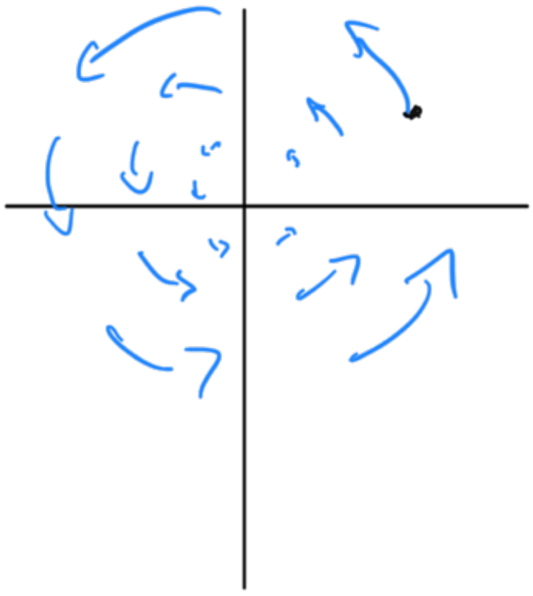
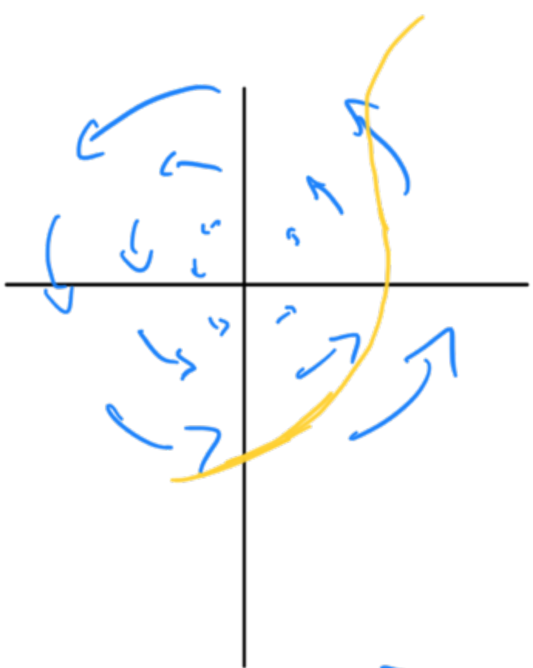
16.6 Parametric Surfaces

Recently have been integrating on paths

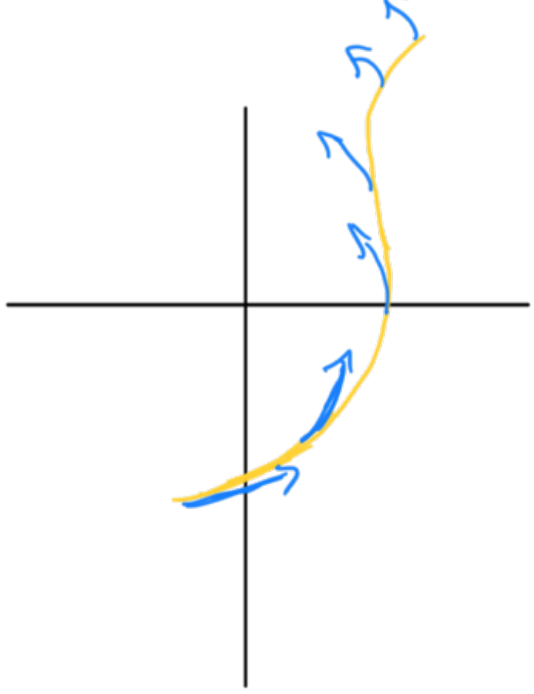
Ex: $\int_C F \cdot dr$ ☆



Have vector field
F defined on a space



Want to "add up"
amount of vector field
along curve C

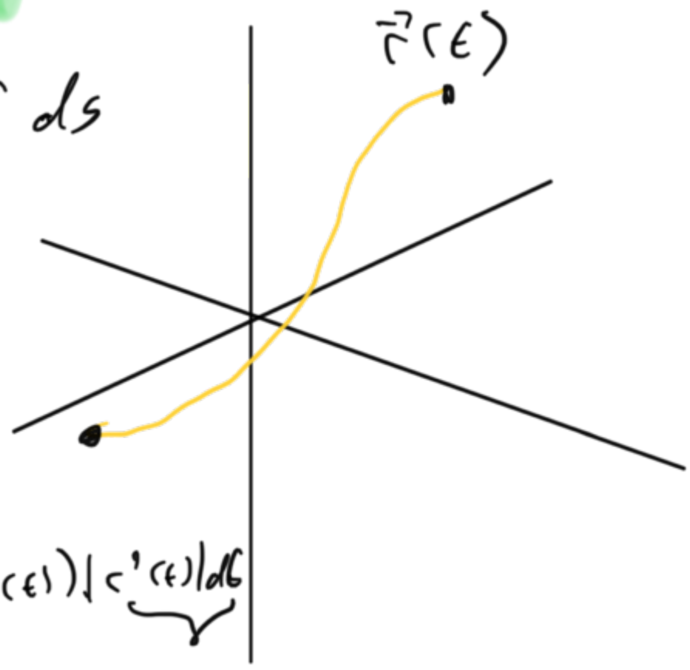


Eventually want to be able to do something similar, but "higher dimension"

Instead of integrating on a curve, want to be able to integrate on a surface

Ex:

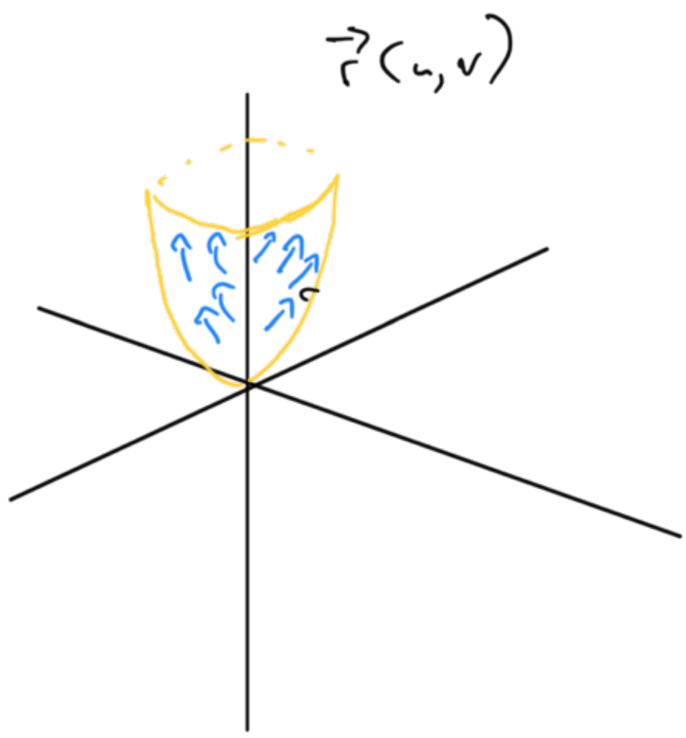
$\int_C f \, ds$



Have vector field F over $\mathbb{R}^3$

Could still do a line integral.

