Section 1.9

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists unique matrix A such that $T(\vec{x}) = A\vec{x}$ for all $x \in \mathbb{R}^n$.

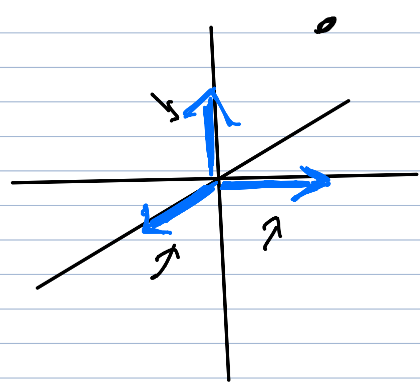
In fact, A is $m \times n$ matrix whose j^{th} column is vector $T(\vec{e}_j)$ where e_j is j^{th} column of identity matrix $I_{n \times n}$.

$$\underline{A} = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{bmatrix}$$

A is standard matrix for linear transformation T

Thoughts: The matrix is completely determined by what T does to identity matrix. Why is $I_{n \times n}$ special?

Think of $\mathbb{R}^3$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ☆



Columns of I are these vectors, the building blocks of $\mathbb{R}^3$

So, matrix really determined by how T acts on these "building blocks"

Ex 1 from book shows why this works

Ex 1 $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

T is the linear transformation such that $T(\vec{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ $T(\vec{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. Find formula for $T(\vec{x})$ for general $\vec{x}$.

☐ Lets say $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}$. Could write $\vec{x}$ in terms of $\vec{e}_1$ and $\vec{e}_2$ (the "building blocks of $\mathbb{R}^2$")

$$\vec{x} = \underline{a\,\vec{e}_1} + \underline{b\,\vec{e}_2}$$

By definition of a linear transformation we know that

$$T(\vec{x}) = T(a\vec{e}_1 + b\vec{e}_2)$$
$$= a\,T(\vec{e}_1) + b\,T(\vec{e}_2)$$
$$= a\begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + b\begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$$

And since a vector equation can be written as a matrix equation, we have

$$= \begin{array}{c} \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \\ \uparrow \quad\quad \uparrow \\ T(\vec{e}_1) \quad T(\vec{e}_2) \end{array} \begin{bmatrix} a \\ b \end{bmatrix}$$

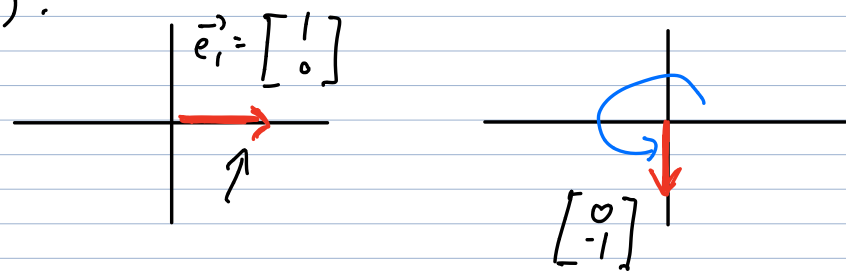
This confirms what theorem told us. Matrix of linear transformation is $[\,\underline{T(e_1)}\ \underline{T(e_2)} \ldots \underline{T(e_n)}\,]$

#3 Find standard matrix for $T: \mathbb{R}^2 \to \mathbb{R}^2$ where T rotates all points $3\pi/2$ radians counterclockwise.

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots \\ | & | & \end{bmatrix}$$

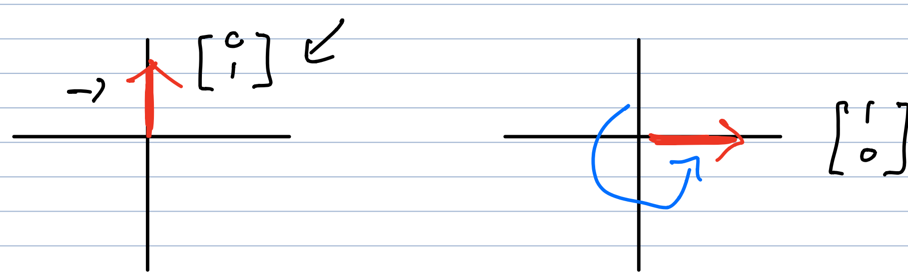
☐ Want to find $T(\vec{e}_1)$ and $T(\vec{e}_2)$.

