13.2 · Calculus for vector valued functions

Working with vector valued functions

$$\vec{s} : \mathbb{R} \to \mathbb{R}^n$$

In particular

$$\vec{s} : \mathbb{R} \to \mathbb{R}^3$$

Recall definition for derivative for

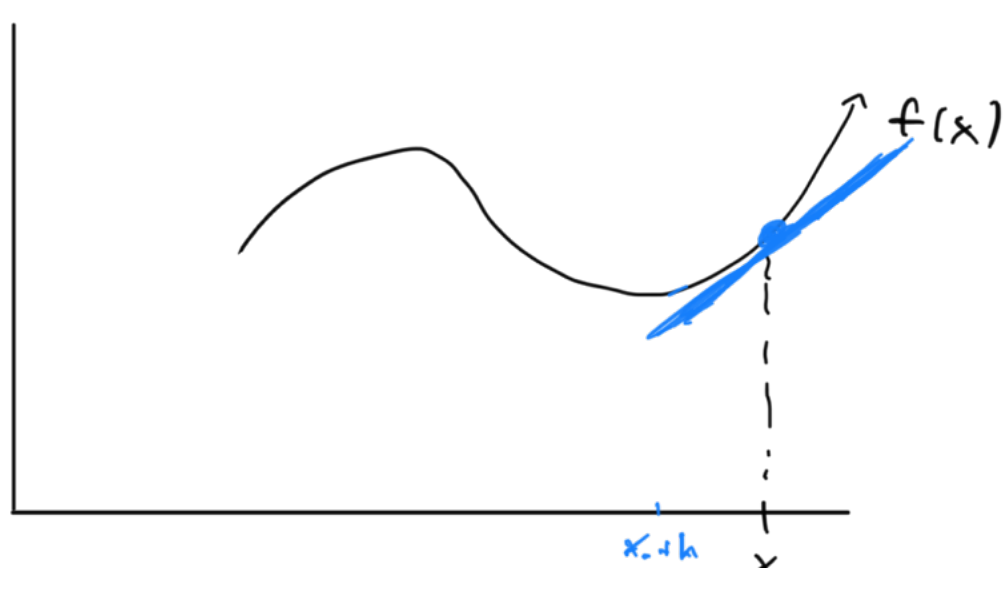
$$f : \mathbb{R} \to \mathbb{R}$$

$$\frac{df}{dx} = \underline{f'(x_0)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$\underline{\text{or}}$

$$\frac{df}{dx} = \underline{f'(x_0)} = \lim_{h \to 0} \boxed{\frac{f(x_0+h) - f(x_0)}{h}}$$

when the limits exist



Can define derivative for $\vec{s}: \mathbb{R} \to \mathbb{R}^3$ the same way

Derivatives

$$\vec{s}: \mathbb{R} \to \mathbb{R}^{n}$$

$$\frac{d\vec{s}}{dt} = \vec{s}\,'(t) = \lim_{h \to 0} \frac{\vec{s}(t+h) - \vec{s}(t)}{h} \quad \text{(when limit exists)}$$

$$\boxed{\frac{\overset{\text{Vector}}{\vec{s}(t+h)} - \overset{\text{Vector}}{\vec{s}(t)}}{h}} \qquad \frac{\overset{\#}{f(x+h)} - \overset{\#}{f(x)}}{h \ \#}$$

