13.1 - Vector Functions

In calc 1, 2 saw functions of form

$$f: \mathbb{R} \to \mathbb{R}$$
$$f(x) = y$$

Single number input, single number output

But could also see:

① $r: \mathbb{R} \to \mathbb{R}^n$ scalar input, vector output

② $s: \mathbb{R}^m \to \mathbb{R}$ vector input, scalar output

③ $t: \mathbb{R}^m \to \mathbb{R}^n$ vector input, vector output

etc.

In chapter 13 focus on easier of the above 3 cases $r: \mathbb{R} \to \mathbb{R}^n$

Since output of r is a vector call it a vector valued function

$$r: \mathbb{R} \to \mathbb{R}^n$$

Notation:

$\vec{r}(t)$ vector output

The dimension of vector output is usually specified in definition of function or is clear from context

$$\vec{r} : \mathbb{R} \to \mathbb{R}^n$$

"n^{th}" dimension

As usual for this course, mainly care about $\mathbb{R}^3$ case $\vec{r} : \mathbb{R} \to \mathbb{R}^3$

$$\vec{r}(t) = \langle r_1, r_2, r_3 \rangle$$

Since r_1, r_2, r_3 depend on t somehow can write:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where $f, g, h : \mathbb{R} \to \mathbb{R}$. They just give scalars, the components of vector $\vec{r}(t)$

So call f, g, h the component functions of $\vec{r}(t)$

Ex:

$$\vec{r}(t) = \langle 0, t, 3-t \rangle$$

② $\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$

③ $\vec{r}(t) = \langle 4t^2 - 7t, e^{t^2}, \ln|t^2+4| \rangle$

etc.

Limits and Continuity — Just like $f: \mathbb{R} \to \mathbb{R}$ we are concerned with limits / continuity for vector valued functions. Luckily, very simple for $\vec{r}: \mathbb{R} \to \mathbb{R}^3$.

Limits

If $\underline{\vec{r}(t)} = \langle f(t), g(t), h(t) \rangle$ then

$$\lim_{t \to t_0} \vec{r}(t) = \langle \underline{\lim_{t \to t_0} f(t)}, \underline{\lim_{t \to t_0} g(t)}, \underline{\lim_{t \to t_0} h(t)} \rangle$$

Can take limit <u>inside</u> vector

☆ Ex: $\underline{\vec{r}(t)} = \langle \underline{e^{3t} + 7t^2}, t, \frac{\sin(3-t)}{3-t} \rangle$

$$\lim_{t \to 3} \vec{r}(t) =$$

$$= \langle e^9 + 63, 3, 1 \rangle$$

Continuity

Recall continuity for $f : \mathbb{R} \to \mathbb{R}$

f is continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Say f is continuous if it is continuous for all t in its domain

Just like with limits, for $\vec{r} : \mathbb{R} \to \mathbb{R}^n$ can just look at components to determine continuity

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$\vec{r}(t)$ continuous if and only if $f(t)$ $g(t)$ and $h(t)$ continuous

Previous example:

Ex: $\vec{r}(t) = \left\langle e^{3t} + 7t^2,\ t,\ \frac{\sin(3-t)}{3-t} \right\rangle$

<u>Not</u> continuous at $t=3$

Not continuous

Ex: $\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$

$-\infty < t < \infty$ ✓ ✓ ✓

$r(t)$ continuous everywhere

(More complicated for $f: \mathbb{R}^m \to \mathbb{R}^n$)

