Section 1.5

Going to look at solution sets of our linear systems again, get some geometric intuition.

First going to look at a special case that is very important.

Homogeneous Systems

A system of form

$$A\vec{x} = \vec{0}$$

is called homogeneous. Very important.

Note:

$A\vec{x} = \vec{0}$ only has at least one solution. The trivial solution, $\vec{x} = \vec{0}$

$$\underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}}_{m \times 1}$$

The bigger question is:

Are there other, non-trivial solutions to equation $A\vec{x}=\vec{0}$? Nonzero vector $\vec{x}$ such that $A\vec{x}=\vec{0}$?

Since $A\vec{x}=\vec{0}$ always has one solution, for there to be a non-trivial solution the system must have at least one free-variable.

Ex Describe <u>all</u> solutions to $A\vec{x}=\vec{0}$ where

$$A=\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$ coefficient matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

Eq. $A\vec{x}=\vec{0}$ is same as augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

No matter what row operation we use, last row always 0's. Its a bit redundant

$$\begin{bmatrix} 1 & 3 & -3 & 7 \\ 0 & 1 & -4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9 & -8 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

$$\begin{cases} x_1 = -9x_3 + 8x_4 \\ x_2 = 4x_3 - 5x_4 \\ x_3 \text{ free} \\ x_4 \text{ free}. \end{cases} \qquad \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ 1 \end{bmatrix}$$

$$x_3\begin{bmatrix}-9\\4\\1\\0\end{bmatrix} + x_4\begin{bmatrix}8\\-5\\0\\1\end{bmatrix} \quad \star$$

Notice that vector of constants we normally have is always $\begin{bmatrix}0\\\vdots\\0\end{bmatrix}$ for equation $A\vec{x}=0$.
So dont bother writing it.

What does $x_3\begin{bmatrix}-9\\4\\1\\0\end{bmatrix} + x_4\begin{bmatrix}8\\-5\\0\\1\end{bmatrix}$ look like?

Linear combination.

But since x_3, x_4 are free variables they can take values

Notice set $x_3\begin{bmatrix}-9\\4\\1\\0\end{bmatrix} + x_4\begin{bmatrix}8\\-5\\0\\1\end{bmatrix}$ for all values x_3, x_4 is exactly

$$\text{span}\left\{\begin{bmatrix}-9\\4\\1\\0\end{bmatrix}, \begin{bmatrix}8\\-5\\0\\1\end{bmatrix}\right\}$$

Set of all solutions to $A\vec{x}=0$ is called Null Space of A

Recap

$A\vec{x}=\vec{0}$ always has trivial solution

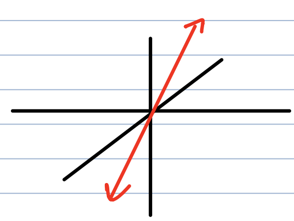
$A\vec{x}=\vec{0}$ has nontrivial solution iff system has free variable

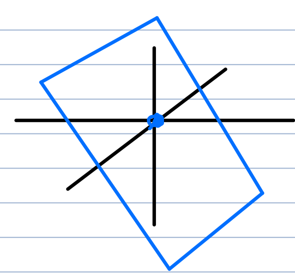
Nontrivial solutions of $A\vec{x}=\vec{0}$ form a span{...} (if they exist)

(Later we will say that "Null space is a subspace of $\mathbb{R}^n$")

Think about what it means for non-trivial solutions of $A\vec{x}=\vec{0}$ to be span$\{\vec{v}_1 \dots \vec{v}_k\}$

Solutions of $A\vec{x}=\vec{0}$ have some nice form passing through origin





Non-Homogeneous Systems

Knowing about solutions to $A\vec{x}=\vec{0}$ gives insight into solutions of $A\vec{x}=\vec{b}$

First recall distributivity of matrix equation

i.e. $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ ☆

Assume $\vec{p}$ is solution to $A\vec{x}=\vec{b}$

Let $\vec{q}$ be any solution to $A\vec{y}=\vec{0}$

Consider $A(\vec{p}+\vec{q})$

$$A(\vec{p}+q) = A(\vec{p}) + A(\vec{q})$$

b.c. $\vec{p}$ solution to $Ax=b$

b.c. q sol. to $A\vec{y}=\vec{0}$

$$= \vec{b} + \vec{0}$$

$$= \vec{b}$$

$A\vec{x}=\vec{b}$

So if we have solution to non-hom. system and add sol. of hom. sys., still get $\vec{b}$.

$Ax=0$

Theorem Suppose $A\vec{x}=\vec{b}$ is consistent for some $\vec{b}$ and let $\vec{p}$ be a particular solution.

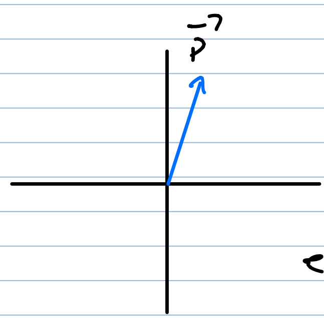
Then the solution set of $A\vec{x}=\vec{b}$ is set of all vectors of form $\vec{w} = \vec{p} + \vec{v}$, where $\vec{v}$ is any solution of $A\vec{x} = \vec{0}$.

Geometric Viewpoint

Consider $A\vec{x} = \vec{0}$

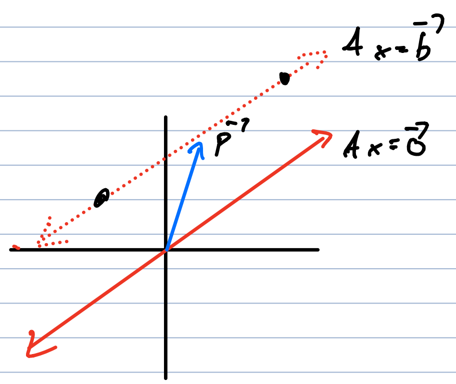
If there are nontrivial solutions to $A\vec{x} = \vec{0}$, they are of form $\text{span}\{\vec{v}_1, \dots \vec{v}_k\}$

Consider $A\vec{x} = \vec{b}$ ☆



Assume we managed to find one particular solution $\vec{p}$

every $\underset{A\vec{x}=\vec{b}}{\text{sol}} = \vec{p} + (\text{Null space})$



vectors of form $\vec{p} + \vec{v}$ (for $\vec{v}$ solution to $A\vec{x}=\vec{0}$) would be on dotted line

This dotted line is solution set $A\vec{x} = \vec{b}$

$A\vec{x} = \vec{b}$

Find
Null space
+
Find one solution

Know <u>all</u> solutions to $A\vec{x} = \vec{b}$