16.7 - Surface Integrals

Will now develop surface integrals, very similar to how we developed line integrals.

Recall two types of line integrals

① $\int_C f \, ds$ $\qquad$ $\iint_M f \, dS$

② $\int_C \underline{F \cdot dr}$

Consider ①. It was very similar to an arc length integral.

$$\int_C \boxed{f}\, \boxed{ds} = \int_a^b f(\vec{r}(t)) \, |r'(t)| \, dt$$

arc length

① Found $\vec{r}(t)$ for curve C

② Given $\vec{r}(t)$, $ds = |r'(t)| \, dt$

③ $f(x, y) = f(\vec{r}(t))$

Only difference is we have $f(\vec{r}(t))$ where we take our parameterization and plug in to our function.

Similar progression for surface integrals.

Not much more theory to develop, straight to formula.

Surface Integral of f on M

$$\iint_M f \boxed{dS}$$

$$\int_c f \boxed{ds} = \int f(r(t)) |r'(t)| dt$$

Integrate on surface M

what we integrate

Integrate w.r.t. surface area

Can we translate this abstract stuff into something real?

Line integral

$$\int_c f \, ds$$

Surface Integral

$$\iint_m f \, dS$$

$$\int_a^b f(\vec{r}(t))\,|r'(t)|\,dt$$

$$\int_c^d\int_a^b f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

Ex: Evaluate $\iint_M x^2 y z \, dS$ where

M is part of plane $z = 1 + 2x + 3y$

above rectangle $[0,6] \times [0,2]$

Step 1: Find parameterization $\vec{r}(u,v)$ for surface M, including bounds

$$\vec{r}(u,v) = \langle u, v, 1 + 2u + 3v \rangle$$

$$0 \le u \le 6 \qquad 0 \le v \le 2$$

Step 2: Find $\vec{r}_u$, $\vec{r}_v$

$$\vec{r}_u = \langle 1, 0, 2 \rangle$$

$$\vec{r}_v = \langle 0, 1, 3 \rangle$$

Step 3: Find $|\vec{r}_u \times \vec{r}_v|$

$$\vec{r}_u \times \vec{r}_v : \begin{bmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$2\vec{i} - 3\vec{j} + \vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4+9+1} = \sqrt{14}$$

$$\Big(\text{So } dS = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \sqrt{14}\, du\, dv \Big)$$

Step 4 Convert integrand from problem statement using $\vec{r}$ from step 1

$$x^2 y z$$

$$\Big\downarrow$$

$$u^2 v\,(1+2u+3v)$$

Step 5 Set up integral, using bounds from step 1

$$\int^{2}\int^{6} u^2 v\,(1+2u+3v)\sqrt{14}\, du\, dv$$

$$\int_0 \int_0 u \; v \,(1 + 2u + 3v)\, \sqrt{14}\; du\,dv$$

Step 6 Integrate

$$\sqrt{14} \int_0^2 \int_0^6 \left(u^2 v + 2u^3 v + 3u^2 v^2\right) du\,dv$$

$$\sqrt{14} \int_0^2 \left(\frac{u^3}{3} v + \frac{u^4}{2} v + u^3 v^2 \Big]_0^6 \right) dv$$

$$= \sqrt{14} \int_0^2 \left(72v + 648v + 216v^2\right) dv$$

$$= \sqrt{14} \left(36v^2 + 324v^2 + 72v^3 \Big]_0^2 \right)$$

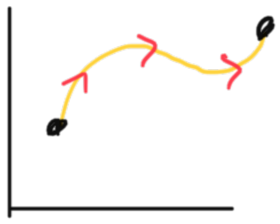
$$= \sqrt{14} \left(36 \cdot 4 + 324 \cdot 4 + 72 \cdot 8\right)$$

