16.4 - Green's Theorem

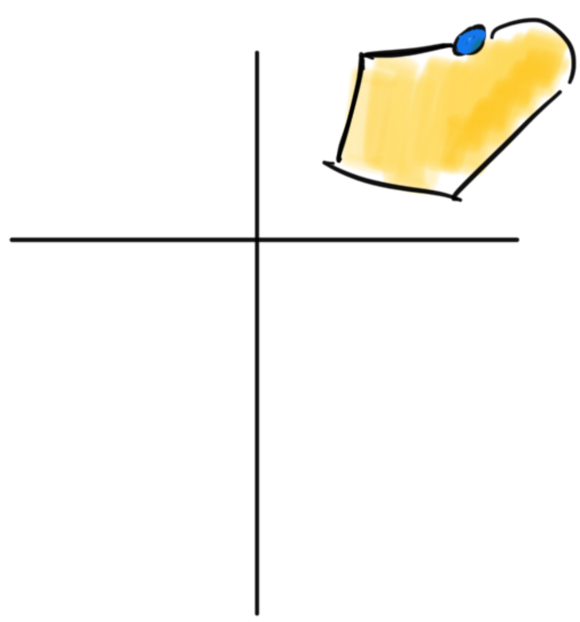
An interesting tool for relating / calculating line integrals / double integrals.

Statement of Green's Theorem: Let C be a positively oriented piecewise smooth, simple closed curve in the plane and let D be region bounded by C. If P, Q have continuous partial derivatives on a region that contains D then:

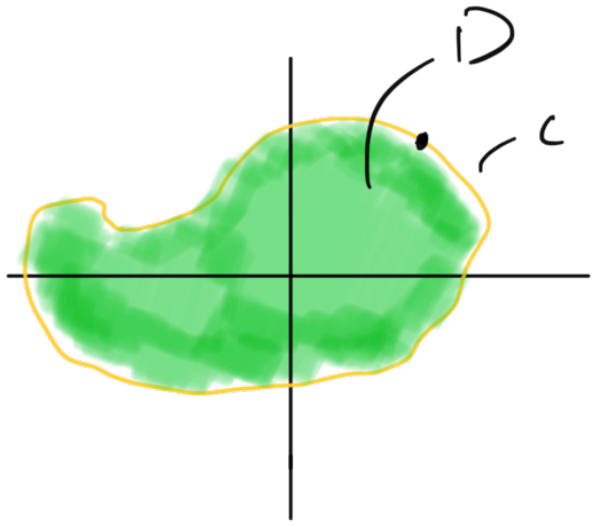
$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

(where

$$\underline{F(x,y)} = \langle P(x,y), Q(x,y) \rangle$$



Explanation: You have some region D inside the boundary C.



You have vector field $\vec{F}$ defined on the region. Then

$$\int_C P\,dx + Q\,dy$$ (line integral of $\vec{F}$ on C)

$$=$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$ (double integral of $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ on D)

Proof: Too Complicated

Significance: Let's say you want to calculate one of these quantities. You now have a choice of <u>how</u> to do it.

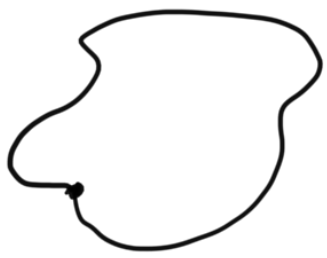
<u>Ex:</u> I want to calculate line integral

of vector field along closed curve.

But $\int_C P\,dx + Q\,dy$ is very difficult.

I can convert to a double integral which may be easier to solve.

$F = P\vec{i} + Q\vec{j}$



$\int_C P\,dx + Q\,dy = 0$

if F conservative

F is not conservative for <u>most</u> problems

Works in opposite direction too with a little finesse.

$$\int_C P\,dx + Q\,dy \longrightarrow \iint_D \left(\partial Q/\partial x - \partial P/\partial y\right) dA \quad \star$$

Converting in this direction is fairly simple, just requires differentiation.