$\int f(\vec{r}(t))|\vec{r}\,'(t)|dt$



Or, maybe I want to integrate on surface

That's our goal. First, have to develop some tools.

Parametric Surfaces

Parametric eqs

$x = x(t)$ $\quad y = y(t)$

$\underline{\vec{r}(t)} = \langle x(t), y(t), z(t) \rangle$

Clear your mind of previous ideas for line integrals. Actually going to start by developing something similar to <u>arc length</u>

$$\int f \, \boxed{ds}$$

Remember when we had curve $\vec{r}(t)$ that arc length from $t=a$ to $t=b$ was

$\int_a^b |r'(t)| \, dt$

$y = f(x)$

$\vec{r}(t) = \langle t, f(t) \rangle$

$$\vec{r}(t) = \langle \quad , \quad , \quad \rangle$$

If we had a curve given by $y = f(x)$ could parameterize curve (rewrite as $\vec{r}(t) = \langle t, f(t) \rangle$) and use same formula.

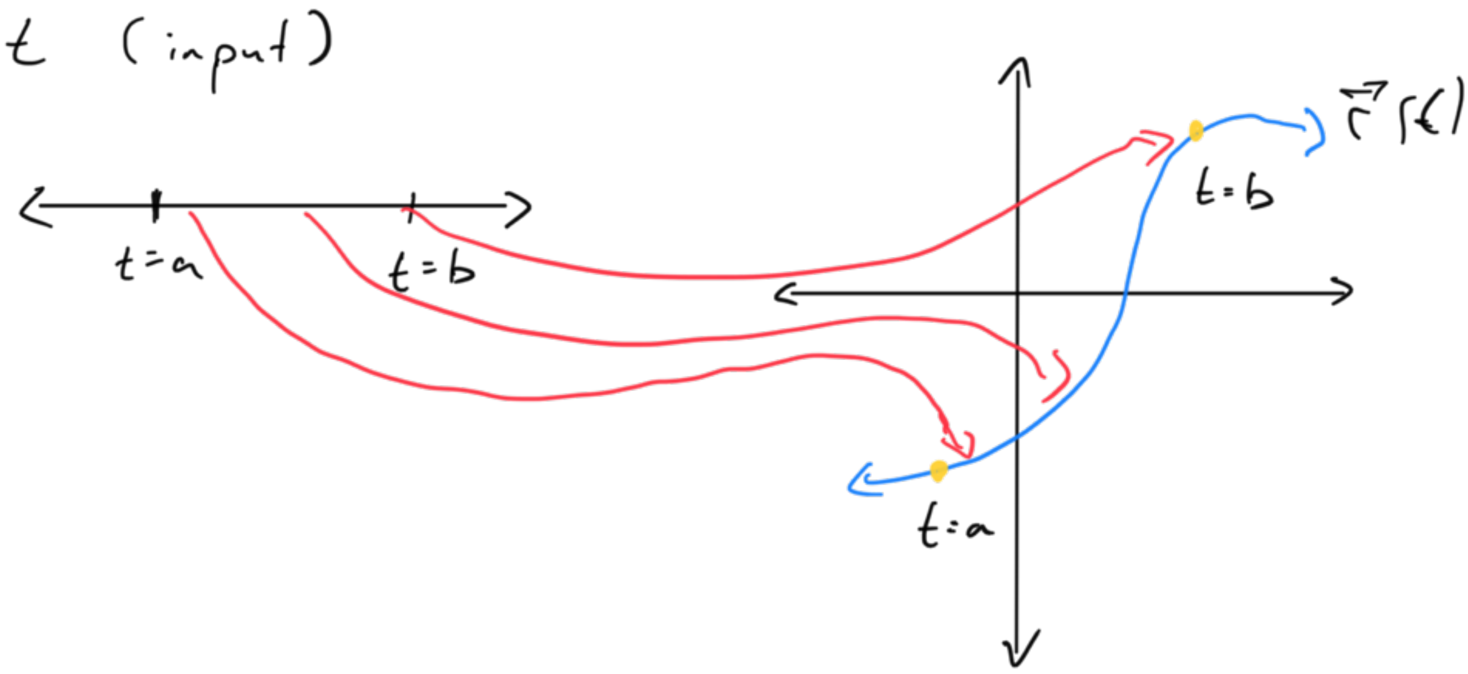
an do similar things for surfaces. First, discuss idea of a parametric surface.

<u>Parametric Surface</u>:

$\vec{r}(t)$ (vector-valued function aka parametric curve)

Can be thought of like this:

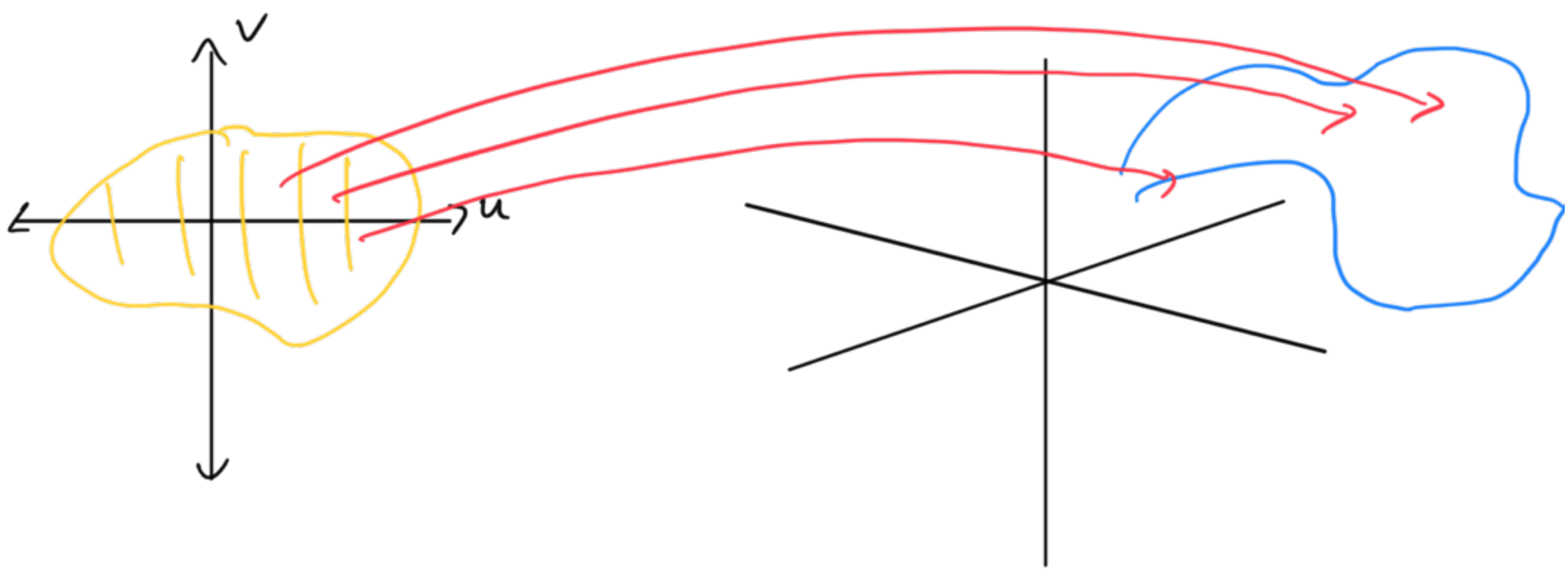
<u>$\vec{r}(t)$</u>



One dimensional input (number line for t) gives a one dimensional output (curve, living in $\mathbb{R}^2$)

Similarly, could define a parametric surface

<u>$\vec{r}(u,v)$</u>



Two dimensional input (uv-plane) gives a two dimensional output (surface, living in $\mathbb{R}^3$)

Ex: $\vec{r}(u,v) = \langle \cos(u), \sin(u), v \rangle$

Don't need this

$z = f(x,y)$ $\qquad$ $\vec{r}(u,v) = \langle \quad , \quad , \quad \rangle$

Given multivariable function
Want to rewrite it as a parametric surface

☆ Turning $z = f(x,y)$ into $\vec{r}(u,v)$

Remember, if wanted to find arc length of $y = f(x)$, could rewrite as

$\rightarrow \vec{r}(t) = \langle x(t), y(t) \rangle$

$$\int_a^b |r'(t)|\, dt$$

$y = f(x)$ $\qquad$ $\vec{r}(t)$

$$\vec{r}(x) = \langle x, f(x) \rangle$$

$$y = x^3$$

$$\vec{r}(x) = \langle x, x^3 \rangle$$

$$\vec{r}(\underline{t}) = \langle t, f(t) \rangle$$

Just rewrite independent variable x, as t

Since $y = f(x)$, the second component should just be $f(t)$

Same process for $z = f(x,y)$. In this set-up x, y are independent variables. Turn x's into u's, y's into v's

☆ $z = f(x,y)$

$$\vec{r}(x,y) = \langle \underline{x}, y, \underline{f(x,y)} \rangle$$

$$\Rightarrow \underline{\vec{r}(u,v)} = \langle u, v, f(u,v) \rangle$$

Ex: Rewrite $z = x^2 + y^2$ as a

parametric surface $\underline{\vec{r}(u,v)}$

$$z = x^2 + y^2$$

$$\vec{r}(x,y) = \langle \underline{x}, \underline{y}, \underline{x^2+y^2} \rangle$$

Surfaces of Revolution:

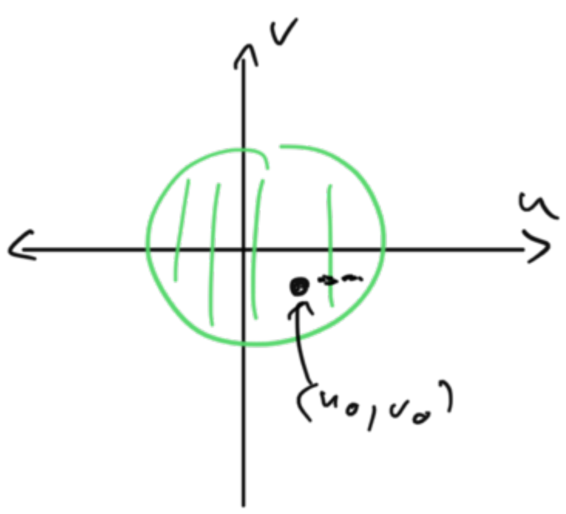
Skip

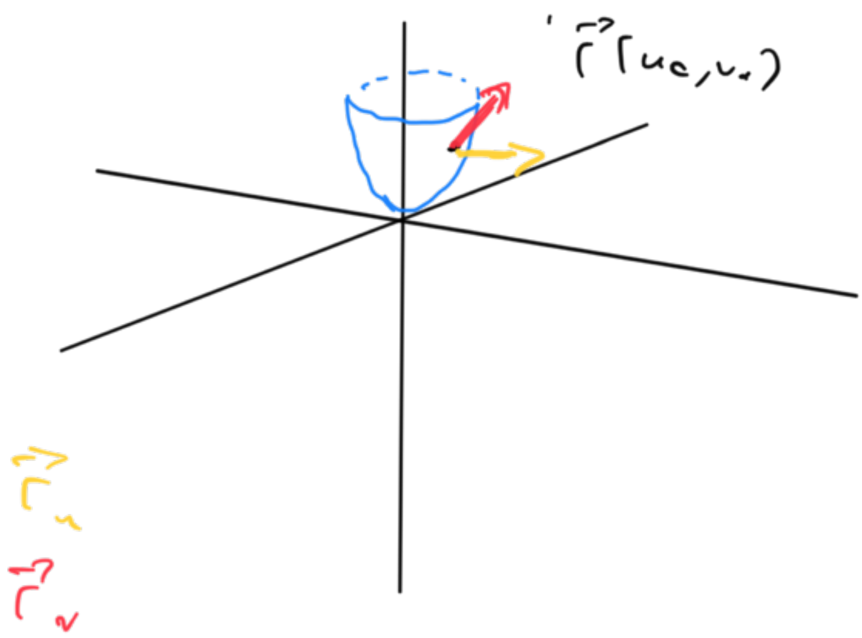
Tangent Planes

$$\vec{r}(u,v) = \langle g_1(u,v), g_2(u,v), g_3(u,v) \rangle$$

$$\vec{r}_u(u,v) = \left\langle \frac{\partial g_1}{\partial u}(u,v), \frac{\partial g_2}{\partial u}(u,v) \ldots \right\rangle$$