scalar

$$f'(x_0)$$

$$\#$$

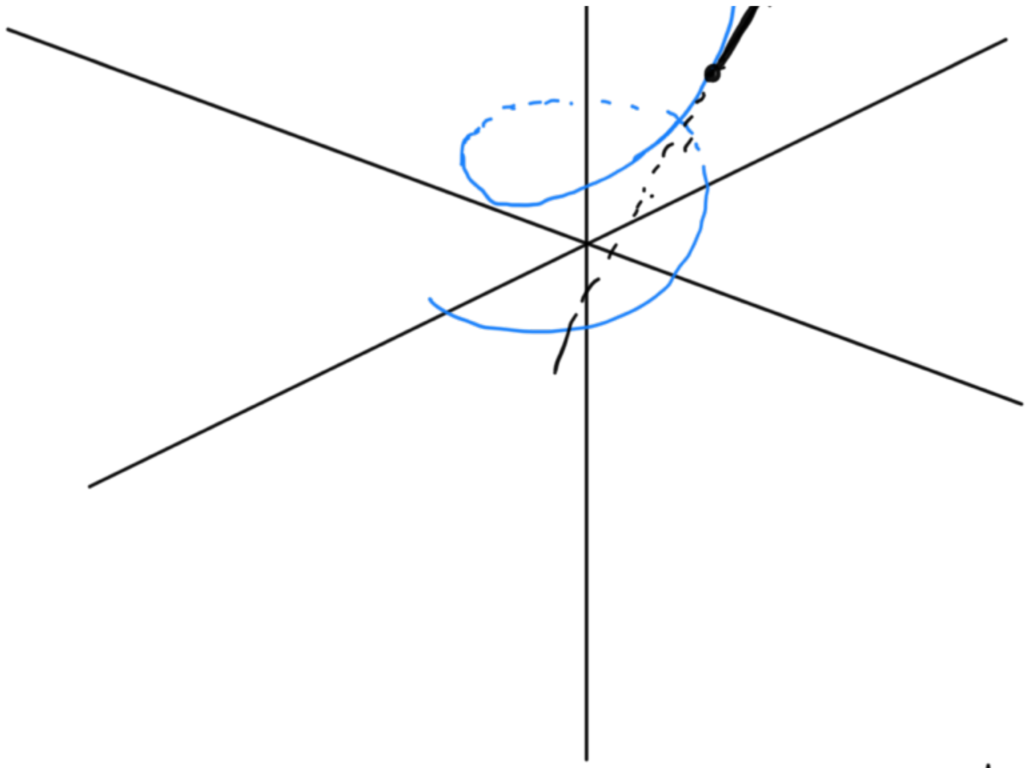
$$= \frac{1}{h}\left(\underbrace{\vec{s}(t+h) - \vec{s}(t)}\right)$$

scalar (vector) = vector

So $\boxed{\vec{s}\,'(t)}$ is a vector. How can we think about it?

$\vec{s}(t)$

$\vec{s}\,'(t) = \langle -1, 1, 1 \rangle$



slope

$$y = b + x \cdot m$$

$$\vec{r}(t) = \vec{r}_0 + t\,\vec{v}$$

slope

$\vec{s}(t)$ $\quad s'(t)$ vector parallel to the tangent line

$\vec{s}\,'(t)$ will be a vector that points in direction of the tangent line

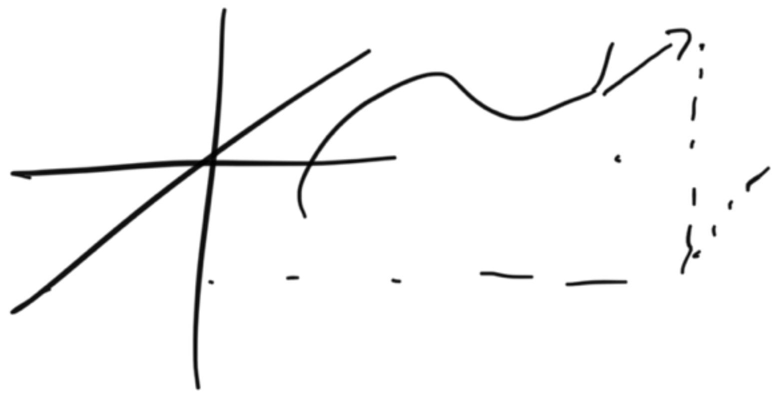
$f : \mathbb{R} \to \mathbb{R}$ $\quad f'$ slope of tangent line

$\vec{s} : \mathbb{R} \to \mathbb{R}^n$ $\quad \vec{s}\,'$ is tangent vector

Similar ideas, both are "direction" function is moving at particular instant

$f : \mathbb{R} \to \mathbb{R}$

$\vec{s} : \mathbb{R} \to \mathbb{R}^3$



Calculating the Derivative

Recall, at its heart the derivative is a limit. Saw in last section that if $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ then

$$\lim_{t \to t_0} \vec{r}(t) = \left\langle \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \right\rangle$$

