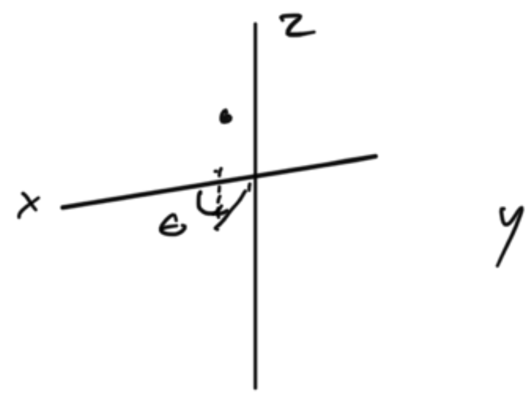
15.8 - Spherical Coordinates

Another extension of polar coordinates to 3-D.

Just going to show spherical coordinates:

$$(\rho, \theta, \phi)$$



Similar to polar coordinates

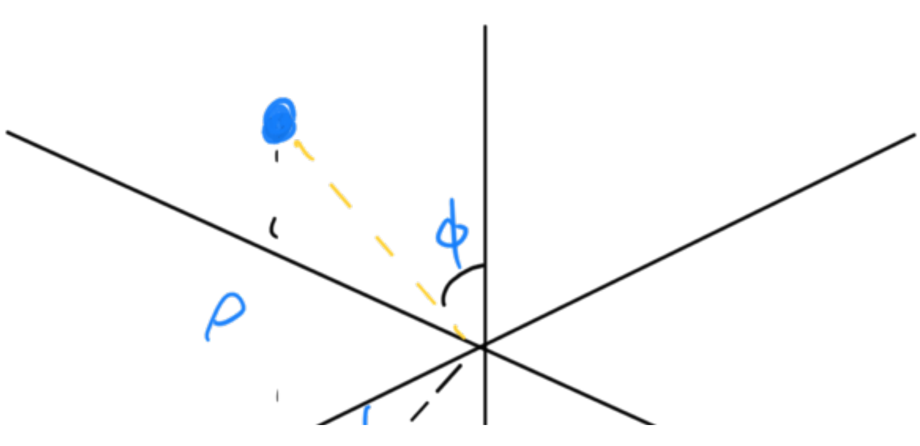
θ = angle with positive x-axis

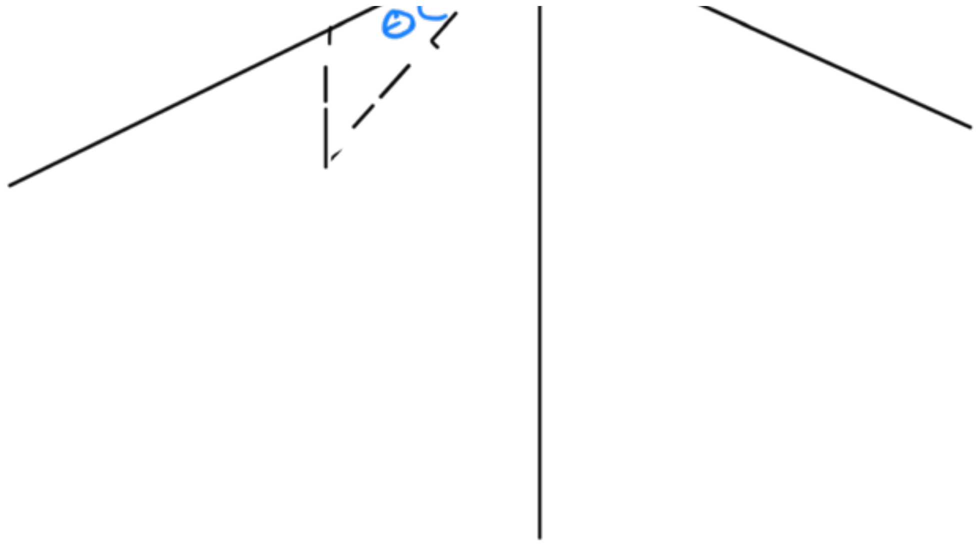
ϕ = angle with positive z-axis

ρ = length / "radius"

