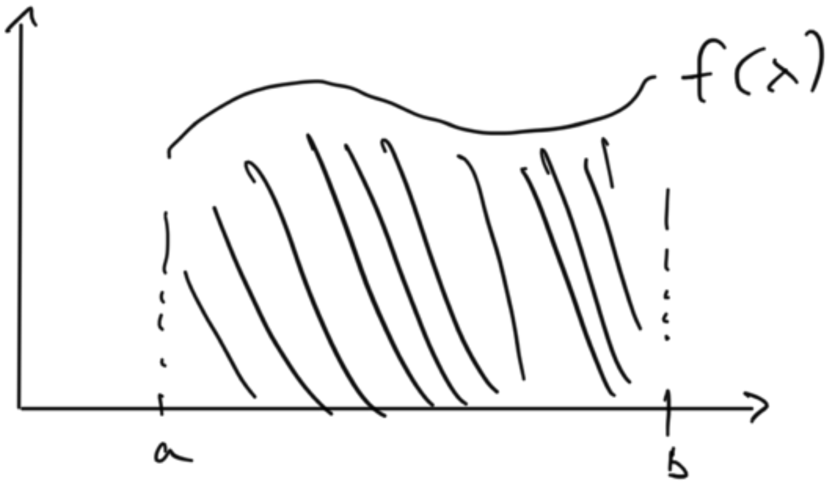
15.1 - Double Integrals

Just as calculus 1, 2 proceeded from limits to continuity to derivatives to integrals we have reached integrals for our multivariable functions.

Intuitive idea

(Definite) Integral of single variable function gave area under a curve.

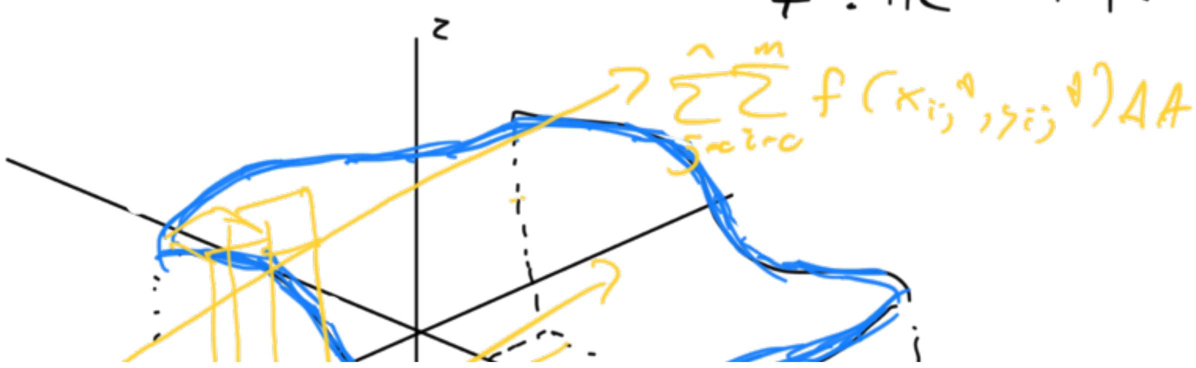


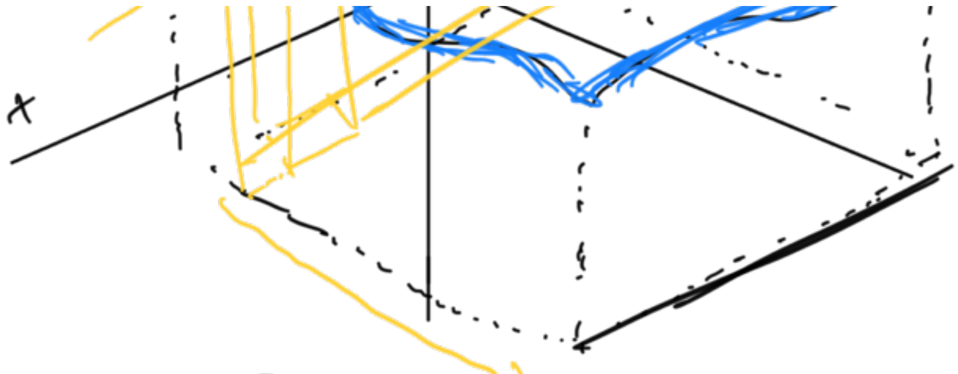
$\int_a^b f(x)dx$

"area beneath the curve"

