 2 & 7 \\ 0 & 1 & -1 & 9 \\ 0 & 0 & 9 & -36 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, lets interpret the results, starting from bottom row, which is the simplest

$$\left[\begin{array}{ccc|c} 1 & 8 & 2 & 7 \\ 0 & 1 & -1 & 9 \\ 0 & 0 & 9 & -36 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow 9x_3 = -36$$

So $\boxed{x_3 = -4}$

$$\left[\begin{array}{ccc|c} 1 & 8 & 2 & 7 \\ 0 & 1 & -1 & 9 \\ 0 & 0 & 9 & -36 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x_2 - x_3 = 9 \\ x_2 - (-4) = 9 \end{array}$$

So $\boxed{x_2 = 5}$

$$\left[\begin{array}{ccc|c} 1 & 8 & 2 & 7 \\ 0 & 1 & -1 & 9 \\ 0 & 0 & 9 & -36 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{array}{l} x_1 + 8x_2 + 2x_3 = 7 \\ x_1 + 8(5) + 2(-4) = 7 \end{array}$$

$\boxed{x_1 = -25}$

So we have a solution to our system

$$\begin{bmatrix} -25 \\ 5 \\ -4 \end{bmatrix}$$

We will refine our solution method more in next section.

A Note on effects of Row Reduction

Remember, our matrix represents system of equations

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & 2 \end{array}\right] \approx \begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + x_2 = 2 \end{cases}$$

We use row operations to find solution to this

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & -1 \end{array}\right]$$

But now this seems like totally different system of equations!

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -5 & -1 \end{array}\right] \approx \begin{cases} x_1 + 2x_2 = 1 \\ -5x_2 = -1 \end{cases}$$

It <u>is</u> different system of equations!
Luckily, if we use elementary row operations all the new matrices/systems we get will have same solutions (they are equivalent)

Easy to understand for 2 of the 3 operations

Think of solution as intersection of lines/planes

1. Switching rows

$$\begin{bmatrix} a_1 & a_2 & a_3 & | & d_1 \\ b_1 & b_2 & b_3 & | & d_2 \\ c_1 & c_2 & c_3 & | & d_3 \end{bmatrix}$$

Does changing order of equations change where they intersect?

2. Multiplying row by constant

$$\begin{bmatrix} a_1 & a_2 & a_3 & | & d_1 \\ k b_1 & k b_2 & k b_3 & | & k d_2 \\ c_1 & c_2 & c_3 & | & d_3 \end{bmatrix}$$

Multiplying just stretches a plane, still have exact same plane

3. Adding multiple of one row to another

$$\begin{bmatrix} a_1 & a_2 & a_3 & | & d_1 \\ b_1 & b_2 & b_3 & | & d_2 \\ c_1 & c_2 & c_3 & | & d_3 \end{bmatrix}$$

Not as clear but still fine