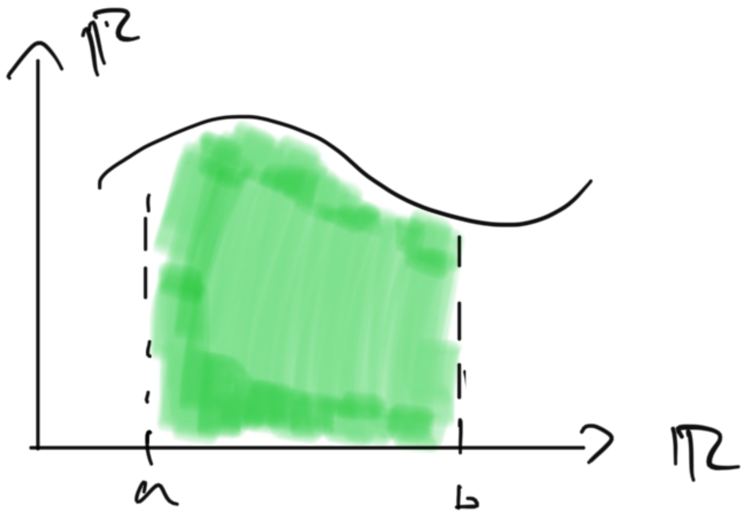
15.6 - Triple Integrals

15.6 is a very quick intro to triple integrals (Now domain will be in $\mathbb{R}^3$, the 3-dimensional space)

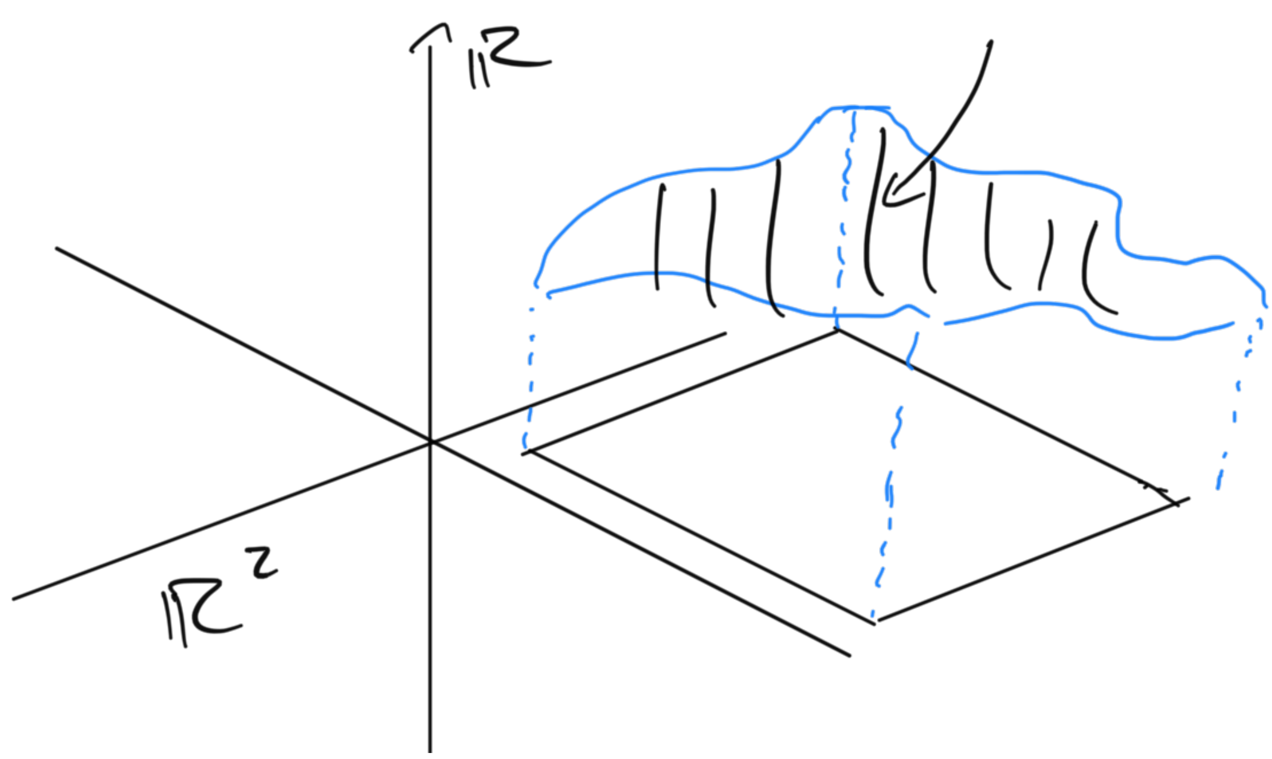
For $f: \mathbb{R} \to \mathbb{R}$, integral could be thought of as "area" beneath a curve