What do we expect from integrals of our multivariable functions

$f : \mathbb{R}^2 \to R$





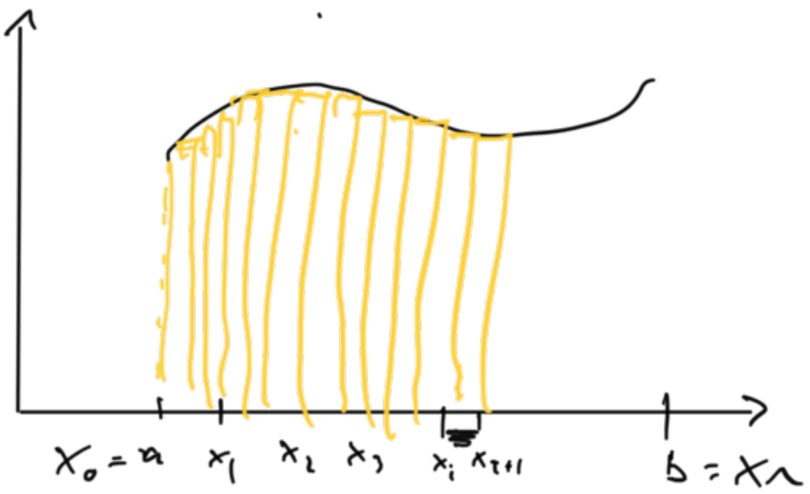
$\int f(x,y)\, d$ " "

Volume under the surface

Mathematical Definition

Recall how we formed integral in Calc 1.

Started with rectangles



$\Delta x = x_{i+1} - x_i$ = width

$f(x_i^*)$ = height

$f(x_i^*)\Delta x$ = area / rectangle

Formed Riemann sums of form

$\sum^{n-1}$

$$\sum_{i=0} f(x_i)\Delta x_i$$

where $\Delta x_i = x_{i+1} - x_i$ (length of little interval)

and $f(x_i) =$ some point in $[x_i, x_{i+1}]$

Then we took limit as $\Delta x_i \to 0$

$$\begin{tikzcd}
\lim\limits_{\Delta x \to 0} \Big(\sum\limits_{i=0}^{n-1} \arrow[d] & \underbrace{f(x_i)} \arrow[d] & \Delta x \Big) \arrow[d] \\
\int_a^b & f(x) & \underbrace{dx}
\end{tikzcd}$$

Have a similar process for $f: \mathbb{R}^n \to \mathbb{R}$

Will show case for $f: \mathbb{R}^2 \to \mathbb{R}$ but holds in general.

Will start by developing a Riemann sum.

Phrase this as an attempt to find volume under a surface.

For simplicity, assume our domain is a rectangle in xy-plane, call it D.

To approximate volume we will use rectangular boxes. Need to decide size of base and height of these boxes.

