14.2 - Limits / Continuity

The basic concept for limits and continuity is same for multivariable functions but a bit more complicated in practice.

- "General concept" for Limits of functions

Say $\lim f = L$ if output gets close to L as input gets close to a.

- What do we mean by close?
- How do we quantify this when input is a vector?

- Formal Definition $f : \mathbb{R}^n \to \mathbb{R}$

Let f be a multivariable function, domain of which contains area around $\vec{a} = \langle a_1, \ldots, a_n \rangle$

vector in $\mathbb{R}^n$

Say that $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ if for every $\varepsilon > 0$ there exists $\underline{\delta > 0}$ such that $|\vec{x} - \vec{a}| < \delta$ implies $|f(\vec{x}) - L| < \varepsilon$

What is $|\vec{x} - \vec{a}|$? $\vec{x} - \vec{a}$ vector $|\vec{x} - \vec{a}|$ mag of vector

What is $|f(\vec{x}) - L|$? absolute of # − #

For specific case $f: \mathbb{R}^2 \to \mathbb{R}$:

Can write $\underline{\vec{a}}$ as (a, b) and $\underline{\vec{x}}$ as $\underline{(x, y)}$. So:

Say $\lim\limits_{\underline{(x,y)} \to \underline{(a,b)}} f(x, y) = L$ if for every $\underline{\varepsilon > 0}$ there exists $\underline{\delta > 0}$ such that $\underline{\sqrt{(x-a)^2 + (y-b)^2}} < \delta$ implies that $|f(x, y) - L| < \varepsilon$

Ex:

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2}$$

$$= 1$$

$f : \mathbb{R} \to \mathbb{R}$

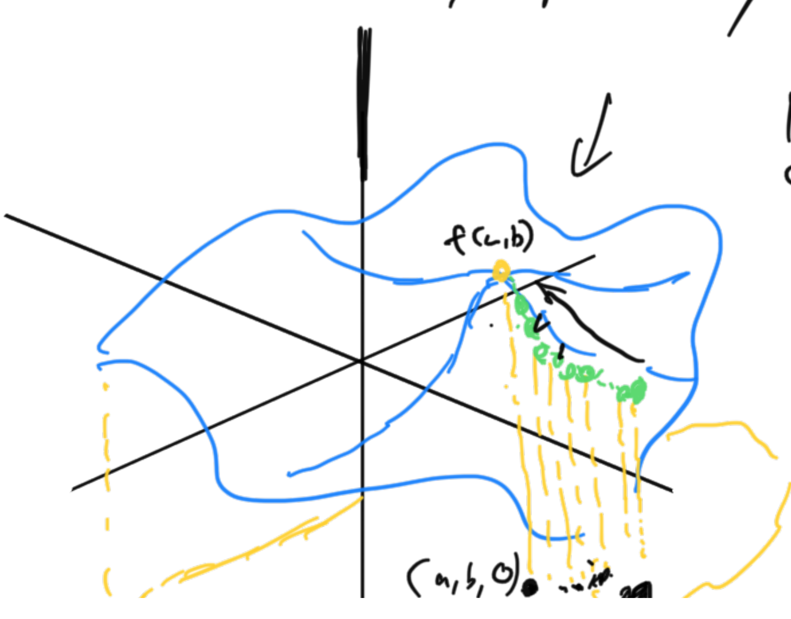
$$\boxed{\lim_{z\to 0} \frac{\sin(z)}{z} = 1}$$

The notation for limit is interesting/important.

$$\lim_{(x,y)\to(0,0)}$$

What does $(x,y) \to (0,0)$ mean?