$\mathbb{R}^2$

For $f: \underline{\mathbb{R}^2} \to \mathbb{R}$, integral is sort of volume beneath a surface

$\mathbb{R}^3$



What is integral for $f: \mathbb{R}^3 \to \mathbb{R}$?
What comes after volume?

Nothing that makes geometric sense.
Can't even graph $f: \mathbb{R}^n \to \mathbb{R}$ nicely for $n \geq 3$.

Could call curve $f: \mathbb{R}^3 \to \mathbb{R}$ a hypersurface and could think of integral of f as giving a hypervolume. But only limited benefit to this way of thinking.

Better to think of it in a more

abstract sense as a limit of Riemann sums.

For $f: \mathbb{R}^3 \to \mathbb{R}$ have

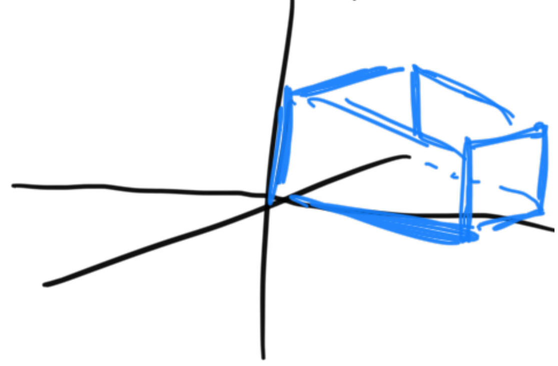
$$\sum_{k=1}^{n} \sum_{j=1}^{m} \sum_{i=1}^{l} f\underbrace{(x_{ijk}, y_{ijk}, z_{ijk})}\Delta V$$

$$\underbrace{\iiint_R f(x,y,z)\, dV}$$

Similar to development of double integral we compute a triple integral by turning it into iterated integral.

Simplest case: Domain

R is a box.



$a \le x \le b$

$c \le y \le d$

$e \le z \le f$

Then by <u>Fubini</u>:

$$\underbrace{\iiint_R f(x,y,z)\,dV}= \int_e^f \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Also by Fubini we can change order of integration as we like (when bounds simple)

Ex: $f(x,y,z) = xyz$

$R = \{ (x,y,z) : 0 \le x \le 1,\ 1 \le y \le 3,$
$-1 \le z \le 0 \}$

$$\iiint_R f(x,y,z)\,dV = \int_0^1 \int_1^3 \int_{-1}^0 xyz\,dz\,dy\,dx$$

$$\int_0^1 \int_1^3 xy \left(-\tfrac{1}{2}\right) dy\,dx$$

$$\int_0^1 -\frac{1}{2}x\left(\frac{y^2}{2}\Big]_1^3\right)dx$$

$$\int_0^1 -\frac{1}{2}x \cdot 4 \quad dx$$

$$-2\int_0^1 x \; dx$$

$$-2\left(\frac{x^2}{2}\Big]_0^1\right)$$

$$-2 \cdot \frac{1}{2} = -1$$

General Regions

Similar to 15.2 progress from simple rectangular regions to those bounded by functions.