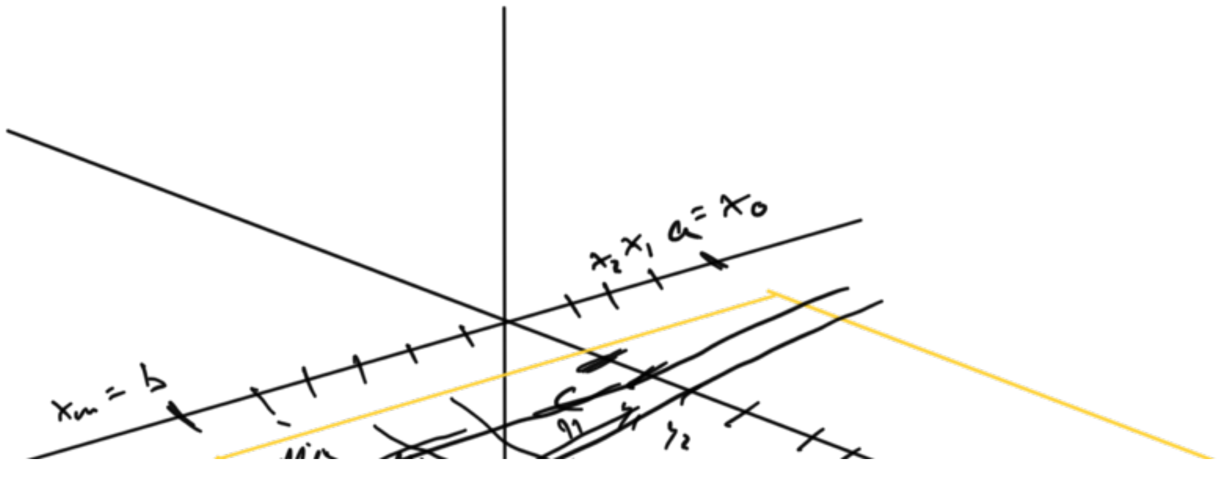
Bases of boxes are rectangles

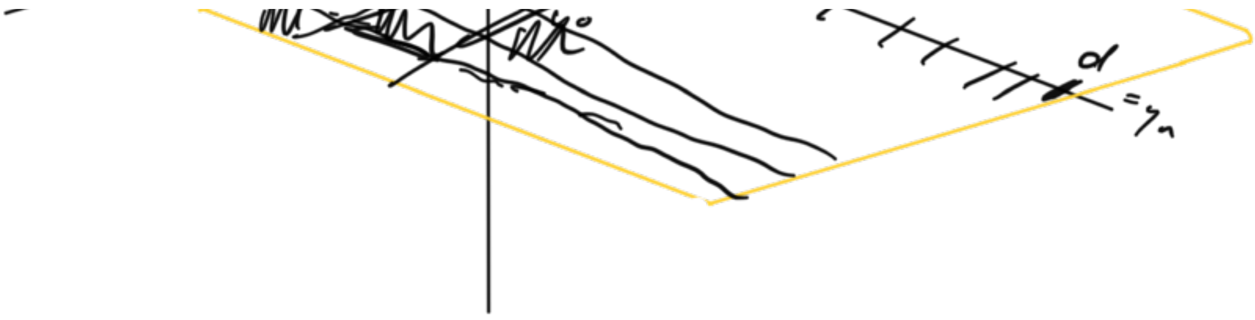
Height

$$\text{Volume of box} = (\underbrace{\text{area of base}})(\underbrace{\text{height}})$$

Start with bases.

Chop domain up into equally sized rectangles





Divide $[a, b]$ into m pieces

$[x_0, x_1]$ $[x_1, x_2]$ $\ldots$ $[x_{m-1}, x_m]$

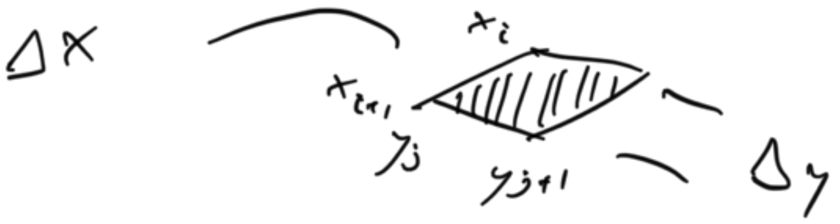
Divide $[c, d]$ into n pieces

$[y_0, y_1]$ $\ldots$ $[y_{n-1}, y_n]$

Gives $(m)(n)$ rectangles

Then have rectangles of form

$$R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$$



The length of x side is $x_{i+1} - x_i = \Delta x$

Length of y side is $y_{j+1} - y_j = \Delta y$

So rectangle R_{ij} has area $\underline{\Delta A} = \underline{\Delta x \Delta y}$

Each of these rectangles will form base of one of my boxes.

What is height of ij-th box?

Pick any point in R_{ij}, call it (x_{ij}^*, y_{ij}^*). Then plug in to f to get $f(x_{ij}^*, y_{ij}^*)$. This will be height of my box.

