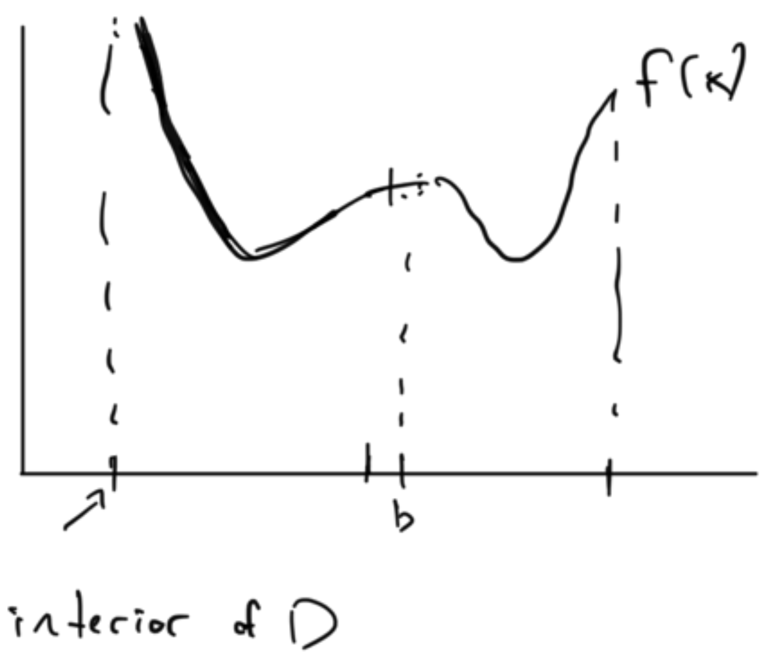
14.7 - Extreme Values

Recall in Calc I we discussed max/min values.

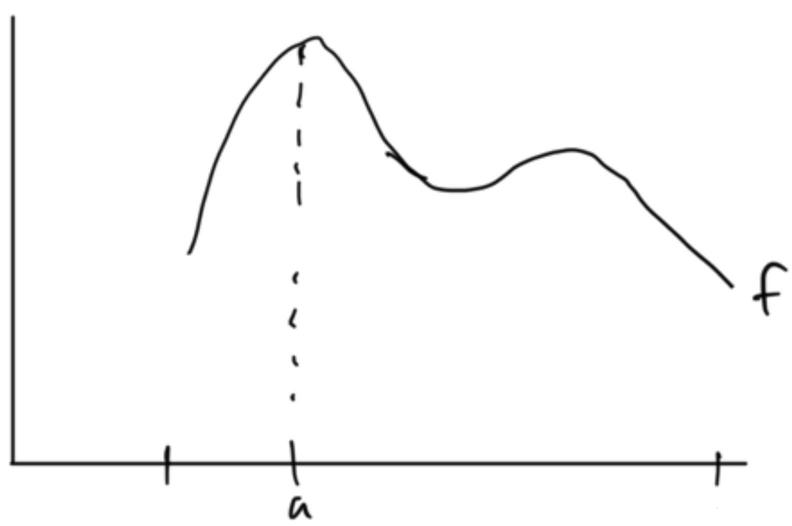
Had idea of ~~global~~ (absolute) maximum/minimums and local max/mins.