Ex: $\oint_C y\,dx - x\,dy$

where <u>C</u> is circle, center origin, radius 4

$\iint (-2)\,dA$

D

$$(-2)\int_0^{2\pi}\int_0^4 r\,dr\,d\theta$$

$$(-2)\int_0^{2\pi}\left(\frac{r^2}{2}\Big]_0^4\right)d\theta$$

$$-16\int_0^{2\pi}$$

$$\boxed{=-32\pi}$$

Could probably solve as line integral easily, but lets practice converting.

$$\int \underbrace{y}_{P}\,dx - \underbrace{x}_{Q}\,dy$$

$$P = y \qquad Q = -x$$

$$\partial P/\partial y = 1 \qquad \partial Q/\partial x = -1$$

$$\iint \left(\partial Q/\partial x - \partial P/\partial y\right) dA$$

$$\iint (-2)\ dA$$

$$\int_0^{2\pi}\int_0^4 (-2)\ r\,dr\,d\theta$$

$$-2\int_0^{2\pi}\int_0^4 r\,dr\,d\theta$$

$$-2\int_0^{2\pi} \frac{r^2}{2}\Big]_0^4\ d\theta$$

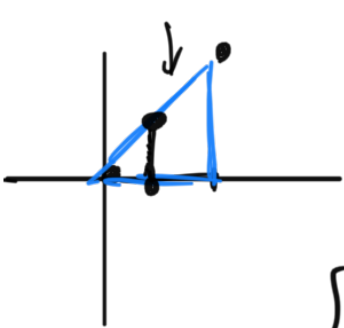
$$(-2)\int_0^{2\pi} 8 \, d\theta$$

$$(-2)(8)\int_0^{2\pi} d\theta$$

$$= -32\pi$$

Ex:

$$\oint_c \underbrace{(xy)}_{P}\,dx + \underbrace{(x^2y^3)}_{Q}\,dy$$



C triangle with vertices $(0,0)$ $(1,0)$ $(1,2)$

$$\int_c P\,dx + Q\,dy = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$\iint (2xy^3 - x)\,dA$$

$$\int_0^1\int_0^{2x} (2xy^3 - x)\,dy\,dx$$

$$\int \frac{2xy^4}{4} - xy \Big]_0^{2x}$$

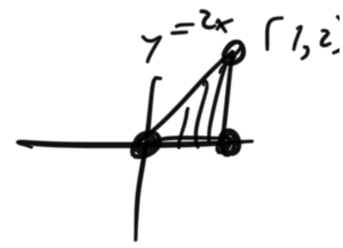
$$\int_0^1 \left[\frac{2x(2x)^4}{4} - x(2x)\right]$$

$$= \int \frac{2x(16x^4)}{4} - 2x^2$$

$$= \int 8x^5 - 2x^2\,dx$$

$$= \boxed{\frac{4}{6}} \qquad \frac{8}{6}x^6 - \frac{2}{3}x^3 \Big]_0^1$$

$P = xy$ $\qquad\qquad$ $Q = x^2 y^3$



$\frac{\partial P}{\partial y} = x$ $\qquad\qquad$ $\frac{\partial Q}{\partial x} = 2xy^3$

$$\int_0^1 \int_0^{2x} (2xy^3 - x)\, dy\, dx$$

$$\int_0^1 \left(\frac{2xy^4}{4} - xy \Big]_0^{2x} \right) dx$$

$$\int_0^1 \left(\frac{2x(16x^4)}{4} - x(2x) \right) dx$$

$$\int_0^1 \left(8x^5 - 2x^2 \right) dx$$

$$= \frac{8}{6}x^6 - \frac{2}{3}x^3 \Big]_0^1$$

$$= \left(\frac{8}{6} - \frac{2}{3}\right) - (0) = \frac{4}{6} = \frac{2}{3}$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \to \oint_C P\,dx + Q\,dy$$

Bit more difficult because we don't know what vector field is. Only know $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$

But we can use any vector field $F = P\vec{i} + Q\vec{j}$ that works.

· Will usually see this in area problems.

