Section 6.7

Have been working almost exclusively in $\mathbb{R}^n$

In chapter 6 we developed idea of orthogonality

which was core concept for many ideas / techniques

"Dot product"

$\downarrow$

Orthogonality

$\downarrow$

Projections

$\downarrow$

Gram - Schmidt

$\downarrow$

Least Squares

Throughout it all, had idea of $\mathbb{R}^n$ in background, that orthogonal = perpendicular

But have been alluding to more abstract spaces throughout the course (spaces of polynomials, continuous functions, etc)

Want to develop similar ideas for these abstract spaces so we have access to these powerful tools

First step, define an analog to dot product

Second step is defining orthogonality in more abstract terms. Don't want to rely on idea of "perpendicular"

Previously, said $\vec{v}, \vec{w}$ were orthogonal when <u>dot product</u> was zero, $\vec{v}\cdot\vec{w} = 0$

For more abstract spaces, have <u>inner product</u>.

Definition: An **inner product** on vector space V is a function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$ that for each pair of vectors $\vec{u}, \vec{v}$ in V associates a real number $\langle \vec{u}, \vec{v} \rangle$ and satisfies following properties for all $\vec{u}, \vec{v}, \vec{w}$ in V and all $c \in \mathbb{R}$

$$\begin{cases} ① & \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \\ ② & \langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle \\ ③ & \langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle \\ ④ & \langle \vec{u}, \vec{u} \rangle \geq 0 \text{ and } \langle \vec{u}, \vec{u} \rangle = 0 \text{ iff } \vec{u} = \vec{0} \end{cases}$$

Note: Consider dot product on $\mathbb{R}^n$. Let $\langle \vec{u}, \vec{v} \rangle$ be $\vec{u} \cdot \vec{v}$. Check properties.

① $u \cdot v = v \cdot u$ ✓

② $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

③ $(c\vec{u}) \cdot \vec{w} = c(\vec{u} \cdot \vec{w})$

④ $\vec{u} \cdot \vec{u} \geq 0$ $\quad \vec{u} \cdot \vec{u}$ ✓

So we see that the dot product <u>is</u> an inner product.

The dot product is just a special case of the more general idea "inner product"

Definition: A vector space equipped with an inner product is called an **inner product space.**

Note: $\mathbb{R}^n$ with inner product $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$ is an inner product space.

A more abstract example:

Let V be the space of polynomials on interval $[0,1]$

The vectors $\vec{u}, \vec{v}$ in V are polynomials

$$\vec{u} = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$$
$$\vec{v} = b_0 + b_1 x + b_2 x^2 + \dots b_k x^k$$

Zero vector $\vec{0}$ is zero polynomial $\vec{0} = 0 + 0x + 0x^2 + \dots = 0$

Define inner product as

$$\langle \vec{u}, \vec{v} \rangle = \int_0^1 (a_0 + a_1 x + \dots a_n x^n)(b_0 + b_1 x + \dots b_k x^k)\, dx$$

This is an inner product!

Check:

① $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ ✓

② $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ ✓

③ $\langle cu, v \rangle = c \langle u, v \rangle$

④ $\langle u, u \rangle \geq 0$ $\quad \langle u, u \rangle = 0$ iff $\vec{u} = \vec{0}$

Recall in $\mathbb{R}^n$, dot product could be used to find "length".

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

Have very concrete idea of what "length" is in $\mathbb{R}^n$.

"Length" can again be generalized to more abstract concept, norm

Idea is same. $\|\vec{v}\|$ is a measure of magnitude of $\vec{v}$. Calculated same way but replace dot product with inner product

$$\|\vec{v}\| = \sqrt{\underline{\langle \vec{v}, \vec{v} \rangle}}$$

If $\|\vec{v}\| = 1$, still refer to $\vec{v}$ as unit vector

Ex: $\vec{v} = \sqrt{3}x$ is vector in $\mathbb{P}[0,1]$

Let $\langle \vec{u}, \vec{v} \rangle = \int_0^1 uv \, dx$ be inner product on $\mathbb{P}[0,1]$

$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$$

$$= \sqrt{\int_0^1 (\sqrt{3}x)(\sqrt{3}x)\, dx}$$

$$= \sqrt{\int_0^1 3x^2 dx}$$

$$= \sqrt{x^3\Big]_0^1}$$

$$= \sqrt{1} = 1$$

Distance between $\vec{u}, \vec{v}$ is $\|\vec{u} - \vec{v}\| = \sqrt{\langle \vec{u}-\vec{v}, \vec{u}-\vec{v} \rangle}$