$f: \mathbb{R} \to \mathbb{R}$

Graphs

What do graphs of vector valued

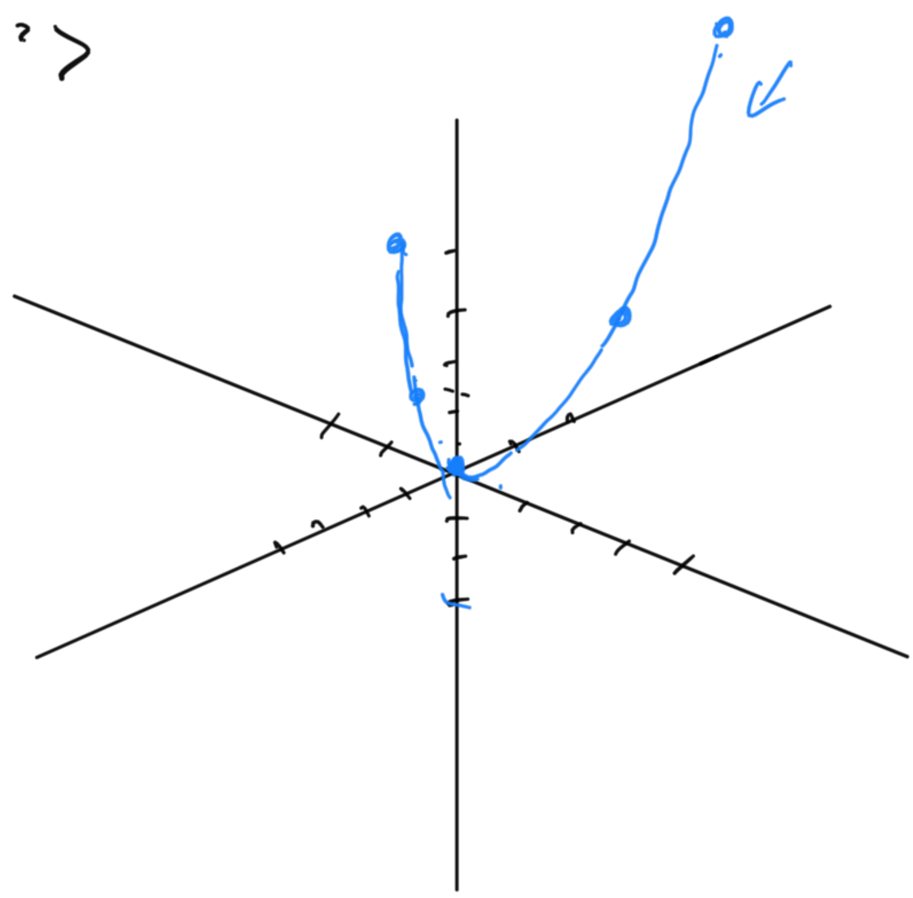
functions look like?

$$\vec{r}(t) = \langle \overset{x}{t}, \overset{y}{t}, \overset{z}{t^2} \rangle \quad ☆$$

Output is in $\mathbb{R}^3$. Lets just graph output. These are essentially parametric equations so use these methods

$$\vec{r}(t) = \langle t, t, t^2 \rangle$$

t	x	y	z
-2	-2	-2	4
-1	-1	-1	1
0	0	0	0
1	1	1	1
2	2	2	4
3	3	3	9



See this is a curve, book calls it a space curve.

May be confused at this point. In 12.5, 12.6 have seen curves and surfaces

Why do some equations give curves and others surfaces?

Curves

$\vec{r} = \vec{r}_0 + t\vec{v}$

$\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$

Surfaces

$x^2 + y^2 + z^2 = 4$

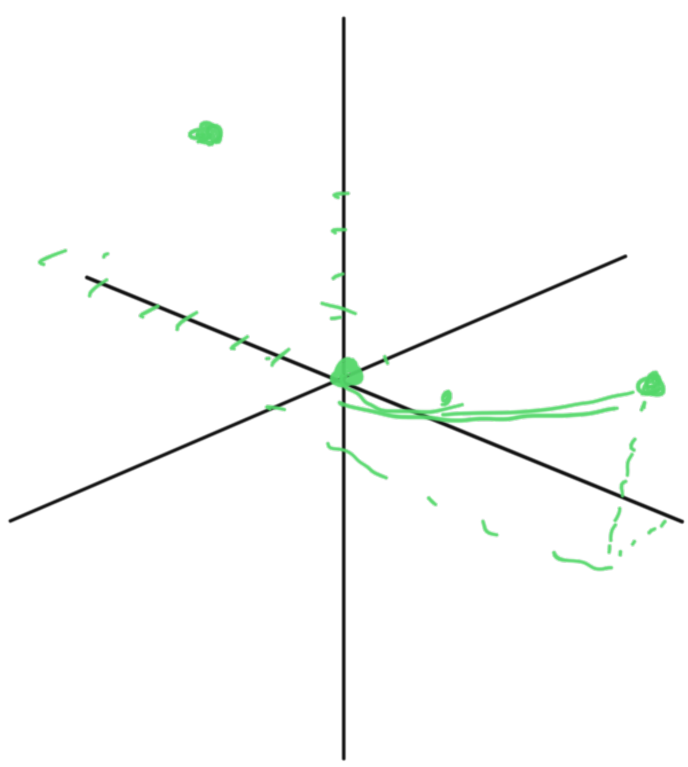
☆ Number of variables a guideline ☆
inputs