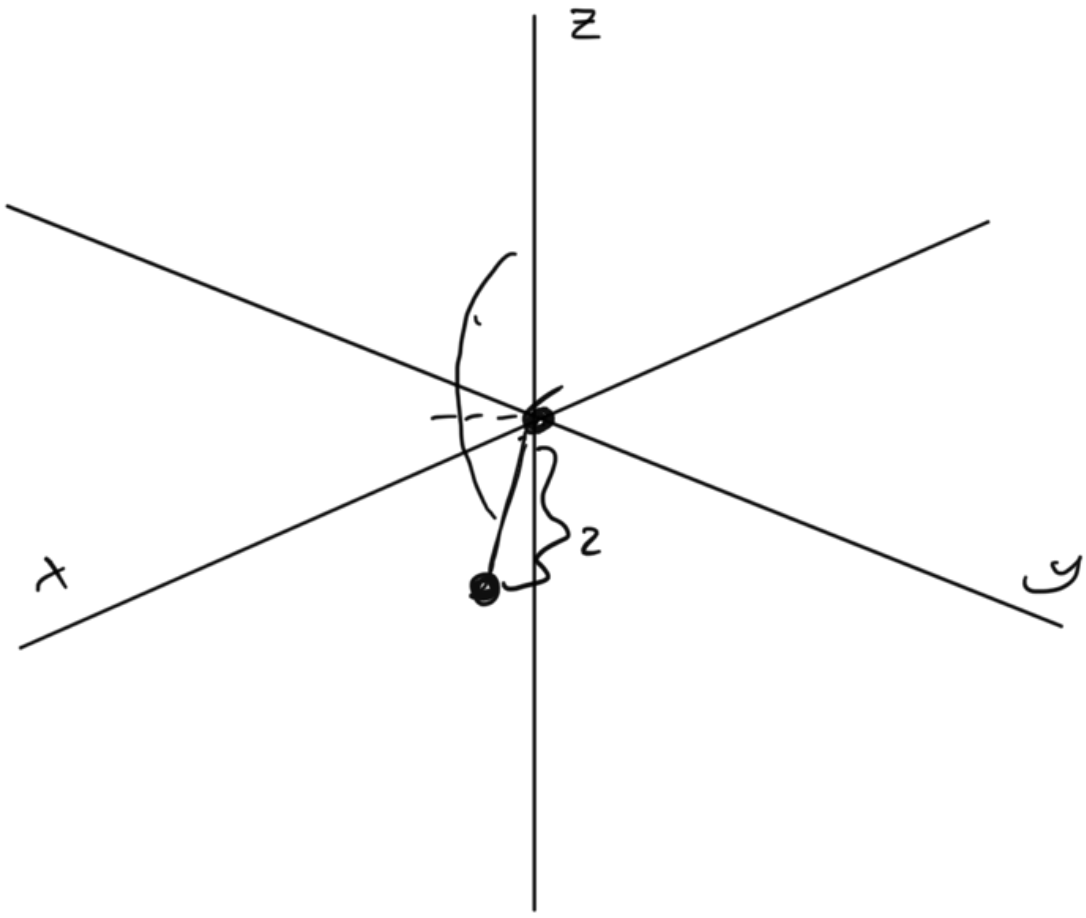
$\underline{\rho \geq 0}$ and $\underline{0 \leq \phi \leq \pi}$

radius





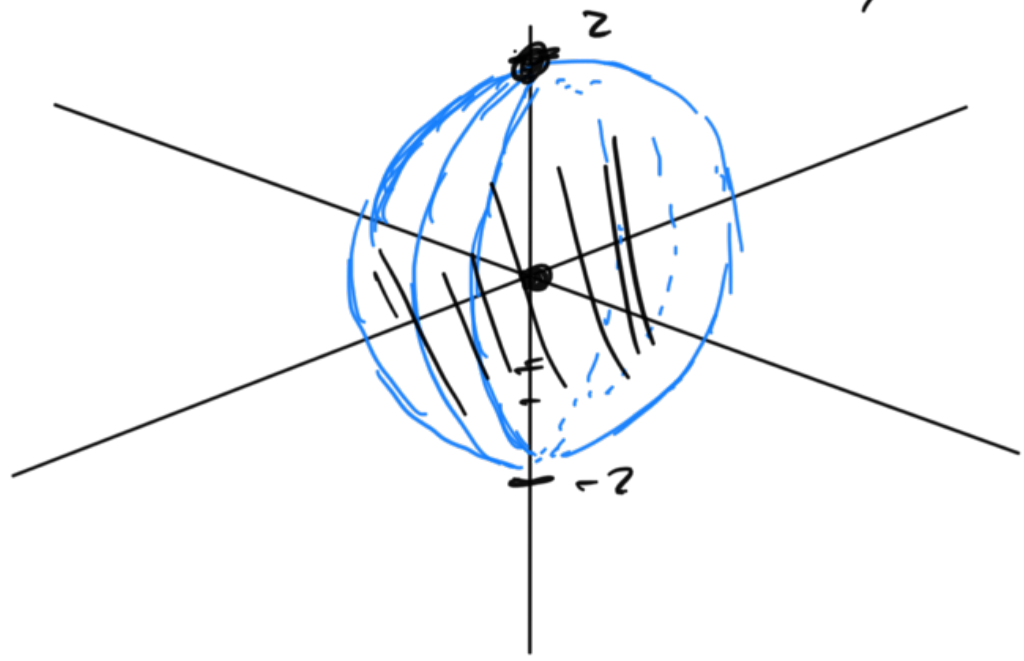
Ex: $\left(2, \frac{7\pi}{4}, \frac{5\pi}{6}\right)$



The power of spherical coordinates is in representing functions with symmetry around <u>origin</u>.