$f'(x) = 0$

critical points

2^{nd} derivative test

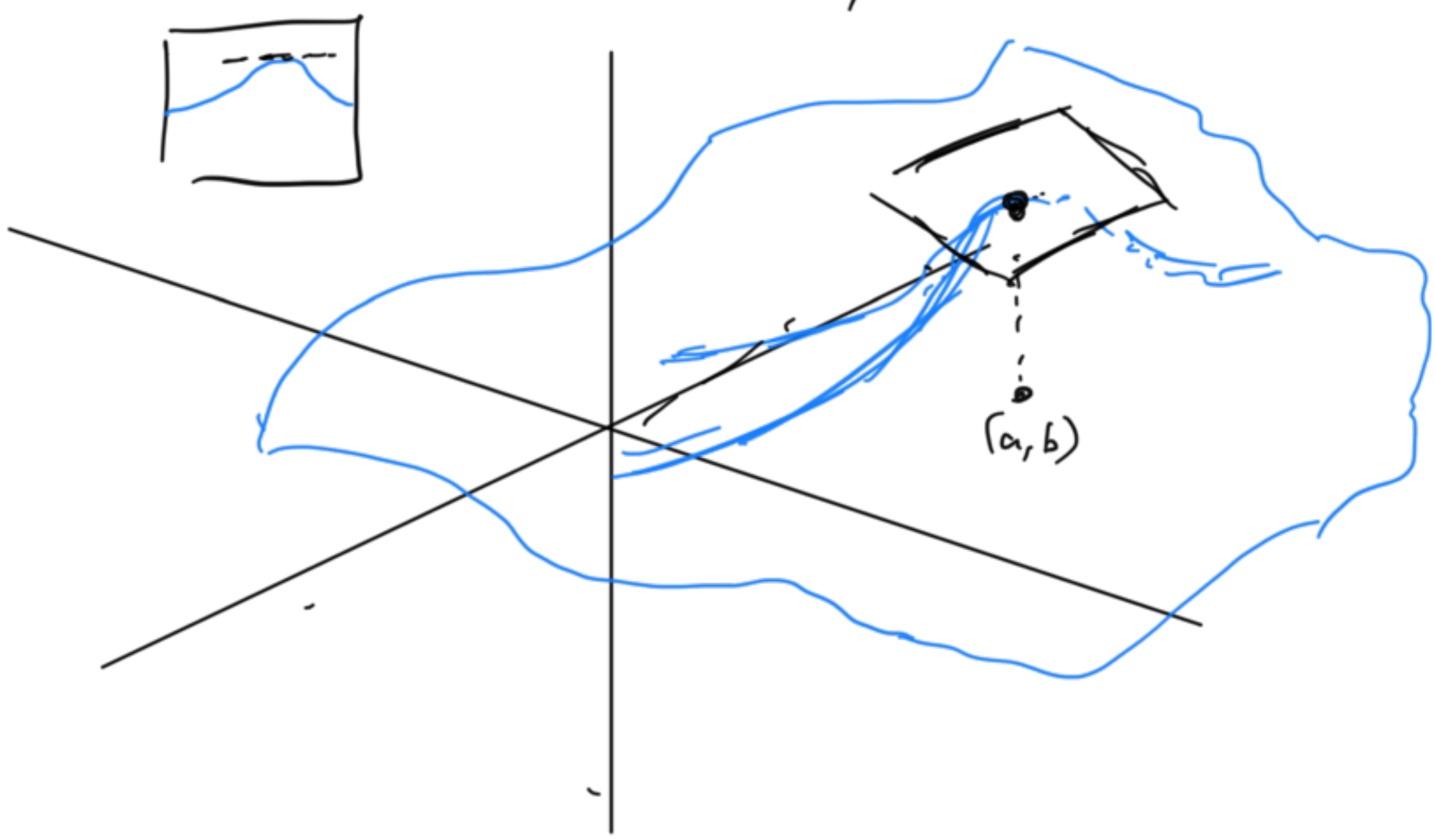


Had a process for finding the x-values where these extreme points occurred. "2nd derivative test"

Want to develop similar techniques for $f: \mathbb{R}^n \to \mathbb{R}$

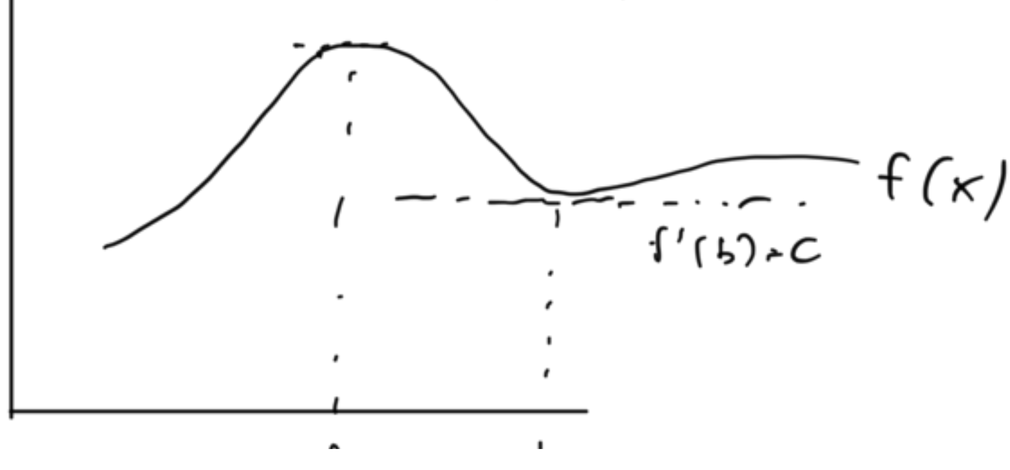
Local min/max

$f_x(a,b) = 0$
$f_y(a,b) = 0$



For $f: \mathbb{R} \to \mathbb{R}$, when we had max/min f' was equal to 0, i.e. tangent line is horizontal

