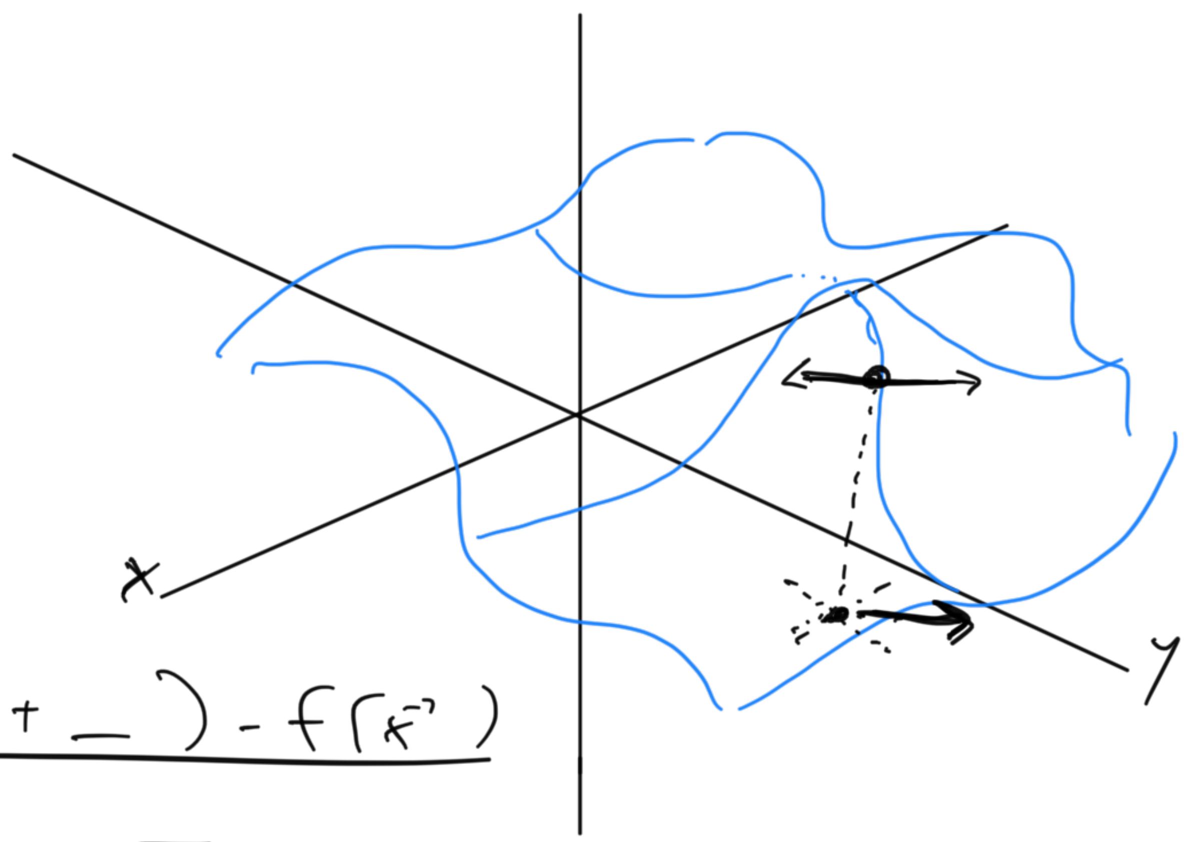
14.6 - Directional Derivatives

Recall for $f:\mathbb{R}^n \to \mathbb{R}$ we are concerned with many, infinite, directions

For derivatives we confined ourselves to the directions along our major axes

For $f:\mathbb{R}^2 \to \mathbb{R}$ have f_x, f_y

But also ...



$$\frac{f(\vec{x} + \_\_) - f(\vec{x})}{\_\_}$$

Have tangents pointing in many different directions, for example in direction of vector $\vec{u}$?

Expect their slopes are given by derivatives of some sort

What if we need to work with these

directional derivatives ?

Can't differentiate $f(x,y)$ in terms of $\vec{u}$ because no u's in my equation

Develop the theory

Assume $f: \mathbb{R}^n \to \mathbb{R}$

Instead of writing $f(x_1, \dots x_n)$, can just write as $f(\vec{x})$

Want to find derivative of f in arbitrary direction $\vec{u}$. Just care about direction of $\vec{u}$ so assume length 1.

