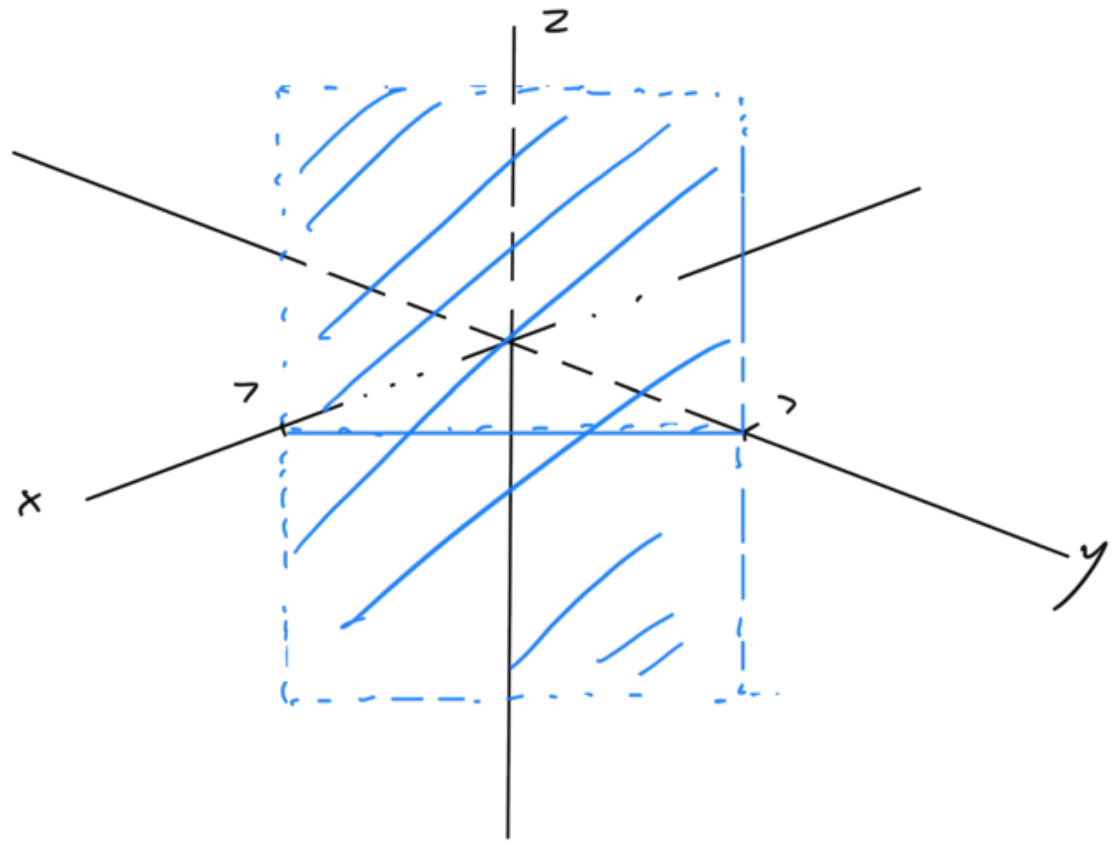
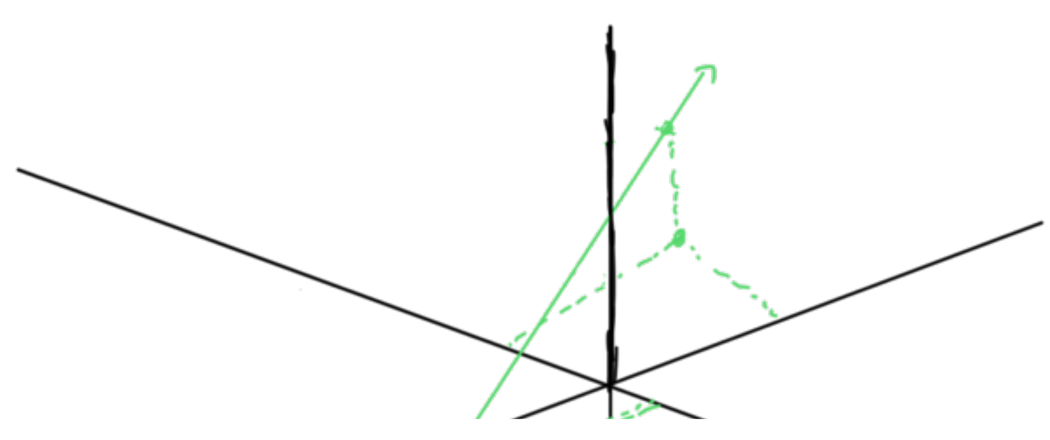
12.5 - Lines / Planes

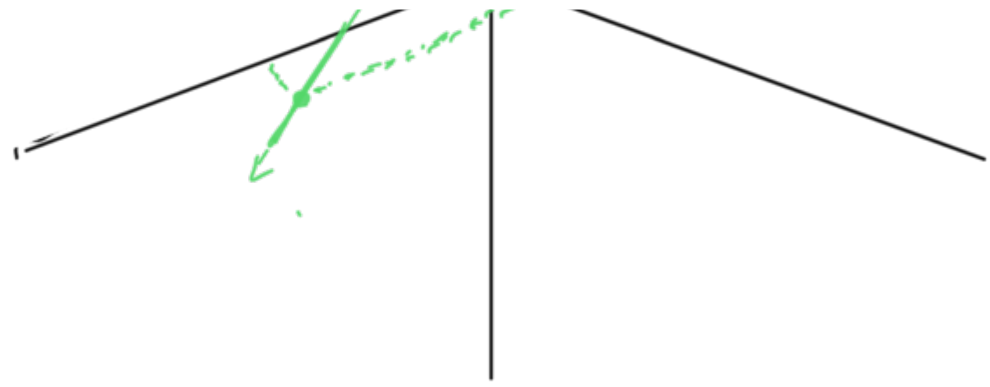
Lines still exist in $\mathbb{R}^3$, but how do we write them?