$f'(a) = 0$



Similarly, for $f: \mathbb{R}^n \to \mathbb{R}$, if we have max / min then all partial derivatives must be zero. So tangent <u>plane</u> will be horizontal

$f: \mathbb{R}^n \to \mathbb{R}$ is differentiable

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$

$$\nabla f(x_1, \dots x_n) = \langle f_{x_1}, \dots f_{x_n} \rangle$$

$f: \mathbb{R}^n \to \mathbb{R}$ $\qquad\qquad$ $f: \mathbb{R} \to \mathbb{R}$

☆ $\nabla f(\vec{x}) = \vec{0}$ $\qquad\qquad$ $f'(x) = 0$

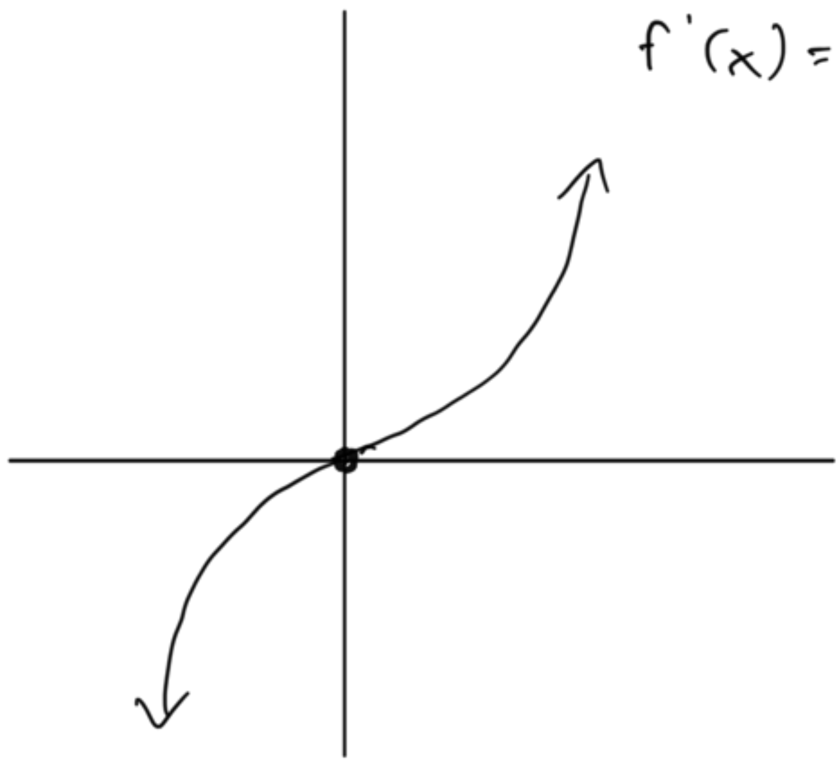
Call the points $\vec{x}$ where all partials are zero the ==critical points==