$$= \sqrt{14} \left(144 + 1296 + 576\right)$$

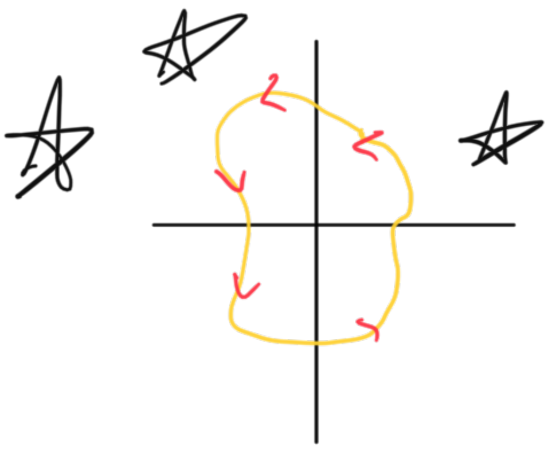
$$= \sqrt{14} \left(2016\right)$$

Oriented Surfaces

For line integrals, direction was important.



May get opposite value if we integrate opposite direction



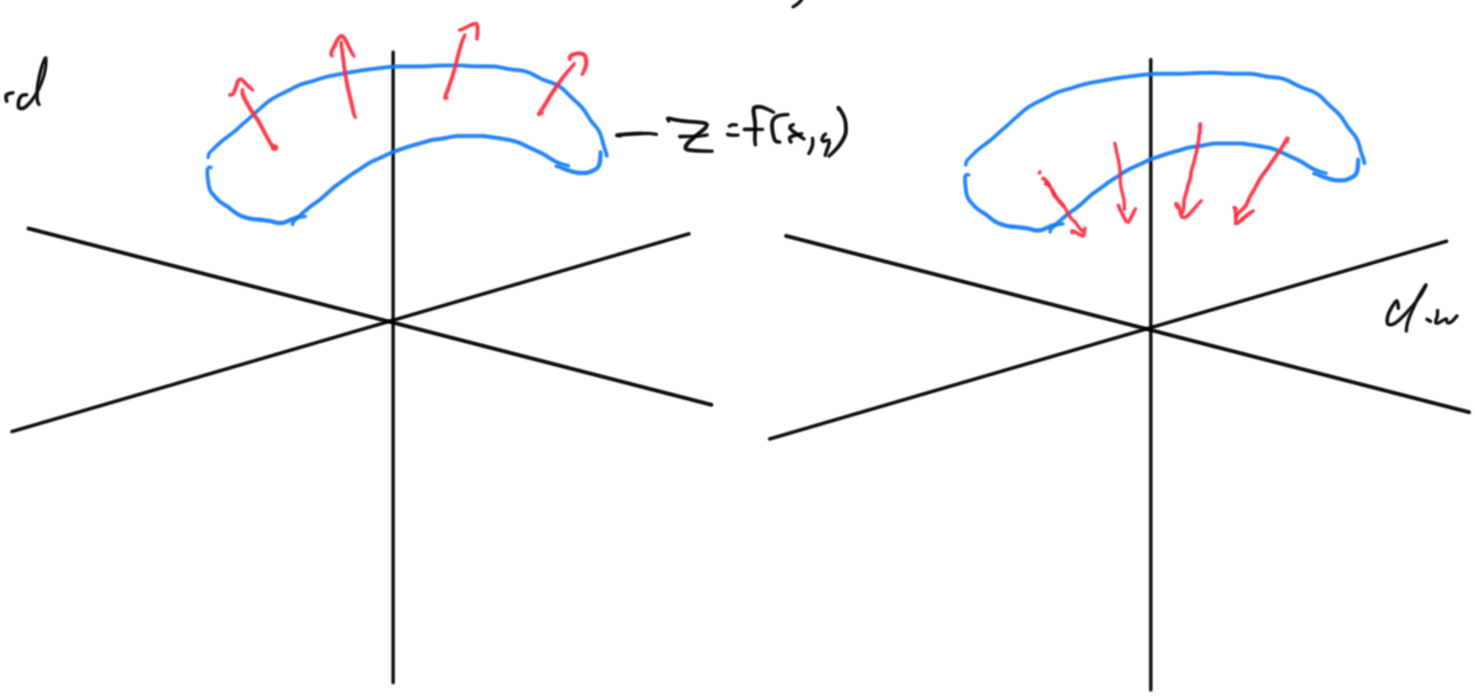
Closed curves had positive/negative orientations

