Section 1.3

Vectors

A vector is an ordered list of numbers
Saw last time a matrix is made up of row/column vectors

When we talk about vectors, vertical/column vectors are the default

$$\bigstar \quad \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$$

To save time/space often give vectors "names" and refer to them that way.

Ex:

$$\vec{v} = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} \qquad \vec{w} = \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix}$$

Scalar Multiplication

Can multiply a vector by a constant, which we will now call a <u>scalar</u>

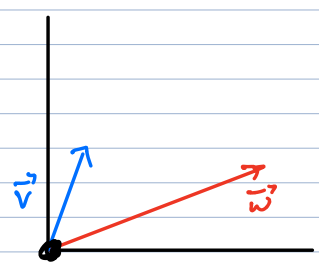
$$c\vec{a} \qquad c\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{vmatrix} ca_1 \\ ca_2 \\ ca_3 \end{vmatrix}$$

Vector Addition

To add 2 vectors simply add their components

$$\vec{v} + \vec{w} = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} + \begin{vmatrix} w_1 \\ w_2 \\ w_3 \end{vmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$$

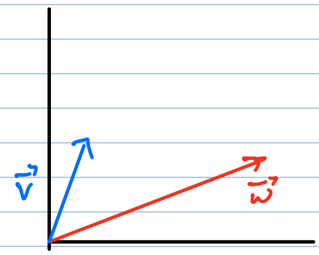
There is also a visual/geometric way to think of vector addition



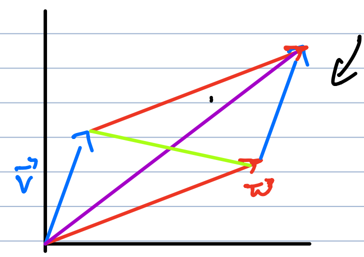
Could look at components of $\vec{v}$, $\vec{w}$ and add up like above.

But also have something called:

Parallelogram Rule



When we want to add/subtract $\vec{v}, \vec{w}$ put them base to base. Form a parallelogram from them



Long diagonal will be $\vec{v}+\vec{w}$
Short will be $\vec{v}-\vec{w}$.

Once we can do scalar multiplication and vector addition, expressions like

$$3\vec{u} - \frac{2}{3}\vec{v} \qquad \star$$

make sense.

The concept of vector is not restricted by dimension. We can have vectors of any arbitrary dimension

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

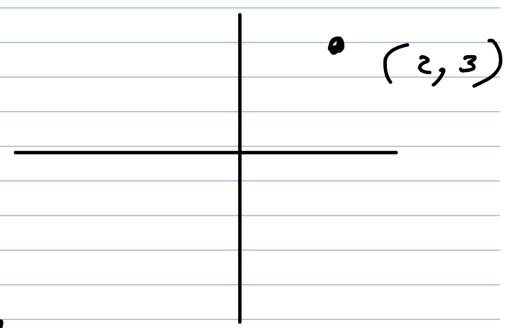
Can't "draw" parallelogram rule for higher dimensions but vector addition and scalar mult. work same in all dimensions.

Algebraic Properties of $\mathbb{R}^n$

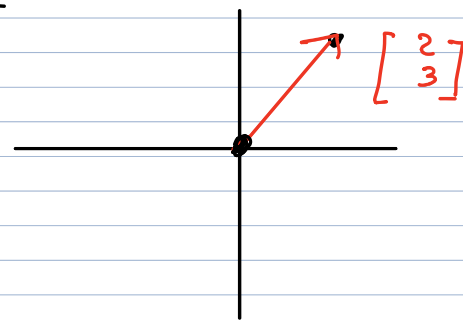
1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ Commutativity
2. $(\vec{u}+\vec{v}) + \vec{w} = \vec{u} + (\vec{v}+\vec{w})$ Associativity
3. $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ (where $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$)
4. $\vec{u} + (-\vec{u}) = -\vec{u} + \vec{u} = \vec{0}$
5. $c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$
6. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
7. $c(d\vec{u}) = (cd)\vec{u}$
8. $1\vec{u} = \vec{u}$

Representing points with vectors

Consider a point in $\mathbb{R}^2$. It is described by coordinates (x,y).