$x + y = 7$ is equation of a line in $\underline{\mathbb{R}^2}$ but in section 12.2 saw this is equation of a plane in $\mathbb{R}^3$



So how to write object like





Remember in $\mathbb{R}^2$ there were multiple ways to find equation of line based on what we knew. One of them:

Point-Slope : Know 1 point on line and slope

(Dont worry about exact form of point-slope form right now)

Can use similar concept for $\mathbb{R}^3$

Assume we know 1 point on the line.

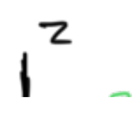
Can think of that point as a vector

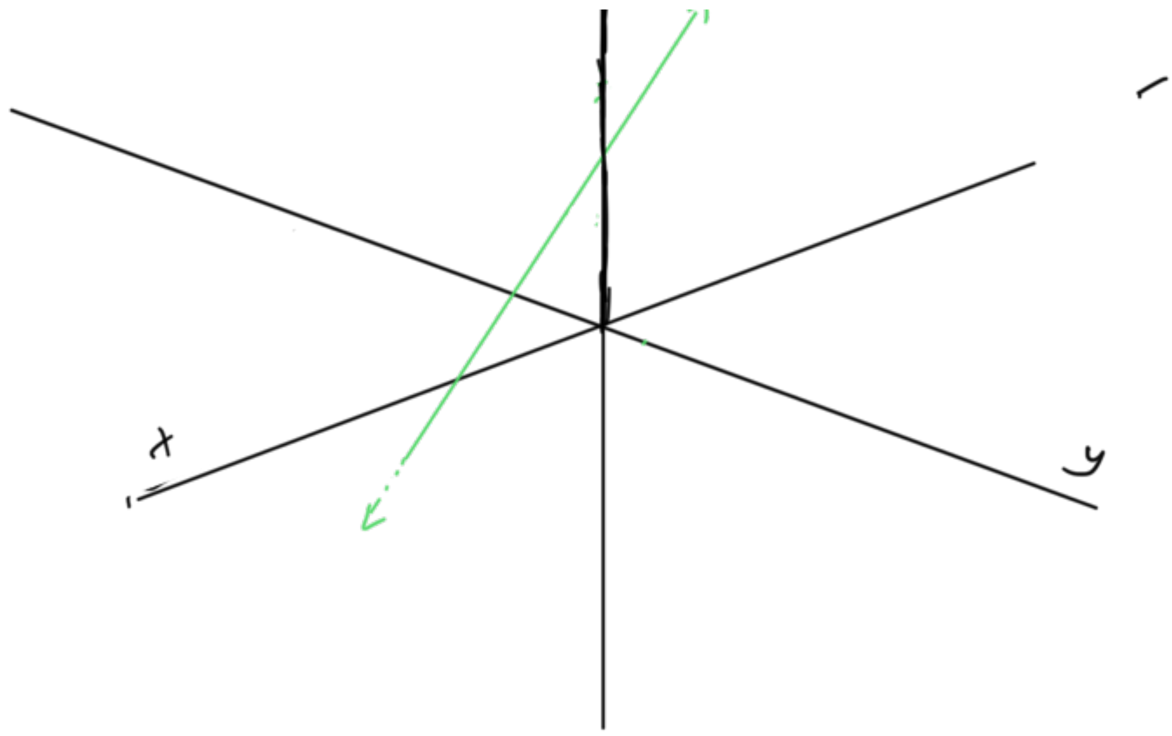
$\vec{r}_0 = \langle 4, 1, 0 \rangle$ $\quad$ $P(4, 1, 0)$

And assume we know a vector parallel to the line

$\vec{v} = \langle -5, -1, 4 \rangle$

t = time





$\vec{r}_0 + \vec{v}$ is on the line.

In fact if I scale $\vec{v}$ up/down (by multiplying by scalar t) can get any/all points on line

So if $\vec{r}$ is arbitrary vector/point on line then

$$\boxed{\vec{r} = \vec{r}_0 + t\,\vec{v}}$$

The above is a vector equation and t is a parameter

Roles of $t, \vec{v}$

- $\vec{v}$ not unique
- values of t at certain points

$-\infty < t < \infty$

Ex: Find vector eq. of line passing through $P = (1, 0, -7)$ and parallel to $\vec{v} = \langle 3, 5, 6 \rangle$

$$\vec{r}_0 = \vec{OP} = \langle 1, 0, -7 \rangle$$

$$\vec{v} = \langle 3, 5, 6 \rangle$$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\langle x, y, z \rangle = \langle 1, 0, -7 \rangle + t\langle 3, 5, 6 \rangle$$

Now think of components of the vectors in the above.

$$\begin{cases} \vec{r} = \langle x, y, z \rangle \\ \vec{r}_0 = \langle x_0, y_0, z_0 \rangle \\ \vec{v} = \langle a, b, c \rangle \end{cases}$$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$

Can write right sides as a single vector

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

Left side equals right side only if all their components are equal

Parametric Equations $\left\{ \begin{array}{l} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{array} \right\}$

May be asked to write them like:

$$(x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$$

Apparently the numbers a, b, c are called **direction numbers**

Eg: Find parametric equations of line passing through point $(-9, 13, 4)$ and parallel to $\langle 4, 6, 1 \rangle$