(If f is continuous all points near $(x_{ij}^*\ y_{ij}^*)$ should give roughly same height)

So have height, area of base for each box. Then volume of ij-th box is

$$\underset{(\text{height})}{f(x_{ij}^*, y_{ij}^*)}\ \underset{(\text{area base})}{\Delta A}$$

Then total volume of <u>all</u> boxes is

$$\sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_{ij}^*, y_{ij}^*) \Delta A$$

This is approximation to volume under our surface

$$V \approx \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_{ij}^*, y_{ij}^*) \Delta A$$

If we take smaller and smaller rectangles the approximation gets better and better

If we take <u>limit</u> as $\Delta A \to 0$ the figure should be exact

$$V = \boxed{\lim_{\Delta A \to 0} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_{ij}^*, y_{ij}^*) \Delta A}$$

Above limit doesn't always exist, but it often does. (ex: Bounded continuous functions)

This limit is a mathematical object in its own right (not just for calculating volumes). We call this limit the integral of f on the region D.

Could be specific and call it the "double integral" on D

Denoted

$$\iint_D f(x,y)\, dA = \boxed{\lim_{\Delta A \to 0} \sum_{j=0}^{n-1} \sum_{i=0}^{m-1} f(x_{ij}^*, y_{ij}^*)\, \Delta A}$$

Note: Could also write just as

$$\boxed{\int_D f(x,y)\, dA} \quad \text{or} \quad \int_D f(\vec{x})\, dA$$

If this limit exist f

If this limit exists for our function f, we say f is integrable on D

Don't need midpoint rule

Previous theory all well and good but how do we actually calculate these integrals?

Have a hugely important theorem that makes life easy.

Fubini's Theorem:

If f is continuous on rectangle $D = \{(x,y) : a \leq x \leq b, \quad c \leq y \leq d\}$ then

$$\iint_D f(x,y)\,dA = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

The right hand side is a iterated integral, meaning we do the inner integral first, as normal, then the outer integral.

Ex: $\int_0^4 \int_1^2 \underbrace{x^2 y}_{} \, dx \, dy$ ★

$\int_1^2 \int_0^4 x^2 y \, dy \, dx$ ★

$$= \int_0^4 \left. y \frac{x^3}{3} \right]_1^2 dy$$

$$= \int_0^4 y \left(\frac{8}{3} - \frac{1}{3} \right) dy$$

$$= \int_0^4 y \, \frac{7}{3} \, dy$$

$$= \frac{7}{3} \int_0^4 y \, dy$$

$$= \frac{7}{3} \left(\left. \frac{y^2}{2} \right]_0^4 \right)$$

$$= \frac{7}{3} (8)$$

$$= \frac{56}{3}$$

Note we could have just as easily said

$$\iint_D f(x,y)\,dA = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

So we see

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b \underbrace{f(x,y)\,dx}\,dy$$

But will need to be careful in later sections if our bounds are more complicated than mere constants

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So Fubini's theorem is big. It allows us to split up these multivariable integrals into simpler pieces.

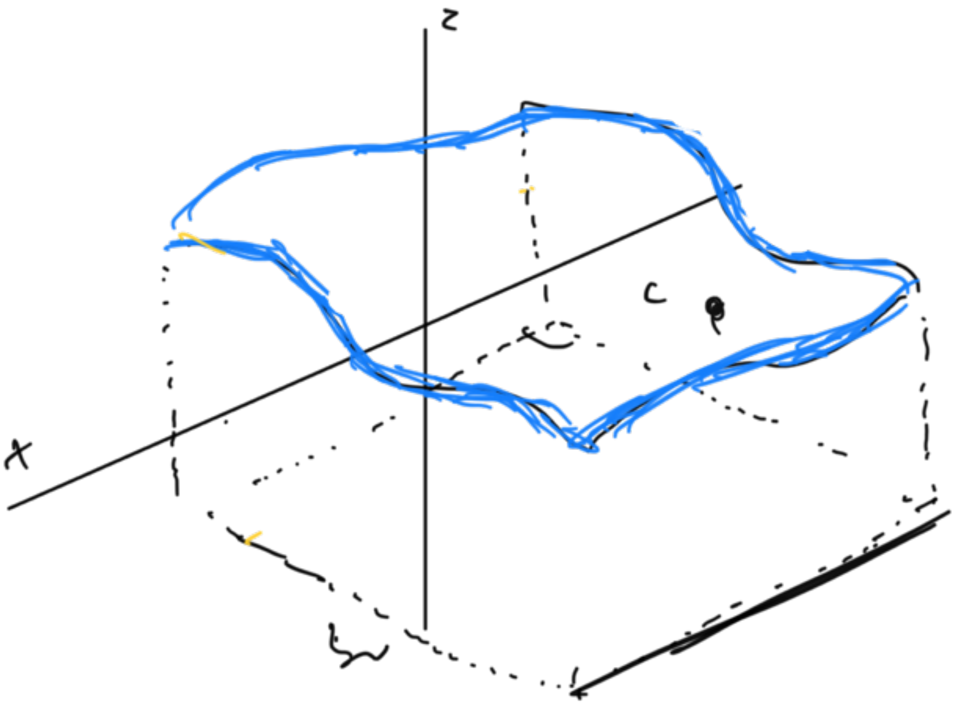
A naive view on why this works:

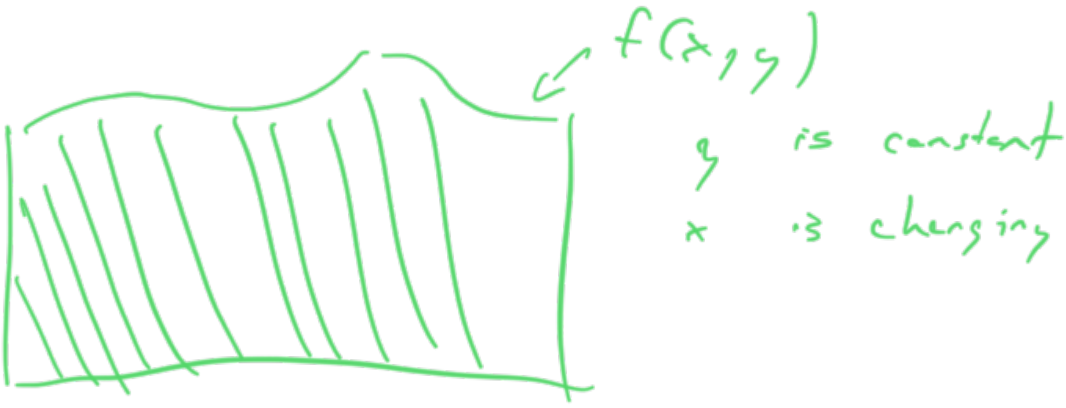
view on why this works.

Want to find volume under a surface. Know

$$V = \iint_D f(x,y)\,dA$$

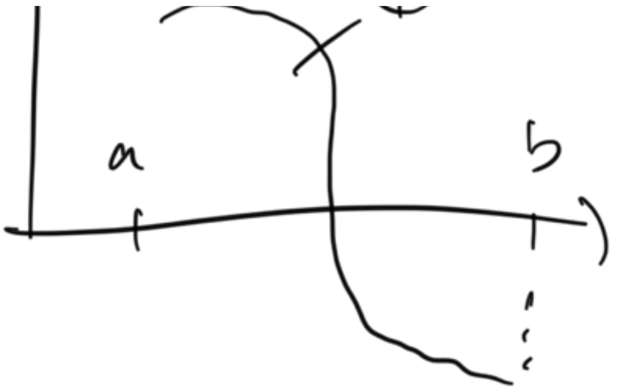
On the other, Fix y and consider $\underline{\int_a^b f(x,y)\,dx}$





$$V = \int_c^d \int_a^b f(x,y)\,dx\ dy$$

integral is



a "signed" area

\+ −

Gives area of a slice. Then integrating with respect to y is like multiplying area by a width to get volume of a slice, adding up slices