Could instead think of point as a vector with base at origin, endpoint at desired (x,y).



So can consider points as vectors.
This will allow us to view our "spaces" (like $\mathbb{R}^2$, $\mathbb{R}^3$) in terms of vectors and linear algebra.

Works in all dimensions.

Linear Combinations

Say we have a group of vectors $\vec{v}_1, \dots, \vec{v}_n$
Then $\vec{y}$ is a linear combination of $\vec{v}_1, \dots, \vec{v}_n$ if

$$\vec{y} = c_1 \vec{v} + \dots + c_n \vec{v}_n$$

where $c_1 \dots c_n$ are scalars (called weight or coefficients)

May help to picture components of vectors

$$\vec{y} = c_1 \vec{v}_1 + \dots + c_n \vec{v}_n$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = c_1 \begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix} + c_2 \begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix} + \dots + c_n \begin{bmatrix} \\ \vdots \\ \\ \end{bmatrix}$$

A common problem: Often given a list of vectors $\vec{v}_1 \dots \vec{v}_n$ and asked if $\vec{y}$ is <u>some</u> linear combination of the $\vec{v}_i$'s.

(Are there constants $c_1 \dots c_n$ such that

$$\vec{y} = c_1\vec{v}_1 + \dots c_n\vec{v}_n \quad)$$

This expression is our vector equation

$$\vec{y} = c_1\vec{v}_1 + \dots c_n\vec{v}_n$$

Just like with matrix/linear system, we are looking for values that make vector equation true.

We will see that we can transform vector equation into matrix, solve using row reduction.

E1 Is $\begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$ a lin. comb. of vectors

$\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$?

Looking for scalars c_1, c_2 such that

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

By algebraic properties, have:

$$\Rightarrow \begin{bmatrix} c_1 \\ -2c_1 \\ -5c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ 5c_2 \\ 6c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Can view this as the system of equations:

$$\begin{cases} c_1 + 2c_2 = 7 \\ -2c_1 + 5c_2 = 4 \\ -5c_1 - 6c_2 = -3 \end{cases}$$

Turn into augmented matrix from here.

Note:

$$c_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Also same as matrix equation

$$\begin{bmatrix} 1 & 2 \\ -2 & 5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

Either way can write down augmented matrix

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right]$$

Now row reduce to find possible solutions

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1 = 3$ $\qquad$ $c_2 = 2$

Theorem

The vector equation

$$x_1 \vec{a}_1 + \dots x_n \vec{a_n} = b$$

the matrix equation

$$\begin{bmatrix} | & & | \\ a_1 & \dots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

and the augmented matrix

$$\left[\begin{array}{ccc|c} | & & | & b_1 \\ \vec{a}_1 & \dots & \vec{a}_n & \vdots \\ | & & | & b_n \end{array}\right]$$

all have same solution.

What does this mean?

We can view a vector equation

$$c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{y}$$

as an augmented matrix

$$\left[\begin{array}{ccc|c} | & & | & \\ v_1 & \dots & v_n & \vec{y} \\ | & & | & \end{array}\right]$$

Span

For a group of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ often want to consider set of <u>all</u> linear combinations of the vectors, every possible combination of form

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

Call all these possible linear combinations the subset/subspace of $\mathbb{R}^n$ spanned by $\vec{v}_1, \ldots \vec{v}_n$ aka $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\}$

Ex: Consider vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$

- $3\vec{v}_1 + (-\pi)\vec{v}_2 + (100{,}000)\vec{v}_3 \in \text{Span}\{v_1, v_2, v_3\}$

$$\vec{v}_1 \in \text{Span}\{v_1, v_2, v_3\}$$

$$\vec{0} = 0\vec{v}_1 + 0\vec{v}_1 + 0\vec{v}_3 \in \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$$

As we have already seen, asking if $y \in \text{Span}\{\vec{v}_1 \ldots \vec{v}_k\}$ is same as asking if

$$\left[\begin{array}{ccc|c} v_1 & \cdots & v_k & \vec{y} \end{array}\right]$$ is consistent.