In other words can take limit inside the vector

Similarly if $\vec{s}(t) = \langle f(t), g(t), h(t) \rangle$

then

$$\vec{s}\,'(t) = \lim_{k \to 0} \left(\frac{\overset{\text{vector} - \text{vector}}{\vec{s}(t+k) - \vec{s}(t)}}{\underset{\text{scalar}}{k}} \right)$$

$$= \lim_{k \to 0} \frac{\langle f(t+k), g(t+k), h(t+k) \rangle - \langle f(t), g(t), h(t) \rangle}{k}$$

$$= \lim_{k \to 0} \left[\frac{1}{k} \langle f(t+k) - f(t), g(t+k) - g(t), h(t+k) - h(t) \rangle \right]$$

$$= \lim_{k \to 0} \left\langle \frac{f(t+k) - f(k)}{k}, \frac{g(t+k) - g(t)}{k}, \frac{h(t+k) - h(t)}{k} \right\rangle$$

$f : \mathbb{R} \to \mathbb{R}$

$$= \left\langle \lim_{k \to 0} \frac{f(t+k) - f(k)}{k}, \lim_{k \to 0} \frac{g(t+k) - g(t)}{k}, \lim_{k \to 0} \frac{h(t+k) - h(t)}{k} \right\rangle$$

$$= \langle f'(t), g'(t), h'(t) \rangle$$

Ex: $\vec{r}(t) = \langle \underline{3e^{-4t}}, \ln(1+2t), 2t\sin(t) \rangle$

$\vec{r}\,'(t) =$

$$\left\langle -12e^{-4t},\ \frac{2}{1+2t},\ 2\sin(t)+2t\cos(t) \right\rangle$$

Ex:

$$\vec{r}(t) = \langle t^2+1,\ 4\sqrt{t},\ e^{t^2-1} \rangle$$

$$\langle 10,\ 4\sqrt{3},\ e^8 \rangle$$

Find vector equation of tangent line

@ at $\boxed{t=3}$

$$\vec{r}\,'(t) = \langle 2t,\ 2t^{-\frac{1}{2}},\ 2te^{t^2-1} \rangle$$

$$\vec{r}\,'(3) = \langle 6,\ \frac{2}{\sqrt{3}},\ 6e^8 \rangle$$

$$\vec{b}(t) = \boxed{\vec{b}_0} + t\,\vec{v}$$

$$\vec{b}(t) = \langle 10, 4\sqrt{3}, e^8 \rangle + \boxed{\langle 6, \frac{2}{\sqrt{3}}, 6e^8 \rangle}$$

(b)