Just like with other derivatives define it as a limit:

$$D_{\vec{u}} f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$$

" Derivative of f in direction $\vec{u}$ "

For $f: \mathbb{R}^2 \to \mathbb{R}$

$f(x, y)$ $\quad \vec{u} = \langle a, b \rangle$

Then $f(\vec{x} + h\vec{u}) = f(x+ha, y+hb)$ and

$$D_u f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$$

becomes

$$D_u f(x, y) = \lim_{h \to 0} \frac{f(x+ha, y+hb) - f(x, y)}{h}$$

Above is just definition of $D_u f(\vec{x})$
Would be cumbersome if we had to compute that limit everytime. Similar to finding f_x, f_y we want a "shortcut"

It can be found in a theorem

from previous section.

Recall, <u>if</u> f is differentiable then

$$\Delta f = f(x+ha, y+hb) - f(x,y)$$

$$= f_x(x,y)\,ha + f_y(x,y)\,hb + \varepsilon_1 ha + \varepsilon_2 hb$$

Then

$$\lim_{h\to 0} \varepsilon_1 = 0$$

$$\frac{\Delta f}{h} = \lim_{h\to}\left(f_x(x,y)\,a + f_y(x,y)\,b + \varepsilon_1 a + \varepsilon_2 b\right)$$

$$f(x,y) \quad \vec{u} \overset{?}{=} \langle a,b \rangle$$

$$D_u f(x,y) = \lim_{h\to 0} \frac{\Delta f}{h} = f_x(x,y)\,a + f_y(x,y)\,b$$

Ex: $f(x,y) = e^x y + y^2 x^3$ }

Find $D_u f(x,y)$ for $\vec{u} = \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$$f_x = e^x y + 3y^2 x^2$$
$$f_y = e^x + 2yx^3$$

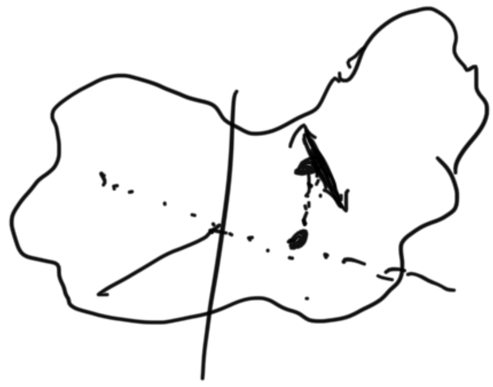
$$D_{\vec{u}} f(x,y) = (e^x y + 3y^2 x^2)\frac{1}{2} + (e^x + 2yx^3)\frac{\sqrt{3}}{2}$$

$D_u f(x,y)$ at $\boxed{(0,2)}$

$$(2+0)\frac{1}{2} + (1+0)\frac{\sqrt{3}}{2}$$

$$= 1 + \frac{\sqrt{3}}{2}$$

$$\boxed{= \frac{\sqrt{3}+2}{2}}$$



For $\vec{u} = \langle a, b\rangle$ have

$$D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

Lets play with notation a bit.

Make a vector out of partial derivatives

$$\nabla f = \langle f_x , f_y \rangle$$

Call this the gradient vector, or gradient

Denote it by ∇f ("del f")

Now note

$$D_u f(x,y) = f_x(x,y)\,a + f_y(x,y)\,b$$

$$D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}$$

This gives nice compact notation that easily scales up to higher dimensions

Directional Derivative for $f: \mathbb{R}^n \to \mathbb{R}$

If $\vec{u}$ in unit vector in $\mathbb{R}^n$, $f(x_1, \dots x_n) = f(\vec{x})$ is differentiable what is $D_{\vec{u}} f(\vec{x})$?

$$D_u f(\vec{x}) = \nabla f \cdot \vec{u}$$

$$\frac{\partial f}{\partial x_1}(\vec{x})\, u_1 + \frac{\partial f}{\partial x_2}(\vec{x})\, u_2 + \dots \frac{\partial f}{\partial x_n}(\vec{x})\, u_n$$

Ex: $f(x, y, z) = zx^2 + yz + \ln(x)\sin(y)$

Find $D_u f(\vec{x})$ for $\vec{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$

