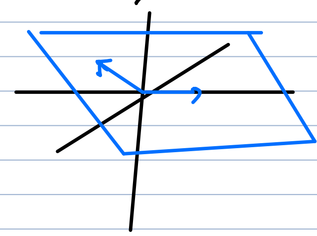
Section 1.7

Linear Independence / Dependence

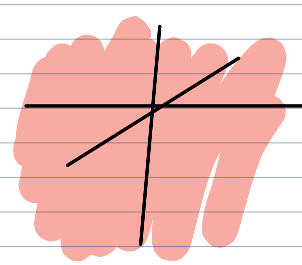
Consider the idea of span again.

$$\text{Span}\left\{\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}}_{\text{lin. dep}},\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$$



Span is a plane in $\mathbb{R}^2$

$$\text{Span}\left\{\underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}}_{\text{lin. ind.}}\right\}$$



Span is all of $\mathbb{R}^3$

What is difference? Linear dependence can help explain it.

Will be useful to use "vector equation viewpoint" to understand the geometric ideas, but will often convert to matrices to do computations.

Def: An indexed set of vectors $\{\vec{v}_1, \dots \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly independent if vector eq.

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_p \vec{v}_p = \vec{0}$$

has only the trivial solution. The set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly dependent if there exist values $c_1, \dots c_p$, not all 0, such that

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots c_p \vec{v}_p = \vec{0}$$

How to determine dep. vs. ind.

Have a list of vectors $\vec{v}_1, \dots \vec{v}_p$. Are they linearly independent or dependent? All depends on if there is nontrivial solution to

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots x_p \vec{v}_p = \vec{0}$$

$$\Big\downarrow \text{ convert to augmented matrix}$$

$$\star \left[\begin{array}{ccc|c} | & & | & 0 \\ \vec{v}_1 & \dots & \vec{v}_p & 0 \\ & & & 0 \\ | & & | & 0 \end{array}\right] \quad \text{Nontrivial solution?}$$

Ex: (a) Determine if $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ ind. or dep.

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right]$$ Homogeneous system

$$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 1 \\ 3 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

pivots < columns = free variable

(b) Determine a dependence relation among the vectors from (a). (A set of specific values, not all 0, such that $c_1 \vec{v}_1 + \dots c_p \vec{v}_p = \vec{0}$)

Continue solving to get some specific solution

$$\sim \begin{bmatrix} 1 & 4 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

If $x_3 = 4$ then we have

$$4 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \\ 4 \end{bmatrix}$$

is solution to homogeneous system.

Thus $$8 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 4 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 4 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \vec{0}$$

This is one of the infinite possible dependence relations

$$\{\vec{v}_1\}$$

Interpretation What **IS** linear dependence?

Lets consider a sets of one, two, three vectors

One: If there is one vector $\{\vec{v}\}$, definition of linear dependence means $\exists\ c \neq \vec{0}$ such that

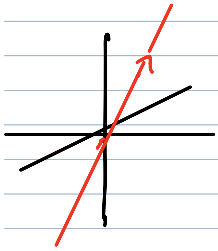
$$c\vec{v} = \vec{0}$$

If $c \neq 0$, must mean $\vec{v} = \vec{0}$

A single vector is linearly dependent iff it is the zero vector.

In fact, any set that contains $\vec{0}$ is linearly dependent. Why?

$$\{\vec{0}, \vec{v}_1, \vec{v}_2 \ldots \vec{v}_p\}$$



$$c_0\vec{0} + c_1\vec{v}_1 + \ldots c_p\vec{v}_p = \vec{0}$$

(can always make $c_0 =$, $c_i = 0$ for all other i. Always nontrivial solution)

Two: $\{\vec{v}_1, \vec{v}_2\}$ (Assume now neither is $\vec{0}$)