Surfaces also have idea of orientation (but not really direction)

Orientation determined by normal vectors

$\vec{n}$ $\vec{n} = \langle x, y, z \rangle$

upward



downward

Recall our standard normal vector for $\boxed{z = f(x,y)}$ is

$$\vec{n} = \langle -f_x, -f_y, 1 \rangle$$

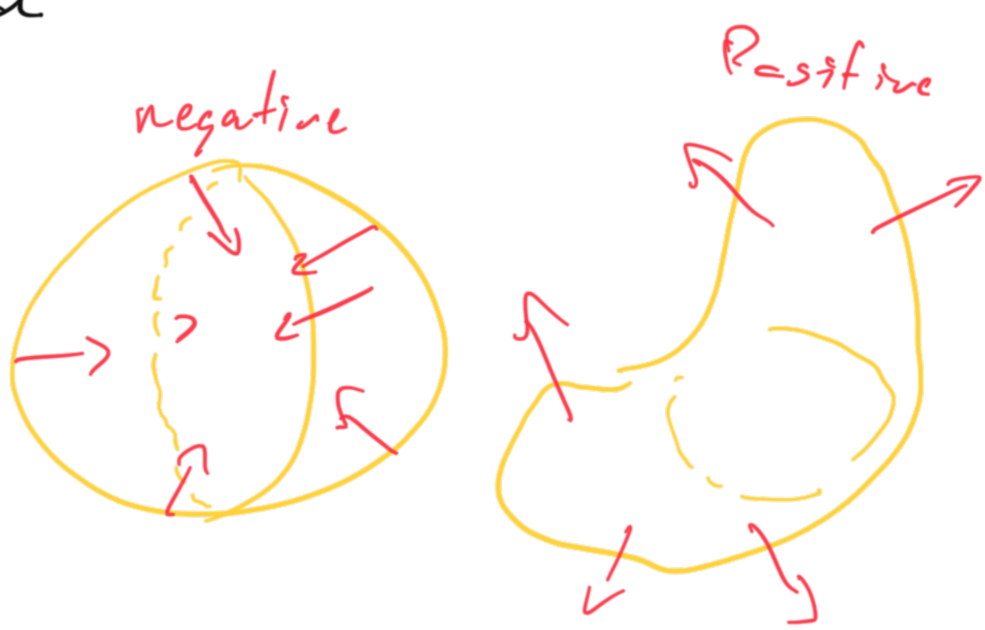
$$\|\vec{n}\| \quad \overline{\sqrt{1+f_x^2+f_y^2}}$$

This can be written in terms of our parameterization as

[rewritten $z = f(x,y)$ $\vec{r}(x,y)$]

$$\boxed{\frac{\vec{n}}{\|\vec{n}\|}} = \boxed{\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}}$$

For closed surfaces (like closed curves) positive orientation is outward, negative is inward



How does orientation affect ... ?

How does orientation affect us?

Surface integrals of vector fields (about to see) are about "how much of vector field is flowing across a surface"

Think of orientation as "direction that the surface <u>wants</u> vector field to flow"