$f_x = 2zx + \frac{1}{x}\sin(y)$

$f_y = z + \ln(x)\cos(y)$

$f_z = x^2 + y$

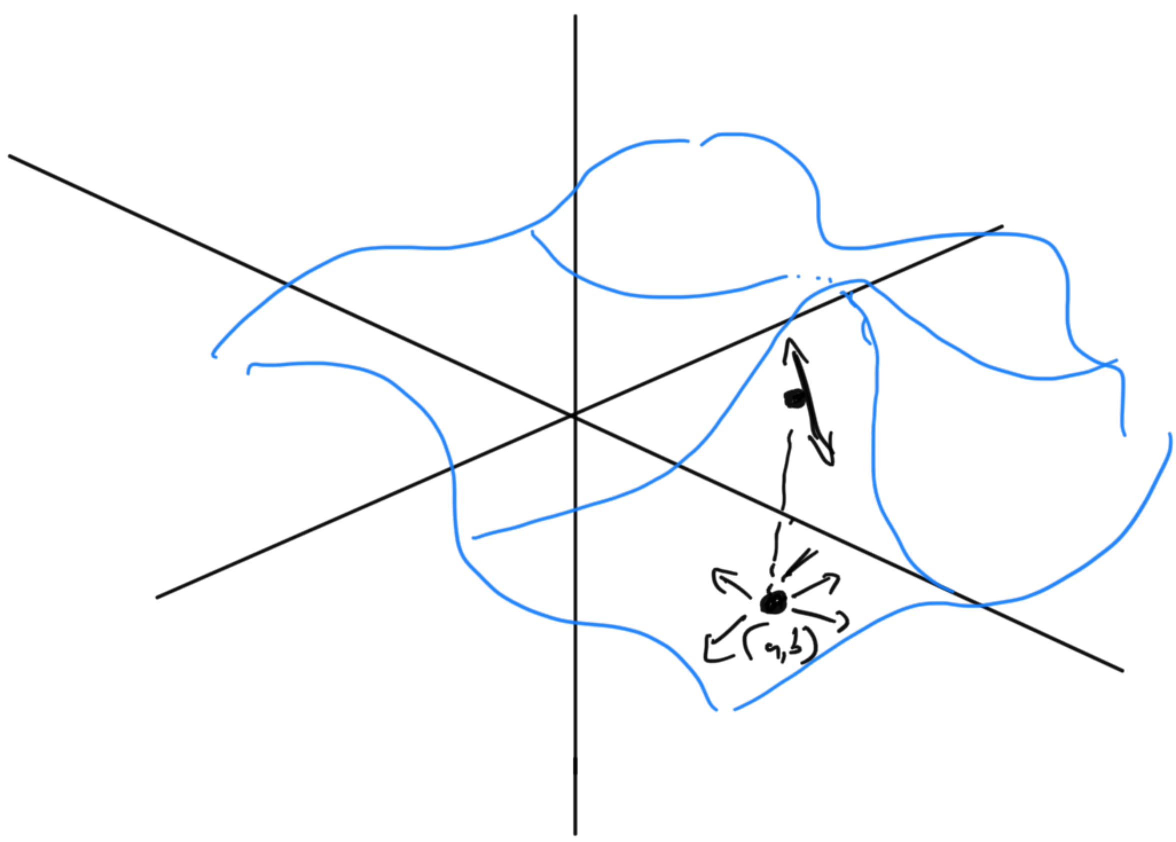
$$\nabla f = \langle 2zx + \tfrac{1}{x}\sin(y),\ z + \ln(x)\cos(y),\ x^2 + y \rangle$$