$$x = -9 + t4$$

$$\begin{cases} y = 13 + t6 \\ z = 4 + t1 \end{cases}$$

$$(x, y, z) = (-9 + t4,\ 13 + t6,\ 4 + t1)$$

Remember in $\mathbb{R}^2$, sometimes we were given two points on line, no slope

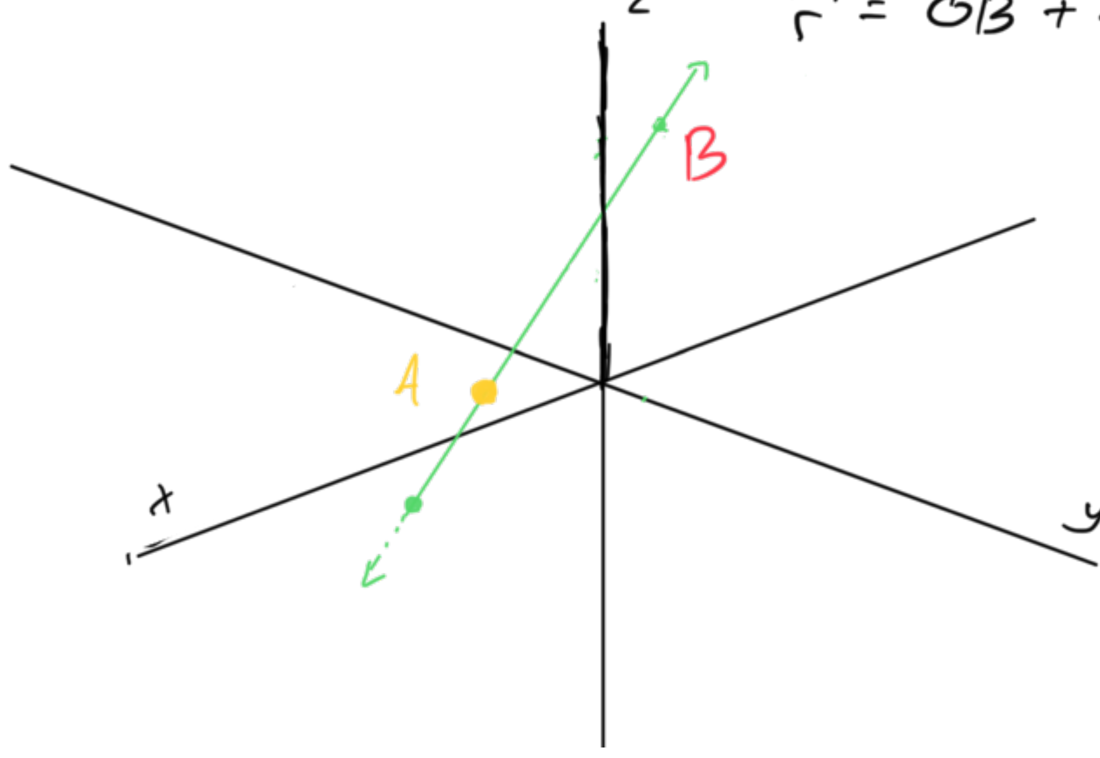
Had to find slope between the points and use that for point-slope formula.

Similarly, might be given two points on line in $\mathbb{R}^3$ but no parallel vector

$$\vec{r} = \vec{r_0} + t\vec{v}$$

$$\vec{r} = \overrightarrow{OA} + t\vec{v}$$

$$\vec{r} = \overrightarrow{OB} + t\vec{v}$$



Vector $\vec{v}$ from A to B parallel to our line. Could use either $\vec{r}_0 = \overrightarrow{OA}$ or $\vec{r}_0 = \overrightarrow{OB}$. So

$$\vec{r} = \overrightarrow{OA} + t\,\overrightarrow{AB}$$

or

$$\vec{r} = \overrightarrow{OB} + t\,\overrightarrow{AB}$$

or

$$\vec{r} = \overrightarrow{OA} + t\,\underline{\overrightarrow{BA}}$$

or

$$\vec{r} = \overrightarrow{OB} + t\,\overrightarrow{BA}$$

Eg: Find equation of line passing through $A = (\underline{4}, \underline{-1}, \underline{16})$ and $B = (\underline{0}, \underline{3}, \underline{-2})$.

$\rightarrow \vec{v} = \langle -4, 4, -18 \rangle$ parallel

$\rightarrow \vec{r}_0 = \langle 4, -1, 16 \rangle$ ✓

$\vec{r} = \langle x, y, z \rangle$

A

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$$

$$\boxed{\langle x, y, z \rangle = \langle 0, 3, -2 \rangle + t \langle -4, 4, -18 \rangle}$$

Symmetric Equations

Don't need them

Lines intersecting planes

Once we have equation of line, may be asked where it intersects a plane

Ex: Line with equation

$$\vec{r} = \langle 7, 2, 9 \rangle + t \langle 4, 1, -3 \rangle$$

$$x = 7 + t \cdot 4$$

$\vdots$

$$\boxed{z = 9 - t \cdot 3}$$

At what point does line intersect xy-plane?

Strategy:

① Determine equation of your plane

$$\boxed{z = 0}$$

② Set parametric equation for component equal to 0, solve for t

$$z = 9 + t(-3) = 9 - 3t$$

$$0 = 9 - 3t$$

$$\boxed{t = 3} \star$$

③ Plug value of t into equation of line to get point/vector

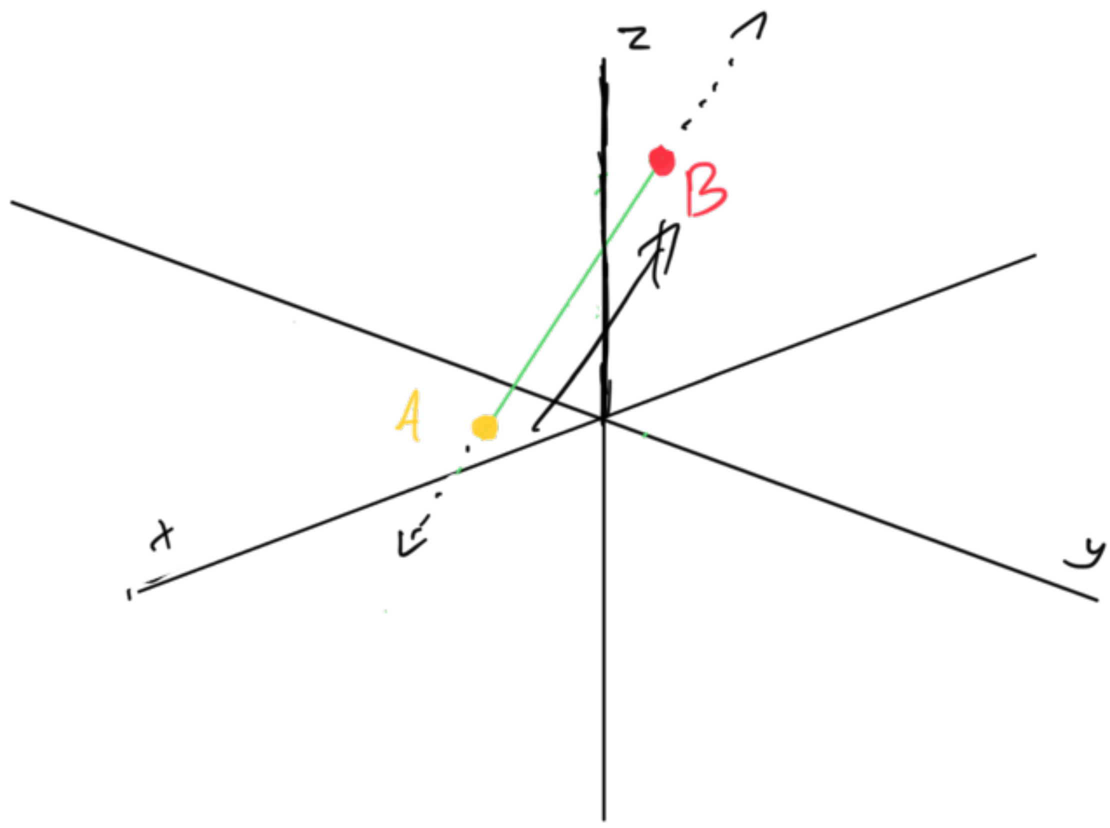
$$\Rightarrow \vec{r} = \langle 7, 2, 9 \rangle + t \langle 4, 1, -3 \rangle$$

$$\vec{r} = \langle 7, 2, 9 \rangle + 3 \langle 4, 1, -3 \rangle$$

$$\boxed{= \langle 19, 5, 0 \rangle} = (19, 5, 0)$$

Line Segments + Rays

What if I want to describe line segment between two points



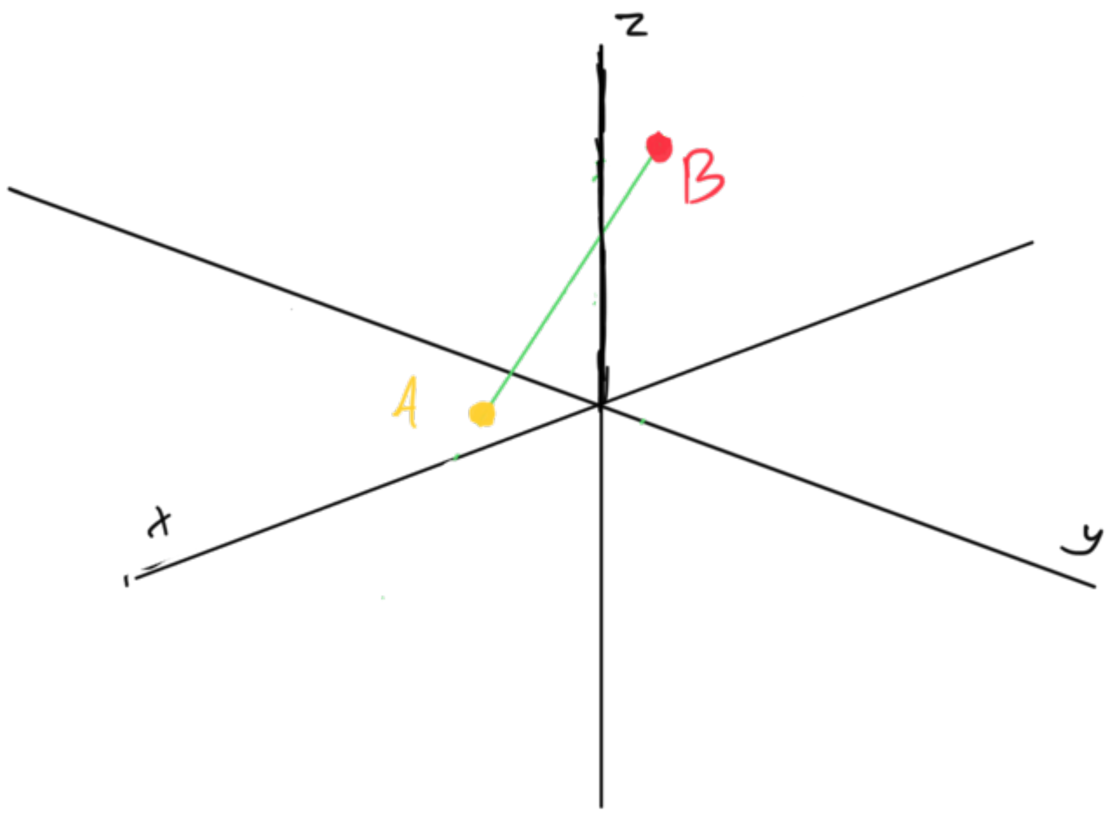
Recall method to find equation of line when we are given two points but no vector

