Section 1.8

Linear algebra may seem totally separate from math we have done before, but idea of linear transformations will remind us of the concept of a function.

Recall a function is essentially a way to take an input and assign it to a single output. Most commonly, $f: \mathbb{R} \to \mathbb{R}$

But in Calc 3 we also saw

$$f: \mathbb{R} \to \mathbb{R}^n \qquad \vec{r}(t) = \langle t, t^2, 3 - e^t \rangle$$

$$f: \mathbb{R}^m \to \mathbb{R} \qquad f(x,y) = x^2 y + y^3$$

$$f: \mathbb{R}^m \to \mathbb{R}^n \qquad F(x,y,z) = \langle x^2 y, z + y, xz \rangle$$

From this point of view consider $A\vec{x}$

$$\underbrace{\begin{bmatrix} & A & \end{bmatrix}}_{m \times n} \underbrace{\overset{\mathbb{R}^n}{\begin{bmatrix} \vec{x} \end{bmatrix}}}_{n \times 1} = \underbrace{\overset{\mathbb{R}^3}{\begin{bmatrix} \vec{y} \end{bmatrix}}}_{m \times 1}$$

So can view $T(\vec{x}) = A\vec{x}$ as a function (transformation) $T: \mathbb{R}^n \to \mathbb{R}^m$

In particular, can call it a matrix transformation.

Like with any transformation, for $T(x) = A\vec{x}$ where A is $m \times n$ matrix, we have:

Domain: Set of all inputs. No restrictions on domain, can plug in any vector. So domain = $\mathbb{R}^n$

Codomain: Not the outputs, but the overall space the outputs "live in". All outputs in $\mathbb{R}^m$, so codomain is $\mathbb{R}^m$

Image (of $\vec{x}$): for a particular $\vec{x}$, image of $\vec{x}$ is the output associated with that $\vec{x}$

Range: The set of all outputs (images). May be all of $\mathbb{R}^m$ or smaller subset of $\mathbb{R}^m$. Range is all possible outputs $A\vec{x}$. But from vector viewpoint $A\vec{x} = x_1 \vec{a}_1 + \dots x_n \vec{a}_n$, a linear combination of columns of A

So range is all linear combinations of columns of A (span of columns of A)

Ex 1 In book

Square matrices are of special interest. If A is $n \times n$ matrix, Domain + Codomain are $\mathbb{R}^n$

Can name these matrices based on their effect on $\mathbb{R}^n$

"Linear" transformations can be thought of as affecting the space

Ex 2 , Ex 3

Def: $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if it satisfies following conditions for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$:

① $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

② $T(c\vec{u}) = cT(\vec{u})$

Alternate def:

...

if for all $\vec{u}, \vec{v} \in \mathbb{R}^n$ and all $c, d \in \mathbb{R}$

① $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

Note: We already know for matrices that

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$$

$$A(c\vec{u}) = cA\vec{u}$$

So every matrix transformation is a linear transformation

A more interesting question: Is every linear transformation able to be represented by a matrix transformation? Next section

Finite vs Infinite Dimension

$T: \mathbb{R}^n \to \mathbb{R}^m$

then there is always matrix A such that $T(\vec{x}) = A\vec{x}$