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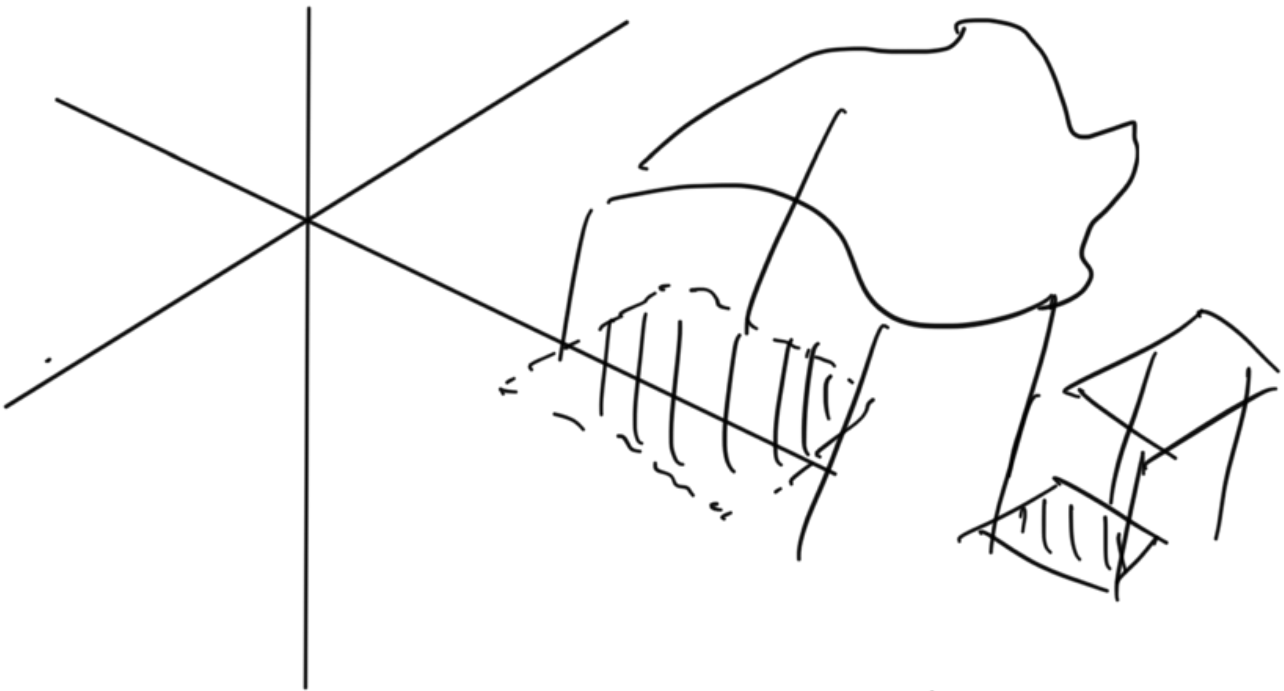
Average value:



volume box / area bas
= height

$$f_{av} = \frac{1}{A(D)} \iint_D f(x,y)\, dA$$

$\overline{}$ (total value of f)
area

= average value of f

① $$\iint_R (f+g)\,dA$$

$$= \iint_R f\,dA + \iint_R g\,dA$$

② $$\iint_R c f\,dA$$

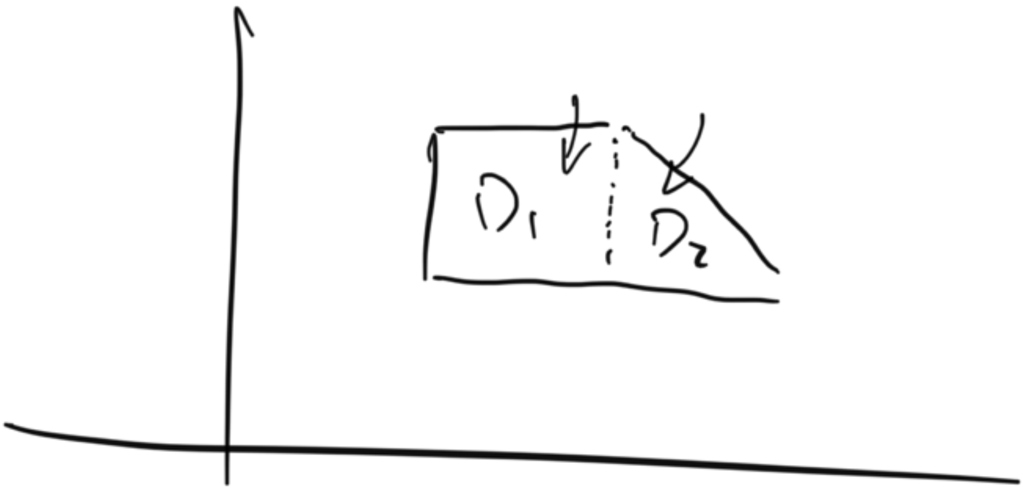
$$c\iint_R f\,dA$$

③ $f \le g$ on R

$$\iint_R f\,dA \le \iint_R g\,dA$$

④ If D_1 and D_2 do not intersect except possibly at a boundary, then

$$\iint_{\underline{D_1} \cup \underline{D_2}} f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA$$