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$$

If $c_1 = 0$, c_2 must be 0 and vice versa. So

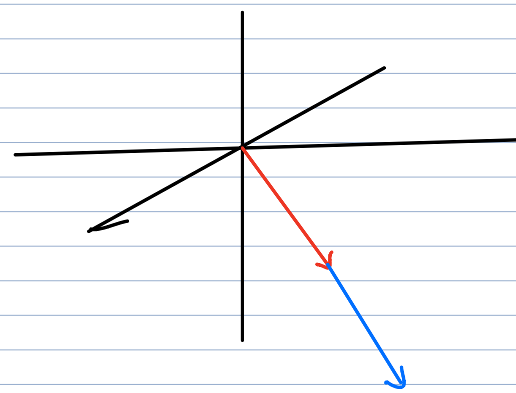
$$c_1\vec{v}_1 = -c_2\vec{v}_2$$

for nontriv. solution to exist, $c_1 \neq 0$ & c_2

$$\vec{v_1} = \left(\frac{-c_2}{c_1}\right)\vec{v_2}$$

So division $-c_2/c_1$ is defined

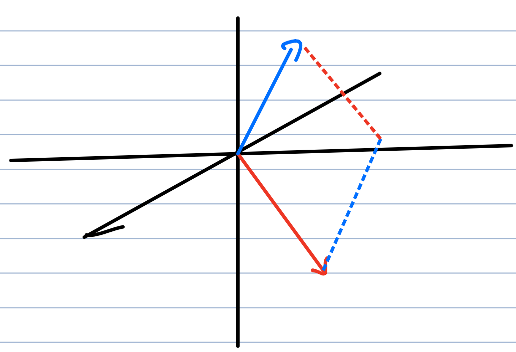
So for two vectors to be linearly dep. one must be a multiple of other.



linear dep: span $\{v_1, \dots, v_p\}$

Span $\{\vec{v_1}, \vec{v_2}\}$ is just a line

If linearly <u>independent</u>, vectors point different directions



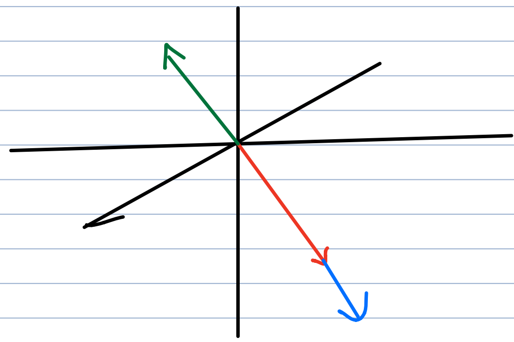
Span $\{\vec{v_1}, \vec{v_2}\}$ forms a plane

linear ind

dim span $\{v_1 \dots v_p\}$ $= p$

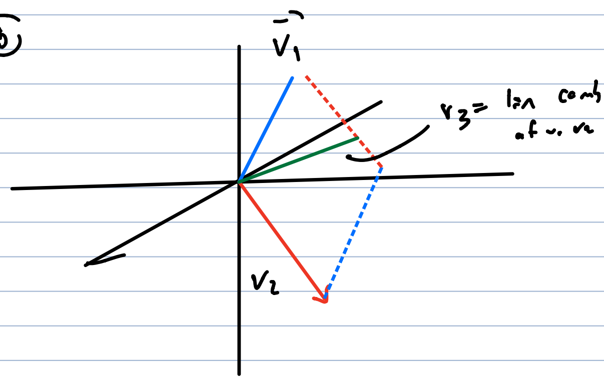
Three: Assume $c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} = \vec{0}$ and none of vectors are $\vec{0}$.

ⓐ All three vectors multiples of one another



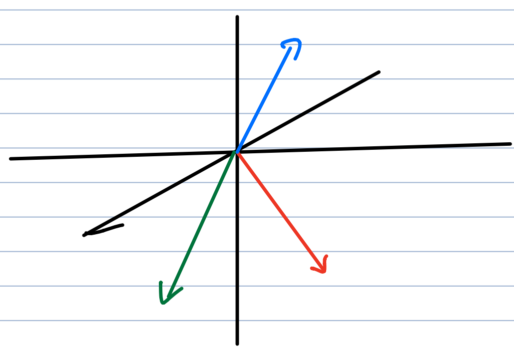
Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a line

(b)



One of vectors is a lin. comb of other two, so span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ a plane

<u>Linear independent</u>: no vector is linear combination of the others



Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is $\mathbb{R}^3$

With these examples in mind, lets generalize.

- A set of vectors is linearly independent if none of them can be written as linear combination of the others.
As a consequence, the span of k lin. ind vectors will have k "dimensions"
Consequence: What if we have more than n vectors in $\mathbb{R}^n$? Must be linearly dependent

- A set of vectors is linearly dependent if some vector or vectors can be written as combination of others. So span of k linearly dependent vectors will have <u>less</u> than k "dimensions"

Have seen that to figure out if $\vec{v}_1, \dots, \vec{v}_p$ are linearly dependent, convert to matrix and check if homogeneous system has nontrivial solution.

From other direction, may start with matrix A. Homogeneous system $A\vec{x} = \vec{0}$ has nontrivial solution iff columns of A are linearly dependent.

Q What does this say about "dimension" of span of columns A.

$$A\vec{x} = \vec{0}$$

$$A \; \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\vec{x}_1 \ldots \vec{x}_n$

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \ldots x_n \vec{a}_n = \vec{0}$$