$$\int_a^b \int_{g_1(x)}^{g_2(x)} dy\, dx$$

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} dz\, dy\, dx$$

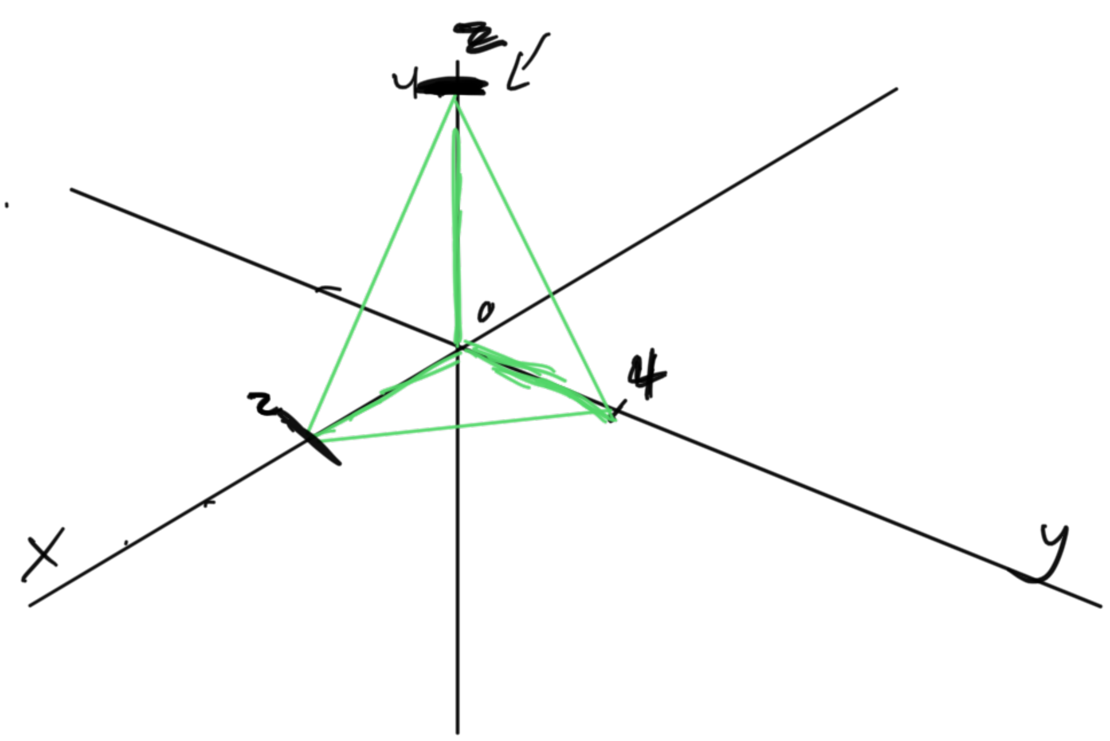
Allow two out of three of integrals to have functions for bounds.

Outer integral should have constants for bounds.

Ex: Find volume of region R bounded by <u>coordinate planes</u> and $2x + y + z = 4$