Ex: $R = \{ (x, y, z) : x^2 + y^2 + z^2 \le 4 \}$

$x^2+y^2+z^2=4$



Cartesian:

$$\underbrace{\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{-\sqrt{4-y^2-z^2}}^{\sqrt{4-y^2-z^2}}}\quad dx\,dy\,dz$$

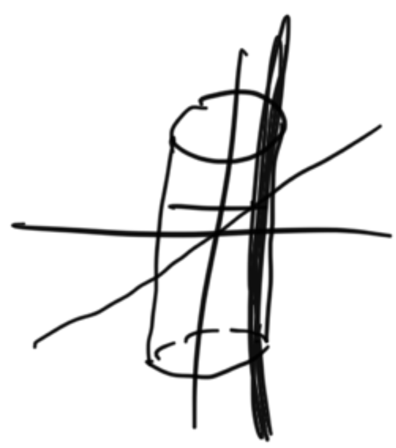
Spherical:

$$\int_0^2\int_0^{\pi}\int_0^{2\pi} \underline{\qquad\qquad} \quad \underbrace{d\theta\, d\phi\, d\rho}$$

$f(x, y)$

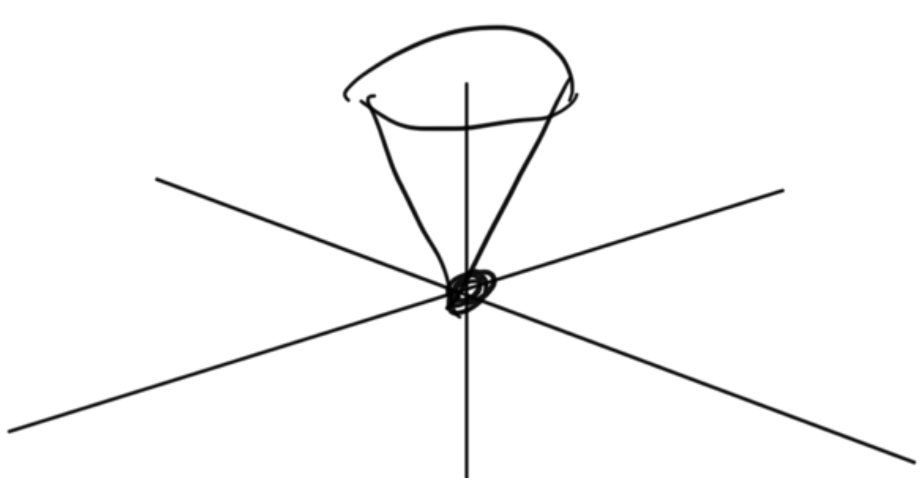
- Polar coordinates (2-D) good for symmetry around origin

- Cylindrical (3-D) good for symmetry around an axis (z-axis)



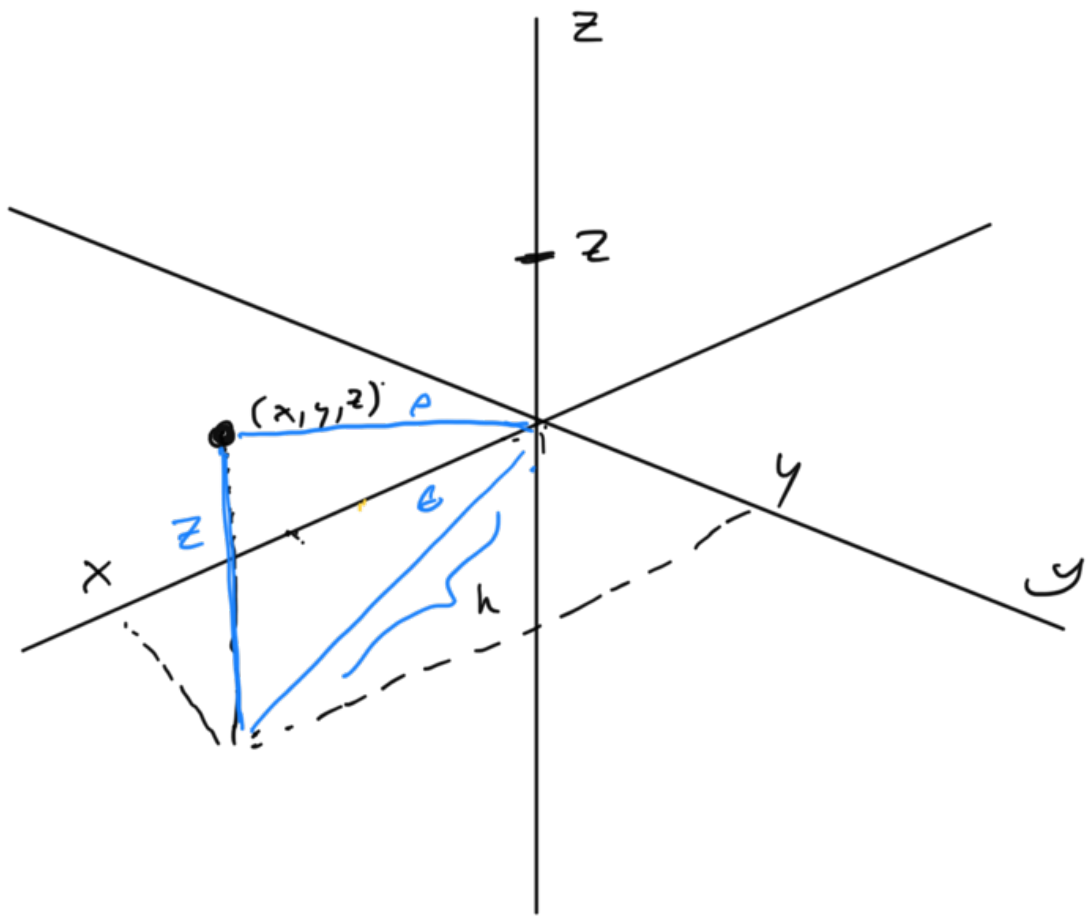
(r, θ, z)

