14.5 - Chain Rule

Simple extension of chain rule for

$f: \mathbb{R} \to \mathbb{R}$

<u>Theorem</u>: Suppose $\underline{f(x,y)}$ is differentiable function of x and y where $\underline{x = g(t)}$ and $\underline{y = h(t)}$ such that g, h are differentiable. Then $\underline{f(x,y)}$ is differentiable in terms of t and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

How does f change in response to a change in t?

Translation: If $\underline{x, y}$ functions of t then $f(x,y) = f(x(t), y(t))$. So changes in t affect both $\underline{x}$ and $\underline{y}$. So to measure df/dt, must consider change in both components.

□ $\lim_{t \to 0} \frac{\Delta z}{\Delta t}$ $\qquad z = f(x(t), y(t))$

$\Delta z = f(x(t+h), y(t+h)) - f(x(t), y(t))$ $\qquad z$ is differentiable

$= f_x(x(t), y(t))\Delta x + f_y(x(t), y(t))\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$\Delta x = x'(t)\Delta(t) + \eta_1 \Delta t$

$\Delta y = y'(t)\Delta(t) + \eta_2 \Delta t$

...

$$\Delta z = f_x(x(t), y(t))(x'(t)\Delta(t) + \eta_1 \Delta t)$$
$$+ f_y(x(t), y(t))(y'(t)\Delta t + \eta_2 \Delta t)$$
$$+ \varepsilon_1 (x'(t)\Delta t + \eta_1 \Delta t)$$
$$+ \varepsilon_2 (y'(t)\Delta t + \eta_2 \Delta t)$$

So $\Delta z / \Delta t =$ $\qquad \lim_{t \to 0} \frac{\Delta z}{\Delta t}$

$$\left\{ \begin{array}{l} f_x(x(t), y(t))(x'(t) + \eta_1) \qquad \underline{f_x(x(t), y(t))x'(t)} \\ + f_y(x(t), y(t))(y'(t) + \eta_2) \qquad \underline{f_y(x(t), y(t))y'(t)} \end{array} \right.$$
$$+ \varepsilon_1 (x'(t) + \eta_1)$$
$$+ \varepsilon_2 (y'(t) + \eta_2)$$

$\underline{dz/dt} = \lim_{t \to 0} \Delta z / \Delta t = \qquad f(x(t), y(t)$

$$f_x(x(t),y(t))\,x'(t) + f_y(x(t),y(t))\,y'(t)$$

Or

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$f(x,y)$ $\quad x(t)$ $\quad y(t)$

Ex:

$$f(x,y) = 2x^2 + 3y^4$$

$$x = e^t$$

$$y = t^3$$

Find dz/dt.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (4x)(e^t) + (12y^3)(3t^2)$$

$$= (4e^t)(e^t) + (12(t^3)^3)(3t^2)$$

$$= 4e^{2t} + 12t^9\,3t^2$$

$$\boxed{= 4e^{2t} + 36t^{11}}$$

Don't be too rigid in requiring x, y to be function of t.

Ex: $f(x,y) = xy + \ln(y)\sin(x)$

Find $\underline{\partial f/\partial x}$ of $f(x^2+1, y)$

Could think of it this way:

$$f(x^2+1, y) = (x^2+1)y + \ln(y)\sin(x^2+1)$$

$$\frac{\partial f}{\partial x} = 2xy + 2x\ln(y)\cos(x^2+1)$$

<u>or</u>:

$$f(\overset{a}{x}, y) = xy + \ln(y)\sin(x)$$

$$f(\overset{a(x)}{x^2+1}, y) \quad \frac{df}{dx} =$$

$$= \underbrace{[y + \ln(y)\cos(x^2+1)]}_{\text{"}\partial f/\partial x\text{"}} \underbrace{(2x)}_{\text{"}df/dt\text{"}}$$

Can make things more complicated.

Assume f is a function of n variables. $f(x_1, x_2, x_3, \dots x_n)$.

Each x_i is function of r variables

$$x_1 = x_1(t_1, \dots t_r)$$
$$\vdots$$
$$x_n = x_n(t_1, \dots t_r)$$

What is $\partial f / \partial t_1$?

Must consider how change in t_1 affects $x_1, x_2, \dots x_n$, then how these change f. So:

$$f(x_1, x_2, x_3, x_4 \dots)$$

$$f(x_1(\underbrace{t_1, \dots t_r}), \dots x_n(t_1, \dots t_r))$$

$$\frac{\partial f}{\partial t_1}$$

$$\frac{\partial f}{\partial t_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_1}$$

$$\partial t_1 \qquad \partial x_1 \quad \partial t_1 \qquad \partial x_2 \quad \partial t_1 \quad + \ldots$$

$$\frac{\partial f}{\partial x_n} \frac{\partial x_n}{\partial t_1}$$

Ex:

$$u(x,y,z) : x^4 y + y^2 z^3$$

$$\begin{cases} x = rse^{t} \\ y = rs^2 e^{-t} \\ z = r^2 s (\sin(t)) \end{cases}$$