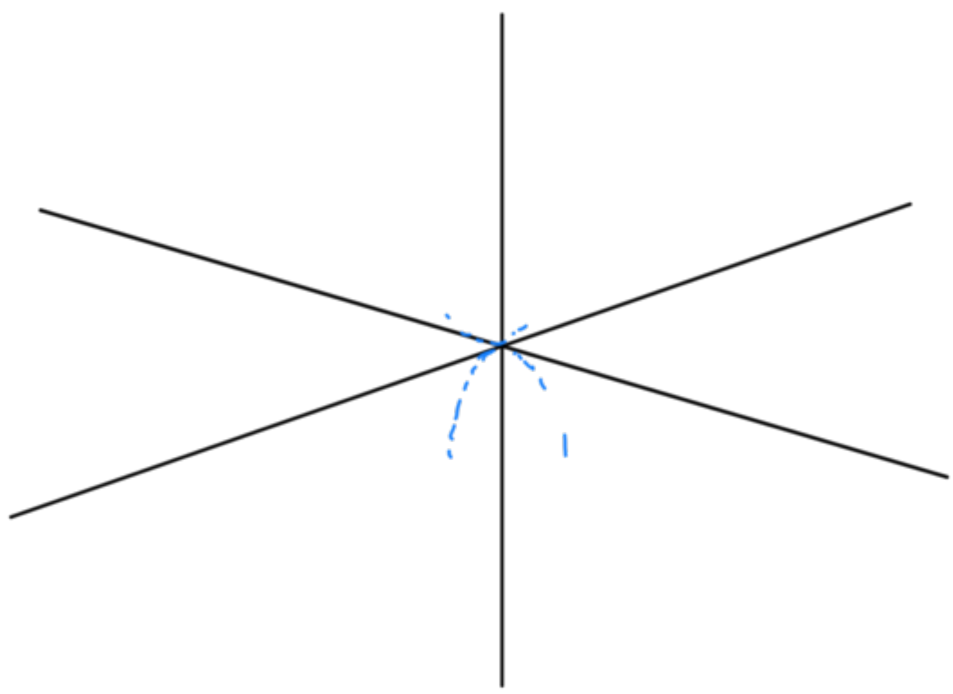
But critical points do <u>not</u> guarantee we have a maximum or minimum

Ex: $f(x) = x^3$

$f'(x) = 3x^2$

$f'(x) = 0$ at $x=0$





Ex: Find critical points for

$f(x,y) = y^2 - x^2$

$$\underline{\nabla f} = \langle -2x, 2y \rangle \overset{?}{=}$$

$$\nabla f(x,y) = \vec{0} \quad \text{at} \quad (0,0)$$

So by previous examples see that having critical point does not guarantee an extreme value, but it is necessary just like for $f: \mathbb{R} \to \mathbb{R}$

For single variable case we went on to test the 2^{nd} derivative

Recall 2^{nd} derivative test:

If $f'(a) = 0$ and

- $f''(a) < 0$, maximum ☆
- $f''(a) > 0$, minimum ☆

(when f'' continuous)

Why did this work?

Recall we discussed idea that derivative is a linear approximation, i.e.

$$f(x+h) - f(x) = \underline{f'(x)h} + \varepsilon(x)h$$

<u>or</u>

$$f(x+h) = f(x) + f'(x)h + \varepsilon(x)h$$

Can make the approximation better by adding the second derivative

$$f(x+h) = f(x) + f'(x)h + f''(x)h^2 + \mathfrak{z}(x)h^2$$

Critical point at $x=a$
(i.e $f'(a)=0$)

$$f''(a) > 0$$

What is up with $f(a+h)$

☆ $f(a+h) = f(a) + f'(a)h + f''(a)h^2 + \underbrace{\mathfrak{z}(x)h^2}_{\text{small}}$

$$\underline{f(a+h)} = \underline{f(a)} + 0 + \underbrace{(\text{positive } \#) + (\text{small } \#)}_{(\text{positive } \#)}$$

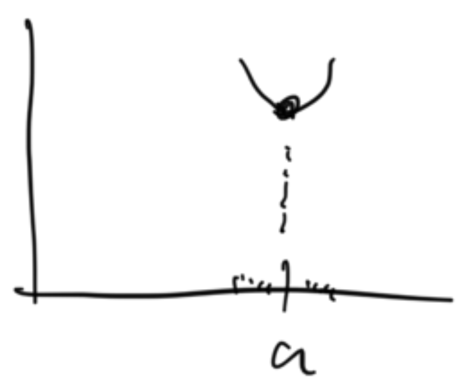
$$f(a+h) = f(a) + \text{a positive } \#$$

For any point "close" to a, can call it $a+h$

$$f(a+h) = f(a) + \text{positive } \#$$

so $$f(a+h) > \boxed{f(a)}$$

$$f(x) > f(a)$$



So $$f'(a) = 0$$

$$f''(a) > 0$$

$\Rightarrow$ minimum (local)