First find vector $\vec{AB}$, then pick either A or B, write

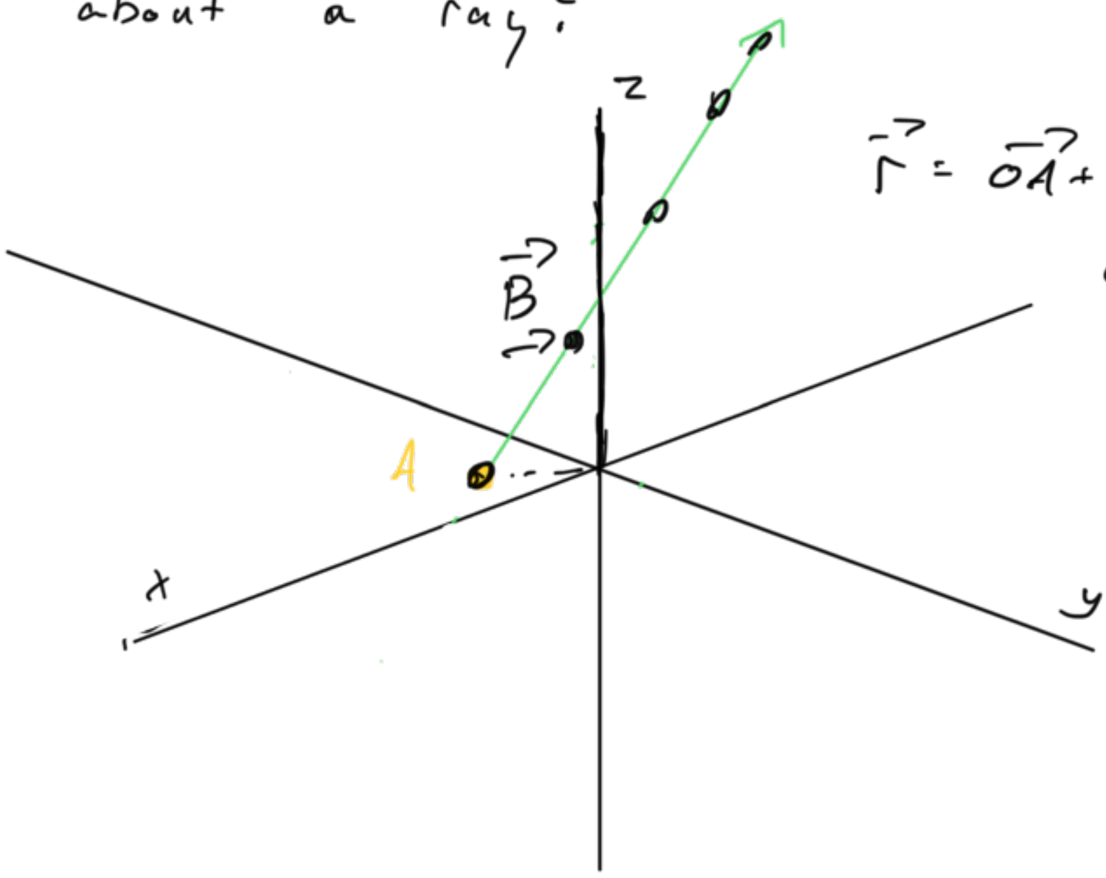
$$\vec{r} = \vec{OA} + t\,\vec{AB} \qquad -\infty < t < \infty$$

For line segment process is same but restrict t

$$0 \le t \le 1$$



What about a ray?



$$\vec{r} = \vec{OA} + t\,\vec{AB}$$

$$0 \le t < \infty$$

$$0 \le t < \infty$$

Planes

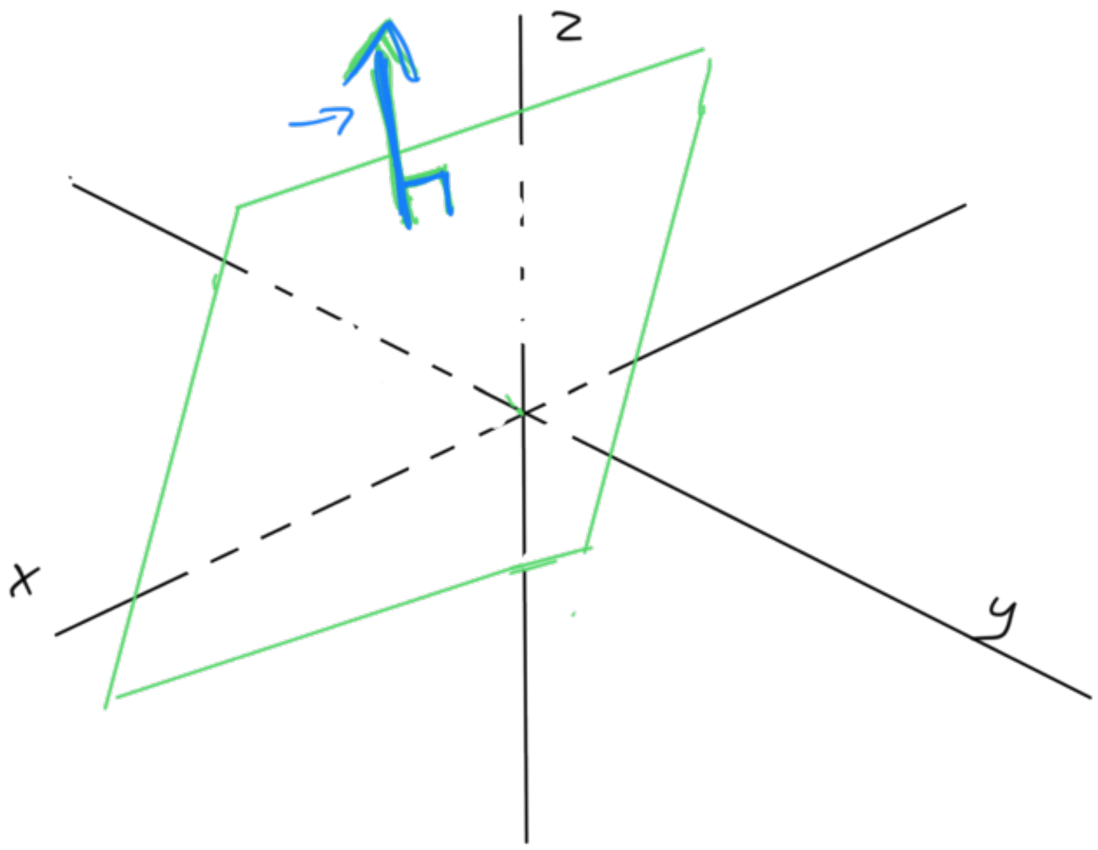
Have seen how to write very simple planes with basic orientations

Ex: $z = 0$

$x + y = 7$

etc.