Geometric idea of span

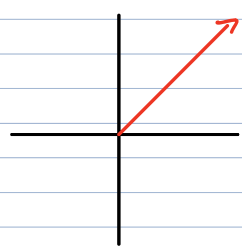
The span of set of vectors has very orderly structure inside the larger space $\mathbb{R}^n$.

Ex Working in $\mathbb{R}^2$. Consider vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

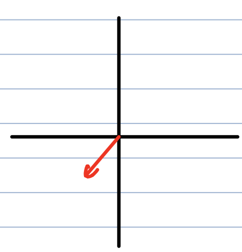
What is $\text{Span}\{\vec{v}\}$?

All vectors $\vec{y}$ of form $\vec{y} = c\,\vec{v}$

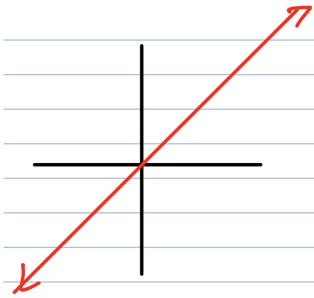
Just $\vec{v}$



$2\vec{v}$ fits definition so in $\text{Span}\{v\}$

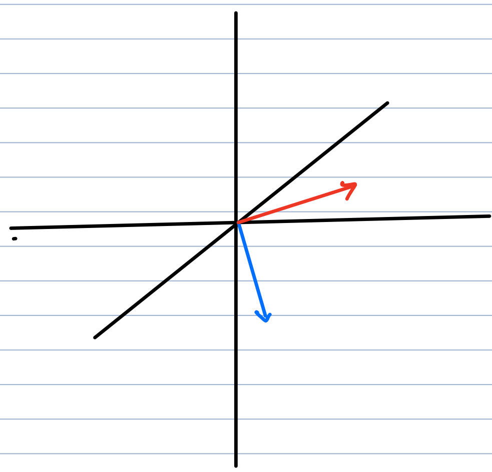


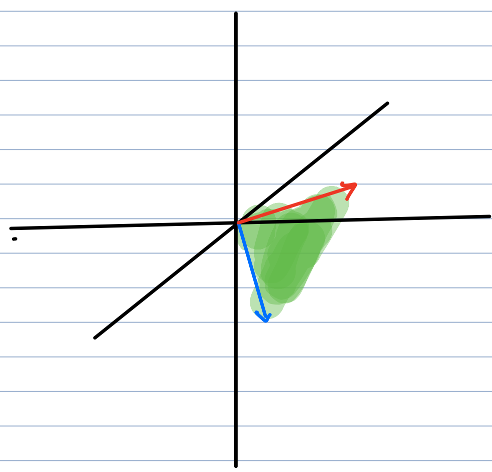
$-\frac{1}{2}\vec{v}$ fits definition so it is in $\text{Span}\{\vec{v}\}$



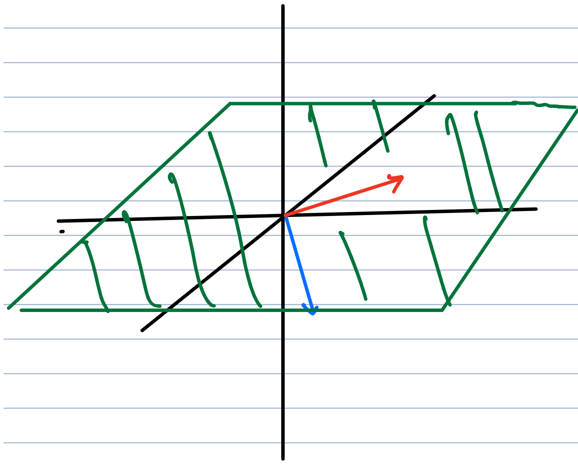
In fact, entire line passing through $\vec{v}$ is in Span$\{v\}$

What about Span$\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ in $\mathbb{R}^3$?





Think about region between the two vectors and then expand in all directions to see plane containing $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\0\end{bmatrix}$



In this case the Span is same as xy-plane

In general Span$\{\vec{v}_1, \dots \vec{v}_k\}$ is "smallest" line, plane, hyperplane, etc. in $\mathbb{R}^n$ that passes through origin and contains all vectors $\vec{v}_1, \dots, \vec{v}_k$.

Will revisit idea when we learn about "linear independence"