<u>Ex.</u> This surface has negative orientation

Surface Integral for Vector Fields (Flux)

$$F = \vec{v}$$

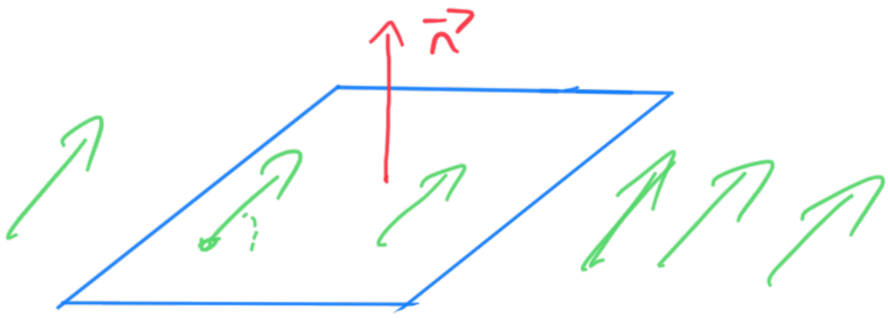
<u>Standard Ex.</u>

Fluid with velocity $\vec{v}$, density ρ

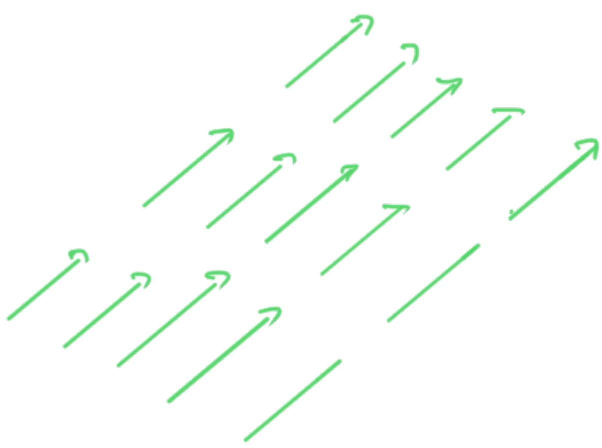
How much mass crosses through our surface S ?

Easy version

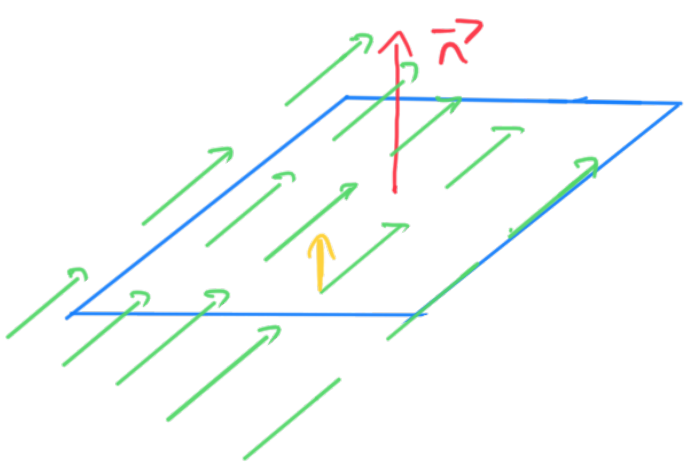
Assume S just a plane (oriented up)