What about:

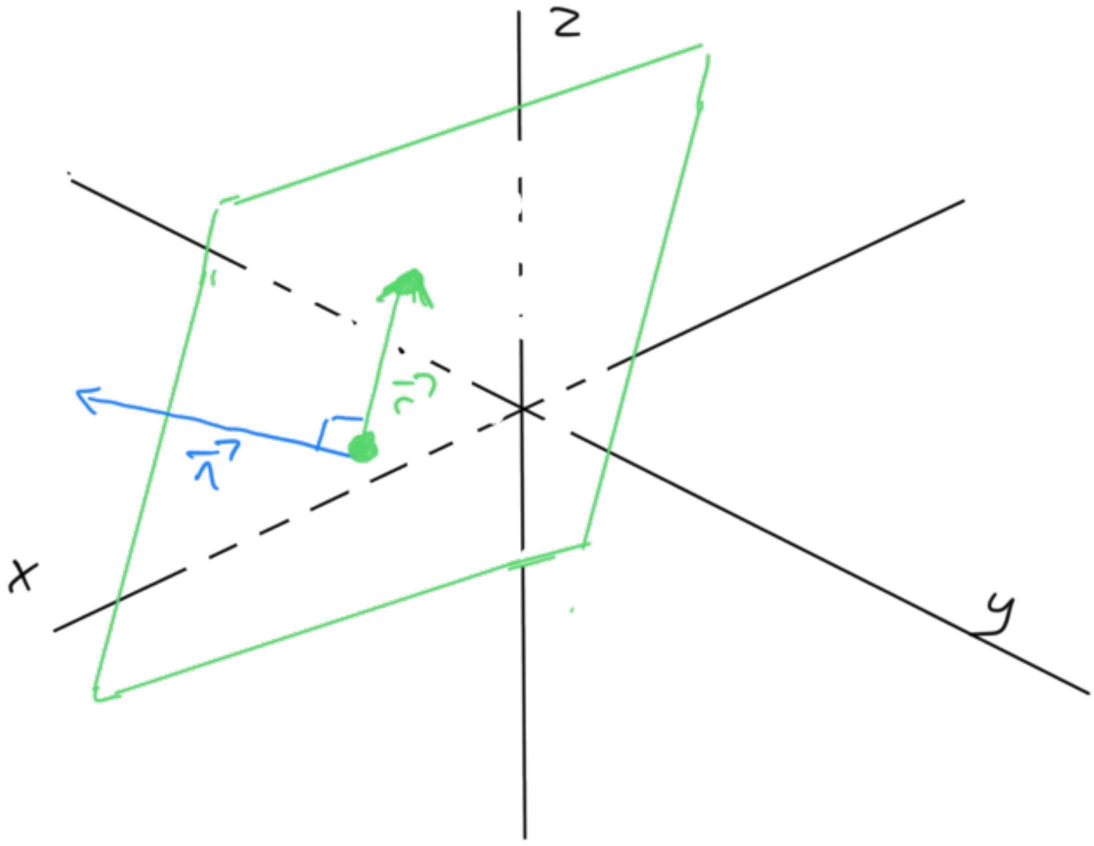


A plane is two-dimensional. Can always draw a vector coming straight out of (orthogonal to) plane to give us 3rd dimension

Such a vector is called a

normal vector $\vec{n}$

When we say $\vec{n}$ is orthogonal to plane, mean it is orthogonal to every vector in plane



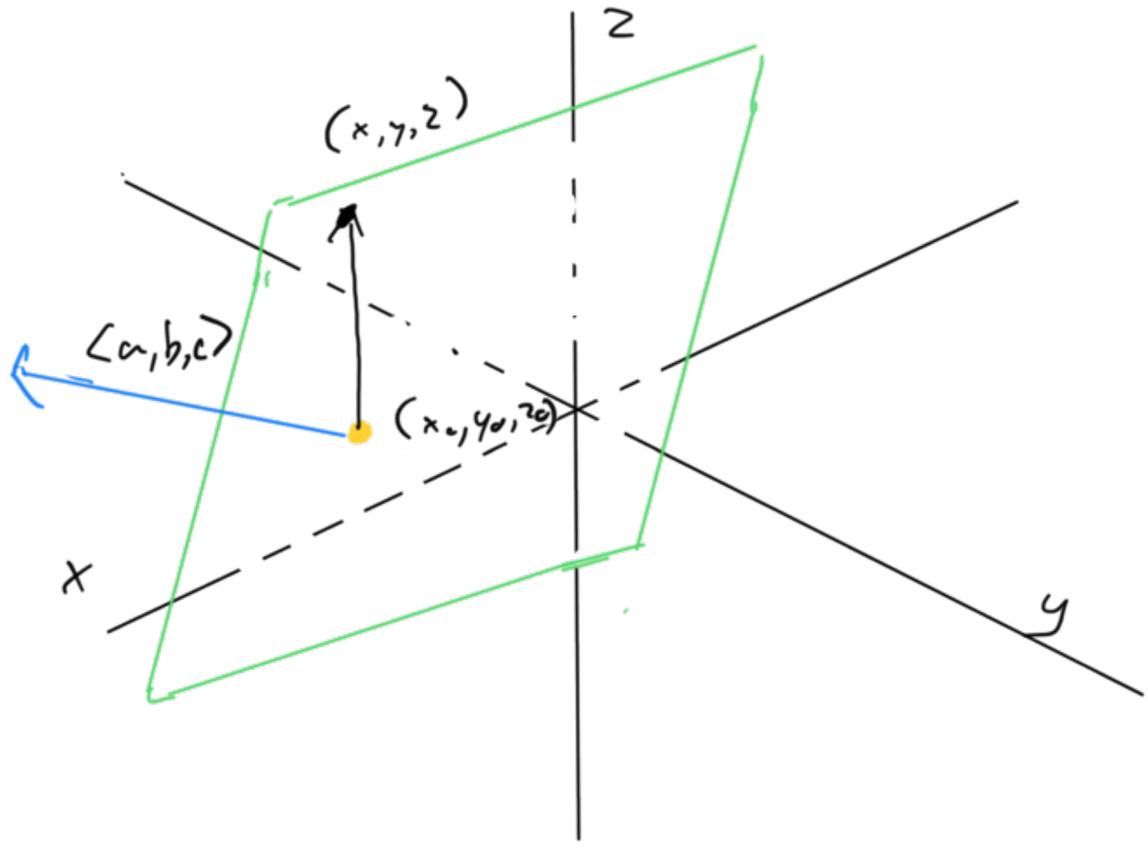
Remember, when two vectors are orthogonal their dot product is 0