at point $\underline{(2,4,1)}$ $t=1$

$t=1$? $\langle 2,4,1\rangle$

$\vec{r}(t) = \langle t^2+1,\ 4\sqrt{t},\ e^{t^2-1}\rangle$

$\vec{r}\,'(t) = \langle 2t,\ 2t^{-\frac{1}{2}},\ 2te^{t^2-1}\rangle$

$\vec{b}(t) = \vec{b_0} + t\vec{v}$

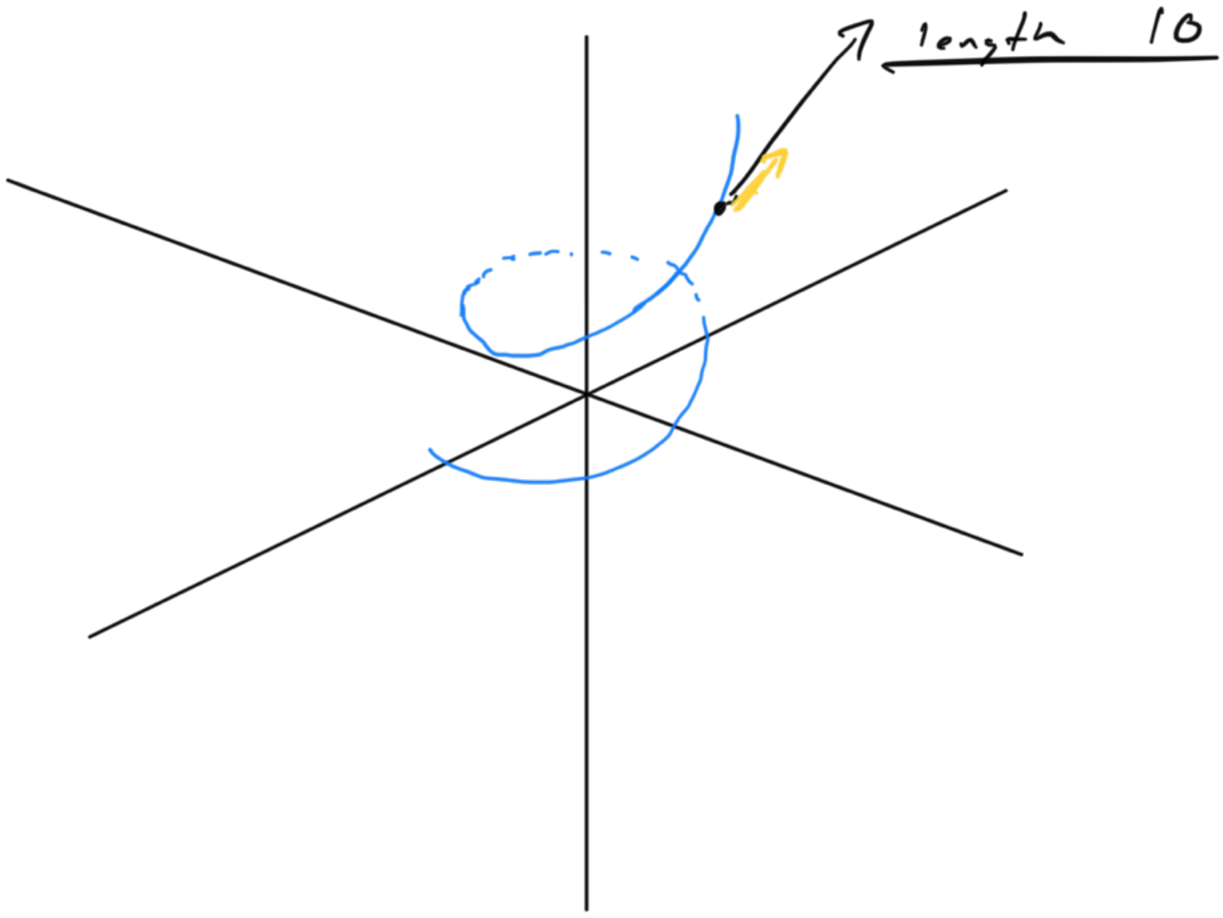
$\vec{b}(t) = \langle 2,4,1\rangle + t\boxed{\langle 2,2,2\rangle}$

Unit Tangent Vector:

Tangent vectors at different points on our curve may have different lengths.