Vector field (fluid) has velocity $\vec{v}$

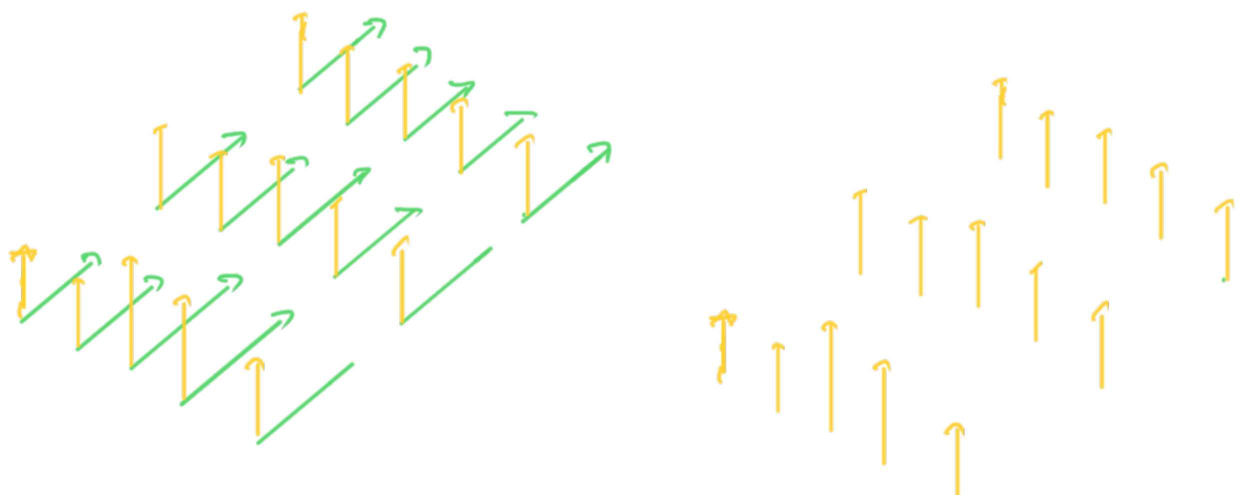


How much is flowing across surface?



Q: How much of vector $\vec{v}$ points in direction of $\vec{n}$?

Projection of $\vec{v}$ onto $\vec{n}$!

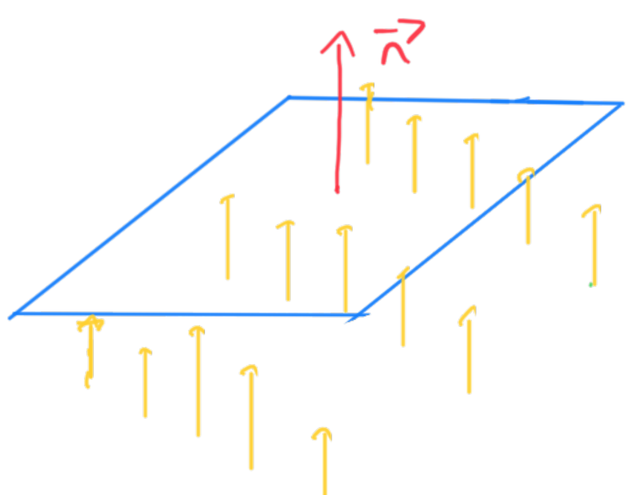


$$\text{Proj}_{\vec{n}}\, \vec{v} = \frac{\vec{v}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\left(\vec{n}\right)$$

$$= \boxed{\frac{(\vec{v}\cdot\vec{n})}{\|n\|}}\ \cancel{\frac{\vec{n}}{\|n\|}}$$

Concerned with amount so just need

$$\frac{(\vec{v}\cdot\vec{n})}{\|\vec{n}\|}$$



For simple plane, amount crossing surface will just be

$$(\text{density}) \underline{\left(\frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|}\right)} \cdot \underline{(\text{Surface area of plane})}$$

Just for this particular problem

$$\iint_M F \cdot \underline{\left(\frac{\vec{n}}{\|n\|}\right)} \quad dS$$

Parameterize M by $\underline{\vec{r}}(x,y)$

$$S = |\vec{r}_x \times \vec{r}_y| \, dx\,dy$$

$$\iint F(\underline{\vec{r}}) \cdot \frac{(\vec{r}_x \times \vec{r}_y)}{\cancel{|\vec{r}_x \times \vec{r}_y|}} \quad \cancel{|\vec{r}_x \times \vec{r}_y|} \, dx\,dy$$

Surface integral of vector field F over surface M

$$= \int^b \int^d F(\vec{r}) \cdot (\vec{\ } \ \vec{\ }) \ \ \ldots$$

$$\left\{ \begin{array}{l} \int_a \ldots (r_x \times r_y)\, dx\, dy \\ \\ \iint_M F \cdot \vec{n}\, dS \end{array} \right.$$

For more complicated surfaces the formula basically same, inside an integral

$$\iint_M \frac{\vec{v} \cdot \vec{n}}{\|\vec{n}\|}\, dS$$

Rewrite using parameterization