Mostly skip. Just need idea of partial derivatives $\vec{r}_u$ and $\vec{r}_v$

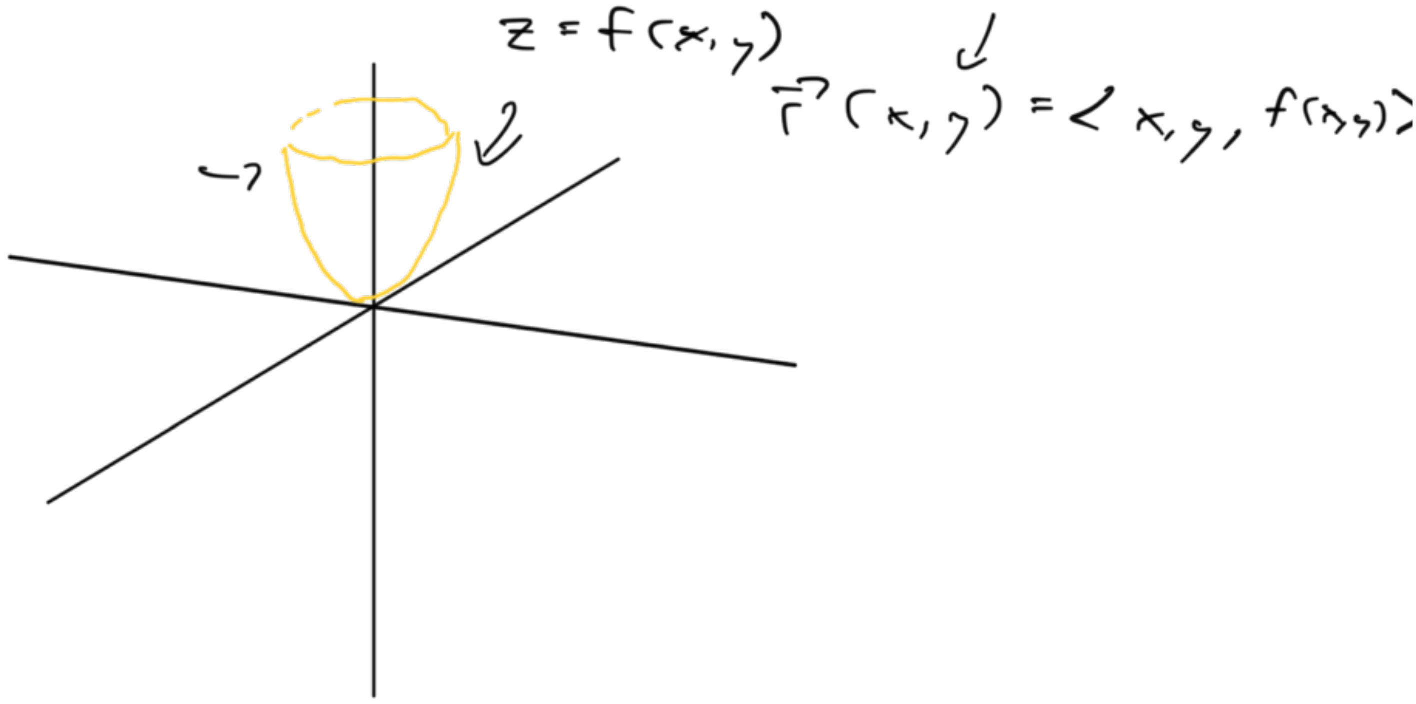




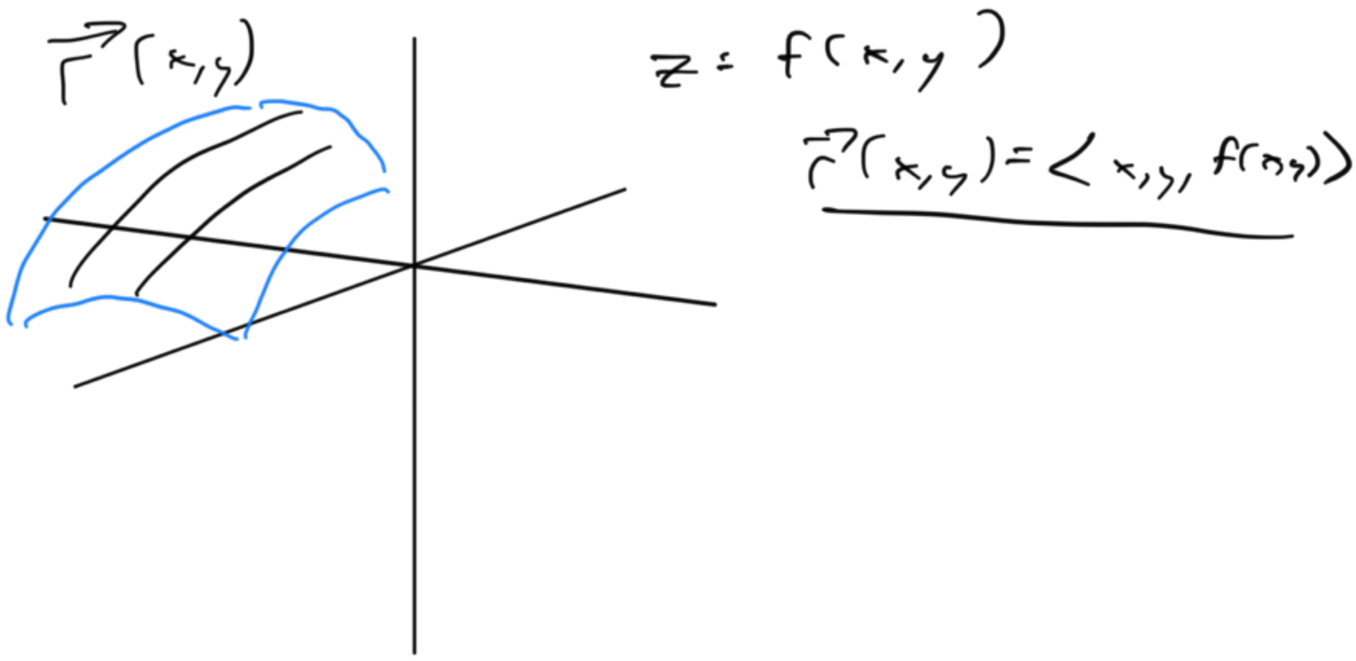
Surface Area

Once we have $\vec{r}(u,v)$

(u, v) (whether given $\vec{r}$ or convert $z = f(x,y)$ to $\vec{r}$) we can find surface area

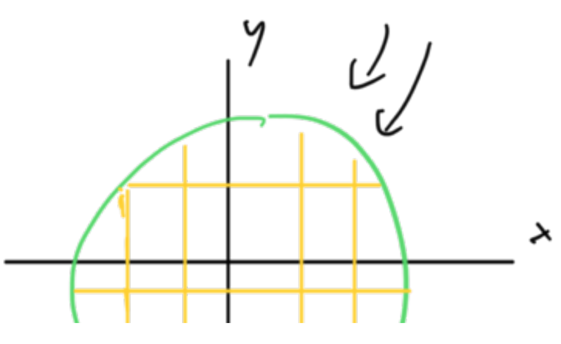


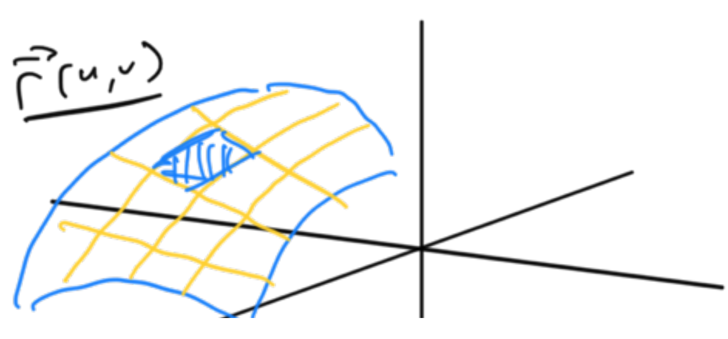
Approximating surface area by Riemann Sums