r, s, t Ind var

Intermediate

Find $\underline{\partial u / \partial s}$ when $r = 2,\ s = 1,\ t = 0$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial u}{\partial s} = (4x^3 y)(re^{t}) + (x^4 + 2yz^3)(2rse^{-t})$$

$$+ (3y^2 z^2)(r^2 \sin(t))$$

$$x(2,1,0) = 2$$
$$y(2,1,0) = 2$$
$$z(2,1,0) = 4$$

$$\frac{\partial u}{\partial s}(2,1,0) = (4 \cdot 8 \cdot 2)(2 \cdot 1) + (16 + 2 \cdot 2 \cdot 64)(2 \cdot 2 \cdot 1 \cdot 1)$$

$$+ (3 \cdot 4 \cdot 16)(0)$$

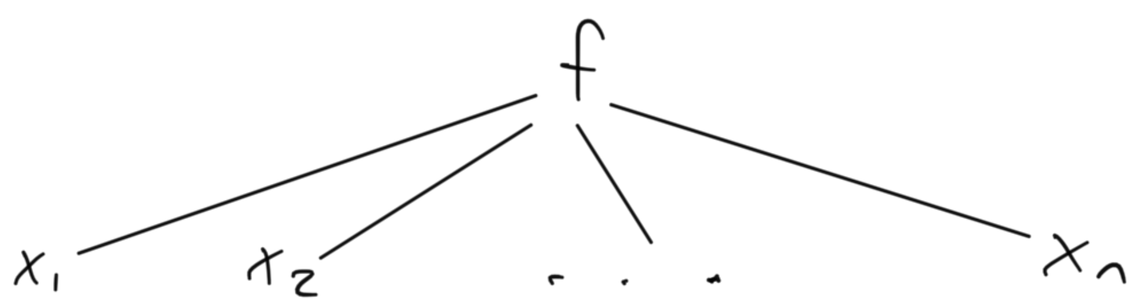
Tree Diagrams

Tree Diagrams are a useful tool for keeping track of all your derivatives in multivariable chain rule

Lets say you have $f(x_1, x_2, \ldots, x_n)$

Start by drawing:

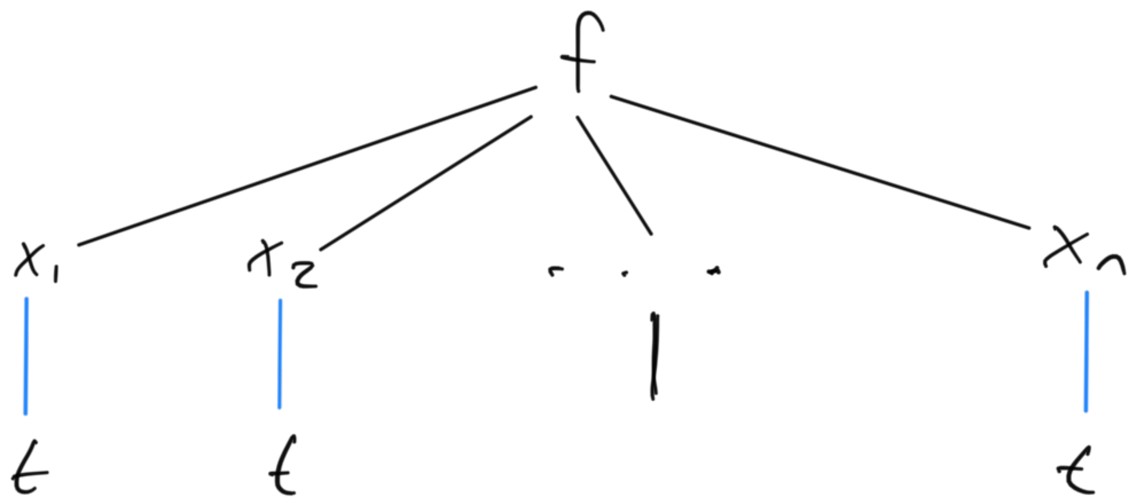
$$f(x_1, \ldots x_n)$$



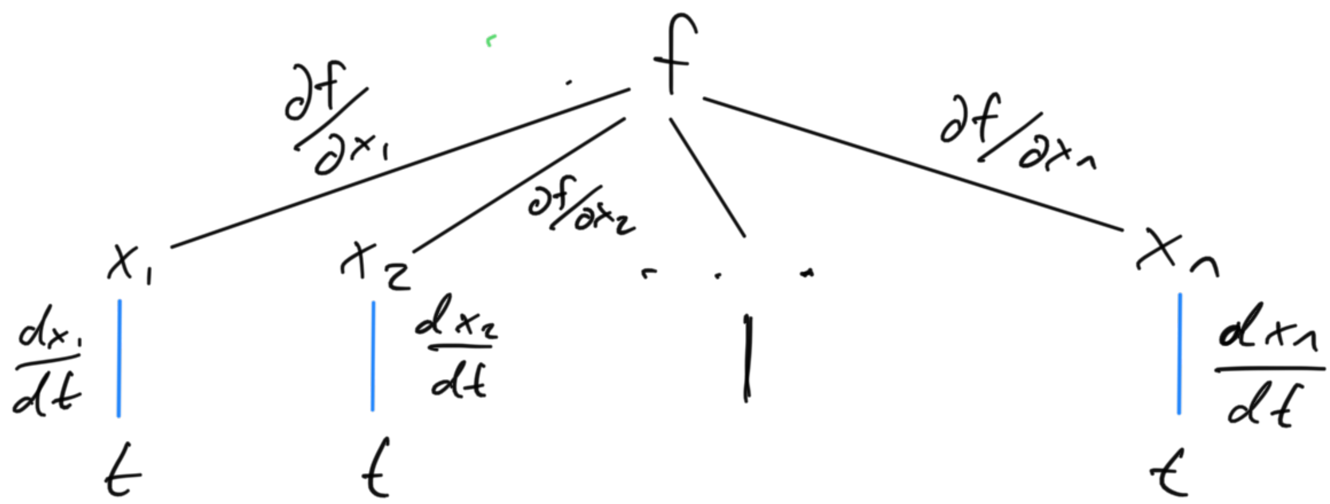
Think of lines between f and x_i as differentiation

$x_1 \quad x_2 \quad \dots \quad x_n$

Then if we assume each x_i is a function of t, we could draw:



Again, think of lines as differentiation



To find $\frac{df}{dt}$, I "add up" all paths to the t's

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \dots \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$$

These trees

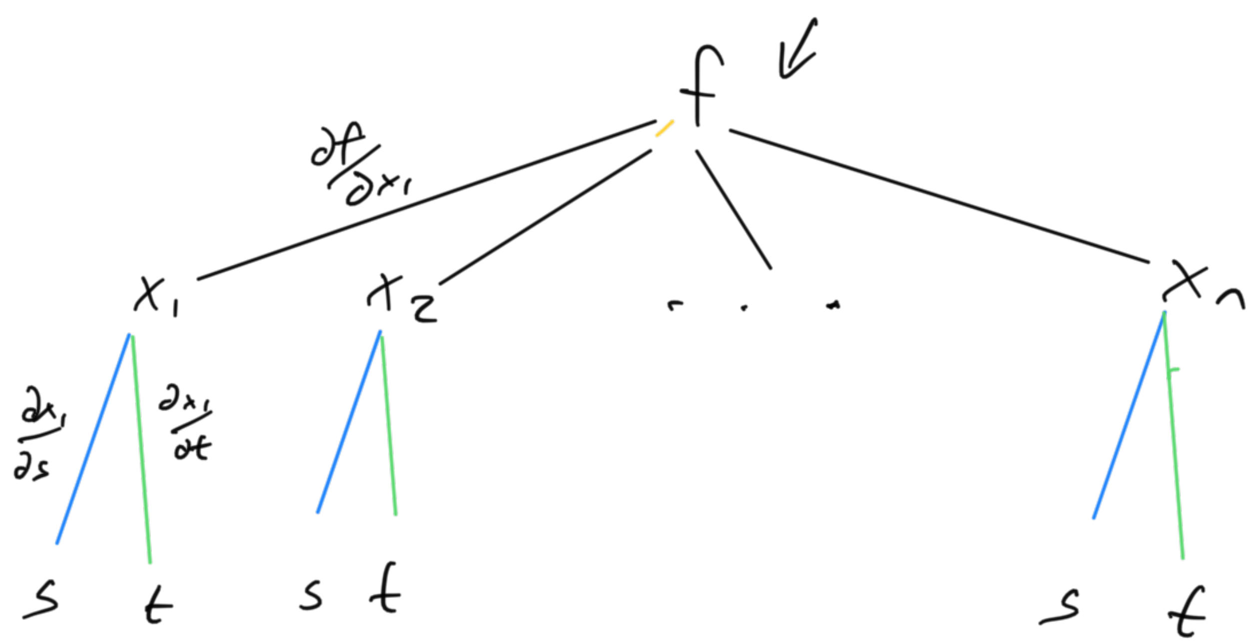
... can also handle the cases where each x_i a function of many variables

$$f(x_1, \dots x_n)$$

$$x_i(t_1 \dots t_r)$$

Ex: $f(x_1, \dots x_n)$ f differentiable

$x_i(s,t)$ x_i differentiable



So $\partial f/\partial s$, "sum" of all paths from f to an s.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial s} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial s} + \dots \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial s}$$

Implicit Function Theorem

Take a look.

$F(x)$ Find dF/dx

$z = f(x,y)$

$\{ x^3 + y^3 + z^3 + 6xyz = 1$

Dont know what $f(x,y)$ is explicitly

Still want to figure out $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

$$3x^2 + 6yz = -3z^2 \partial z/\partial x - 6xy\, \partial z/\partial x$$

$$3x^2 + 6yz = \frac{\partial z}{\partial x}(-3z^2 - 6xy)$$

$$\frac{3x^2 + 6yz}{-3z^2 - 6xy} = \partial z/\partial x$$