16.1 - Vector Fields

In this course we have seen functions

$h: \mathbb{R} \to \mathbb{R}$ (single variable functions)

$y = h(x)$ Ex: $y = x^2$

$f: \mathbb{R} \to \mathbb{R}^n$ (vector valued functions)

$\vec{r}(t) = \langle f(t), g(t) \rangle$

$\vec{r}(t) = \langle t^2, t^3 + 1 \rangle$

$g: \mathbb{R}^n \to \mathbb{R}$ (multivariable functions)

$f(x, y) = z$

$z = x^2 + y^3 \ln(x)$

Now let's consider $F: \mathbb{R}^n \to \mathbb{R}^n$. In particular will look at $F: \mathbb{R}^2 \to \mathbb{R}^2$ or

$F: \mathbb{R}^3 \to \mathbb{R}^3$

What does a function like this look like?

$$F: \mathbb{R}^3 \to \underline{\mathbb{R}^3}$$

$$\langle \ , \ , \ \rangle$$

$$F(\underline{x, y, z}) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$$

$$F(\underline{x,y,z}) = \langle x^2y + z, \ 3z - \ln(x)y, \ 1 - x^2zy \rangle$$

(similar to how $\underline{j}: \mathbb{R} \to \mathbb{R}^3$ can be written as $\langle f(t), g(t), h(t) \rangle$)

$$F: \mathbb{R}^2 \to \mathbb{R}$$

Ex: $F(x,y) = \langle x+y, \ xy \rangle$

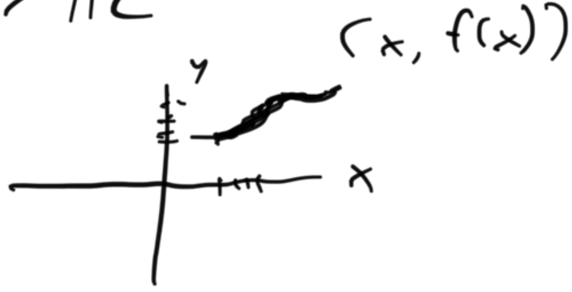
$$F(0,0) = \langle 0, 0 \rangle$$

$$F(1,2) = \langle 3, 2 \rangle$$

$$F(-3,1) = \langle -2, -3 \rangle$$

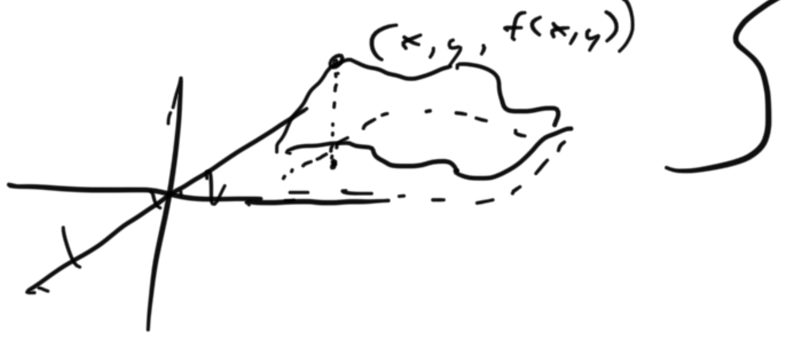
Can't graph these nicely like we can with $f: \mathbb{R} \to \mathbb{R}$ or $f: \mathbb{R}^2 \to \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$



