Section 4.2

In last section we saw that if $\vec{v}_1 \dots \vec{v}_k \in V$ then Span$\{\vec{v}_1 \dots \vec{v}_k\}$ subspace of V.
We'll see some special examples of this type.

Consider the homogeneous equation $A\vec{x} = \vec{0}$.

$$\underbrace{A}_{m\times n}\ \underbrace{\vec{x}}_{n\times 1} = \underbrace{\vec{0}}_{m\times 1}$$

Recall we discussed set of all solutions to this eq. called it the Null Space

$\vec{0} \in \mathbb{R}^n$

$$\text{Nul } A = \{ x \in \mathbb{R}^n : A\vec{x} = \vec{0} \}$$

- $\vec{0} \in \text{Nul } A$ ✓

There is <u>always</u> the <u>trivial</u> solution but there may be nontrivial solutions as well. In that case we can write solutions like so:

$$A\vec{x} = \vec{0}$$

$$\rightarrow \quad x_i \begin{bmatrix} \vec{v}_i \end{bmatrix} + x_j \begin{bmatrix} \vec{v}_j \end{bmatrix} + \dots \; x_p \begin{bmatrix} \vec{v}_p \end{bmatrix}$$

$$\{ \vec{v}_i, \dots, \vec{v}_p \}$$

Free variables

Argue that Nul A is a subspace of $\mathbb{R}^n$

From previous know that if we take $\vec{v}_i \dots \vec{v}_p$ from the above then Span $\{\vec{v}_i \dots \vec{v}_p\}$ is a subspace of $\mathbb{R}^n$

But does $\underline{\text{Nul}}\ A = \text{Span}\{\vec{v}_i \dots \vec{v}_p\}$?

Assume $\vec{y} \in \text{Span}\{\vec{v}_i \dots \vec{v}_p\}$. Then $\vec{y}$ is a linear combination of the vectors.

$$\vec{y} = x_i \vec{v}_i + \dots \; x_p \vec{v}_p \qquad A\vec{y} = \vec{0}$$

By linearity of matrix equation we have

$$\begin{aligned} A\vec{y} &= x_i A\vec{v}_i + \dots \; x_p A\vec{v}_p \\ &= x_i \vec{0} + \dots \; x_p \vec{0} \\ &= \vec{0} \end{aligned}$$

So $\vec{y}$ also a solution to $A\vec{x} = \vec{0}$

Thus every element in Span$\{\vec{v}_i \dots \vec{v}_p\}$ is a solution to $A\vec{x} = \vec{0}$!!!

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to the system $A\vec{x} = \vec{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Describing Nul A:

Have seen that if we solve $A\vec{x} = \vec{0}$ get something like:

$$x_i \begin{bmatrix} \vec{v}_i \end{bmatrix} + x_j \begin{bmatrix} \vec{v}_j \end{bmatrix} + \dots x_p \begin{bmatrix} \vec{v}_p \end{bmatrix}$$

And Span$\{\vec{v}_i \dots \vec{v}_p\}$ is same as Nul A.

$\{\vec{v}_i \dots \vec{v}_p\}$ is the key information here, call it the spanning set. This is often most convenient way to describe Nul A.

Note: $\{\vec{a}_1, \dots, \vec{a}_k\}$ is a spanning set for H of subspace H

if $H = \text{span}\{\vec{a}_1, \dots \vec{a}_k\}$

We can associate another important subspace with the $m \times n$ matrix A. Assume A is made up of columns $\vec{a}_1 \dots \vec{a}_n$

$$A = \underbrace{\begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}}_{n} \Big\} m$$

Can consider $\text{Span}\{\vec{a}_1, \dots \vec{a}_n\}$. This will also be a subspace, called Column Space of A

$$\text{Col } A = \text{Span}\{\vec{a}_1 \dots \vec{a}_n\}$$

What does element of Col A look like?

A linear combination of $\vec{a}_1 \dots \vec{a}_n$

$\vec{y} \in \text{Col } A = \text{Span}\{\vec{a}_1 \dots \vec{a}_n\}$

Vector eq. $\vec{y} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots x_n \vec{a}_n$

$\downarrow$ Rewrite as matrix transformation

$$\begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$A \quad \vec{x} = \vec{y}$$

So elements in Column Space are the vectors you get when you multiply A by some input vector $\vec{x}$

In other words Col A is all the possible vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$

$$\text{Col } A = \{ \vec{b} \in \mathbb{R}^m : A\vec{x} = \vec{b} \}$$

Col A may be some or all of $\mathbb{R}^m$.

The Column Space of $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if $A\vec{x} = \vec{b}$ has solution for all $\vec{b} \in \mathbb{R}^m$ (i.e. A is Onto)

Like Column Space, there is also Row Space, but is not very often used. Will omit it here.

What is difference between Nul A, Col A?

☆ Table, page 217 ☆

Remember there is a close relationship between linear transformations and matrices (matrices $A\vec{x}$ are linear transformations and can represent most linear transformations by matrices)

Slightly different terminology.

Matrix Transformations	Linear Transformations
$A\vec{x}$	$T(\vec{x})$
Nul A	Kernel T
Col A	Range T