To write an equation of a plane need:

$$\begin{cases} ① \text{ A point in the plane } (x_0, y_0, z_0) \\ ② \text{ A normal vector } \vec{n} = \langle a, b, c \rangle \end{cases}$$

The points (x, y, z) in plane will be ones such that $\langle x - x_0, y - y_0, z - z_0 \rangle$ orthogonal to $\vec{n}$

(x, y, z)



So

$$\langle \underline{x - x_0}, \underline{y - y_0}, \underline{z - z_0} \rangle \cdot \langle \underline{a}, \underline{b}, \underline{c} \rangle = 0$$

$$\Rightarrow \; ☆ \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad ☆$$

Call this Scalar Equation of plane

<u>Note</u>: May distribute scalar equation to rewrite as $ax + by + cz + d = 0$

<u>or</u>

$ax + by + cz = f$

Basically, linear equation in x, y, z is a plane

$$ax + 0y + cz = 0$$

Ex: Write scalar equation of plane that contains point $(1, 1, -9)$ and has normal vector $\langle 4, 6, 4 \rangle$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$4(x - 1) + 6(y - 1) + 4(z + 9) = 0$$

$$4x - 4 + 6y - 6 + 4z + 36 = 0$$

$$4x + 6y + 4z + 26 = 0$$

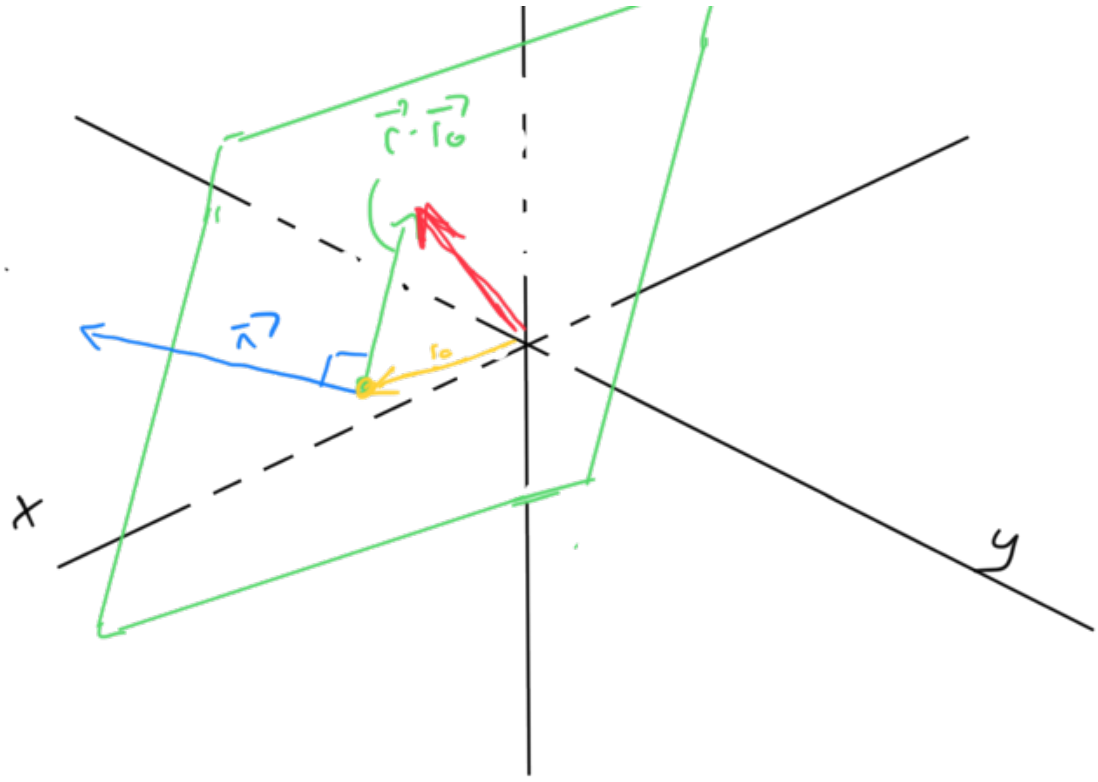
$$4x + 6y + 4z = -26$$

Can also write using vector notation

Let $\vec{r_0}$ be vector with endpoint on plane.

$\vec{r}$ will also have endpoint on plane if

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0$$



Call this **vector equation** of the plane

$$\begin{cases} \vec r = \langle x, y, z\rangle \\ \vec r_0 = \langle 1, 4, 6\rangle \\ \vec n = \langle 0, 0, 3\rangle \end{cases}$$