Most importantly we can use inner product to define orthogonality. Just like with dot product in $\mathbb{R}^n$, say $\vec{u}, \vec{v}$ orthogonal if

$$\langle \vec{u}, \vec{v} \rangle = 0$$

Ex Consider space of polynomials (degree ≤ 3) with inner product

$\langle \vec{u}, \vec{v} \rangle = \int_{-1}^1 \vec{u}\vec{v}\, dx$

$$\vec{v}_0 = 1 \quad \vec{v}_1 = x \quad \vec{v}_2 = x^2 \quad \vec{v}_3 = x^3$$

$\{\vec{v}_0, \vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is basis for $\mathbb{P}_3[-1, 1]$

(can make any 3^{rd} degree polynomial out of them)

But <u>not</u> orthogonal

$$\langle \vec{v}_0, \vec{v}_2 \rangle = \int_{-1}^{1} 1\,x^2\,dx$$
$$= \frac{x^3}{3}\Big]_{-1}^{1}$$
$$= 2/3$$ Not orthogonal

OTOH Have orthogonal polynomials

Several types.

Legendre Polynomials:

$\vec{L}_0 = 1$ $\quad \vec{L}_1 = x$ $\quad \vec{L}_2 = \frac{1}{2}(3x^2-1)$

$\vec{L}_3 = \frac{1}{2}(5x^3-3x)$

Basis for $\mathbb{P}_3\,[-1,1]$ and orthogonal

$$\langle \vec{L}_1, \vec{L}_2 \rangle = \int_{-1}^{1} x\left(\tfrac{1}{2}\right)(3x^2-1)\,dx$$
$$= \frac{1}{2}\int_{-1}^{1} (3x^3-x)\,dx$$
$$= \frac{1}{2}\left[\left(\frac{3}{4}x^4 - \frac{x^2}{2}\right)\right]_{-1}^{1}$$
$$= \frac{1}{2}(0)$$
$$= 0$$ Orthogonal

With general idea of orthogonality we can do projections, Gram-Schmidt, least squares directly. Just replace dot products with whatever inner product we are given

Ex: $\mathbb{P}_2$, space polynomials degree ≤ 2

If $\vec{p}, \vec{q}$ polynomials and $\vec{p}'(x_i)$ is the polynomial evaluated at x_i can define inner product: Pick finite # of x values $x_0 \ldots x_n$

$$\langle p, q \rangle = \vec{p}'(x_0)\vec{q}'(x_0) + \vec{p}(x_1)q(x_1) + \ldots \vec{p}(x_n)\vec{q}(x_n)$$

(4) Compute $\langle \vec{p}, \vec{q} \rangle$ for

$$\vec{p} = 3t - t^2 \qquad \vec{q} = 3 + 2t^2$$

evaluated at points $-1, 0, 1$

(8) Compute $\text{proj}_{\vec{p}}\, \vec{q}$

Best Approximations

The best approximation of $\vec{v}$ onto subspace $W = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n\}$ with orthogonal basis is still projection of $\vec{v}$ onto basis vectors

$$\mathrm{proj}_W \vec{v} = \mathrm{proj}_{\vec{v}_1} \vec{v} + \mathrm{proj}_{\vec{v}_2} \vec{v} + \dots \mathrm{proj}_{\vec{v}_n} \vec{v}$$

Big Inequalities!

Cauchy-Schwartz - Have seen before with dot products

$$\star \quad |\vec{u}\cdot\vec{v}| \leq \|\vec{u}\| \; \|\vec{v}\|$$

Euclidean norm - what we're used to

Similarly for inner products have

$$|\langle \vec{u},\vec{v}\rangle| \leq \|\vec{u}\| \; \|\vec{v}\|$$

(Here norm is defined by $\|\cdot\| = \sqrt{\langle \cdot,\cdot \rangle}$)

Triangle Inequality:

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

Equal iff $\langle \vec{u},\vec{v}\rangle = 0$, in other words if $\vec{u},\vec{v}$ are orthogonal

C[a b]

$f, g \in C[a,b]$ $\quad \langle f, g\rangle$

$$\int_a^b f g \, dx \qquad (L^1 \text{ inner product})$$

$C[0, \pi]$ with L^1 inner product

$f = \sin(x)$ $\quad g = \cos(x)$

$$\langle f, g \rangle = \int_0^{\pi} \underline{\overset{h = \sin}{\sin(x)\cos(x)}\, dx}$$

$$= \int_0^{\pi} \frac{1}{2} \sin(2x)\, dx$$

$$= -\frac{1}{4} \left. \underline{\cos(2x)} \right]_0^{\pi}$$

$$= -\frac{1}{4} \left(\cos(2\pi) - \cos(0) \right)$$

$$= -\frac{1}{4} (1 - 1)$$

$$= -\frac{1}{4} (0)$$

$$= 0$$

$\sin(x)$, $\cos(x)$ are orthogonal on $[0, \pi]$ w.r.t the L^1 inner product