- Spherical (3-D) good for symmetry around origin



Conversion to Spherical

Points / Functions:



$$(x, y, z) \rightarrow (\rho, \theta, \phi)$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\rho = \sqrt{x_0^2 + y_0^2 + z_0^2}$$

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

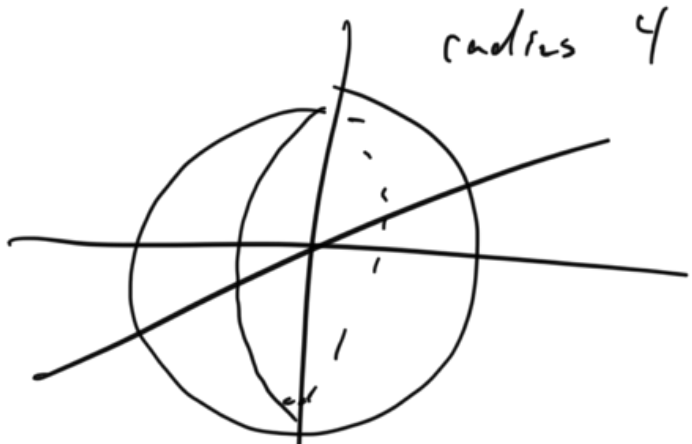
Ex. Convert $(\overset{\rho}{4}, \overset{\theta}{\pi/6}, \overset{\phi}{\pi/6})$ to cartesian coordinates.

$$x = 4 \sin(\pi/6)\cos(\pi/6)$$
$$= 4 \cdot \tfrac{1}{2} \cdot \tfrac{\sqrt{3}}{2} = \sqrt{3}$$

$$y = 4 \sin(\pi/6)\sin(\pi/6)$$
$$= 4 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 1$$

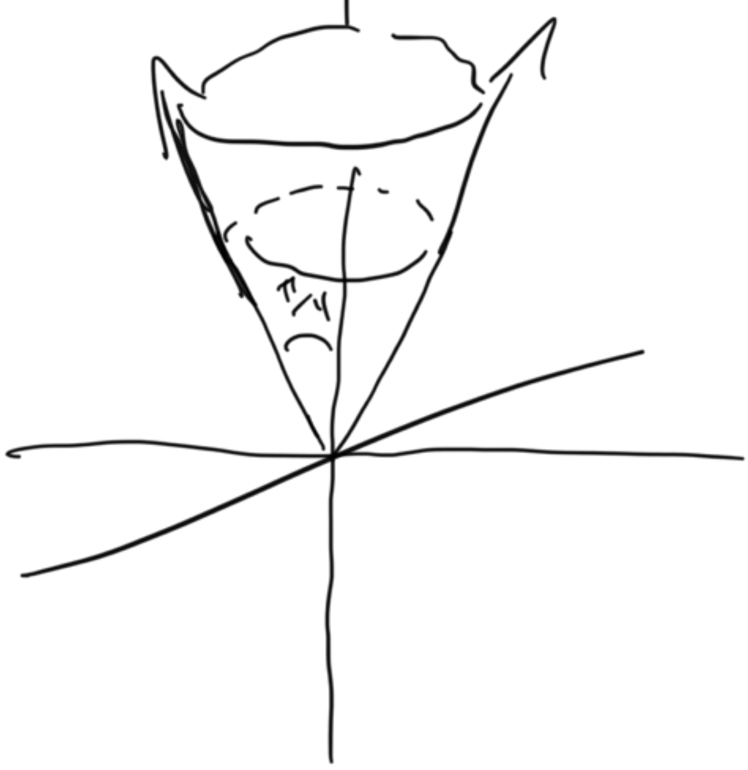
$$z = 4\cos(\pi/6) = 2\sqrt{3}$$

☆ $(\sqrt{3}, 1, 2\sqrt{3})$



$\rho = 4$

θ, ϕ anything



$\phi = \pi/4$

Integrating in Spherical Coordinates

Want to know how to integrate in spherical coordinates so we can then switch from cartesian to spherical.

In cartesian, start by taking Riemann sums over boxes of form

$a \le x \le b \qquad c \le y \le d \qquad e \le z \le f$

$$\sum \sum \sum f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \underbrace{\Delta V_{ijk}}$$

