Section 4.6

Return to idea of coordinates. Recall that once we have basis for vector space, $\underline{\mathcal{B} = \{\vec{b}_1, \dots \vec{b}_n\}}$ we can write any vector as linear combination of these basis vectors.

$$\vec{x} = \underline{c_1} \vec{b}_1 + \dots \underline{c_n} \vec{b}_n$$

Coefficients $c_1 \dots c_n$ are coordinates of $\vec{x}$ with respect to basis $\mathcal{B}$.

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

What if we have coordinates in one basis $\mathcal{B}$ but need to convert to another basis $\mathcal{C}$?

$$\mathcal{B} = \{\vec{b}_1 \dots \vec{b}_n\} \qquad \mathcal{C} = \{\vec{c}_1 \dots \vec{c}_n\}$$

Start with $\vec{x}$ written in terms of $\mathcal{B}$, $[\vec{x}]_{\mathcal{B}}$

$$[\vec{x}]_{\vec{\mathcal{B}}} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \qquad \vec{x} = r_1 \underline{\vec{b}_1} + \underline{r_2 \vec{b}_2} + \dots \underline{r_n \vec{b}_n}$$

Lets say we know how to write each $\vec{b}_i$ in terms of $\{\vec{c}_1 \dots \vec{c}_n\}$ as well

$$[\vec{b}_1]_{\mathcal{C}}, [\vec{b}_2]_{\mathcal{C}} \dots [\vec{b}_n]_{\mathcal{C}}$$

Make matrix out of these, consider multiplication

The amount c I need to make $\vec{b}_1$

$$\begin{bmatrix} | & & | \\ [b_1]_c & \dots & [b_n]_c \\ | & & | \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$[\vec{x}]_B$

Think of as vector equation:

$$\star \quad r_1 [\vec{b}_1]_c + r_2 [\vec{b}_2]_c + \dots r_n [\vec{b}_n] = [\vec{x}]_c$$

r_1 times b_1 but $\vec{b}_1 = s_1 \vec{c}_1 + \dots s_n \vec{c}_n$

$r_1 (s_1 \vec{c}_1 + \dots s_n \vec{c}_n)$

$= r_1 s_1 \vec{c}_1 + \dots r_1 s_n \vec{c}_n$

Can do same for all other terms $r_2 [\vec{b}_2]_c, \dots$

Rearrange everything and we get "how many" $\vec{c}_1 \dots \vec{c}_n$ it takes to make $\vec{x}$.

In other words $[\vec{x}]_c$

Theorem: Let $B = \{\vec{b}_1 \dots \vec{b}_n\}$ and $C = \{\vec{c}_1 \dots \vec{c}_n\}$ be bases of vector space V. Then there is unique $n \times n$ matrix $\underset{C \leftarrow B}{P}$ such that

$$\underset{C \leftarrow B}{P} [\vec{x}]_B = [x]_C$$

Columns of $\underset{C \leftarrow B}{P}$ are C-coordinate vectors of $\{\vec{b}_1, \dots \vec{b}_n\}$, i.e. $[\vec{b}_1]_C$, ... $[\vec{b}_n]_C$

$\underset{C \leftarrow B}{P}$ is the change-of-coordinates matrix from B to C

or change-of-basis matrix

If we can change basis in one direction can just as easily change back in other direction and undo our work. So no surprise that

$$\left(\underset{C \leftarrow B}{P}\right)^{-1} = \underset{B \leftarrow C}{P}$$

$$[x]_B = \underset{B \leftarrow C}{P} \; \underset{C \leftarrow B}{P} \; [x]_B$$

$$\left(\underset{C \leftarrow B}{P}\right)^{-1} = \left(\underset{B \leftarrow C}{P}\right)$$

(9) $b_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix} \quad b_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad c_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad c_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

$$\underset{C \leftarrow B}{P} \qquad \left[\begin{array}{cc|cc} 2 & 6 & -6 & 2 \\ -1 & -2 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 3 & -3 & 1 \\ -1 & -2 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 3 & -3 & 1 \\ 0 & 1 & -4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{array}\right]$$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$$

$$\underset{B \leftarrow C}{P} \qquad \left[\begin{array}{cc|cc} 9 & -2 & 1 & 0 \\ -4 & 1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 4 & 9 \end{array}\right]$$

$$\underset{B \leftarrow C}{P} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$$