$$\iiint_D 1 \, dV$$

$$2x + y + z = 4$$
$$y + z = 4$$
$$z = -y + 4$$



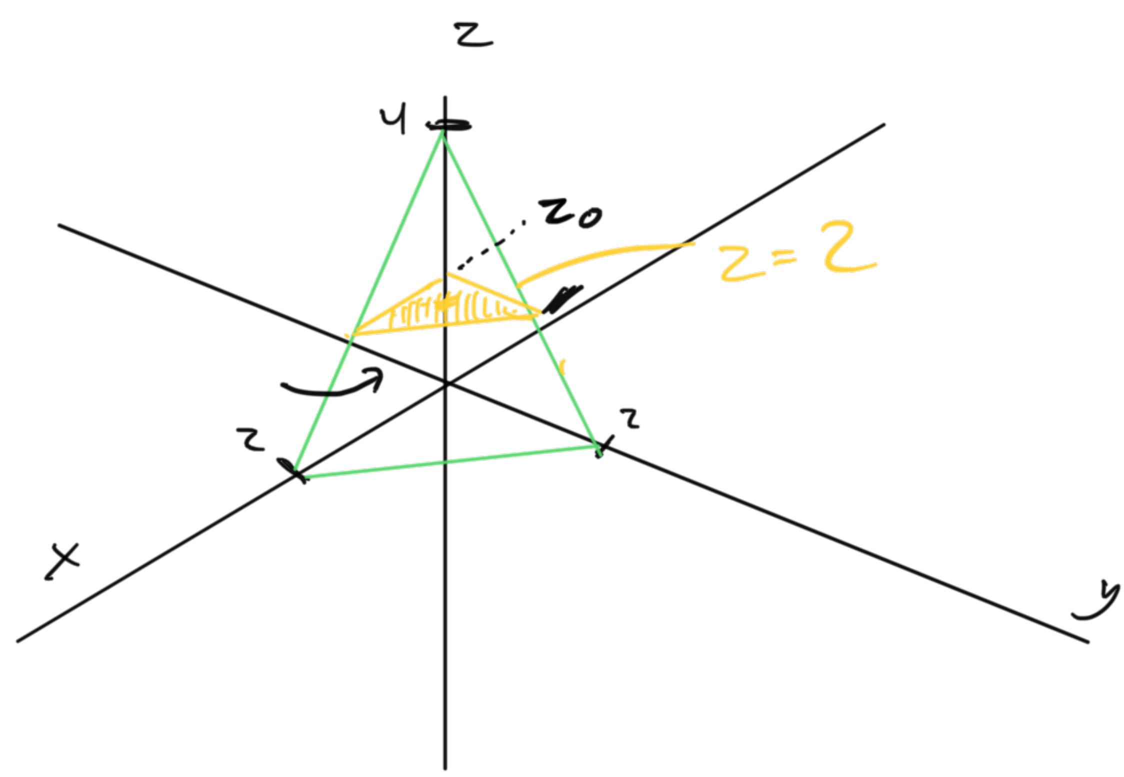
① Draw region of integration

$$\int\int\int$$

Strategy: ② Pick one direction to be constants.

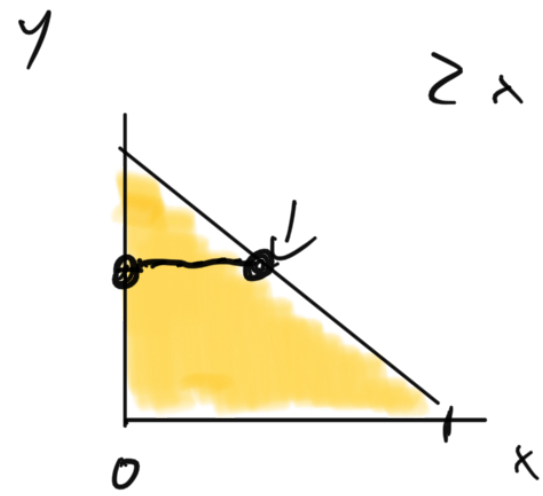
$$\int_0^4 \qquad dz$$

③ Consider cross sections of region along your chosen line



$$\int_0^4 \int_0^{4-z} \int_0$$

$$2x+y+z=4$$



$$2x+y+z=4$$

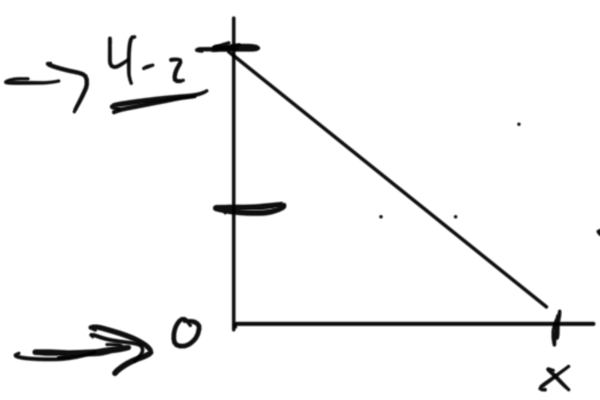
$$2x+y$$

$$\cancel{y=4}-z$$

$$2x+y+z=4$$

$$x=\boxed{\frac{4-y-z}{2}}$$

③ Now write bounds for 2-D regions pretending chosen parameter is a constant.



$$y=-2x+(4-z_0)$$

$$x=\frac{-y+(4-z)}{2}$$

$$\int_0^4\int_0^{4-z}\int_0^{\boxed{\frac{-y}{2}+\frac{(4-z)}{2}}} dx\,dy\,dz$$

④ Take 1-D cross section along your 2nd direction

$$\int_0^4 \int_0^{4-z} \int_0^{-\frac{y}{2} + \frac{(4-z)}{2}} dx\,dy\,dz$$

$$= \int_0^4 \int_0^{4-z} -\frac{y}{2} + \frac{(4-z)}{2} \; dy\,dz$$

$$\rightarrow \quad = \int_0^4 \left(-\frac{y^2}{4} + \frac{(4-z)y}{2} \Big]_0^{4-z} \right) dz$$

