Section 2.2

In this section we discuss the inverse of a matrix A

The concept is similar to the multiplicative inverse of a real number

For $c \in \mathbb{R}$, $1/c = c^{-1}$ is the multiplicative inverse because

$I_{n \times n}$

$$\underbrace{c \cdot c^{-1} = c^{-1} \cdot c}_{\text{order doesn't matter}} = 1$$

order doesn't matter

identity element for multiplication

For certain matrices a similar concept (inverse) exists.

Def: A square $n \times n$ matrix $\underline{A}$ is invertible if there exists some matrix C such that:

$$CA = AC = I_{n \times n} \quad \cancel{A}$$

Thoughts: Everything has to be square, $n \times n$, otherwise dimensions won't match up for either CA or AC

Def: A matrix C such that $AC = CA = I_{n \times n}$ is called inverse of A

Thoughts: The inverse of A is unique. Why?

Imagine B, C are both inverses of A. Then:

$$AB = I_{n \times n}$$

$$\Rightarrow \; C(AB) = C(I_{n\times n})$$

$$\Rightarrow \; (CA)B = C$$

$$\Rightarrow \; I_{n\times n} B = C$$

$$\Rightarrow \; B = C$$

Notation / Terminology:

- Will denote inverse of A as A^{-1}
- An invertible matrix may also be called nonsingular
- A non-invertible square matrix may also be called singular

Calculating the inverse

Theorem 4 of section 2.2 gives formula for special case of a 2×2 matrix.

Thm Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $(ad-bc) \neq 0$ then A invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$ then A not invertible

I have never bothered using above formula. I prefer more general method which works for any size matrix. But need to build up to it.

Def: An elementary matrix is one obtained by performing a single row operation on the identity matrix. (Usually denoted E)

Ex: $\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}$ Added 3 (1st row) to the 3rd row $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$EA =$

When we multiply a matrix A by an elementary matrix, it has effect of performing row operation on A

Elementary matrix is a way to represent a row operation

Ex:

$$\overset{E}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}} \overset{A}{\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 4 & 0 \end{bmatrix}} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 6 & 7 & 9 \end{bmatrix}$$

★ Note ★ Order matters! Always put most recent elementary matrix on left

We can easily undo any of our basic row operations. If we think of the <u>inverse</u> of matrix as something that "cancels out" then seems obvious that every elementary matrix $\underline{E}$ has inverse $\underline{E^{-1}}$

Ex: $\nearrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ Represents row operation of adding 3(row 1) to row 3

How would we undo that?

Subtract back 3(row 1) from row 3

$\nearrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

Let's check. Should be able to multiple either order and get $I_{3\times 3}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

Brings us to our theorem/method for finding general inverses

Theorem: An $n \times n$ matrix A is invertible if and only if it is row equivalent to $I_{n\times n}$. And in this case a sequence of row operations that transforms A into $I_{n\times n}$ also transforms $I_{n \times n}$ into $\underline{A^{-1}}$.

The method

Glue matrix A and $I_{n \times n}$ together

Ex: $A = \begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$\to \left[\begin{array}{ccc|ccc} 2 & 4 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

Now row reduce until left side is $I_{n \times n}$.

$$E_2 \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$E_3 \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$E_3 \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$E_3 \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{3}{2} & 0 & -2 \\ 0 & 1 & 0 & \frac{1}{2} & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$E_3 \sim \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_{3 \times 3}} & \underbrace{\begin{matrix} \frac{1}{2} & -2 & 0 \\ \frac{1}{2} & 1 & -1 \\ -\frac{1}{2} & 0 & 1 \end{matrix}}_{A^{-1}} \end{array}\right]$$

Left side is now $I_{n \times n}$

Right side should be A^{-1}

Check

$$\begin{bmatrix} 2 & 4 & 4 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -2 & 0 \\ \frac{1}{2} & 1 & -1 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$A \qquad A^{-1} \qquad = \qquad I$$

Why it works:

Assume A is row equivalent to $I_{n\times n}$. This means there are elementary matrices $E_1 \dots E_k$ s.t.

$$\underbrace{E_k \dots E_1}_{} A = I_{n\times n} \qquad \underline{BA = I}$$

These multiplied together are just some matrix B
So have $BA = I$

$$\bigstar \quad E_k \dots E_1 \, A = I_{n\times n}$$

Each E is invertible
Multiply on left by inverses

$$E_1^{-1} \dots E_k^{-1} E_k \dots E_1 \, A = E_1^{-1} \dots E_k^{-1} I_{n\times n}$$

$$A = E_1^{-1} \dots E_k^{-1}$$

Now multiply on right by E's.

$$A E_k \dots E_1 = E_1^{-1} \dots E_k^{-1} E_k \dots E_1$$

$$A \underbrace{E_k \dots E_1}_{} = I_{n\times n}$$

This is same matrix B as before

So have a matrix B s.t

$$AB = BA = I_{n \times n}$$

This fits definition of inverse. $B = A^{-1}$.

Theorem: If A is invertible $n \times n$ matrix then for each $\underline{b \in \mathbb{R}^n}$, equation $A\vec{x} = \vec{b}$ has unique solution $\vec{x} = A^{-1}\vec{b}$

☆ **Thoughts:** By above, $\underline{A\vec{x}}$ is one-to-one and onto.

Ex: use inverse to solve system

$$\begin{cases} 3x_1 + 4x_2 = 3 \\ 5x_1 + 6x_2 = 7 \end{cases} \qquad \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} \end{array}\right] \qquad \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Theorem: ① If A invertible, then so is A^{-1} and

$$(A^{-1})^{-1} = A$$

② If A, B invertible $n \times n$ matrices then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$

③ If A is an invertible matrix, so is A^T and

$$(A^T)^{-1} = (A^{-1})^T$$

Theorem: Product of $n \times n$ invertible matrices is invertible and

$$(A_1 \dots A_k)^{-1} = A_k^{-1} \dots A_1^{-1}$$