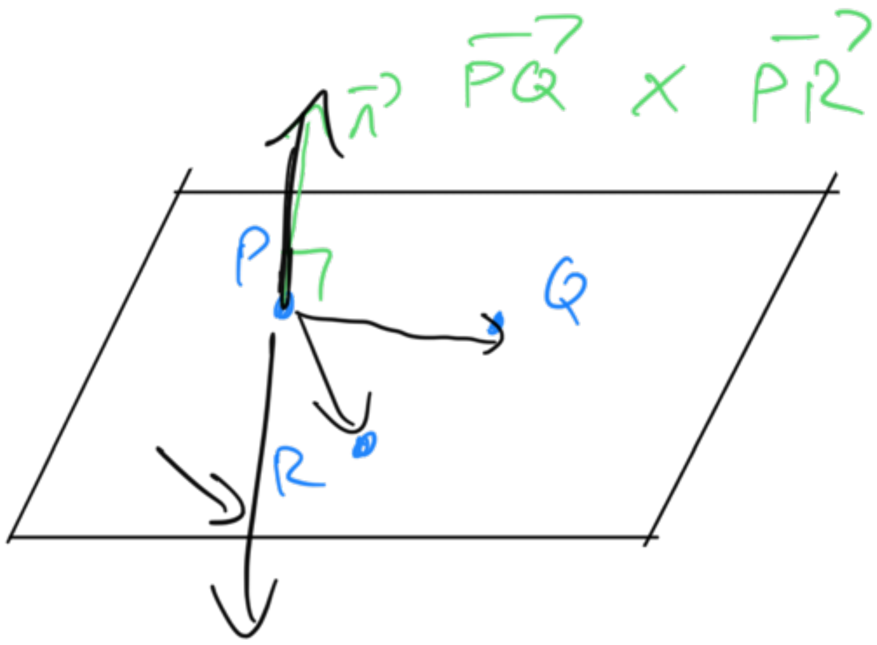
$$n \cdot (\vec r - \vec r_0) = 0$$

$$\langle 0, 0, 3\rangle \cdot (\langle x, y, z\rangle - \langle 1, 4, 6\rangle) = 0$$

$$\langle 0, 0, 3\rangle \cdot \langle x-1, y-4, z-6\rangle = 0$$

Special Case: Given 3 points on plane but no normal vector

Ex: Find equation of plane containing points $P(1,2,1)$ $Q(0,0,1)$ and $R(-1,0,-3)$



$\vec{PR} \times \vec{PQ}$

Solution: Find $\vec{PQ}$ and $\vec{PR}$.

Both vectors in plane.

$\vec{PQ} \times \vec{PR} = \vec{n}$ will be orthogonal to both (a normal vector)

Then use one vector in plane and $\vec{n}$ to form equation

$\boxed{\text{Ex}}$ $P(1,3,2)$ $Q(-1,-4,6)$

$R\ (2,0,2)$

$\vec{PQ} = \langle -2, -7, 4 \rangle$ ☆

$\vec{PR} = \boxed{\langle 1, -3, 0 \rangle}$ ☆

$\vec{PQ} \times \vec{PR}$

$$\begin{bmatrix} i & j & k \\ -2 & -7 & 4 \\ 1 & -3 & 0 \end{bmatrix}$$

$i\,(12) - j\,(-4) + k\,(6+7)$

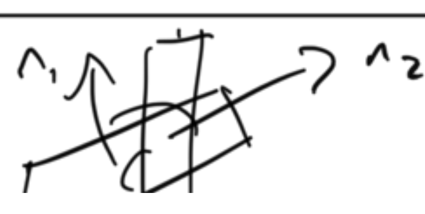
$12i + 4j + 13k$

$\vec{n}\ \langle 12, 4, 13 \rangle$

☆ $\vec{n} \cdot (\vec{r} - \vec{r_0})$

$$\boxed{\langle 12, 4, 13 \rangle \cdot (\langle x, y, z \rangle - \langle 1, -3, 0 \rangle) = 0}$$

$12(x-1) + 4(y+3) + 13(z-0) = 0$



Notes

- Angle between planes is the angle between their normal vectors

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1|\,|\vec{n}_2| \cos(\theta)$$

$$\cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|}\right) = \theta$$

- Planes are parallel only if their normal vectors are parallel

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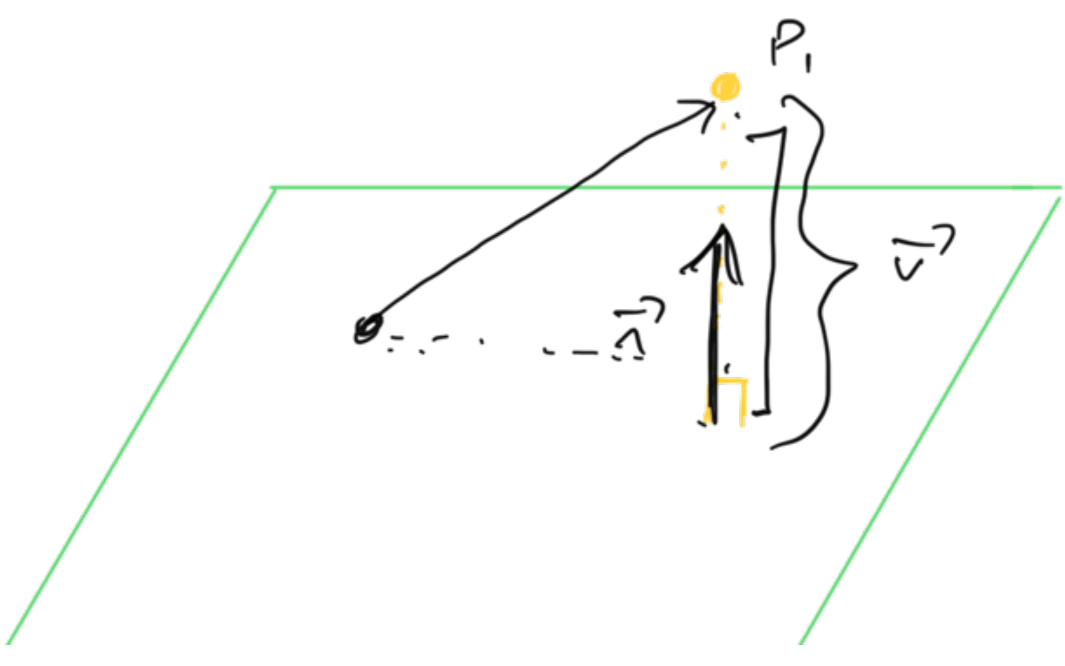
Distances

Concerned with distance from given point to given plane. (Shortest distance)

$$\overrightarrow{P_0P_1}$$



Some part of yellow dotted line can be thought of as plane's normal vector $\vec{n}$

Pick point P_0 in plane, project vector $\overrightarrow{P_0P_1}$ onto $\vec{n}$.

$$\underbrace{\frac{\overrightarrow{P_0P_1} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n}} = \vec{v}$$

$$\left| \frac{\overrightarrow{P_0P_1} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} \right|$$

$$\left| \frac{\overrightarrow{P_0P_1} \cdot \vec{n}}{|n|^2} \vec{n} \right|$$