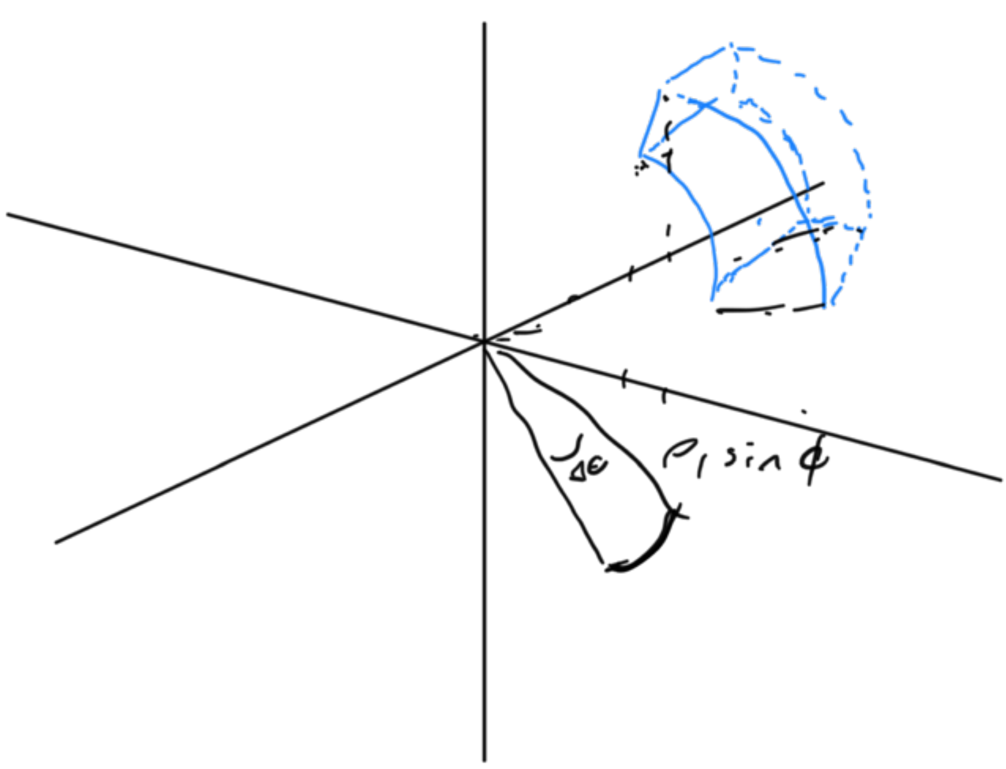
$$= \underbrace{\Delta x_{ijk}} \ \underbrace{\Delta y_{ijk}} \ \underbrace{\Delta z_{ijk}}$$

Want to do same for spherical.
Start by breaking domain up into pieces

$\underbrace{\alpha \le \rho \le} \beta \qquad \gamma \le \theta \le \delta \qquad \varepsilon \le \phi \le \zeta$

Gives a "wedge"

rectangles in spherical coordinates



$$\cancel{\Delta}\rho\,(\cancel{\Delta}\phi\,\underline{\rho_1})(\underline{\rho_1 \sin\phi}\ \cancel{\Delta\theta})$$

$$\Big(\rho_1^2 \underbrace{\sin\phi\,\Delta\rho\,\Delta\theta\,\Delta\phi}\Big)$$

$$\sum\sum\sum f(\rho,\theta,\phi)\,\rho_1^2 \sin\phi\,\Delta\rho\Delta\theta\Delta\phi$$

$$\downarrow \qquad \downarrow$$

$$\iiint f(\rho,\theta,\phi)\boxed{\rho^2 \sin\phi}\,d\rho\,d\theta\,d\phi$$

$$\Delta\rho\,(\Delta\phi\,\cancel{\rho_1})\,\cancel{\rho_1}\,\cancel{\sin\phi}\,\Delta\theta$$

$$\rho_1^2 \sin\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$$

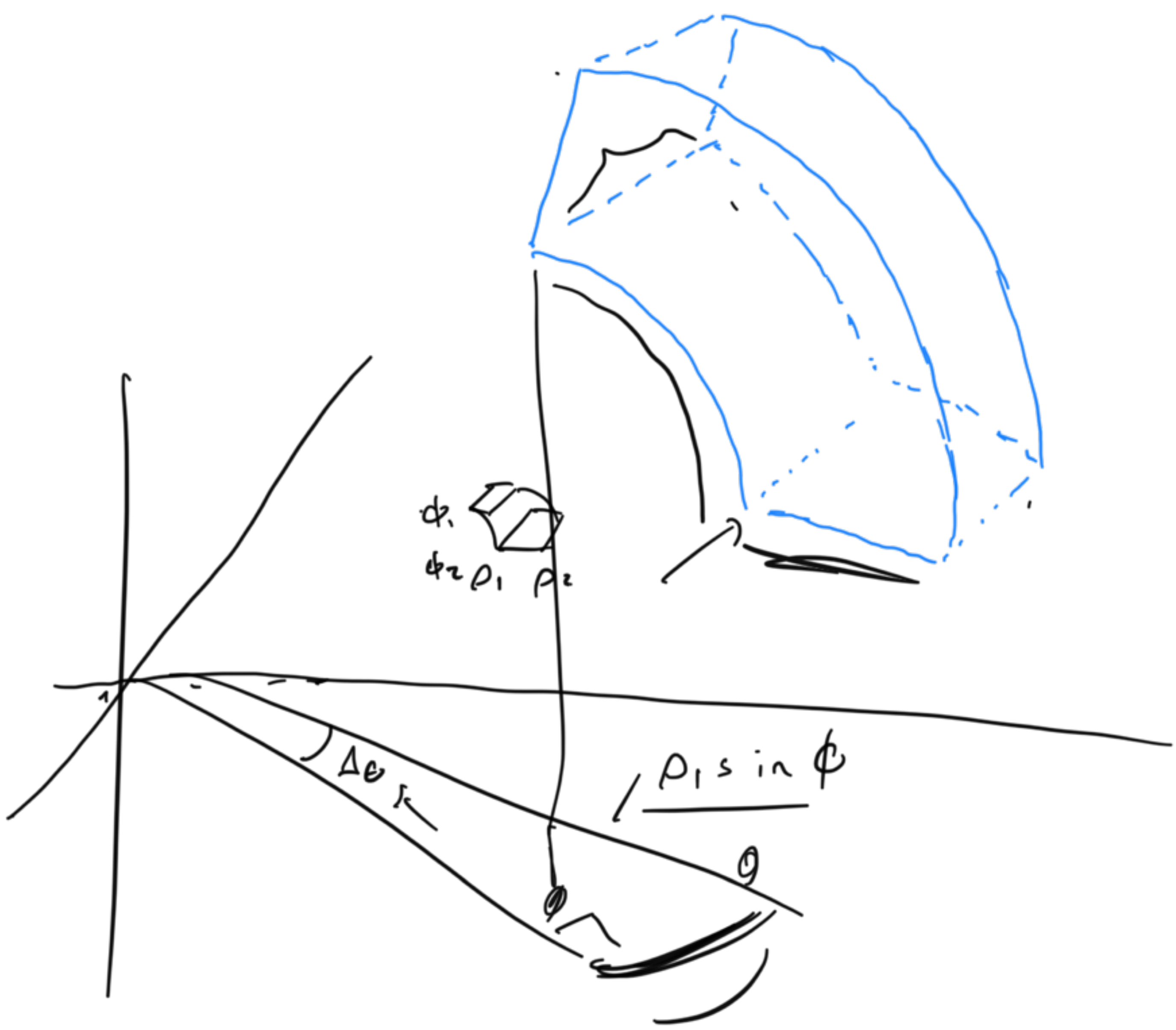
$$\sum_\phi\sum_\theta\sum_\rho \underline{f(\rho,\theta,\phi)}\quad \rho_1^2 \sin\phi\,\underline{\Delta\rho\Delta\phi\Delta\theta}$$