$$= \int_0^4 \left(-\frac{(4-z)^2}{4} + \frac{(4-z)^2}{2} \right) dz$$

$$\rightarrow \quad = \int_0^4 \frac{(4-z)^2}{4} \, dz$$

$$= \int_0^4 \frac{16 - 8z + z^2}{4} \, dz$$

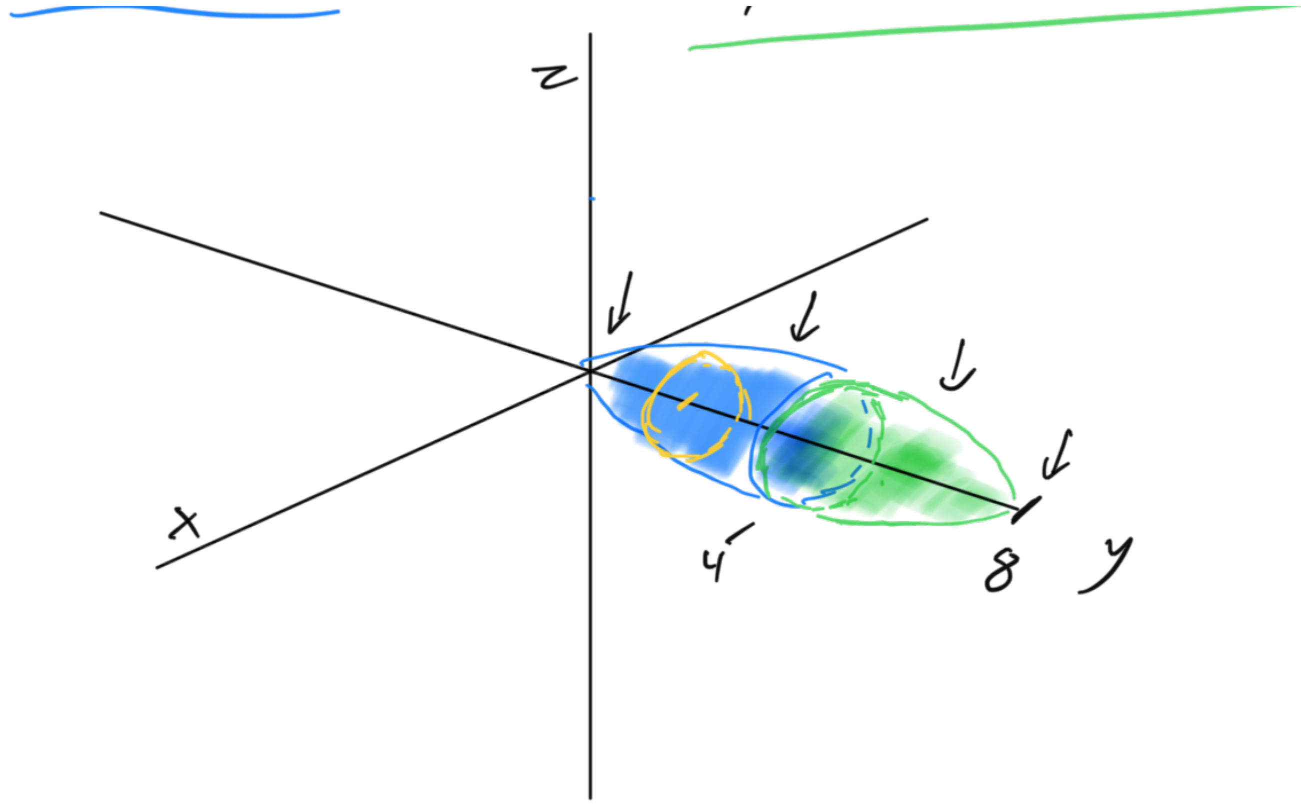
$$= \int_0^4 \left(4 - 2z + \frac{z^2}{4} \right) dz$$