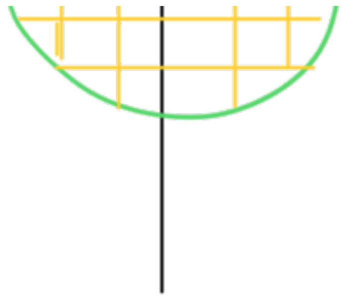


Want to break surface up into pieces, find area of each piece

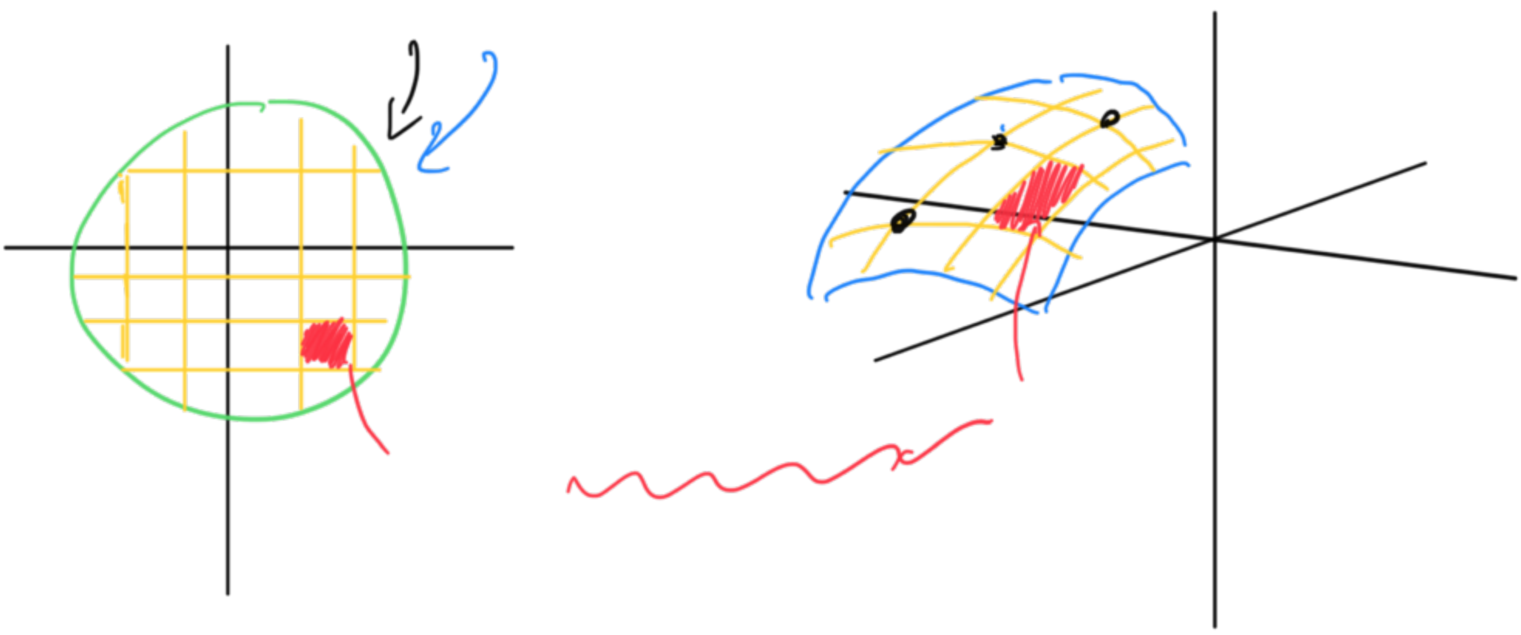
Instead of slicing up surface directly, slice up domain. This will also split up surface.