$$\iiint f(\rho,\theta,\phi)\boxed{\rho^2 \sin\phi}\,d\rho\,d\theta\,d\phi$$

$f(x, y, z)$

For Riemann sums, need to know volume of a wedge

Wedge is approximately a box



$\rho_1 \sin\phi \, \Delta\theta$

$\Delta V_{ijk} \approx \Delta\rho \, (\rho_1 \Delta\phi) \, (\rho_1 \sin(\phi) \Delta\theta_k)$

$$= \rho_1^2 \sin(\phi_1)\, \Delta\rho\, \Delta\theta\, \Delta\phi$$

So Riemann sum is

$$f(\rho, \theta, \phi)\, \Delta V$$

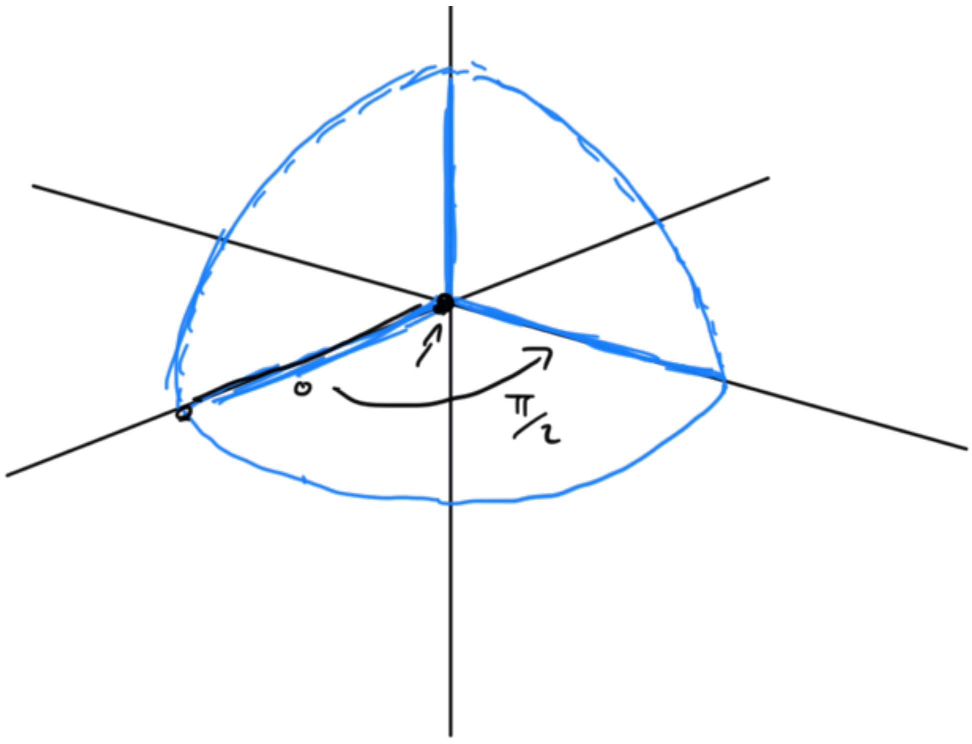
$$\sum\sum\sum f(\rho_{ijk}, \theta_{ijk}, \phi_{ijk}) (\rho_{ijk})^2 \sin(\phi_{ijk})\, \Delta\rho\, \Delta\theta\, \Delta\phi$$

$$\iiint f(\rho, \theta, \phi)\, \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

So what?