$\downarrow$

$f: \mathbb{R}^2 \to \mathbb{R}$



$$F: \mathbb{R}^2 \to \mathbb{R}^2$$

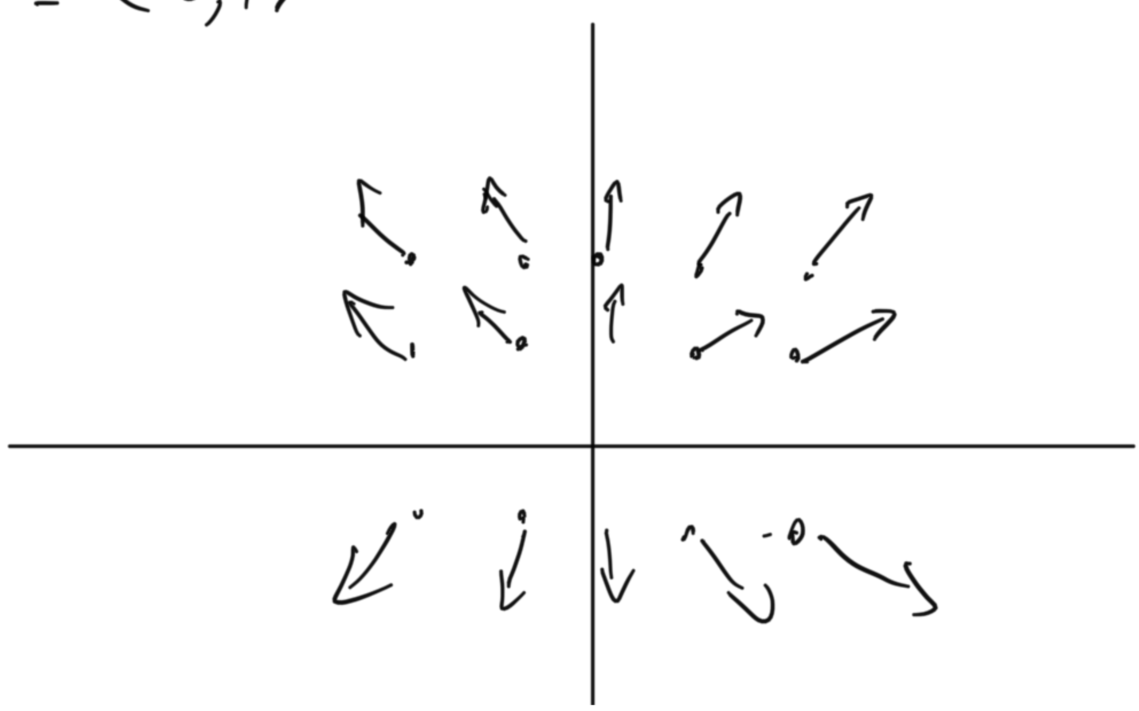
$\nearrow \quad \nearrow$

$\mathbb{R}^4$

One way to visualize: ($\underline{F: \mathbb{R}^2 \to \mathbb{R}^2}$)