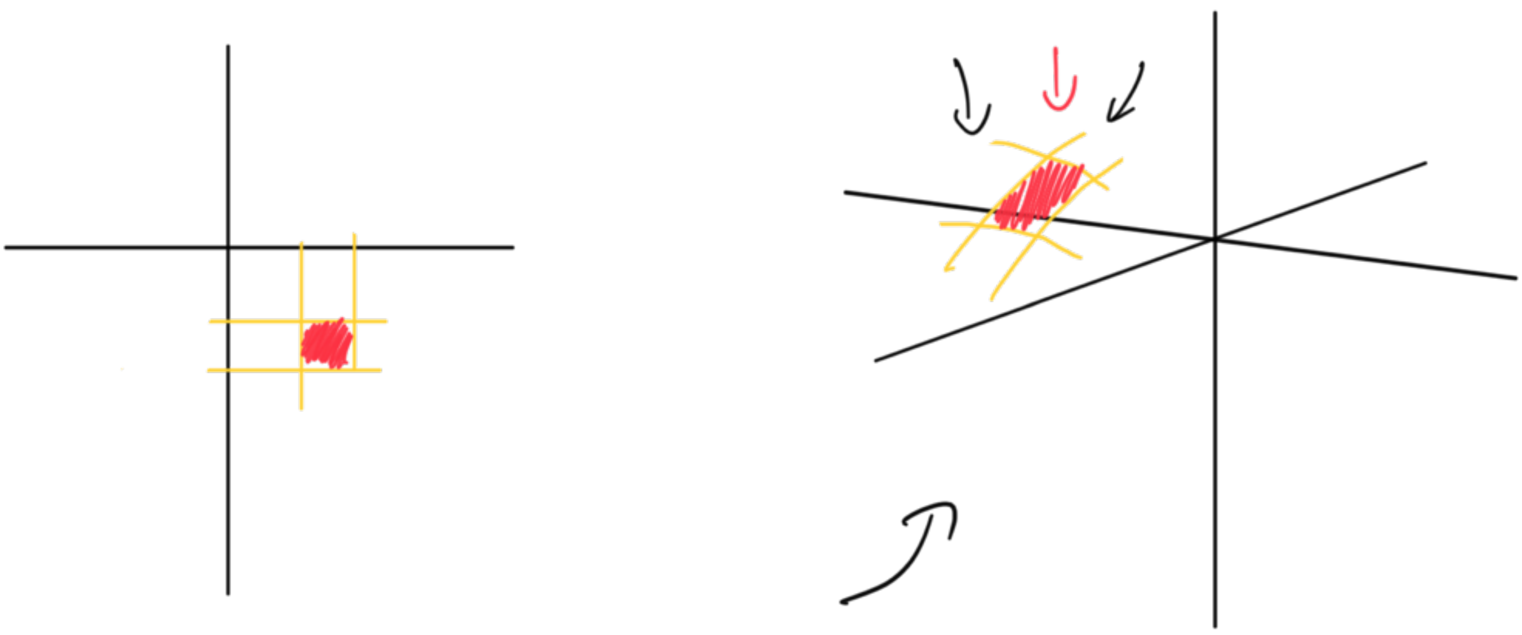


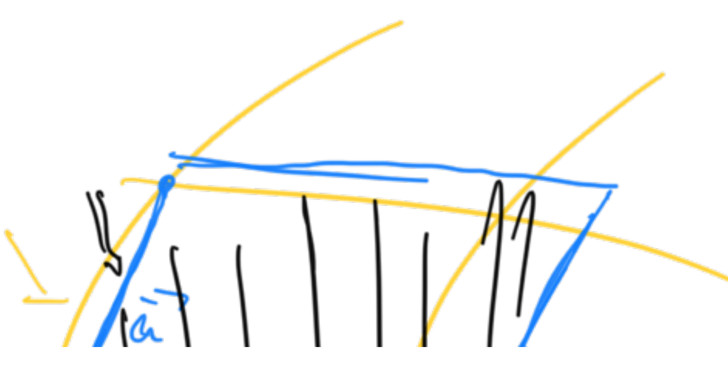




Lets pick one section and approximate its area





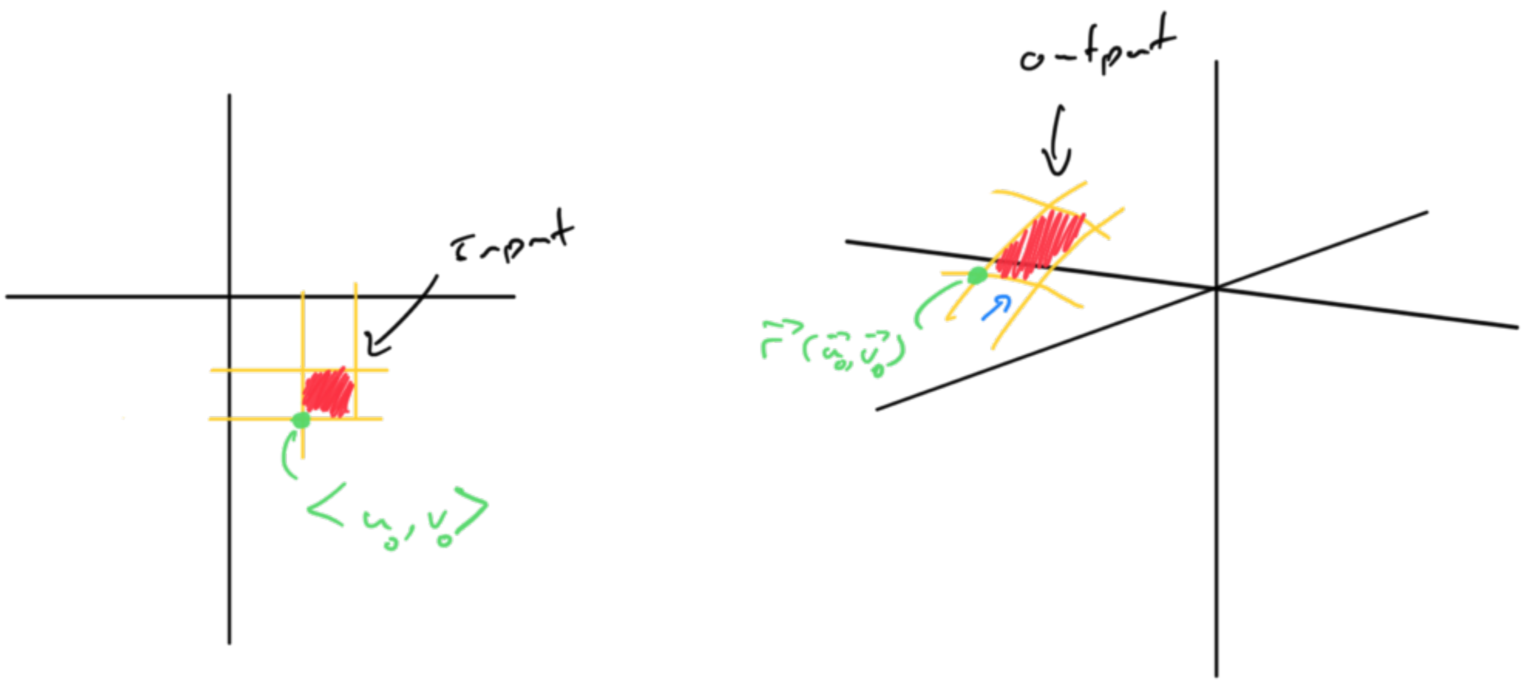


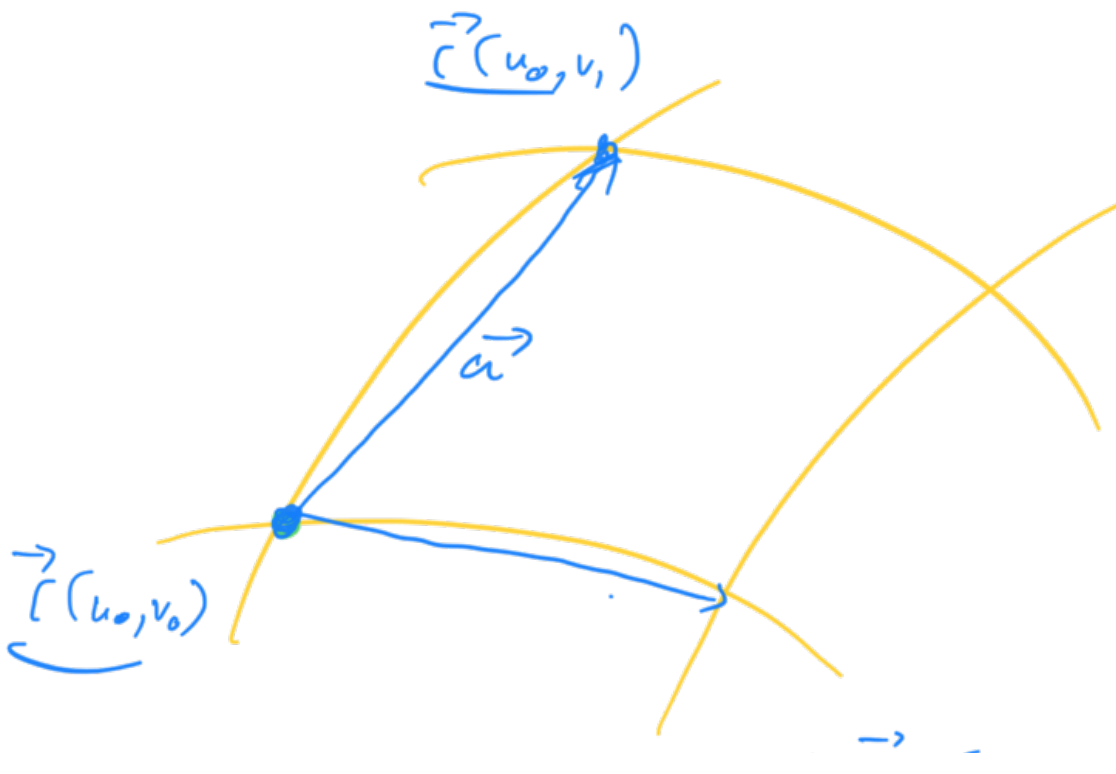
What should approximating shape be?