This may give some <u>false</u> impression about rate of change of function

To dispel these notions, often work with <u>unit tangent vector</u>



To calculate

① Find $\vec{s}\,'(t)$

② Find magnitude, $|s'(t)|$

③ $\dfrac{s'(t)}{|s'(t)|} = \mathbf{T}(t)$

Ex: $\vec{s}(t) = \langle \sin(t), \cos(t), t^3 \rangle$

① Find $\vec{s}\,'(t)$

$$\vec{s}\,'(t) = \langle \cos(t), -\sin(t), 3t^2 \rangle$$

② Find $T(t)$

① Done

② $\|s'(t)\| =$

$$\sqrt{\cos^2(t) + \sin^2(t) + 9t^4}$$

$$|s'(t)| = \sqrt{1 + 9t^4}$$

$$\vec{T}(t) = \frac{1}{\sqrt{1+9t^4}} \langle \cos(t), -\sin(t), 3t^2 \rangle$$

③ Find $\underline{\vec{T}(1)}$

$$T(1) = \frac{1}{\sqrt{10}} \langle \cos(1), -\sin(1), 3 \rangle$$

Properties of Derivative

① $\frac{d}{dt}[\vec{u}(t) + \vec{v}(t)] = \vec{u}'(t) + \vec{v}'(t)$

② $\frac{d}{dt}[c\,\vec{u}(t)] = c\,\vec{u}'(t)$

$$f, g : \mathbb{R} \to \mathbb{R}$$

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

$f: \mathbb{R} \to \mathbb{R}$ $\quad \vec{u}: \mathbb{R} \to \mathbb{R}^n$

(3) $\frac{d}{dt}\left[f(t)\, \vec{u}(t) \right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

scalar vector $\quad$ scalar vector + scalar vector

= vector

(4) $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t) \right] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$

vector • vector $\quad$ scalar + scalar

scalar $\quad$ $\boxed{\text{scalar}}$

(5) $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t) \right] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

vector × vector $\quad$ vector + vector

vector $\quad$ $\boxed{\text{vector}}$

$\frac{d}{dt}\left[\underline{a} \circ \underline{b} \right] = \underline{a}' \circ \underline{b} + a \circ b'$

(6) $\frac{d}{dt}\left[\vec{u}(f(t)) \right] = \underline{f'(t)}\ \underline{u'(f(t))}$

$\to \vec{u}(t) = \langle 3t, 4t^3, \sin(t) \rangle$

$\vec{u}(t) = \langle 3,\ 12t^2,\ \cos(t) \rangle$

$f(t) = \ln(1+3t)$

$$\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\underline{\vec{u}'(f(t))}$$

$$\boxed{\left(\frac{3}{1+3t}\right)\left\langle 3,\ 12(\ln(1+3t))^2,\ \cos(\ln(1+3t)) \right\rangle}$$

Ex:

$f(t) = e^{2t}$

$\vec{s}(t) = \langle \ln(1+t),\ t^2,\ \cos(t) \rangle$

Find $\frac{d}{dt}\left[f(t)\,\vec{s}(t)\right]$

$$= 2e^{2t}\langle \ln(1+t),\ t^2,\ \cos(t) \rangle$$

$$+\ e^{2t}\left\langle \frac{1}{1+t},\ 2t,\ -\sin(t) \right\rangle$$

$= e^{2t} \langle 2\ln(1+t) \ldots$

$$= \left\langle \ln(1+t) + \frac{\quad}{1+t},\ 2t^{2} + 2t,\ 2\cos(t) - \sin t \right\rangle$$

Ex:

$$\vec{r}(t) = \langle e^{2t},\ t,\ 1-t^2 \rangle$$

$$\vec{s}(t) = \langle \tan(t),\ 3t^3,\ 1 \rangle$$

Find $\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{s}(t)\right]$

$$\langle 2e^{2t},\ 1,\ -2t \rangle \cdot \langle \tan(t),\ 3t^3,\ 1 \rangle$$

$$+ \langle e^{2t},\ t,\ 1-t^2 \rangle \cdot \langle \sec^2(t),\ 9t^2,\ 0 \rangle$$

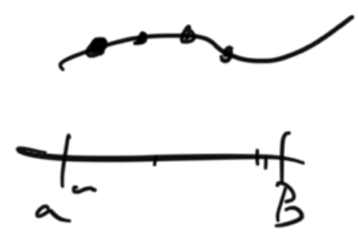
Integrals

unsurprisingly, integral behaves similarly to limit, derivative. (can be taken inside vector)

First, let go of intuitive ideas of integral for a moment

For $f : \mathbb{R} \to \mathbb{R}$, recall mathematical definition of integral

$f : \mathbb{R} \to \mathbb{R}$



$$\boxed{\int_a^b f(x)\,dx} = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \text{scalar}$$

function value

change in $\boxed{\text{ind.}}$ variable

Can make similar definition for

$\vec{s} : \mathbb{R} \to \mathbb{R}^3$

$$\int_a^b \vec{s}(t)\,dt = \lim_{\Delta t \to 0} \boxed{\sum_{i=1}^{n} \vec{s}(t_i^*)\,\Delta t_i}$$

$$= \cancel{\lim_{\Delta t \to 0}} \left\langle \underbrace{\lim \sum_{i=1}^{n} f(t_i^*)\Delta t_i}_{\int_a^b f(x)dx},\ \lim \sum_{i=1}^{n} g(t_i^*)\Delta t_i,\ \lim \sum_{i=1}^{n} h(t_i^*)\Delta t_i \right\rangle$$

$$= \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

$$\vec{s}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\int_a^b \vec{s}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