$\underline{\mathbb{R}^2}$

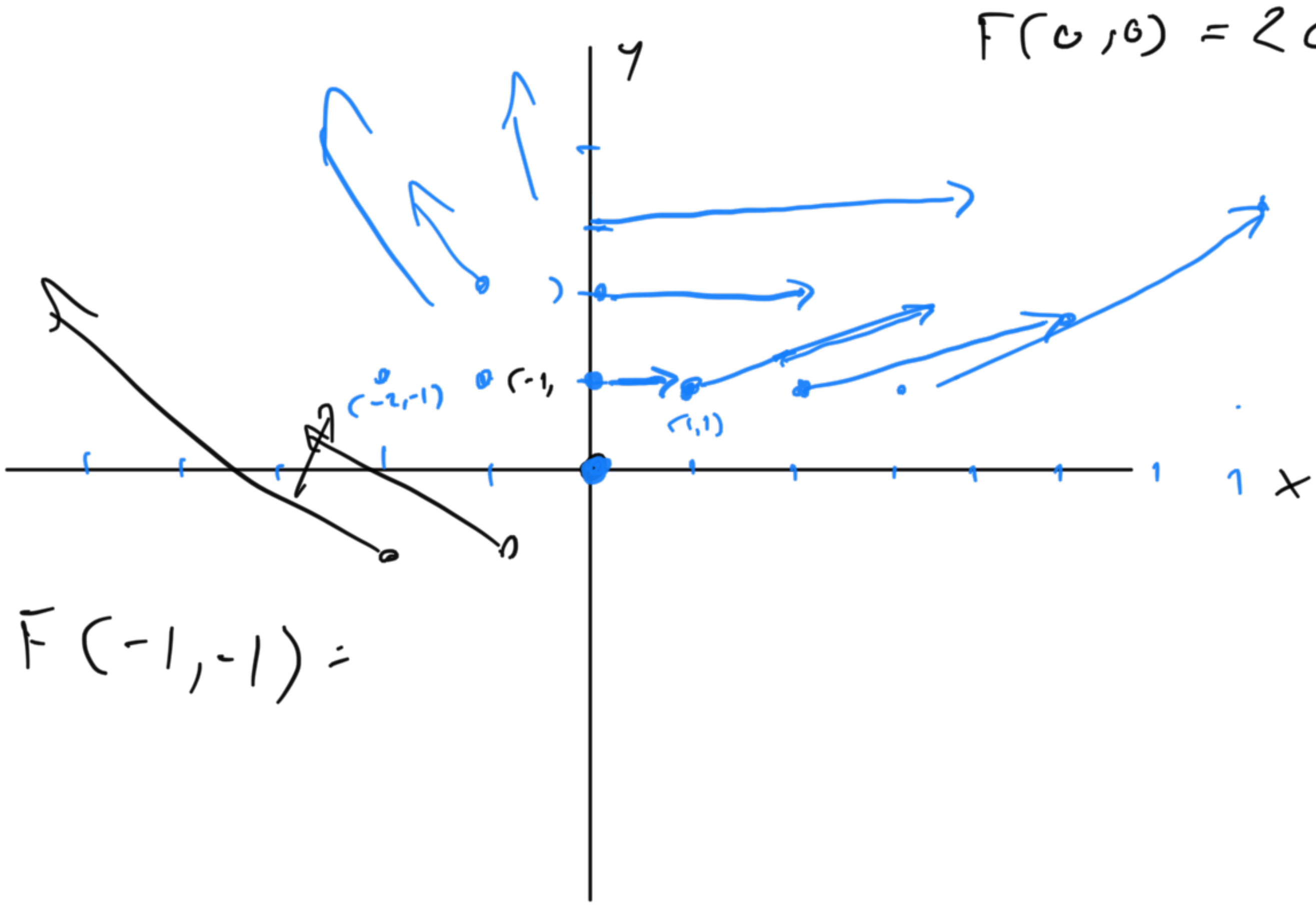
$F(1,1) = \langle 0, 4 \rangle$



$$F: \mathbb{R}^2 \to \mathbb{R}^2$$

Ex: $F(x,y) = \langle x+y, xy \rangle$

$F(0,0) = \langle 0\ 0 \rangle$



$F(-1,-1) =$

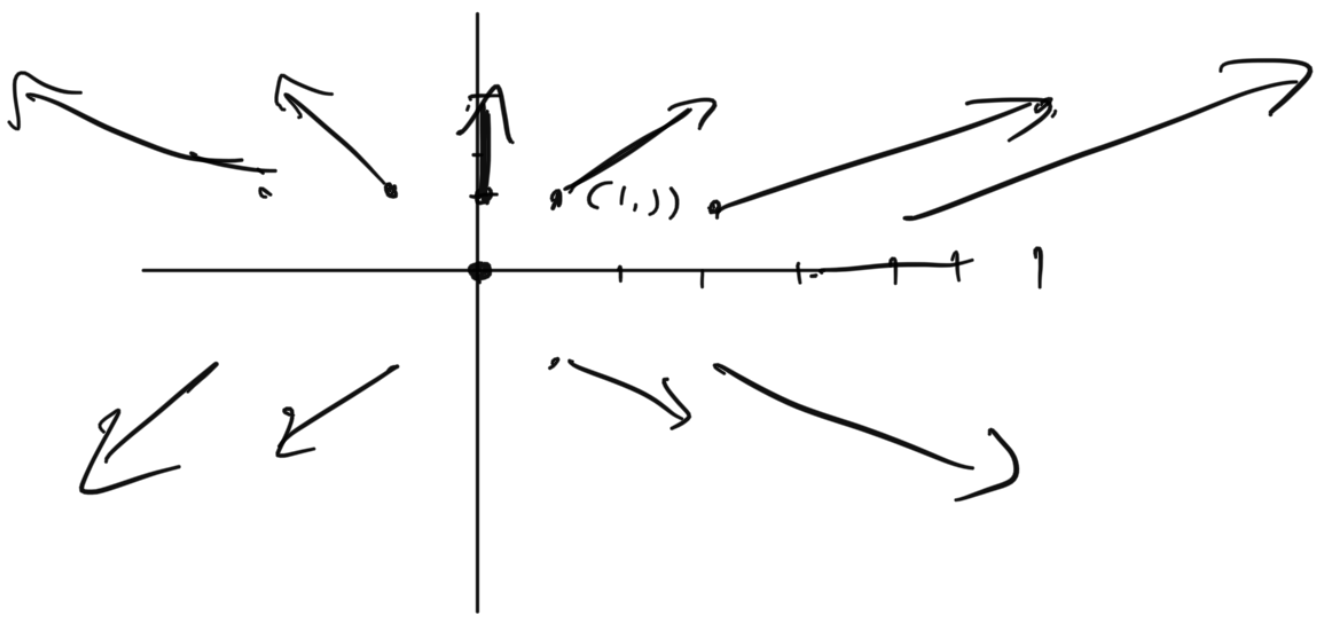
Maybe now it is clear why we call this type of function $F: \mathbb{R}^n \to \mathbb{R}^n$ a vector field

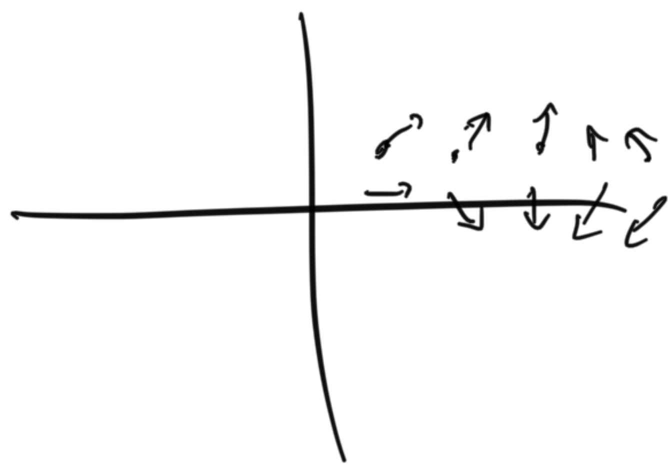
Many real world examples of vector fields, and that is what most of this section shows

Specialized

specialized names but mathematically they are all the same

$F(\vec{x,y}) = \langle 2x, 2y \rangle$ $|v| = \text{speed}$





- Velocity field ☆
- Force field ☆
- Electric field ☆

☆☆ Gradient field ☆☆

Vector Fields

$f: \mathbb{R}^2 \to \mathbb{R}$ a multivariable function. Recall that ∇f is a vector, gradient vector

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$$

So $\nabla f: \mathbb{R}^2 \to \mathbb{R}^2$, gradient function is a vector field