$$D_u f(\vec{x}) = \nabla f \cdot \vec{u}$$

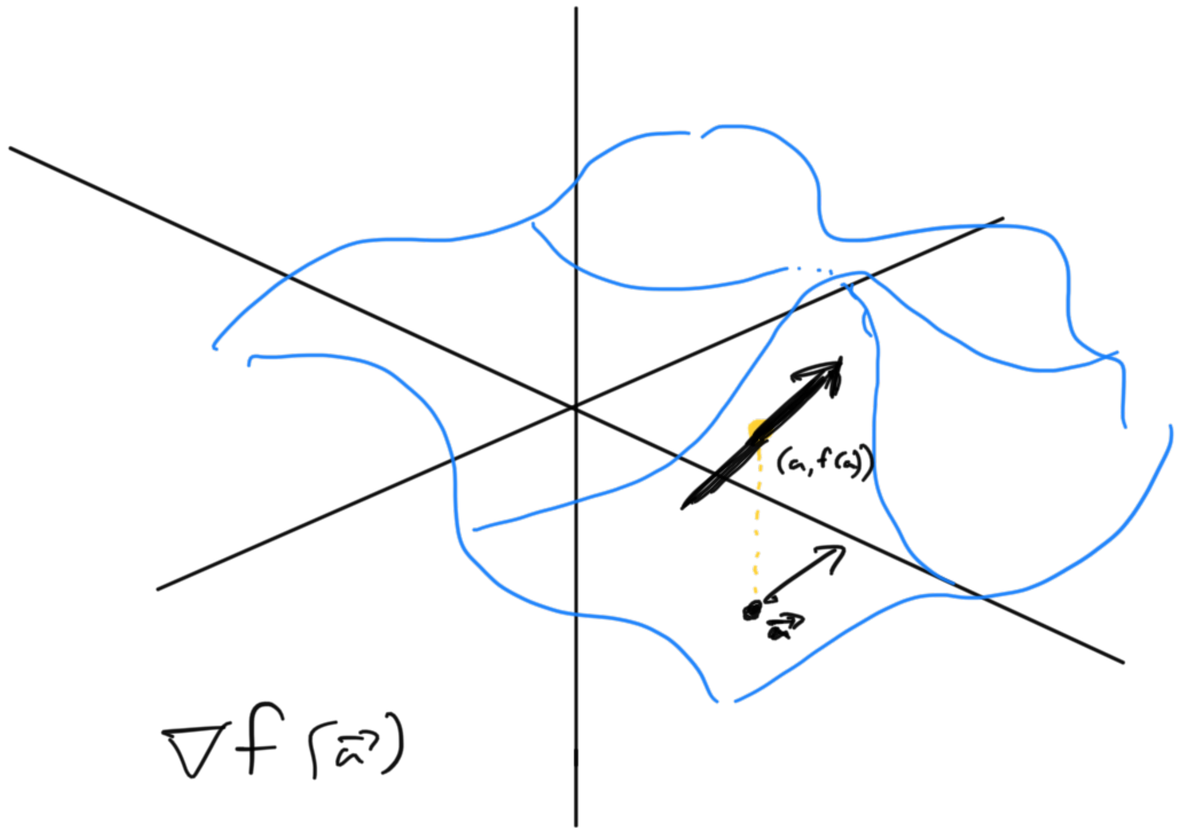
Maximizing directional derivative

Given $f: \mathbb{R}^n \to \mathbb{R}$, may want to find in which direction f is increasing/decreasing the most

For $f: \mathbb{R}^2 \to \mathbb{R}$ perhaps can think of graph of f is a mountain. In which direction is mountain steepest?



Lets say we are "at" point $\vec{a}$ in our domain our domain.



Turns out direction of greatest change will be in direction of $\nabla f(\vec{a})$

(The vector we get when we plug $\vec{a}$ in to gradient)

Our vector notation/dot product provides a simple proof.

$\square$ $$D_u f(\vec{a}) = \nabla f(a) \cdot \vec{u}$$

So

$$\underline{|D_u f(\vec{a})|} = |\nabla f(\vec{a}) \cdot \vec{u}|$$

$$\leq |\nabla f(\vec{a})| \; |\vec{u}|$$

But $\vec{u}$ is unit vector so $|\vec{u}| = 1$

Thus

$$|D_u f(\vec{a})| \leq |\nabla f(\vec{a})|$$

Amount of change in direction $\vec{u}$

Magnitude of $\nabla f(\vec{a})$

$$|D_{\vec{u}} f| \leq |\nabla f(a)|$$

OTOH

$\nabla f(a)$ is a vector

$$\vec{v} = \frac{\nabla f(a)}{|\nabla f(a)|}$$

$$D_{\vec{v}} f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{v}$$

$$= \nabla f(a) \cdot \frac{\nabla f(a)}{|\nabla f(a)|}$$

So $$|D_v f(\vec{a})| = \left| \nabla f(a) \cdot \frac{\nabla f(a)}{|\nabla f(a)|} \right|$$

$$= \frac{1}{|\nabla f(a)|} \left| \nabla f(\vec{a}) \cdot \nabla f(\vec{a}) \right|$$

$$= \frac{1}{|\nabla f(\vec{a})|} |\nabla f(a)|^2$$

$$= |\nabla f(\vec{a})|$$

So $|D_{\vec{u}} f| \leq |\nabla f(a)|$

OTOH in direction $\nabla f(a)$

$$|D_u f| = |\nabla f(\vec{a})|$$

Ex: Find direction in which tangent line steepest for

$$f(x,y,z) = x^2 y + z x^3 + \cos(z) y$$

at point $(x,y,z) = (1,0,1)$

$\underline{\nabla f}\ (1,0,1)$

$$\nabla f(x,y,z) = \langle 2xy + 3zx^2,\ x^2 + \cos(z),\ x^3 - \sin(z) y \rangle$$

$$\nabla f(1,0,1) = \langle 3,\ 1+\cos(1),\ 1 \rangle$$

$$\vec{u} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$D_u f \quad \lim_{t \to 0} \frac{f(\vec{x} + t\vec{u}) - f(\vec{x})}{t}$$

$$D_u f = \nabla f \cdot \vec{u}$$

unit vector