Ex: $\vec{s}(t) = \langle 3t^3, \ln(t), e^t \rangle$

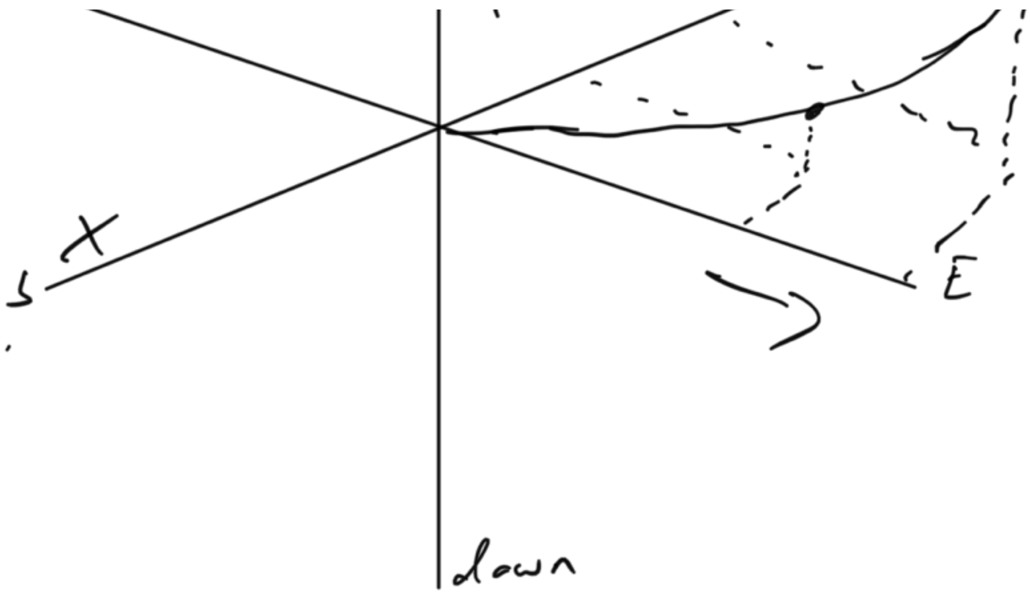
$$\int_2^4 \vec{s}(t)\,dt$$

$$= \left\langle \int_2^4 3t^3\,dt,\ \int_2^4 1\cdot\ln(t)\,dt,\ \int_2^4 e^t\,dt \right\rangle$$

$$= \left\langle \frac{3}{4}t^4 \Big]_2^4,\ t\ln(t) - t \Big]_2^4 \right.$$

$$\int_2^4 \vec{s}(t)\,dt = \left\langle \frac{3}{4}(240),\ (4\ln(4) - 4) - (2\ln(2) - 2),\ e^4 - e^2 \right\rangle$$

Ex: Plane travelling through air.



$$\vec{v}(t) = \langle -20t,\ 400t^2,\ 30t \rangle$$

Distance travelled in each direction after 30 minutes?

$$\int_0^{30} \vec{v}(t)\,dt = \left\langle \int_0^{30} -20t\,dt,\ \int_0^{30} 400t^2\,dt,\ \int_0^{30} 30t\,dt \right\rangle$$

$$\langle a,\ b,\ c \rangle$$

a distance travelled north/south

b distance east west

c distance up/down

$|\langle a,b,c\rangle| =$ <u>total</u> distance

$$\int \vec{r}(t)\,dt = \left\langle \int f(t)\,dt,\ \int g(t)\,dt,\ \int h(t)\,dt \right\rangle + \vec{c}$$