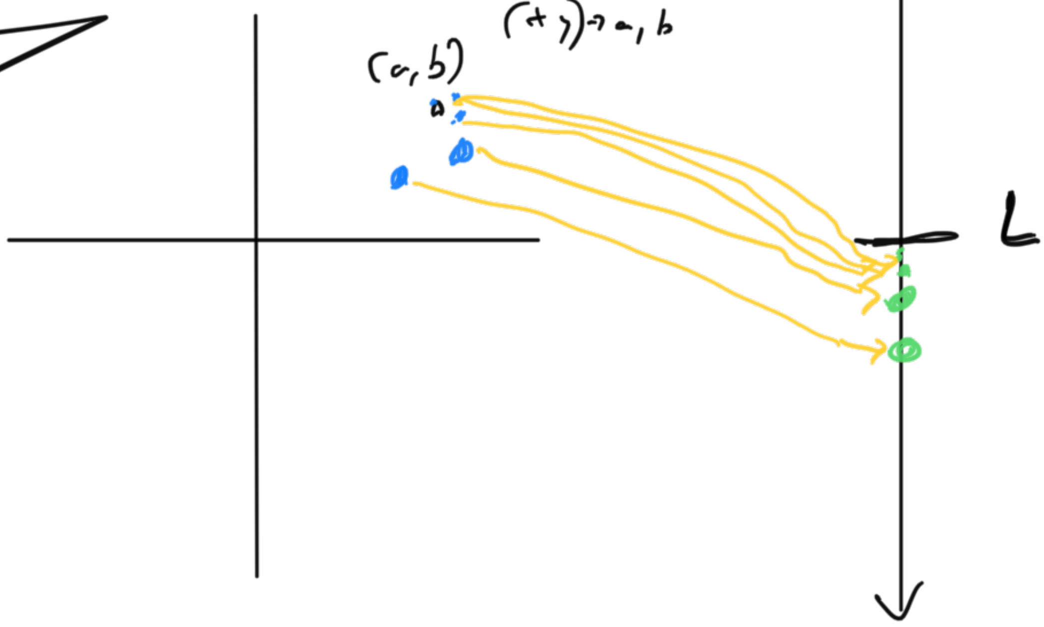
(x,y) approaches $(0,0)$. Could think of it in terms of ε, δ but try to see it graphically



$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

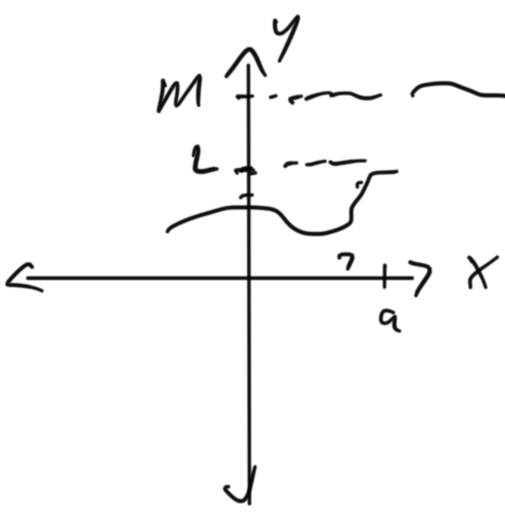
$$f : \underline{\mathbb{R}^2} \to \underline{\underline{\mathbb{R}}}$$

$$\lim_{(x,y)\to(a,b)} f(x,y)=L$$



Compare to limit of $f:\mathbb{R}\to\mathbb{R}$

$$\lim f(x)=L$$



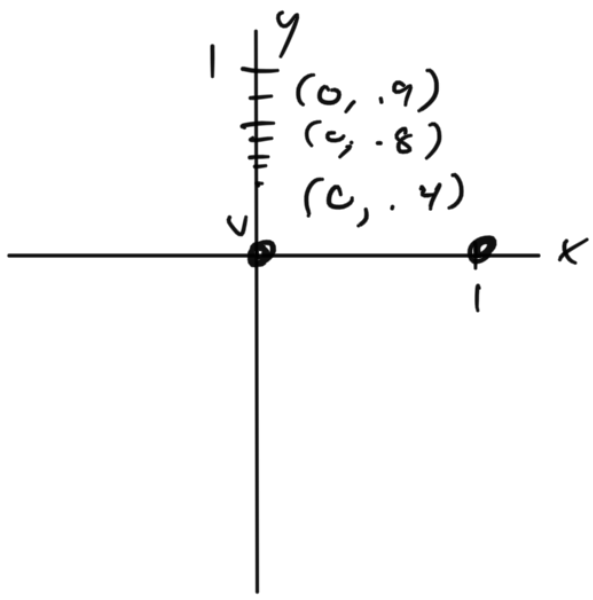
$$\lim_{x\to a^-} f(x)=L$$

$$\lim_{x\to a^+} f(x)=M$$

$$\lim_{x\to a} f(x) \text{ DNE}$$

Ex: $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}=L$$

Inputs



$$\frac{0-y^2}{0+y^2} = \frac{-y^2}{y^2} = -1$$

$$\frac{x^2-0}{x^2+0} = \frac{x^2}{x^2} = 1$$

Along x-axis, limit equals 1

$$\boxed{\lim_{(x,0)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = 1}$$

Along y-axis, limit equals -1

$$\boxed{\lim_{(0,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = -1}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) \quad \text{DNE}$$

Important concept. For limit to exist, it must be same from all possible directions. So:

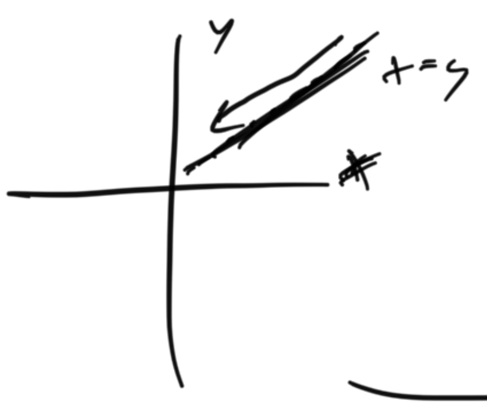
To show limit exists: Need to guarantee limit is same from all directions (hard)

To show limit doesn't exist: Show that there are two different directions with two different limits

This concept of "direction of approach" is really at the heart of calc 3. After all, continuity and derivatives defined directly in terms of limits

☆ Examples ☆

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad \lim_{(x,y)\to(0,0)} \text{ exist?}$$



$$\frac{x \cdot x}{x^2+x^2} = \frac{x^2}{2x^2} = \frac{1}{2}$$

$\lim$ $f(x,y) =$

$$\lim_{(x,0)\to(0,0)}$$

$$\lim_{(x,x)\to(0,0)} f(x,y) = \frac{1}{2}$$

Limit Laws

Assume $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$.

① $\lim\limits_{(x,y)\to(a,b)} f(x,y) + \lim\limits_{(x,y)\to(a,b)} g(x,y)$

$= L + M$

② $\left(\lim\limits_{(x,y)\to(a,b)} f(x,y)\right)\left(\lim\limits_{(x,y)\to(a,b)} g(x,y)\right)$

$= LM$

③ If $\lim\limits_{(x,y)\to(a,b)} g(x,y) \neq 0$ then

$\lim\limits_{(x,y)\to(a,b)} f(x,y) \Big/ \lim\limits_{(x,y)\to(a,b)} g(x,y) = L/M$

④

If $\displaystyle\lim_{(x,y)\to(a,b)} \underline{f(x,y) = L}$ and $\displaystyle\lim_{(x,y)\to(a,b)} \underline{h(x,y) = L}$

$$L \qquad\qquad L$$

$\underline{\text{and}}$ $\quad \underline{f(x,y)} \le g(x,y) \le \underline{h(x,y)}$

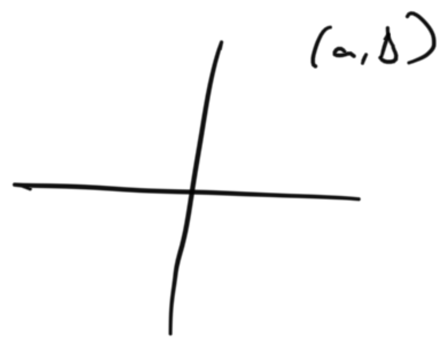
then $\displaystyle\lim_{(x,y)\to(a,b)} g(x,y) = L$

⑤ $\displaystyle\lim_{(x,y)\to(a,b)} c = c \qquad$ (constant)

⑥ $\displaystyle\lim_{(x,y)\to(a,b)} c\, f(x) = c\, L$

⑦ $\displaystyle\lim_{(x,y)\to(a,b)} x = a$

⑧ $\displaystyle\lim_{(x,y)\to(a,b)} y = b$



Continuity

Once we have idea of limits, concept of continuity is familiar

Def: Say $f(x,y)$ continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

f continuous on D if continuous at every point in D.

f is continuous
(continuous at every point in its domain)

$\mathbb{R}^2$ $\mathbb{R}^n$

Polynomials in multiple variables

$\mathbb{R}$ $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x^1 + a_0$

$\mathbb{R}^2$ $4x^5y^3 + 3x^2y^7 + 7$

Always continuous

Rational functions

$\mathbb{R}$ $\frac{p(x)}{q(x)}$ $\frac{\text{poly}}{\text{poly}}$

$$\frac{4x^2 + 7x - 1}{3x - 6}$$

$x = 2$ not in domain

$D = (-\infty, 2) \cup (2, \infty)$

Continuous on domain

$$\frac{4x^2y^3+7}{xy-1}$$

For continuous functions

Can plug and chug limits

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \sin(x)$$

$$\boxed{\lim_{x \to \pi/2} f(x) = 1}$$

$$f(x,y) = \frac{x^4+y}{x^2+y^2}$$

$$\lim_{(x,y)\to(2,1)} f(x,y) = \frac{2^4+1}{2^2+1^2} = \boxed{\frac{17}{5}}$$

$f: \mathbb{R}^1 \to \mathbb{R}$

$f: \mathbb{R}^2 \to \mathbb{R}$