Section 5.3

Diagonal matrices are very easy to work with.
For example can take powers very easily

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

A^2 easy = B^2 hard

A^5 still easy B^5 super hard

For general matrix A, may be <u>similar</u> to diagonal matrix. Then powers will still be easy to find

Assume $A \sim D$ where D diagonal, $P^{-1}AP = D$
Thus $A = PDP^{-1}$

$$A^k = (PDP^{-1})^k$$

$$= (PDP^{-1})(PDP^{-1}) \dots (PDP^{-1})$$

$$= PD\underbrace{P^{-1}P}D\underbrace{P^{-1}P}D\underbrace{P^{-1}} \dots PDP^{-1} \quad \star$$

these terms cancel

$$= P\overbrace{D \cdot D \cdot D \dots D}P^{-1}$$

$$= PD^kP^{-1}$$

Not a power

Power but easy to calc

Not a power

Definition: If $n \times n$ matrix A is similar to diagonal matrix D, say that A is **diagonalizable**, i.e.

$$P^{-1}AP = D \quad \text{(where } D \text{ diagonal)}$$

or

$$A = PDP^{-1}$$

Ex

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \qquad P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Diagonalization Theorem: An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors

$A = PDP^{-1}$ (where D diagonal) iff **columns** of P are n lin. ind. eigenvectors of A.

Entries on diagonal of D are eigenvalues corresponding to columns of P.

$\square$

Assume A is diagonalizable. So $P^{-1}AP = D$

or $AP = PD$

Right side: Assume P has column vectors $\vec{v}_1 \dots \vec{v}_n$

$$PD = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \vdots \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} | & & | \\ \lambda_1 \vec{v}_1 & \cdots & \lambda_n \vec{v}_n \\ | & & | \end{bmatrix}$$

Left side:

$$AP = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{bmatrix}$$

Since A diagonalizable $AP = PD$

$$\begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{bmatrix} = AP = PD = \begin{bmatrix} | & & | \\ \lambda_1 \vec{v}_1 & \cdots & \lambda_n \vec{v}_n \\ | & & | \end{bmatrix}$$

For two matrices to be equal, each column equal. So:

$$\begin{bmatrix} | & & | \\ A\vec{v}_1 & \cdots & A\vec{v}_n \\ | & & | \end{bmatrix} = AP = PD = \begin{bmatrix} | & & | \\ \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \\ | & & | \end{bmatrix}$$

$$A\vec{v}_1 = \lambda_1 \vec{v}_1$$

$\vec{v}_1$ eigenvector for eigenvalue λ_1

$$\begin{bmatrix} | & | & & | \\ A\vec{v}_1 & A\vec{v}_2 & & A\vec{v}_n \\ | & | & & | \end{bmatrix} = AP = PD = \begin{bmatrix} | & | & & | \\ \lambda_1\vec{v}_1 & \lambda_2 v_2 & & \lambda_n\vec{v}_n \\ | & | & & | \end{bmatrix}$$

$$A\vec{v}_2 = \lambda_2 \vec{v}_2$$

$\vec{v}_2$ eigenvector for eigenvalue λ_2

etc.

Assumed A diagonalizable, i.e. $AP = PD$. If this is case columns of P are eigenvectors of A, and values in D are eigenvalues of A.

Thus, if A similar to diagonal matrix D:

- there are n lin. ind eigenvectors
- the columns of P are eigenvectors of A
- entries on diagonal of D are A's eigenvalues.

To complete proof we would do opposite direction.

Assume A has n lin. ind. eigenvectors

then show this implies A diagonalizable.

But this is straightforward, can see book for details.

$\blacksquare$

Definition: To **diagonalize** an $n \times n$ matrix A means to find invertible P, P^{-1} and diagonal D such that

$$P^{-1} A P = D$$

or

$$A = P D P^{-1}$$

(<u>if</u> possible)

Know its possible if A has n lin. ind eigenvectors. But can't usually tell if this is the case. Just have to try and diagonalize, see if it works.

Given $n \times n$ matrix A, this is how to diagonalize.

How to diagonalize, Step 1: Find eigenvalues of A.

- If there are no eigenvalues we can't diagonalize.
- If there is at least one, may/may not

be possible. Have to keep going.

How to diagonalize, Step 2: Find eigenbasis for each eigenvalue

- Recall, for eigenvalue λ_i, this means finding basis for $\mathrm{Nul}(A - \lambda_i I)$

How to diagonalize, Step 3: Count all the basis vectors from all different eigenspaces.

Ex: A is $\overset{n \times n}{5 \times 5}$ matrix, eigenvalues/vectors:

$\lambda = 1$: $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 3$: $\begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 6 \end{bmatrix}$

$\lambda = -1$: $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 6 \\ 0 \\ 0 \end{bmatrix}$

In total, <u>5 eigenvectors</u>

Since number is equal to n, we know it is possible to diagonalize, because have n linear independent eigenvectors.

☆ How do we know lin. ind? ☆

If less than n vectors, not possible to diagonalize. Stop process here.

How to diagonalize, step 4: Make diagonal matrix D out of eigenvalues. If an eigenvalue has multiple eigenvectors, repeat it in D for each vector.

Ex

$\lambda = 1$ $\qquad$ $\lambda = 3$ $\qquad$ $\lambda = -1$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

How to diagonalize, step 5: Make matrix P out of eigenvectors, in same order as eigenvalues

Ex

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

eigenvalue = 1

eigenvectors for 1

$$P = \begin{bmatrix} 1 & 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 & 3 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

How to diagonalize, Step 6: From P, calculate P^{-1}

How to diagonalize, Step 7: Write answer all in one place

Once you figure out P, P^{-1}, D you should have a final line of form

$$A = P D P^{-1}$$

Write out matrices, but <u>don't</u> multiply the right side. Just leave it as 3 matrices.

Done

Miscellaneous

Theorem: If A has n distinct eigenvalues, it is diagonalizable.

□ Remember distinct eigenvalues ⇒ lin. ind eigenvectors. So n distinct values ⇒ n lin ind vectors. By diagonalization theorem, this means diagonalizable.

But

Matrix may have less than n eigenvalues and still be diagonalizable (as long as # of vectors from eigenbasis add up to n.)

The dimension of an eigenspace is called its **geometric multiplicity**

Theorem: Let A be $n \times n$ matrix with p distinct eigenvalues $(1 \leq p \leq n)$

① <u>Geometric</u> multiplicity of λ_i less than or equal to <u>algebraic</u> multiplicity

② A diagonalizable iff sum of geometric multiplicities $= n$. For this, need

- Char polynomial to factor completely into something like $(\lambda - a_1)(\lambda - a_2)\ldots(\lambda - a_p)$
- Algebraic multiplicity = geometric multiplicity

③ If A diagonalizable, and β_k basis for eigenspace of λ_k, collection of all $\beta_1 \ldots \beta_p$ will form basis of $\mathbb{R}^n$