<u>Parallelogram.</u> (flat, easy to find area)

Remember, if $\vec{a}$ and $\vec{b}$ are two sides of parallelogram P then

$$\text{area}(P) = |\vec{a} \times \vec{b}|$$





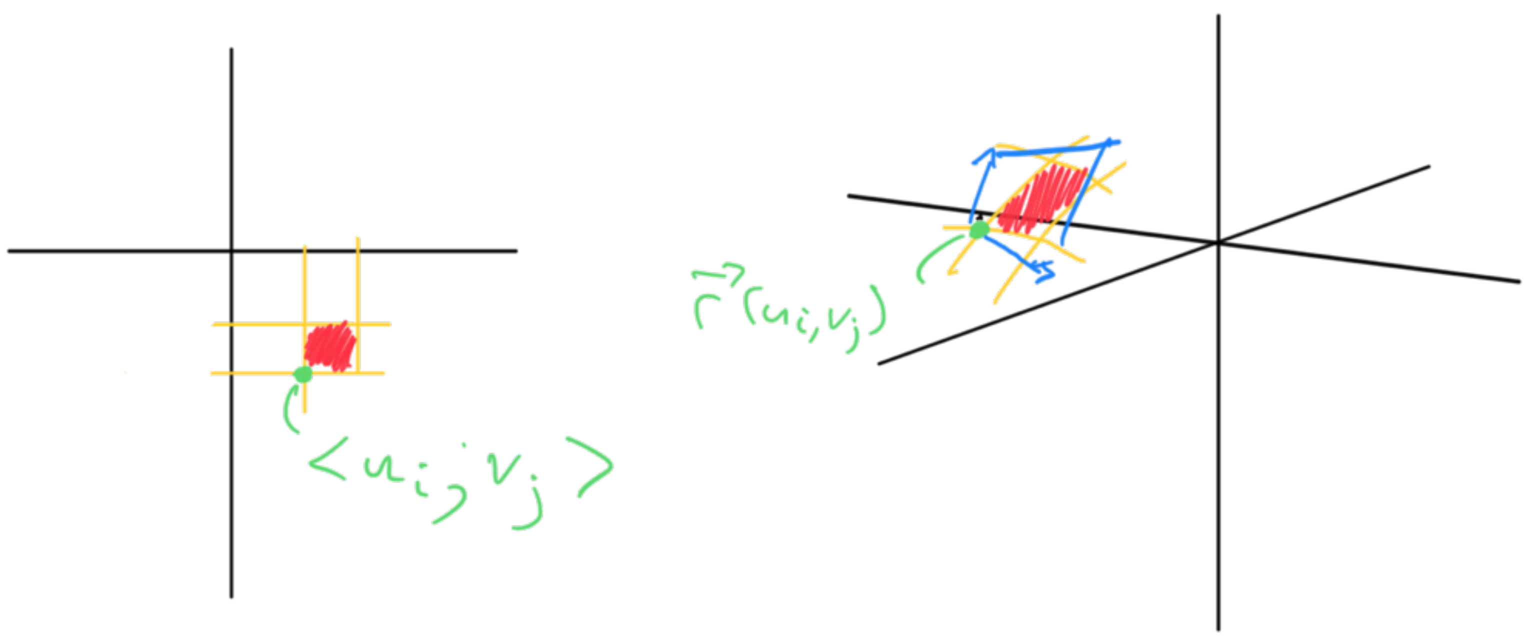
$$\begin{cases} \vec{r}_u(u_0, v_0)\Delta u \\ \vec{r}_v(u_0, v_0)\Delta v \end{cases}$$

Can we find two vectors based at $\vec{r}(u,v)$?

Yes!

$\vec{r}_u(u_i, v_j)$ and $\vec{r}_v(u_i, v_j)$ will point in right direction. To get right length, use

$$\vec{r}_u(u_i, v_j)\,\Delta u_i \qquad \vec{r}_v(u_i, v_j)\,\Delta v_j$$



So what is area of parallelogram?

$$\text{Area} = \left\| \vec{r}_u(u_i, v_j)\Delta u \times \vec{r}_v(u_i, v_j)\Delta v \right\|$$

$$= \left\| \vec{r}_u(u_i, v_j) \times \vec{r}_v(u_i, v_j) \right\| \Delta u \, \Delta v$$

Add up areas

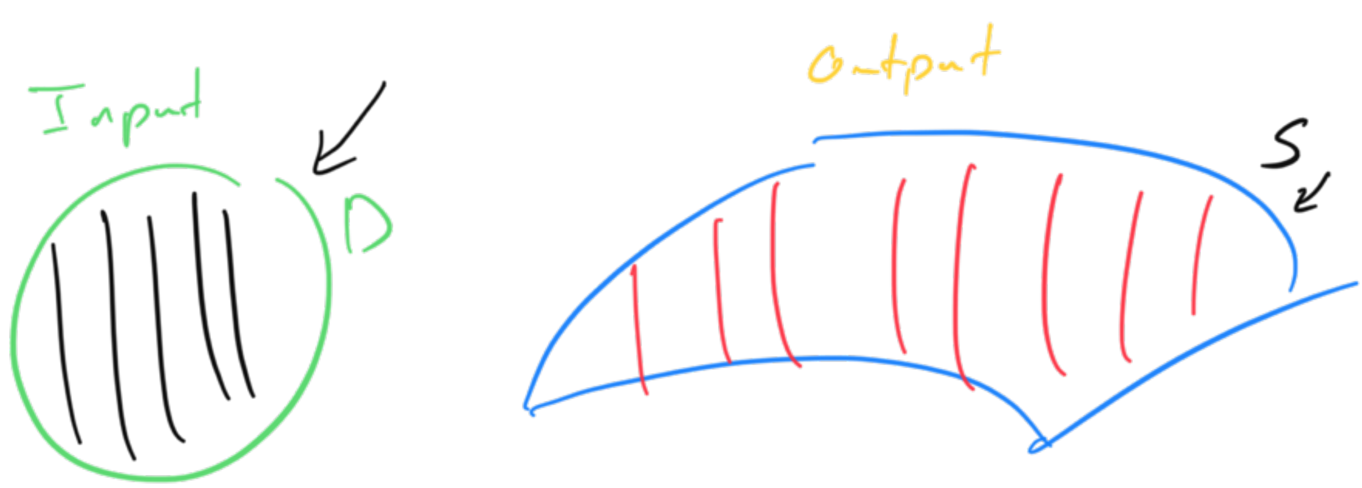
$$\sum_{i=1}^{m} \sum_{j=1}^{n} \left| \vec{r}_u(u_i, v_j) \times \vec{r}_v(u_i, v_j) \right| \Delta u \, \Delta v$$