Whenever we integrate a function in spherical coordinates, need to multiply by $\rho^2 \sin(\phi)$

Ex: Integrate $f(\rho, \theta, \phi) = (\rho^3 + 3)\theta$ on region in first octant bounded by $x^2 + y^2 + z^2 = 9$



$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} (\rho^3+3)\,\theta \quad \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$$

$$\iiint (\rho^5+3\rho^2)\,\theta \sin\phi \quad d\rho\, d\theta\, d\phi$$

$$\left(\frac{\rho^6}{6}+\rho^3\right)\Big]_0^3$$

$$\iint \left(\frac{297}{2}\right) \theta \sin\phi \quad d\theta$$

$$\frac{297}{2}\int \frac{\theta^2}{2}\Big]_0^{\pi/2}$$

$$\frac{297}{2}\cdot\frac{\pi^2}{8}\int \sin\phi\, d\phi$$

$$\boxed{\frac{297}{2}\cdot\frac{\pi^2}{8}\cdot 1}$$

Cartesian → Spherical

$$\iiint f(x, y, z)\, dx\,dy\,dz$$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$dx\,dy\,dz =$
$\rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi$

$$\iiint f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\theta)\ \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

Difficult part:

Bounds

Ex: $\displaystyle\iiint_E (x^2+y^2)\, dV$ #23

$dV = dx\,dy\,dz$

E is region between spheres

$x^2+y^2+z^2 = 4$

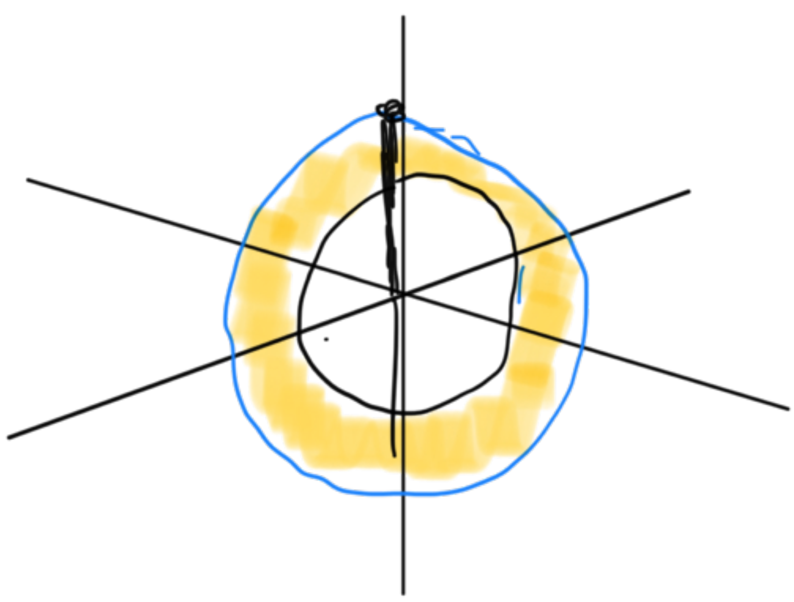
$x^2+y^2+z^2=4 \qquad x^2+y^2+z^2=9$

$$\iiint [(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2]\rho^2\sin\phi \, d\rho\, d\theta\, d\phi$$

$$\iiint \rho^4[\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta]\sin\phi\, d\rho\, d\theta\, d\phi$$

$$\iiint \rho^4\sin^3\phi[\cos^2\theta+\sin^2\theta]\,d\rho\, d\theta\, d\phi$$

$$= \iiint \rho^4 \sin^3\phi \, d\rho\, d\theta\, d\phi$$



$$\int_2^3\int_0^{2\pi}\int_0^{\pi} \rho^4\sin^3\phi\, d\phi\, d\theta\, d\rho$$

$$\iiint \rho^4 \sin^2\phi\sin\phi\, d\phi\, d\theta\, d\rho$$

$$\iiint \rho^4(1-\cos^2\phi)\sin\phi\, d\phi\, d\theta\, d\rho$$

$$u = \cos\phi$$

$$du = -\sin\phi\, d\phi$$

$$\iiint \rho^4 (1-u^2)\,(-du)\,d\phi\,d\theta\,d\rho$$

$$-\iint \rho^4 \left(u-\frac{u^3}{3}\right)\Big]\,d\theta\,d\rho$$

$$\iint \rho^4 \left(\cos\phi-\frac{\cos^3\phi}{3}\right)\Big]_0^{\pi}\,d\theta\,d\rho$$

$$\iint \rho^4 \left[\left(\underline{-1+\tfrac{1}{3}}\right)-\left(\underline{1-\tfrac{1}{3}}\right)\right] d\theta\,d\rho$$

$$-\left(-\frac{4}{3}\right)\int_2^3\int_0^{2\pi} \rho^4 \,d\theta\,d\rho$$

$$-\left(-\frac{4}{3}\right)\cdot 2\pi\int_2^3 \rho^4\,d\rho$$

$$\left(-\frac{8\pi}{3}\right)\left(\frac{\rho^5}{5}\Big]_2^3\right)$$

$$-\left(-\frac{8\pi}{3}\right)\left(\frac{243}{5}-\frac{32}{5}\right)$$

$$-\left(-\frac{8\pi}{3}\left(\frac{211}{5}\right)\right)=\boxed{\frac{1688}{15}\pi}$$

Ex: $\iiint_E y^2 z^2 \, dV$ # 22 ?