Drawing Graphs

Making tables

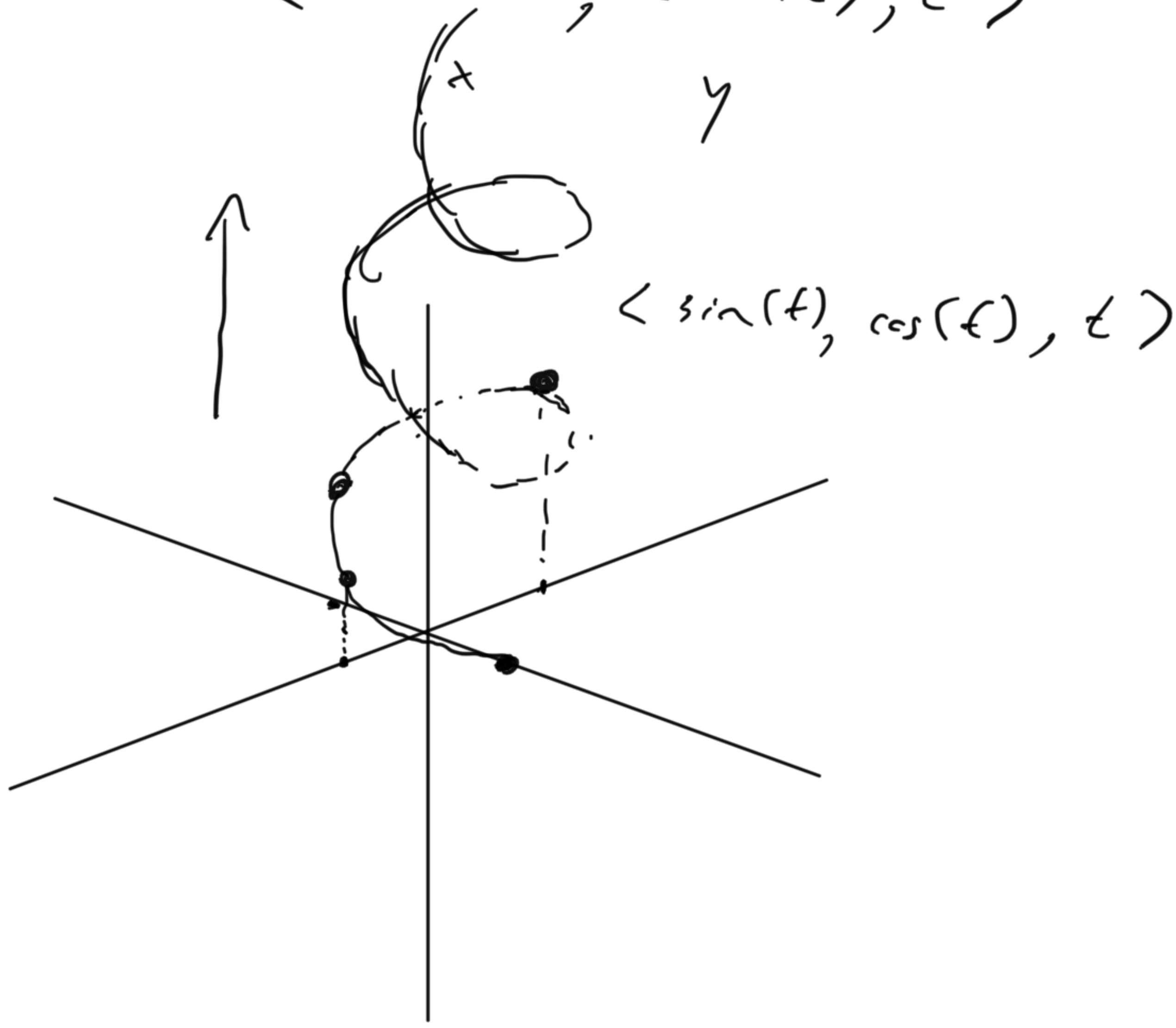
$\vec{r}(t) = \langle t, 3t, t^2 \rangle$

t	x	y	z
-2	-2	-6	4
-1	-1	-3	1
0	0	0	0
1	1	3	1
2	2	6	4



2-D, then 3-D

$\vec{r}(t) = \langle \sin(t), \cos(t), t \rangle$



Finding Equations

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$z = 5x^2 + y^2 \qquad y = 2x^2$

Find $\vec{r}(t)$ that represents the intersection of these two surfaces.

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$z = 5x^2 + y^2$ $y = 2x^2$

$x = t$

$y = 2t^2$

$z = 5(t)^2 + (2t^2)^2$

$z = 5t^2 + 4t^4$

45 $z = x^2 - y^2$ $\underline{x^2 + y^2 = 1}$

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\boxed{\begin{array}{lll} x = & \cos(t) & \sin(t) \\ y = & \sin(t) & \cos(t) \\ z = & \cos^2(t) - \sin^2(t) & \sin^2 - \cos^2 \end{array}}$$

$\vec{r}(t) = \langle \cos(t), \sin(t), \cos^2(t) - \sin^2(t) \rangle$

$$\cos^2 t - \sin^2 t$$

$$= \cos(2t)$$

$$\langle \cos(t), \sin(t), \cos(2t) \rangle$$

$$\cos(t)\,\vec{i} + \sin(t)\,\vec{j} + \cos(2t)\,\vec{k}$$