$T(\vec{e}_1)$:



So $T(\vec{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 0 & ? \\ -1 & ? \end{bmatrix}$

$T(e_2)$:



So $T(e_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

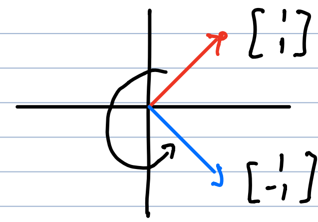
Then the standard matrix should be

$$A = \begin{array}{c} \begin{array}{cc} T(e_1) & T(e_2) \end{array} \\ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \end{array}$$

Lets test it. How about $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \checkmark$$

Does that match idea of rotating $3\pi/2$?

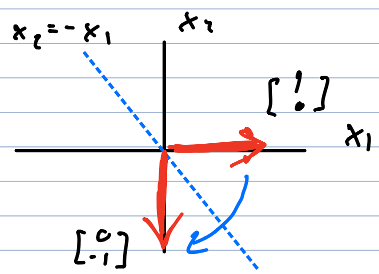


✓

#9 $T: \mathbb{R}^2 \to \mathbb{R}^2$ that first performs shear transformation $\vec{e_2} \to \vec{e_2} - 3\vec{e_1}$ and $\vec{e_1}$ unchanged, <u>then</u> reflects all points across line $x_2 = -x_1$

$$A = \begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix}$$

1) $T(\vec{e_1})$:

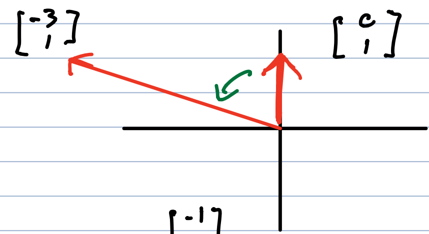


e_1 unchanged by shear transformation but then goes to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ when reflected across line

$$T(\vec{e_1}) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

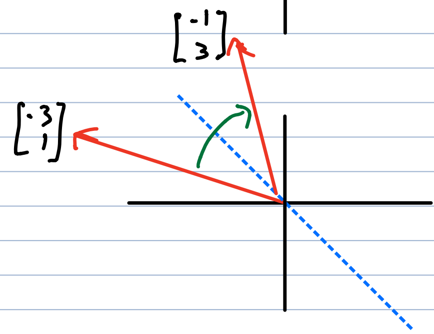
$$A = \begin{bmatrix} 0 & ? \\ -1 & ? \end{bmatrix}$$

$T(\vec{e_2})$:



First $\vec{e_2} \to \underline{\vec{e_2} - 3\vec{e_1}}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} -3 \\ 1 \end{bmatrix}$



Then $\vec{e_2} - 3\vec{e_1}$ reflected around line

$$\text{So} \quad \vec{e_2} \to e_2 - 3e_1 \to 3e_2 - \vec{e_1}$$

$$T(\vec{e_2}) = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \checkmark$$

So standard matrix is:

$$A = \begin{array}{c} \begin{array}{cc} T(\underline{e_1}) & T(\underline{e_2}) \end{array} \\ \begin{bmatrix} 0 & -1 \\ -1 & 3 \end{bmatrix} \end{array}$$

More on Existence/Uniqueness

Will define some properties of transformations that will give us insight into existence and uniqueness transformation, function

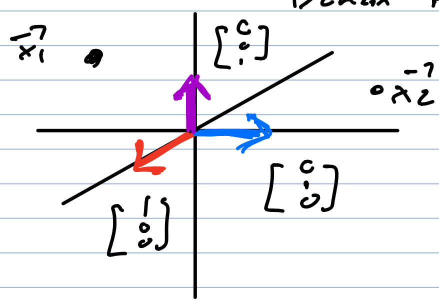
Def: A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is image of some $\vec{x} \in \mathbb{R}^n$