E region above cone $\underline{\phi=\pi/3}$

and below sphere $\underline{\rho=1}$

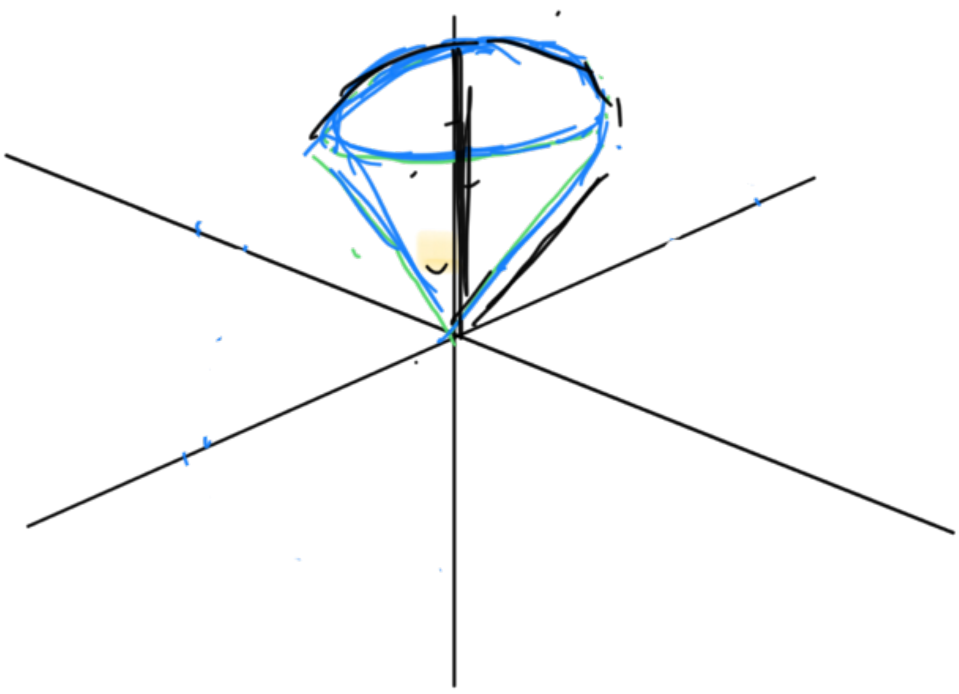
$\sin\phi\sin\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$$\iiint y^2 z^2 \, dV = \iiint \rho^2 \sin^2\phi \sin^2\theta \, \rho^2 \cos^2\phi \, \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho$$

$$= \iiint \rho^6 \sin^3\phi \cos^2\phi \sin^2\theta \, d\phi \, d\theta \, d\rho$$



$$\int_0^1 \int_0^{\pi/3} \int_0^{2\pi} \rho^6 \sin^3\phi \cos^2\phi \sin^2\theta \quad d\theta \, d\phi \, d\rho$$

A) $\int_0^1 \rho^6 \, d\rho$

B) $\int_0^{\pi/3} \sin^3\phi \cos^2\phi \, d\phi$

C) $\int_0^{2\pi} \sin^2\theta \, d\theta$

(A) $\int_0^1 \rho^6 \, d\rho = \frac{\rho^7}{7} \Big]_0^1 = \frac{1}{7}$

(B) $\int_0^{\pi/3} \sin^3\phi \cos^2\phi \, d\phi$

$\int_0^{\pi/3}$

$$\int_0^{} \sin\phi \, \sin^2\phi \, \cos^2\phi \, d\phi$$

$$\int_0^{\pi/3} \sin\phi \left(1-\cos^2\phi\right) \cos^2\phi \, d\phi$$

$$u = \cos\phi \qquad du = -\sin\phi \, d\phi$$

$$-\int u^2 - u^4 \, du$$

$$= \quad -\left[\frac{u^3}{3} - \frac{u^5}{5}\right)\Big]$$

$$= \quad \frac{\cos^5\phi}{5} - \frac{\cos^3\phi}{3}\Big]_0^{\pi/3}$$

$$\left(\frac{\left(\frac{1}{2}\right)^5}{5} - \frac{\left(\frac{1}{2}\right)^3}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)$$

$$\left(\frac{1}{32\cdot 5} - \frac{1}{8\cdot 3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)$$

$$\left(\frac{1\cdot 3}{8\cdot 20\cdot 3} - \frac{20}{8\cdot 3\cdot 20}\right)$$

$$\left(\frac{3}{480} - \frac{20}{480}\right) - \left(\frac{-2}{15}\right)$$

$$\left(-\frac{17}{480}\right) + \frac{2\cdot 4\cdot 8}{5\cdot 3\cdot 4\cdot 8}$$

$$\left(\frac{-17}{480}\right) + \frac{64}{480}$$

$$\frac{47}{480}$$

(c) $\int_0^{2\pi} \sin^2\theta \, d\theta$

$$\int \underline{\frac{1}{2}} \left(1 - \cos(2\theta)\right)$$

$$\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\Big]_0^{2\pi}\right)$$

$$= \frac{1}{2}\left((2\pi - \underline{0}) - (0-0)\right)$$

$$\pi$$

$$\int_0^1 \int_0^{\pi/3} \int_0^{2\pi} \rho^6 \sin^3\phi \cos^2\phi \sin^2\theta \quad d\theta \, d\phi \, d\rho$$

$$= \boxed{\pi \cdot \frac{1}{7} \cdot \frac{47}{480}}$$