We must have a local minimum at a

Have similar idea for $f: \mathbb{R}^n \to \mathbb{R}$

but need to account for extra

dimensions

$$\vec{x}$$

☆ $$f(\vec{x}+\vec{h}) = f(\vec{x}) + \nabla f(x)\cdot\vec{h} + \_\_\_\_$$

$$f(x+h) = f(x) + f'(x)h + \underline{f''(x)h^2}$$

$$f(x,y)$$

1st order partials: f_x, f_y

2nd order partials: $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$$f(x_1, \dots x_n)$$

$$\nabla f(\vec{x}) = \langle f_{x_1}, \dots f_{x_n} \rangle$$

$$\begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ \vdots & & & \end{bmatrix}$$

$$\begin{bmatrix} & & \\ f_{x_n x_1} & \cdots & f_{x_n x_n} \end{bmatrix}$$

Want to add a "second derivative term"

But recall have several 2^{nd} derivatives

If f is function of x, y then

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$

2^{nd} order partial derivatives are

$$f_{xx}, f_{xy}, f_{yx}, f_{yy}$$

Organize into a matrix call it $\nabla^2 f$

$$\nabla^2 f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$\star\ f(\vec{x} + \vec{h}) = \underline{f(\vec{x})} + \nabla f(\vec{x}) \cdot \vec{h} + \nabla^2 f(\vec{x})\, \vec{h} \cdot \vec{h}$$

$$\underline{+ 3(\lambda) h^2}$$

$$\left\{ \begin{array}{l} \underline{\vec{a}} \quad \text{critical point} \quad (\nabla f(a) = \vec{0}) \\ \nabla^2 f(a) \quad \text{"positive definite"} \end{array} \right.$$

$$f(\vec{a}+\vec{h}) = f(\vec{a}) + 0 + (\text{positive number}) + (\text{small error})$$

$$f(\vec{a}+\vec{h}) \approx f(\vec{a}) + (\text{positive } \#)$$

$$f(\vec{a}+\vec{h}) > f(\vec{a})$$

So if $\nabla f(a) = 0$ and $\nabla^2 f$ "positive definite"

these will be minimum at $\vec{a}$

Too technical! Why do we care?

Well, this way works for any dimension,

so we see there is a sort of 2nd derivative test for any dimension

For $f: \mathbb{R}^2 \to \mathbb{R}$ can use idea of "positive definite" to get nice criteria

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$$f_{xx} f_{yy} - [f_{xy}]^2$$

when $f_{xx} > 0$ and

$f_{xx} f_{yy} - (f_{xy})^2 > 0$

$\nabla^2 f(x)$ is positive definite

$f_{xx} < 0$ and $f_{xx} f_{yy} - [f_{xy}]^2 > 0$

$\nabla^2 f(x)$ negative definite

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

For critical points:

- $f_{xx} > 0$ and $\det(\nabla^2 f) > 0$

  local minimum ("positive definite")

- $f_{xx} < 0$ and $\det(\nabla^2 f) > 0$

  local maximum ("negative definite")

- $\det(\nabla^2 f) < 0$

neither

To recap:

Steps to find local min/max:

$f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y)$

Has continuous 2^{nd} order partials

① Calculate $\nabla f(x)$

② Determine where $\nabla f(x) = \vec{0}$ to find critical points

$$\nabla f(x) = \langle x^2 - 1, y \rangle$$

$x^2 - 1 = 0$ $\qquad y = 0$

$x = +1, x = -1$

$\langle 1, 0 \rangle$

$\langle -1, 0 \rangle$

Then test whether critical points are maximums, minimums, or neither by:

③ Calculate $\nabla^2 f(x)$ (the matrix of 2^{nd} order partial derivatives)

$$f(x,y) = \_\_\_\_\_\_\_\_$$

$$\nabla f(x,y) = \langle x^2-1, y \rangle$$

$$\rightarrow \nabla^2 f = \begin{bmatrix} 2x & 0 \\ 0 & 1 \end{bmatrix}$$

$(1,0)$: $$= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$(-1,0)$: $$= \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

④ Calculate $\det \nabla^2 f(x)$ i.e:

$$f_{xx} f_{yy} - [f_{xy}]^2 = 2$$

⑤ Classify using these rules:

$\times$ • $f_{xx} > 0$ and $\det(\nabla^2 f) > 0$

local minimum

$\times$ • $f_{xx} < 0$ and $\det(\nabla^2 f) > 0$

local maximum

✓ • $\det(\nabla^2 f) < 0$

neither

A local minimum occurs at point $(1,0)$

Saddle point at $(-1,0)$

Ex: $f(x,y) = x^2 + xy + y^2 + y$

① $\nabla f(x,y) = \langle 2x + y,\ x + 2y + 1 \rangle$

②

$$
\begin{aligned}
2x + y &= 0 \\
-2x - 4y &= 2 \\
\hline
-3y &= 2
\end{aligned}
$$

$y = -\frac{2}{3}$

$x = \frac{1}{3}$

critical point $= \langle \frac{1}{3}, -\frac{2}{3} \rangle$

(3) $\nabla^2 f(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

(4) $f_{xx} f_{yy} - [f_{xy}]^2 = 4 - 1 = \boxed{3}$

So ...

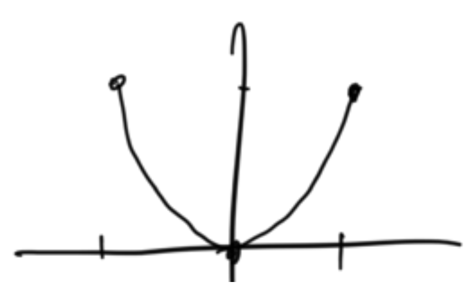
(5) $\underbrace{\underline{f_{xx}} > 0 \quad \det(\nabla^2 f) = 3 > 0}$

So have a local minimum at the point $(1/3, -2/3)$

Absolute Max/Min

Remember, for $f: \mathbb{R} \to \mathbb{R}$ the second derivative test only worked on the interior of our domain, didn't work on the boundary (endpoints)

Ex $f(x) = x^2$ on $\underline{[-1, 1]}$



$2x$

$x = 0$

"2nd derivative test" – method on the interior

Test endpoints individually

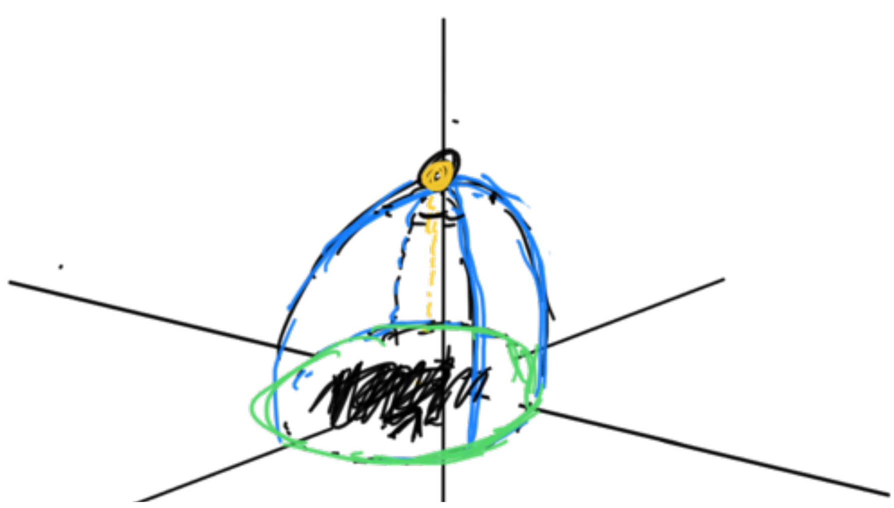
Similar idea for multivariable functions

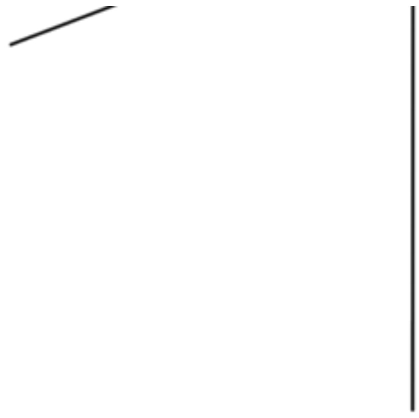
For function f with continuous 2nd order partials on domain D, the "2nd derivative test" works inside D but not on boundary

