Section 3.1

Would like to be able to look at a matrix and determine if it is invertible or not without having to go all the way through row reduction

Note: Calculating inverse via row reduction becomes very "expensive" in terms of computation / run time. Especially as matrix becomes large.

We would like to figure out some quantity that will be 0 if matrix <u>not</u> invertible, nonzero otherwise. This will be exactly the **determinant**.

The theory behind the existence of the determinant is somewhat complicated, will not go through it all here. For now, just trust the fact that for every <u>square</u> matrix we can calculate the determinant, which will be zero if matrix invertible, nonzero otherwise. And the determinant is <u>unique</u>.

Calculating:

The simplest case is a 2x2 matrix. Method for larger matrices builds from here.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = ad - bc$$

Ex $A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ $\quad \det A = (1)(4) - (-2)(3)$

$= 10$

$$A = \begin{bmatrix} 3 & 1 & 0 & 2 \\ 1 & 4 & 2 & 1 \\ 2 & 6 & 1 & 3 \\ 5 & 1 & 1 & 2 \end{bmatrix}$$

$\begin{cases} 0 \Rightarrow \text{not invertible} \\ \text{non-zero} \Rightarrow \text{invertible} \end{cases}$

Steps for $n \times n$ matrix:

① "Label" entries of A with alternating $+/-$ starting with $+$ in top left corner

$$\rightarrow \begin{bmatrix} {}^{+}3 & {}^{-}1 & {}^{+}0 & {}^{-}2 \\ {}^{-}1 & {}^{+}4 & {}^{-}2 & {}^{+}1 \\ {}^{+}2 & {}^{-}6 & {}^{+}1 & {}^{-}3 \\ {}^{-}5 & {}^{+}1 & {}^{-}1 & {}^{+}2 \end{bmatrix}$$

② Pick a row/column

1st row

③ Take first entry in column/row along with its sign

$$3(+)$$

④ Temporarily eliminate/ignore the row and column holding that entry

$$\begin{bmatrix} {}^{+}3 & 1 & 0 & 2 \\ 1 & 4 & 2 & 1 \\ 2 & 6 & 1 & 3 \\ 5 & 1 & 1 & 2 \end{bmatrix}$$

⑤ Make a submatrix from remaining entries

$$\begin{bmatrix} {}^{+}3 & 1 & 0 & 2 \\ 1 & 4 & 2 & 1 \\ 2 & 6 & 1 & 3 \\ 5 & 1 & 1 & 2 \end{bmatrix}$$

Submatrix obtained by eliminating i^{th} row j^{th} column is A_{ij}.

⑥ Multiply term from step ③ by determinant of this new submatrix

$$3(+)(\det A_{11})$$

⑦ Move down your chosen row/column from step ①, summing results together

$$\det A = 3(+)(\det A_{11}) + 1(-)(\det A_{12}) + 0(+)(\det A_{13}) + 2(-)(\det A_{14})$$

⑧ To find determinant of each submatrix, repeat steps 1-7.

Note: Each submatrix gets new +/- labels

$$A_{11} = \begin{bmatrix} 4 & 2 & 1 \\ 6 & 1 & 3 \\ 1 & 1 & 2 \end{bmatrix} \qquad A_{12} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 5 & 1 & 2 \end{bmatrix} \end{array} \qquad A_{13} = \text{doesn't matter}$$

$$A_{14} = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & 4 & 2 \\ 2 & 6 & 1 \\ 5 & 1 & 1 \end{bmatrix} \end{array}$$

Once you get down to a 2x2 matrix we can actually calculate

$$\det A_{11} = 4(2-3) - 2(12-3) + 1(6-1)$$
$$= -4 - 18 + 5 = \underline{-17}$$

$$\det A_{12} = 1(2-3) - 2(4-15) + 1(2-5)$$
$$= -1 + 22 - 3 = 18$$

$\det A_{13} =$ doesn't matter

$$\det A_{14} = 1(6-1) - 4(2-5) + 2(2-30)$$
$$= 6 + 12 - 56 = -38$$

So:

$$\det A = 3(-17) - 1(18) + 0(\det A_{13}) - 2(-38)$$
$$= -54 - 18 + 76$$
$$= 4$$

Theorem: If A is a triangular matrix then $\det A$ is just product of the entries on the main diagonal.