Ex: Plot gradient field for function $f(x,y) = x^2 y - y^3$

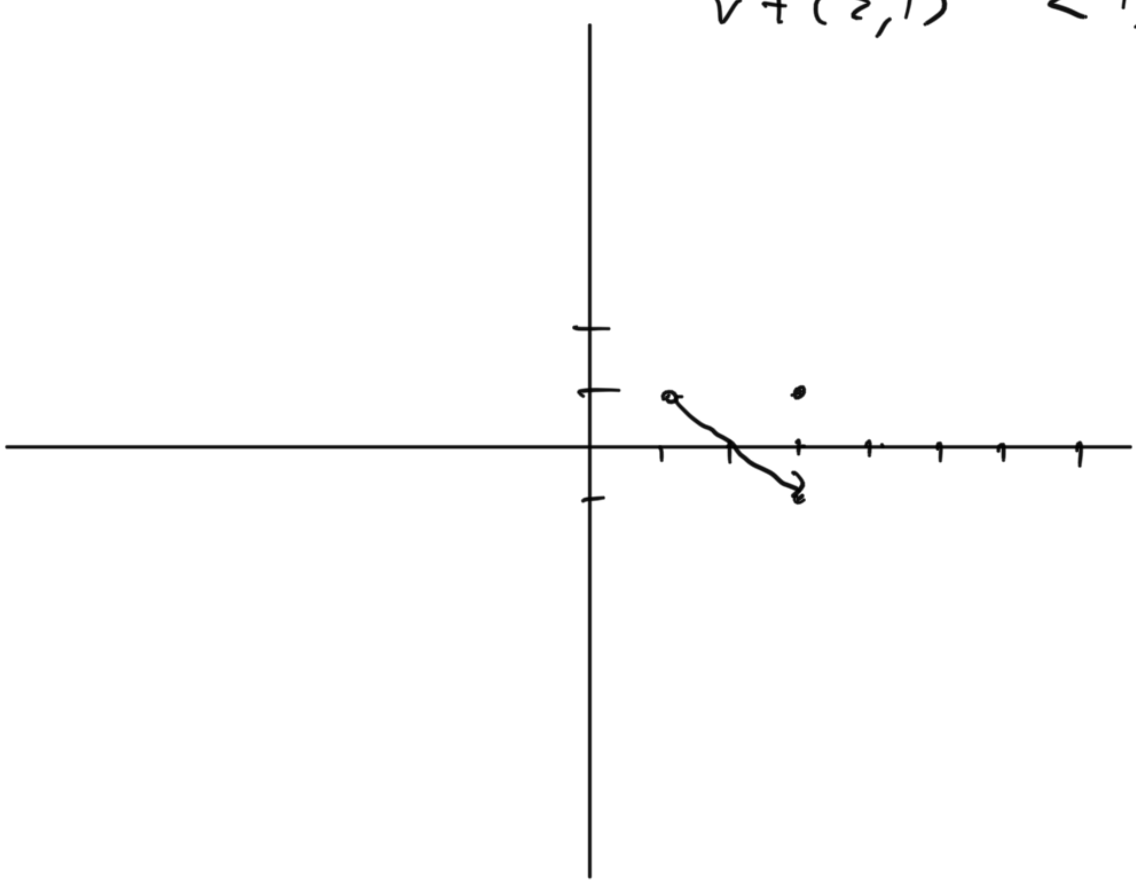
$\mathbb{R}^2 \to \mathbb{R}$

$F: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \nabla f(x,y)$

$$\nabla f(x,y) = \langle 2xy, x^2 - 3y^2 \rangle$$

$\nabla f(1,1) = \langle 2, -2 \rangle$

$\nabla f(2,1) = \langle 4, 1 \rangle$



The gradient vector fields are particularly important.

Some vector fields, but not all, are gradient fields

$$\nabla f \overset{?}{=} F : \mathbb{R}^2 \to \mathbb{R}^2$$

If we are just given vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, it may be unclear whether this is gradient field for some other function $f : \mathbb{R}^n \to \mathbb{R}$

In later sections we will develop a test for this.

If vector field F is the gradient of some function f, we call F a conservative vector field

Call the "source" $\boxed{f}$ the

potential function for IF

$$\nabla f = F \quad ?$$

f

potential function

conservative vector field

16.2, 16.3 delayed

16.1 last one due tomorrow