Integration examples:

Ex: $$\boxed{\int_1^4 \int_0^2 \underline{6x^2y} - \underline{2x}) \, dy\, dx}$$

$$= \int_1^4 \left(\frac{6x^2y^2}{2} - 2xy \Big]_0^2 \right) dx$$

$$= \int_1^4 \left(\left[\frac{6x^2 4}{2} - 4x\right] - [0] \right) dx$$

$$= \int_1^4 \left(12x^2 - 4x\right) dx$$

$$= 4x^3 - 2x^2 \Big]_1^4$$

$$= (4\cdot 64 - 2\cdot 16) - (4\cdot 1 - 2)$$

$$224 - 2$$

$$\boxed{= 222}$$

Ex: $$\underline{\iint_R} \frac{xy}{x^2+1}\,dA$$

where $R = \underline{\underline{[0,1]}} \times \underline{\underline{[-3,3]}}$

$$\int_{-3}^{3} y\left(\int_0^1 \frac{x}{x^2+1}\,dx\right)dy$$

$u = x^2 \qquad du = 2x\,dx$

$$\frac{du}{2x} = dx$$

$$\int_{-3}^{3} y\left(\int_{-}^{-} \frac{1}{2}\,\frac{1}{1+u}\,du\right)dy$$

$$\int_{-3}^{3} y\left(\frac{1}{2}\ln|1+u|\Big]_{-}^{-}\right)dy$$

$$\int_{-3}^{3} y\left(\frac{1}{2}\ln|1+x^2|\Big]_0^1\right)dy$$

$$\frac{1}{2}\ln(2)\int_{-3}^{3} y\,dy$$

$$\frac{1}{2}\ln(2)\left(\frac{y^2}{2}\Big]_{-3}^{3}\right)$$

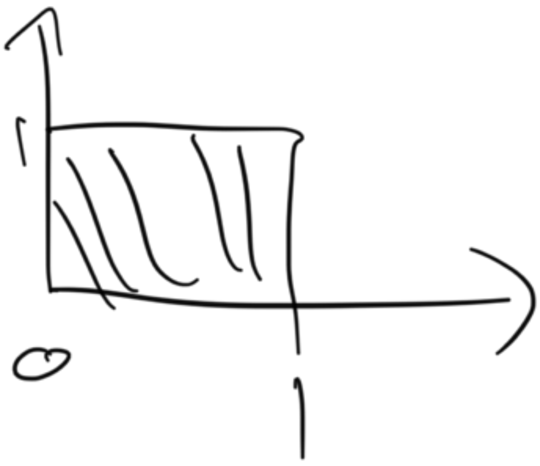
$$\frac{1}{2} \ln(2) \left(\frac{9}{2} - \frac{9}{2} \right)$$

Ex:

$$\iint_{R} \frac{x}{1+xy}$$

on

$$R = \{(x,y) ; 0 \le x \le 1, \quad 0 \le y \le 1\}$$



$$\int_0^1 \int_0^1 \frac{x}{1+xy} \, dy \, dx$$

$$u = xy \qquad du = x\,dy$$

$$\frac{du}{x} = dy$$

$$\int_0^1 \int_{-}^{-} \frac{1}{1+u} \, du \, dx$$

$$\int_0^1 \ln|1+u| \Big] \, dx$$

$$\int_0^1 \ln|1+xy| \Big]_0^1 \, dx$$

$$= \int_0^1 \overset{dv}{1} \cdot \overset{u}{\ln}(1+x) \, dx$$

$$= x \ln(1+x)\Big]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$$x \ln(1+x)\Big]_0^1 - \int \frac{v-1}{v}\,dv$$

$$x \ln(1+x)\Big]_0^1 - \int 1 - \frac{1}{v}\,dv$$

$$x \ln(1+x)\Big]_0^1 - \left(v - \ln v\Big]\right)$$

$$x \ln(1+x)\Big]_0^1 - \left((1+x) - \ln(1+x)\Big]_0\right.$$

$$\ln(2) - \left((2 - \ln(2)) - (1-0)\right.$$

$$\ln(2) - 2 + \ln(2) + 1$$

$$\boxed{2\ln(2) - 1}$$