Section 6.4

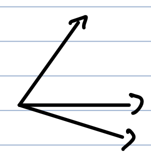
Have said that orthogonal / orthonormal vectors are convenient to word with.
Previously saw how to turn orthogonal into orthonormal.

<u>But</u> what if we dont even have orthogonal?

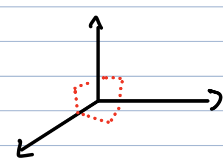
The <u>Gram - Schmidt Process</u> is a way to take linear independent vectors and get orthogonal / orthonormal

linear ind $\Rightarrow$ orthogonal / orthonormal

Takes basis like this



Turns it into this



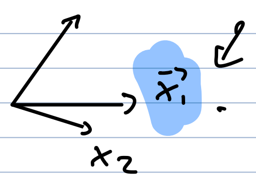
<u>Linear independent $\rightarrow$ Orthogonal</u>

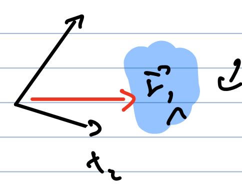
Theorem (Gram-Schmidt): Given basis $\{\vec{x}_1, \ldots, \vec{x}_p\}$ for nonzero subspace W of $\mathbb{R}^n$, let

$$\left.\begin{aligned}
\vec{v}_1 &= \vec{x}_1 \\
\vec{v}_2 &= \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1\right) \\
\vec{v}_3 &= \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} v_2\right) - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1\right) \\
&\cdots \\
\vec{v}_n &= \vec{x}_n - \frac{\vec{x}_n \cdot \vec{v}_{n-1}}{\vec{v}_{n-1} \cdot \vec{v}_{n-1}}\vec{v}_{n-1} - \cdots - \frac{\vec{x}_n \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1
\end{aligned}\right\}$$

Then $\{\vec{v}_1 \ldots \vec{v}_p\}$ is orthogonal basis for W and $\operatorname{span}\{\vec{v}_1 \ldots \vec{v}_p\} = \operatorname{span}\{\vec{x}_1 \ldots \vec{x}_p\}$.

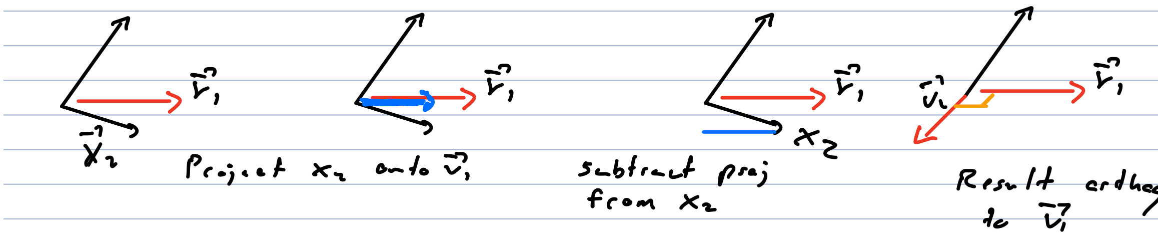
Translation: Pick one vector as your starting point ($\vec{x}_1$). It will remain unchanged. ($\vec{x}_1 = \vec{v}_1$) Everything else will move to become orthogonal to it and each other.





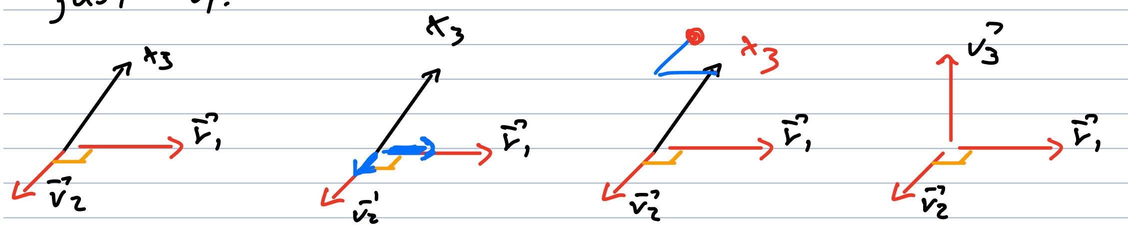
Now take next vector x_2. Take projection of x_2 onto $\vec{v}_1$. Then subtract projection from $\vec{x}_2$. Call the result $\vec{v}_2$, will be orthogonal to $\vec{v}_1$

$$\vec{v}_2 = \vec{x}_2 - \operatorname{proj}_{\vec{v}_1} x_2$$



Project x_2 onto $\vec{v}_1$ — Subtract proj from x_2 — Result orthog to $\vec{v}_1$

Now we want to "straighten out" $\vec{x}_3$. Repeat process, but need to project onto all $\vec{v}_i$'s, not just $\vec{v}_1$.



Take projections — Subtract projections — Result is $\vec{v}_3$

$$\underline{\vec{x}_3} - \text{proj}_{v_1}\vec{x}_3 - \text{proj}_{v_2}\vec{x}_3$$

straightens w.r.t $\vec{v}_1$ — straightens w.r.t. $\vec{v}_2$

Call result $\vec{v}_3$

$$\underline{\vec{v}_3} = x_3 - \text{proj}_{v_1}\vec{x}_3 - \text{proj}_{v_2}\vec{x}_3$$

(2) $\begin{array}{cc} & \vec{x}_1 \\ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, & \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} \\ \vec{x}_2 & \end{array}$ Turn them into orthogonal basis

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

$$\vec{v} = \vec{x}_2 - \text{proj}_{\vec{v}_1} \vec{x}_2$$

$$\begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{10}{20} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$$

Linear Independent → Orthonormal

① Gram-Schmidt (lin. ind → orthogonal)

② Divide new vectors by their own lengths

Will make them all unit vectors

Orthogonal and Unit vectors = orthonormal

QR

A has linearly ind. columns, can write

$$A = QR$$

where columns of Q are orthonormal basis for Col A, R upper triangular