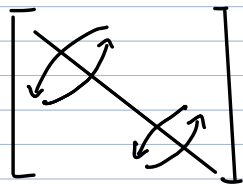
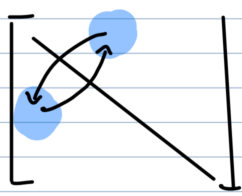
Section 7.1

Definition: An $\underline{n \times n}$ matrix A is symmetric if $A^T = A$.



For square matrix taking transpose is essentially "reflecting" across diagonal.

For A to be symmetric, need $a_{ij} = a_{ji}$



These terms must be equal

Ex:

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 2 & 4 & 9 \\ -3 & 4 & -1 & 5 \\ 2 & 9 & 5 & 6 \end{bmatrix}$$

Note entries on diagonal are unrestricted

$$\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 2 & 4 & 9 \\ -3 & 4 & -1 & 5 \\ 2 & 9 & 5 & 6 \end{bmatrix}$$

$A_{ij} = A_{ji}$

Symmetric matrices are nice to work with especially when it comes to diagonalization. Next two theorems will explain why.

Theorem 1: If A is symmetric then any two eigenvectors from <u>different</u> eigenspaces are orthogonal.

□ Recall dot product of two vectors

$\vec{a} \cdot \vec{b}$ dot product

is same as matrix multiplication

$\vec{a}^T \vec{b}$ matrix multiplication

$1 \times n$ $n \times 1$

With that in mind, assume $\vec{v}_1, \vec{v}_2$ are eigenvectors from different eigenspaces. (so they have different eigenvalues λ_1, λ_2).

Consider the dot product $(A\vec{v}_1) \cdot \vec{v}_2$. On one hand:

$$(A\vec{v}_1) \cdot v_2 = (\lambda_1 \vec{v}_1) \cdot v_2 = \underline{\lambda_1 (\vec{v}_1 \cdot \vec{v}_2)}$$

On the other hand:

$$(A\vec{v}_1) \cdot \vec{v}_2 = (A\vec{v}_1)^T \vec{v}_2$$ rewrite dot prod as matrix mult.

$$= \vec{v}_1^T A^T \vec{v}_2$$ Remember $(AB)^T = B^T A^T$

$$= \vec{v}_1^T A \vec{v}_2$$ A symmetric so $A^T = A$

$$= \vec{v}_1^T (A\vec{v}_2)$$ $ABC = A(BC)$

$$= \vec{v}_1 \cdot (A\vec{v}_2)$$ Can write the matrix mult as

$= \vec{v}_1 \cdot (\lambda_2 \vec{v}_2)$ — $\vec{v}_2$ eigenvector w/ value λ_2 (dot prod.)

$= \underline{\lambda_2 (\vec{v}_1 \cdot \vec{v}_2)}$ — can pull out constant

So we see that $(A\vec{v}_1)\cdot \vec{v}_2 = \lambda_1 (v_1 \cdot v_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$

Focus on fact that

$$\lambda_1 (\vec{v}_1 \cdot \vec{v}_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$$

Since $\lambda_1 \neq \lambda_2$, only way this can be done is if $\vec{v}_1 \cdot \vec{v}_2 = 0$. So $\vec{v}_1, \vec{v}_2$ are orthogonal.

$\blacksquare$

Note: We already knew vectors from different eigenspaces were linearly independent. Now we know a bit more (for symmetric matrices).

Caution: Eigenvectors from <u>same</u> eigenspace may not be orthogonal.

Ex: A is 3×3 symmetric matrix with eigenvalues λ_1, λ_2

$\underline{\lambda_1}$	$\underline{\lambda_2}$
$\begin{bmatrix} \vec{v}_1 \end{bmatrix}, \begin{bmatrix} \vec{v}_2 \end{bmatrix}$	$\begin{bmatrix} \vec{v}_3 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_3 = 0$ and $\vec{v}_2 \cdot \vec{v}_3 = 0$ but maybe $\vec{v}_1 \cdot \vec{v}_2 \neq 0$

<u>However</u>, once we have basis for eigenspace can use Gram-schmidt to get orthogonal basis. Result will still be eigenvectors for λ_1, but will be orthogonal to each other <u>and</u> to $\vec{v}_3$.

Can apply same principle in general.

Assume symmetric A is diagonalizable.

$$A = PDP^{-1}$$

Remember, P is matrix of eigenvectors. By above we see we can modify eigenvectors a bit so they're all orthogonal.

Once we have orthogonal vectors, can just scale them so all have length one, <u>othonormal vectors</u>

So can write $A = QDQ^{-1}$ where Q is a matrix of orthonormal vectors (remember, this is confusingly called an <u>orthogonal matrix</u>).

Finally, recall that $Q^{-1} = Q^T$ for orthogonal matrices. So

$$A = QDQ^T$$

Definition: An $n \times n$ matrix A is **orthogonally diagonalizable** if there exist $n \times n$ matrices P, D (where D diagonal) such that

$$A = PDP^T$$

Assume A is orthogonally diagonalizable, i.e.

$$A = PDP^T$$

Then

$$\begin{aligned} A^T &= (PDP^T)^T \\ &= (P^T)^T D^T P^T \\ &= PDP^T \end{aligned}$$

$(ABC)^T = C^T B^T A^T$

Two transposes cancel, and diagonal matrices D are symmetric

Thus $A^T = A$. Thus,

- If A is orthogonally diagonalizable then it is symmetric

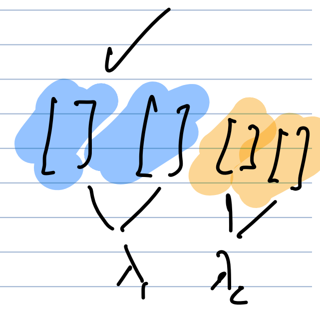
And by what we have done so far we know

- If matrix is symmetric <u>and</u> diagonalizable then it is orthogonally diagonalizable.

Turns out symmetric matrices are <u>always</u> diagonalizable so we actually have an "if and only if" statement

Theorem 2: An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

Steps to orthogonally diagonalize matrix are very similar to regular diagonalizing. Can skip some steps (counting eigenvectors) but add others (in orange)



Step 1: Find eigenvalues.

Step 2: Find basis for each eigenspace

Step 3: Check if each basis is orthogonal

Step 4: If basis not orthogonal, make it so using Gram-Schmidt

Step 5: Check if each vector has length one

Step 6: If not unit vector, divide by its own length

Step 7: Make D out of eigenvalues. Repeat eigenvalue as necessary to match # of eigenvector

Step 8: Make P out of eigenvectors (the orthogonal unit vectors)

Step 9: Find P^T (P^T replaces P^{-1} in this method)

Step 10: Write all together as

$$A = PDP^T$$

$$A \Rightarrow PDP^T$$

Problems / Exercises

Next theorem doesn't tell us anything too surprising. Note that spectrum of A is the set of eigenvalues of A

Spectral Theorem for Symmetric Matrices:

An $n \times n$ symmetric matrix has following properties:

1. A has n real eigenvalues (may be some repeats)
2. Algebraic multiplicity = geometric multiplicity
3. Eigenspaces are mutually orthogonal
4. A is orthogonally diagonalizable.

Spectral Decomposition - Skip.