Take limit

$$A(S) = \iint_D \left| \vec{r}_u(u,v) \times \vec{r}_v(u,v) \right| du\,dv$$

Surface Area

Domain of surface (in uv-plane)



$A(S)$

Ex: Find surface area of the part

of $z = x^2 + y^2$ <u>that is under plane</u> $z = 1$.

$$\overbrace{A(S)} = \underbrace{\iint_D \left| \vec{r}_u(u,v) \times \vec{r}_v(u,v) \right| \, du\,dv}$$

Step 1: Rewrite $z = f(x,y)$ as

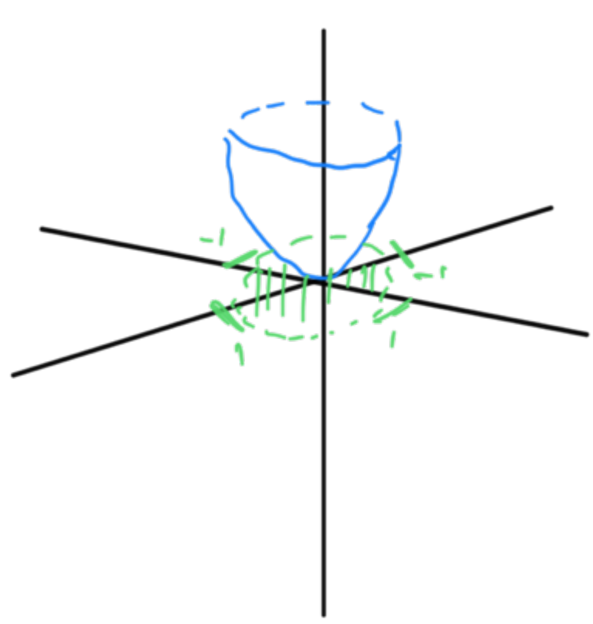
$\vec{r}(x,y) = \langle x, y, f(x,y) \rangle$

Find bounds for x, y

$$z = x^2 + y^2$$

$$\vec{r}(x,y) = \langle x, y, x^2 + y^2 \rangle$$

$$-1 \le x \le 1 \qquad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$



Step 2: Find $\vec{r}_u$, $\vec{r}_v$

$$\vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle$$

$$\vec{r}_x = \langle 1, 0, 2x \rangle$$

$$\vec{r}_y = \langle 0, 1, 2y \rangle$$

Step 3: Take cross product, find its magnitude

$$\vec{r}_x \times \vec{r}_y \qquad \begin{bmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{bmatrix}$$

$$= \langle -2x, -2y, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{4x^2+4y^2+1}$$

Step 4:

$$\star \quad \underbrace{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2+4y^2+1}\; dy}_{}\, dx$$

$$\int_0^1 \int_0^{2\pi} \sqrt{4r^2+1} \; r \, d\theta \, dr$$

$$2\pi \int_0^1 \sqrt{4r^2+1} \; r \, dr$$

$$u = 4r^2+1$$
$$du = 8r$$

$$2\pi \left[\int \frac{1}{8} u^{\frac{1}{2}} \, du \right]$$

$$2\pi \left(\frac{2}{24} u^{\frac{3}{2}} \Big] \right)$$

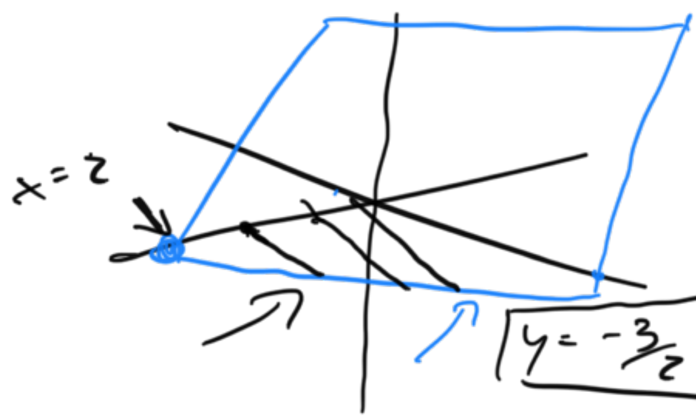
$$2\pi \left(\frac{2}{24} (4r^2+1)^{\frac{3}{2}} \Big]_0^1 \right)$$

$$\frac{2\pi}{12} \left((4r^2+1)^{\frac{3}{2}} \Big]_0^1 \right)$$

$$\boxed{\frac{\pi}{6} \left(5^{\frac{3}{2}} - 1 \right)}$$

#3

Find area of the part of plane $3x+2y+z=6$ that lies in the first octant



$$\boxed{y=-\tfrac{3}{2}x+3}$$

$$z = \underline{6-3x-2y}$$

Step 1: $\boxed{\vec{r}(x,y) = \langle x, y, 6-3x-2y \rangle}$

$$0 = 6-3x-2y$$

$$2y = 6-3x$$

$$y = 3-\tfrac{3}{2}x$$

or $y = -\frac{3}{2}x+3$

x int: $0 = -\frac{3}{2}x+3$

$$\tfrac{3}{2}x = 3$$

$$x = 2$$

Bounds $\begin{cases} 0 \le x \le 2 \\ 0 \le y \le -\frac{3}{2}x+3 \end{cases}$

Step 2:

$$\boxed{\vec{r}(x,y) = \langle x, y, 6-3x-2y \rangle}$$

$$\vec{r}_x = \langle 1, 0, -3 \rangle$$

$$\vec{r}_y = \langle 0, 1, -2 \rangle$$

Step 3:

$$\begin{bmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

$$r_x \times r_y = \langle +3, +2, 1 \rangle$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{9+4+1}$$

$$= \sqrt{14}$$

Step 4:

$$\int_0^2 \int_0^{-\frac{3}{2}x+3} \sqrt{14} \, dy \, dx$$

$$\sqrt{14} \int_0^2 \int_0^{-\frac{3}{2}x+3} dy \, dx$$

$$\sqrt{14}\int_0^2\left(-\frac{3}{2}x+3\right)dx$$

$$\sqrt{14}\left(-\frac{3}{4}x^2+3x\Big]_0^2\right)$$

$$\sqrt{14}\left(\left[-3+6\right]-0\right)$$

$$3\sqrt{14}$$