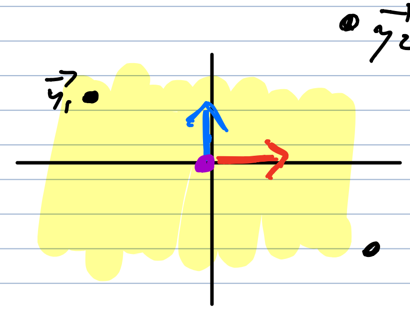
a.k.a. <u>surjective</u>

Ex: $T(x) = \underset{(2\times 3)}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}} \underset{(3\times 1)}{\begin{bmatrix} \vec{x} \end{bmatrix}}$

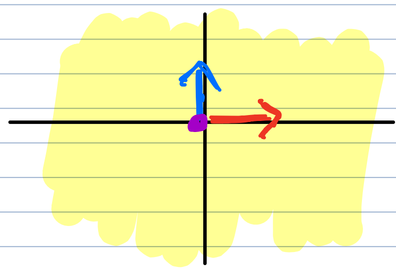
$T: \mathbb{R}^3 \to \mathbb{R}^2$

Domain is $\mathbb{R}^3$





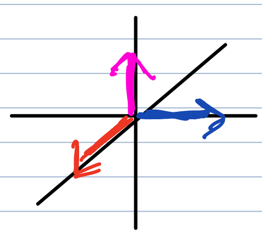
The range of T covers all of $\mathbb{R}^2$

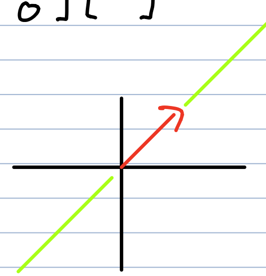


So T is <u>onto</u> $\mathbb{R}^2$

or "T is onto"

Ex $T(x) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{x} \end{bmatrix}$





Range of T is just the line, not all of $\mathbb{R}^n$

<u>Not</u> onto.

Significance: If $T(\vec{x}) = A\vec{x}$ is <u>onto</u> then <u>$A\vec{x} = \vec{b}$</u> always has a solution!

☆ m pivots (in coefficient matrix) ☆

$m = \#$ rows $\left\{ \begin{bmatrix} \blacksquare & & & \bigg| & \\ & \blacksquare & & \bigg| & \\ & & \blacksquare & \bigg| & \end{bmatrix} \right.$ always consistent

Def: A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each $\vec{b} \in \mathbb{R}^m$ is image of <u>at most one</u> (possibly none) $\vec{x} \in \mathbb{R}^n$

aka **injective**

Figure 4 in textbook

Ex: $f(x) = x^2$ is <u>not</u> one-to-one

because $\boxed{1} = (1)^2$ <u>and</u> $\boxed{1} = (-1)^2$

Ex $A = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

One-to-one? Onto?

✓ Onto: Pick arbitrary $\vec{b} \in \mathbb{R}^m$. Is $A\vec{x} = \vec{b}$ consistent?

✗ One-to-one: Any free variables? Yes So not one-to-one

Thm: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. T is one-to-one if and only if $T(\vec{x}) = \vec{0}$

has only the <u>trivial solution.</u>

Importance: Recall if we have infinite solutions to matrix eq $A\vec{x} = b$ can write them like this (for example)

$$\begin{cases} x_1 = 2 - 3x_2 \\ x_2 = \text{free} \\ x_3 \; 5 \end{cases} \qquad \begin{bmatrix} \vec{v} \end{bmatrix} + x_i \begin{bmatrix} \vec{w} \end{bmatrix}$$

A particular solution to $A\vec{x} = \vec{b}$

The "null space" solutions to $A\vec{x} = \vec{0}$

On other hand, if only solution to $\underline{A\vec{x} = \vec{0}}$ is trivial then <u>this</u> part doesn't exist / is just $\vec{0}$.

$$\begin{bmatrix} \vec{v} \end{bmatrix} + x_i \begin{bmatrix} \vec{w} \end{bmatrix}$$

That leaves only this one particular solution, at most. (there may be no solution).

☆ n pivots ☆

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear transformation with standard matrix A. Then:

① T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff columns of A span $\mathbb{R}^m$

② T is one-to-one iff columns of A are linearly independent.

Q: Can a matrix / transformation be one-to-one and onto?

Yes.

A is $m \times n$ matrix, one-to-one / onto.

Then

one-to-one

$$m = \# \text{ pivots} = n$$

onto

$$m = n$$

So, A must be square, have a pivot in every column / row.

A is $m \times m$

(or $n \times n$)