Ex: Find area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\iint_D \left(1 \right) dA$$

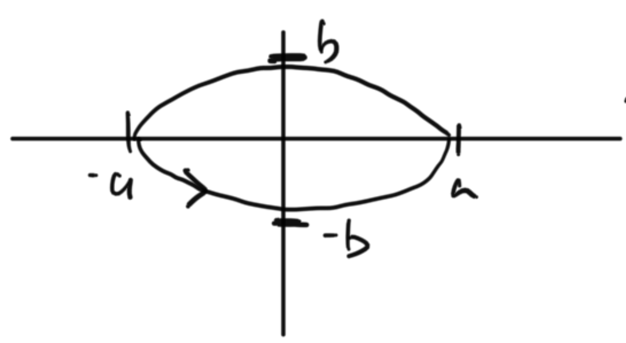
$$\iint \left(\overset{\frac{1}{2}}{\frac{\partial Q}{\partial x}} - \overset{(-\frac{1}{2})}{\frac{\partial P}{\partial y}} \right) dA$$

$$\int_C P\,dx + Q\,dy$$

$$\frac{\partial Q}{\partial x} = \frac{1}{2} \qquad \frac{\partial P}{\partial y} = -\frac{1}{2}$$

$\star$ $Q = \frac{1}{2}x$ $\quad\star$ $P = -\frac{1}{2}(y)$

$$\int_C \left(-\frac{1}{2}y\right) dx + \left(\frac{1}{2}x\right) dy$$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\vec{r}(\theta)$ $\langle a\cos(\theta), b \ldots \rangle$

$\langle a\cos(\theta), b\sin(\theta)\rangle$

$0 \le \theta \le 2\pi$

$\vec{r}\,'(\theta) = \langle -a\sin(\theta), b\cos(\theta)\rangle$

$\int -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy$

$\int_0^{2\pi} \langle -\frac{1}{2} b\sin(\theta), \frac{1}{2}a\cos\theta\rangle \cdot \langle -a\sin\theta, b\cos\theta\rangle \, d\theta$

$= \int_0^{2\pi} \frac{1}{2}ab\sin^2\theta + \frac{1}{2}ab\cos^2\theta \; d\theta$

$\int_0^{2\pi} \frac{1}{2}ab = \boxed{ab\pi}$ ☆

$\oint_C P\,dx + Q\,dy$

$\oint_C -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy$

$\vec{r}(t) = \langle a\cos(t), b\sin(t)\rangle$

$\vec{r}\,'(t) = \langle -a\sin(t), b\cos(t)\rangle$

$P = -\frac{1}{2}b\sin(t)$

$Q = \frac{1}{2}a\cos(t)$

$\oint_C \langle P, Q\rangle \cdot r'(t)\,dt$

$\oint_C \left(-\frac{1}{2}b\sin(t)\right)\left(-a\sin(t)\right) + \left(\frac{1}{2}a\cos(t)\right)b\cos(t)\,dt$

$\oint_C \frac{1}{2}ab\sin^2(t) + \frac{1}{2}ab\cos^2 t \quad dt$

$\oint \frac{1}{2} \ldots$

$\int_C \frac{1}{2} ab(\sin^2 t + \cos^2 t)\,dt$

$$\int_0^{2\pi} \frac{1}{2} ab \; dt$$

$$\frac{1}{2} ab \cdot 2\pi = \boxed{\pi ab}$$

Practice

$$\int_C \underbrace{ye^x}_{P}\,dx + \underbrace{2e^x}_{Q}\,dy$$

C rectangle w/ vertices

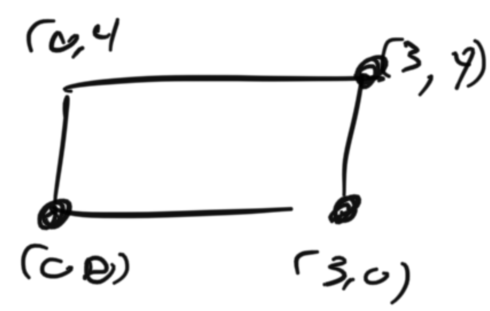
$(0,0) \quad (3,0) \quad (3,4) \quad (0,4)$

$P = ye^x \qquad Q = 2e^x$

$\partial P/\partial y = e^x \qquad \partial Q/\partial x = 2e^x$

$$\partial Q/\partial x - \partial P/\partial y = 2e^x - e^x = e^x$$

$$\int_0^4 \int_0^3 e^x \, dx\,dy$$



$$\int_0^4 (e^3 - 1)\,dy$$

$$\boxed{4(e^3-1)}$$

$\int_c + ds$

$\int_c F \cdot dr$