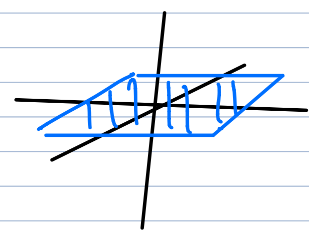
Section 4.3

This section is about how to represent vector spaces and subspaces in the most efficient way possible

Ex: $\underline{\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$ $\quad$ $\text{span}\underline{\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$

Spans of both these give same subspace of $\mathbb{R}^3$



$H = \text{span}$

This idea of redundancy may be familiar from linear independence, which is tool we need to discuss this.

Previously defined linear independence/dependence for vectors in $\mathbb{R}^n$

$\begin{bmatrix} \vec{v}_1 \end{bmatrix}, \begin{bmatrix} \vec{v}_2 \end{bmatrix}, \dots, \begin{bmatrix} \vec{v}_k \end{bmatrix}$ lin. dep if $\exists$ $c_1, \dots, c_n$ not all 0 s.t.

$$c_1\begin{bmatrix} \vec{v}_1 \end{bmatrix} + \dots c_k\begin{bmatrix} \vec{v}_k \end{bmatrix} = \vec{0}$$

Same definition for vectors in general vector spaces except may not have convenient way to represent vectors so just need to keep it general

$$c_1 \underline{\vec{v_1}} + \cdots c_n \underline{\vec{v_k}} = \vec{0}$$

Recall informal idea that linear dependent has some redundancy, linear independent more efficient

Encapsulated in idea of "basis"

Definition: Let H be subspace of V. A set of vectors B in V is a **basis** for H if

1. B is linearly independent set
2. H is spanned by B, i.e.

$$H = \text{Span } B$$

So a basis spans H in an efficient manner

Note that the H in the definition can be V itself. Meaning we can discuss basis for V or any other subspace H of V

Ex: Let A be $n \times n$ invertible matrix. Then columns of A are basis for $\mathbb{R}^n$.

1. Recall from 2.3 that invertible $\Rightarrow$ columns A are lin ind.
2. Invertible $\Rightarrow$ onto. So every $\vec{b} \in \mathbb{R}^n$ is lin comb of columns A. So columns span $\mathbb{R}^n$

Some bases are simpler than others

Standard Basis: Let $V = \mathbb{R}^n$. The vectors

☆ $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \dots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$ are <u>standard basis</u> of $\mathbb{R}^n$

- Any vector in $\mathbb{R}^n$ can be written as lin. comb of these: i.e. they span $\mathbb{R}^n$
- Can easily check lin ind.

If we start with a spanning set (which may be lin. dependent) we can whittle set down to make it lin. ind, keep spanning, and so end up with a basis.

Spanning Set Theorem: Let $S = \{\vec{v}_1 \dots \vec{v}_p\}$ be a set in vector space V and $H = \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$

① If one of vectors $\vec{v}_k$ in S is linear comb of others then can remove $\vec{v}_k$ and set still spans H

② If $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

☆ Example 7, pg 224 ☆

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Bases for Nul A, Col A:

When solve $A\vec{x} = \vec{0}$ get

$$\vec{x}_1 \begin{vmatrix} \vec{v}_1 \end{vmatrix} + \dots x_k \begin{vmatrix} \vec{v}_k \end{vmatrix}$$ Soln. fren set

Know $\{\vec{v}_1, \dots \vec{v}_k\}$ span Nul A

Similarly know for $m \times n$ matrix A, columns $\vec{a}_1, \dots \vec{a}_n$ span Col A

Question is, how to make these linearly independent?

For any list of vectors there is simple process to eliminate vectors to make linearly independent while maintaining the span

Given $\{\vec{v}_1, \dots, \vec{v}_k\}$ (or matrix made up of these columns)
→ ① Row reduce to echelon form
② Pivot columns will be linearly ind. and span same space.

Very direct process to find basis of Col A

Theorem: Pivot columns of matrix A form basis for Col A.

☆ **Warning!** Must use pivot columns of $\underline{A}$, not pivot columns of its echelon form.

For **Nul A**, first find spanning set by writing solution to $A\vec{x}=\vec{0}$ in parametric vector form. Collect these vectors into matrix, then determine pivot columns, similar to method for Col A. These will be basis.

Ex: Find basis for Col A, Nul A

$$A = \begin{vmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{vmatrix}$$

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2 views of basis:

- Largest possible linearly independent set we can make
- ☆ Smallest possible spanning set we can make