$$= 4z - z^2 + \frac{z^3}{12} \Big]_0^4$$

$$\rightarrow \quad = \frac{16}{3}$$

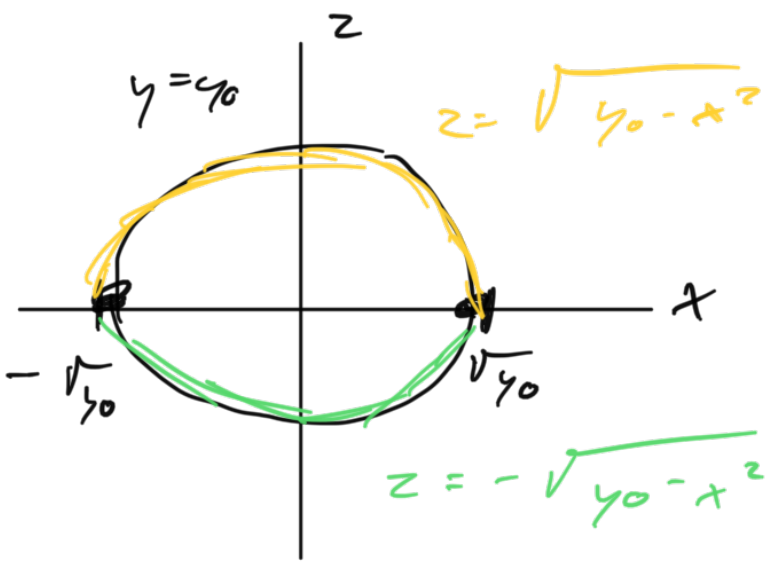
Ex: Find volume enclosed by surfaces

$y = x^2 + z^2$ and $y = 8 - x^2 - z^2$



① $2\int_0^4 \qquad dy$

$y_0 = x^2 + z^2$



$$2\int_0^4 \int_{-\sqrt{y_0}}^{\sqrt{y_0}} \int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} dz\,dx\,dy$$

$$2\int_0^4 \int_{-\sqrt{y_0}}^{\sqrt{y_0}} 2\sqrt{y-x^2}\,dx\,dy$$

$$x = \sqrt{y}\ \sin\theta$$

$$dx = \sqrt{y}\ \cos\theta\, d\theta$$

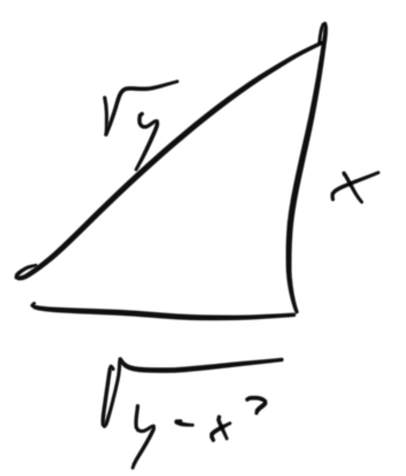
$$\int_0^4 \int_{-}^{-} 2\sqrt{y - y\sin^2\theta}\ \cos\theta\ d\theta$$

$$= \int_0^4 2\sqrt{y}\cos^2\theta\ d\theta$$

$$\int_0^4 \sqrt{y} \cdot \quad (1+\cos(2\theta))$$

$$\int_0^4 \sqrt{y}\left(\theta + \frac{\sin(2\theta)}{2}\right)$$

$$\sqrt{y}\,(\theta + \sin\theta\cos\theta)$$



$$x = \sqrt{y}\ \sin\theta$$

$$\frac{x}{\sqrt{y}} = \sin\theta$$

$$\theta = \sin^{-1}\left(\frac{x}{\sqrt{y}}\right)$$

$$2\int_0^4 \sqrt{y}\left(\sin^{-1}\left(\frac{x}{\sqrt{y}}\right) + \frac{x\sqrt{y-x^2}}{y}\right)\Bigg]_{-\sqrt{y}}^{\sqrt{y}}\ dy$$

$$2\int_0^4 \sqrt{y}\left[\left(\overset{\pi/4}{\sin^{-1}(1)} + \cancel{\frac{x\cdot 0}{4}}\right) - \left(\overset{-\pi/4}{\sin^{-1}(-1)} + \cancel{0}\right)\right] dy$$

$$2\int_0^4 \sqrt{y} \cdot \pi/2$$

$$= \pi \int_0^4 \sqrt{y}\, dy$$

$$\frac{2\pi}{3} \quad y^{\frac{3}{2}}\Big]_0^4$$

$$= \frac{2\pi}{3} \cdot 8$$

$$= \boxed{16\pi/3}$$

$$f : \mathbb{R}^2 \to \mathbb{R}$$