$f(x,y) = 4 - x^2 - y^2$

for $D = \{(x,y) : \underbrace{x^2+y^2}_{} \leq 4\}$

$\nabla f(x) = \langle -2x, -2y \rangle$





Steps for finding Absolute max/min

① Find local max/min inside domain using steps outlined before

② Find max/min on boundary. Usually by

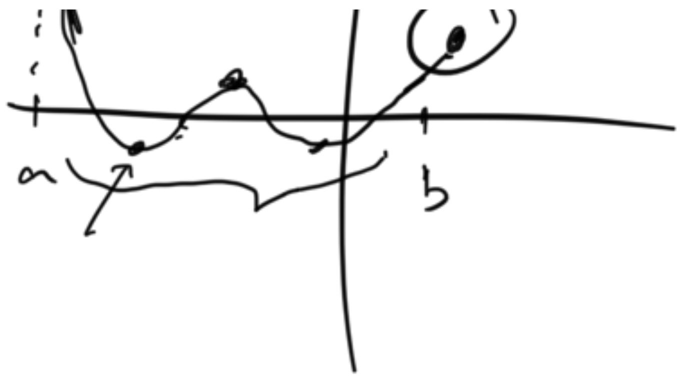
(2a) Break into pieces if necessary and consider pieces individually

(2b) Write the pieces as functions of a single variable

or $f(x)=z$
$f(y)=z$

(2c) Use Calc I methods to find max/min on pieces

③ Compare max/min on boundary with those in the interior

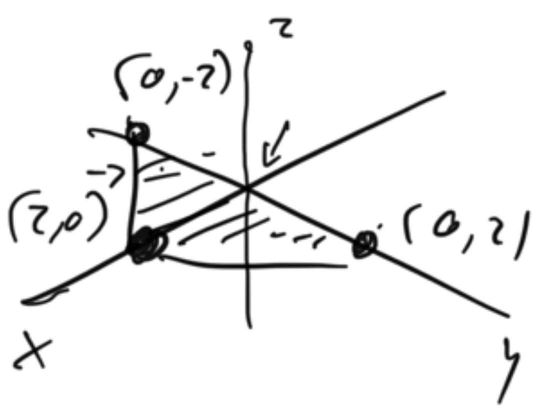


Ex: Find absolute max/min of function

$$f(x,y) = x^2 + y^2 - 2x$$

on closed triangular region with vertices

$$(2,0) \quad (0,2) \quad (0,-2)$$



① $f = x^2 + y^2 - 2x$

$$\nabla f = \langle 2x - 2, 2y \rangle$$

Critical points

$(1,0)$

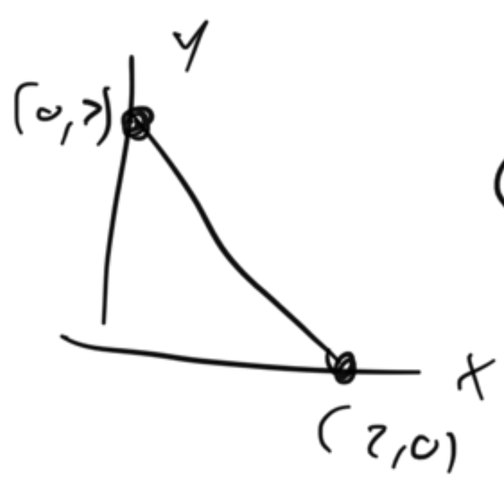
$$\nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) = 4$$

$f_{xx} > 0 \qquad \det = 4 > 0$

local minimum at $(1,0)$

$(2,0) \qquad (0,2)$



$y = -x + 2$

$$f(x,y) = x^2 + y^2 - 2x$$

$$f(x) = x^2 + (-x+2)^2 - 2x$$

$$= x^2 + x^2 - 4x + 4 - 2x$$

$= 2x^2 - 6x + 4$

For $0 \le x \le 2$

~~Min~~ max of $f = 2x^2 - 6x + 4$

$f' = 4x - 6$

$= 0$ at $x = \frac{6}{4}$

$f'' = 4$

Minimum at $x = 6/4$

$y = 2/4$

$(6/4, 2/4)$

What is smaller, the function $f(x,y)$ at $(1,0)$ or

at $(6/4, 2/4)$

$f(x,y) \; x^2 + y^2 - 2x$

$f(1,0) = -1$

$f(6/4, 2/4)$

$$f(-4,-4) = \frac{36}{16} + \frac{4}{16} - \frac{48}{16}$$

$$= \frac{-8}{16}$$

$$= -\frac{1}{2}$$