Section 1.2

More on row reduction

Previously, did forward phase and used elementary row operations to make our matrix upper triangular.

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 2 & 7 & -2 & 2 \\ 0 & 3 & 6 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & -1 & -4 & 2 \\ 0 & 3 & 6 & 1 \end{array}\right] =$$

$$\searrow \quad \left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & -1 & -4 & 2 \\ 0 & 0 & -6 & 7 \end{array}\right]$$

We have put our matrix in to ...

Echelon Form

① Any nonzero rows are above all-zero rows
② Each leading entry in a row is in column to right of leading entry of the row above
③ All entries in a column below a leading entry are zero.

After putting our matrix in echelon form we then interpreted" matrix by thinking of our system of equations.
But we can use same row operations to do same thing

Now "backward phase" and eliminate all entries above first term in bottom row

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & -1 & -4 & 2 \\ 0 & 0 & -6 & 7 \end{array}\right] \Leftarrow$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & -1 & -4 & 2 \\ 0 & 0 & -6 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & -1 & 0 & -16/6 \\ 0 & 0 & \underline{-6} & \underline{7} \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 0 & 7/6 \\ 0 & -1 & 0 & -16/6 \\ 0 & 0 & -6 & 7 \end{array}\right]$$

Now repeat same process with next row up

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -57/6 \\ 0 & -1 & 0 & -16/6 \\ 0 & 0 & \underline{-6} & 7 \end{array}\right]$$

If we make leading entries 1's, solution very easy to see

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -57/6 \\ 0 & 1 & 0 & 16/6 \\ 0 & 0 & 1 & -7/6 \end{array}\right]$$

Think of this in terms of list of equations

$$1x_1 = -57/6$$

$$1x_2 = 5/6$$

$$1x_3 = -7/6$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -57/6 \\ 0 & 1 & 0 & 6/6 \\ 0 & 0 & 1 & -7/6 \end{array}\right]$$

This form of a matrix is called

Reduced Echelon Form

(or Reduced Row Echelon Form, RREF, etc)

Definition of Reduced Echelon Form is same as echelon form:

1. Any nonzero rows are above all-zero rows ✓
2. Each leading entry in a row is in column to right of leading entry of the row above ✓
3. All entries in a column below a leading entry are zero.

with added:

4. Leading entry in each nonzero row is +1
5. Each leading entry 1 is only nonzero entry in that column

Different Order = Different Matrix?

$$\underbrace{\left[\begin{array}{ccc|c} 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 3 \end{array}\right]} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 1 & 3 & 2 & 3 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ \underbrace{0 \quad 0 \quad -1} & & & 1 \end{array}\right] \Leftarrow$$

Echelon form

$-x_3 = 1$

$x_3 = -1$

~~Or~~

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] \quad x_3 = -1$$

Both forms give same values of x_1, x_2, x_3

OTOH

Each matrix is equivalent to only <u>one</u> unique reduced echelon form

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Pivots

Echelon forms may be different but the leading entries in each row/column all have same positions

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 3 & 2 & 3 \\ 0 & -2 & 7 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

■ any nonzero value

★ any value

$$\left[\begin{array}{ccc|c} \blacksquare & \star & \star & \star \\ 0 & \blacksquare & \star & \star \\ 0 & 0 & \blacksquare & \star \end{array}\right]$$

Leading entries are called **pivots**

Columns with a pivot are **pivot columns**

Rows with a pivot are **pivot rows**

Ex:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \checkmark$$

pivot column   pivot column

pivot rows

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We see where pivots are when we find echelon form of matrix.

Even though echelon forms may differ, pivots always in <u>same place.</u>

Say we have matrix A

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & y_1 \\ b_1 & b_2 & b_3 & y_2 \\ c_1 & c_2 & c_3 & y_3 \end{array}\right] \quad A$$

Equivalent to echelon form

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 \end{array}\right]$$

These are pivot columns. All echelon forms have same pivot columns. Even when considering original matrix we say these are pivot columns

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & y_1 \\ b_1 & b_2 & b_3 & y_2 \\ c_1 & c_2 & c_3 & y_3 \end{array}\right]$$

More on interpreting solutions

We've said linear system may have

$$\begin{cases} \bullet \ 1 \text{ (unique) solution } \star \\ \bullet \ \text{Infinite solutions } \star \\ \bullet \ \text{No solutions } \star \end{cases}$$

But have only seen unique solution.

Unique Solutions

This is when every variable x_i can be solved for a value

Ex

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{array}\right]$$

$\star$ Will explore more later, but system has one solution when every column of coefficient matrix is pivot column

Infinite Solutions

Will have a solution set

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix places no restrictions on x_3

$$1x_1 + 1x_3 = 3$$

So x_3 could be <u>anything</u>! Call it a

free variable

Continuing, $x_2 = 2$ (no problem here)

$$x_1 + x_3 = 1 \quad \text{i.e} \quad x_1 = 1 - x_3$$

So solution is $\begin{cases} x_1 = 1 - x_3 \\ x_2 = 2 \\ x_3 = \text{free} \end{cases}$

How I prefer to write solution set:

Write constants in one vector $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$ Free variables in their own vectors

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Can factor out free variable

Ex

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x_1 = 1 - 1x_3 \\ x_2 = 3 - 2x_4 \\ x_3 = \text{free} \\ x_4 = \text{free} \\ x_5 = 4 \end{cases}$$

x_3, x_4 are free. We can tell because there are no pivots in 3^{rd}, 4^{th} columns

$$\begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2x_4 \\ 0 \\ x_4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Will have infinite numbers of solutions when # pivots is less than # of columns (and pivots only in coefficient matrix)

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 6 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

No Solutions

☆ Pivot in last column of augmented matrix

When there is no solution, row reduction will lead to a row that doesn't make sense

Ex:

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 1 & 2 & 5 & 6 \\ 2 & 0 & 8 & 7 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

How can we interpret last row?

$$0x_1 + 0x_2 + 0x_3 = 1$$

Not possible!

There are no values of x_1, x_2, x_3 that can make sense of last row of matrix. This means there are no values of x_1, x_2, x_3 that are a solution to our system.

No Solution

<u>A rule:</u> If an augmented matrix has a pivot in last column, then the system is inconsistent.

This rule is restated more formally in the Existence and Uniqueness Theorem

A note about ideas of Existence and Uniqueness:

These are fundamental questions in math.

Does a solution to the problem exist?. (Existence)
Is this the only way to solve the problem? (Uniqueness)

There are many existence/uniqueness theorems scattered throughout different fields of math

Existence/Uniqueness Theorem

A linear system is consistent <u>if and only if</u> the rightmost column of augmented matrix is <u>not</u> a pivot column.

If a linear system <u>is</u> consistent then solution set contains either

1. unique solution with no free variables
2. infinite solution with at least one free variable