Section 6.3

In previous section discussed projection of vector onto a subspace. Formalize it in next theorem.

Orthogonal Decomposition Theorem: Let W be a subspace of $\mathbb{R}^n$. Then each $\vec{y}$ in $\mathbb{R}^n$ can be written uniquely as:

$$\vec{y} = \hat{y} + \vec{z}$$

where $\hat{y}$ is in W and $\vec{z}$ in $W^\perp$. If $\{\vec{u}_1, \dots \vec{u}_p\}$ orthogonal basis for W then

$$☆ \quad \hat{y} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}u_2 + \dots \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p \quad ☆$$

and $\vec{z} = \vec{y} - \hat{y}$.

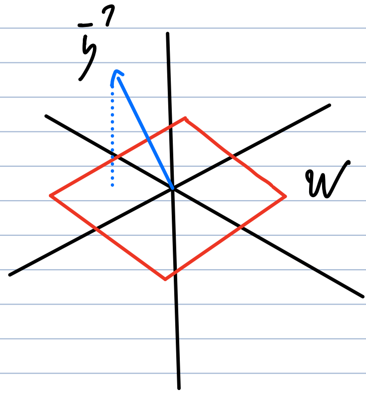
□

Fact

- Linear independent set can be made orthogonal

W has basis $\mathcal{B} = \{\vec{w}_1, \dots \vec{w}_k\}$. Contained inside basis for V, $\{\vec{w}_1 \dots \vec{w}_k, \vec{v}_{k+1} \dots \vec{v}_n\}$. Linear independent. Can make it orthogonal, $\{\hat{w}, \dots \hat{w}_k, \hat{v}_{k+1} \dots \hat{v}_n\}$. $\vec{y}$ has unique representation as

$$\vec{y} = \underbrace{c_1\hat{w}_1 + \dots c_k\hat{w}_k}_{\text{in } W} + \underbrace{c_{k+1}\hat{v}_{k+1} + \dots c_n\hat{v}_n}_{\text{in } W^\perp}$$



When projecting $\vec{v}$ onto a subspace W, still have idea of "shadow" cast by $\vec{v}$

From this picture, clear that $\text{proj}_W \vec{v}$ is "best approximation to $\vec{v}$ in W"

Best Approximation Theorem: W subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$. Let $\hat{y}$ be orthogonal projection of $\vec{y}$ onto W. Then $\hat{y}$ is closest point in W to $\vec{y}$, meaning for any <u>other</u> $\vec{\omega} \in W$

$$\| \vec{y} - \hat{y} \| < \| \vec{y} - \vec{\omega} \|$$

$\square$ Consider $\| \vec{y} - \vec{\omega} \|$.

$$\| \vec{y} - \vec{\omega} \| = \| \vec{y} - \hat{y} + \hat{y} - \vec{\omega} \|$$

Add/subtract $\hat{y}$
Doesn't change anything

$$= \| (\vec{y} - \hat{y}) + (\hat{y} - \vec{\omega}) \|$$

Consider this as two vectors

By def. of orthog. projection $y - \hat{y}$ is orthogonal to all of W

$\hat{y}, \vec{\omega}$ both in W, then so is $\hat{y} - \vec{\omega}$

Remember if $\vec{a}, \vec{b}$ orthogonal that $\|\vec{a}+\vec{b}\| = \|\vec{a}\| + \|\vec{b}\|$

$(\vec{y} - \hat{y})$ and $(\hat{y} - \vec{w})$ orthogonal so

$$\|\vec{y} - \vec{w}\| = \|(\vec{y} - \hat{y}) + (\hat{y} - \vec{w})\| = \|\vec{y} - \hat{y}\| + \|\hat{y} - \vec{w}\|$$

Greater than 0

So

$$\|\vec{y} - \vec{w}\| = \|y - \hat{y}\| + \|\hat{y} - \vec{w}\| > \|y - \hat{y}\|$$

Written in more standard way:

$$\|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{w}\|$$

What if our orthogonal vectors give us all of $\mathbb{R}^n$, not just subspace? (i.e. form basis)

Then ortogonal projection of $\vec{y}$ is just $\vec{y}$ itself, but have nice formula for coordinates of $\vec{y}$. ($\vec{v}_1 \ldots \vec{v}_n$ orthogonal basis)

$$\vec{y} = \underline{\frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}} \vec{v}_1 + \underline{\frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}} \vec{v}_2 + \ldots \underline{\frac{\vec{y} \cdot \vec{v}_n}{\vec{v}_n \cdot \vec{v}_n}} \vec{v}_n$$

Have similar if basis is orthonormal but $v_i \cdot v_i = 1$, so denominators all one.

$$y = (\vec{y} \cdot \vec{v}_1)\vec{v}_1 + \ldots + (\vec{y} \cdot \vec{v}_n)\vec{v}_n$$

In general have:

Theorem: If $\vec{u}_1, \dots \vec{u}_p$ is an orthonormal basis for subspace W of $\mathbb{R}^n$ then

$$\text{proj}_W \vec{y} = (\vec{y}\cdot\vec{u}_1)\vec{u}_1 + \dots (\vec{y}\cdot\vec{u}_p)\vec{u}_p$$

If $U = \begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_p \\ | & & | \end{bmatrix}$ then:

★ $\text{proj}_W \vec{y} = UU^T\vec{y}$ for all $\vec{y} \in \mathbb{R}^n$

$$\underbrace{\begin{bmatrix} — \vec{u}_1 — \\ — \vec{u}_2 — \\ — \vec{u}_n — \end{bmatrix}}_{U^T}\underbrace{\begin{bmatrix} \vec{y} \end{bmatrix}}_{\vec{y}} = \begin{bmatrix} \vec{u}_1\cdot\vec{y} \\ \vdots \\ \vec{u}_n\cdot y \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} — \vec{u}_1 — \\ — \vec{u}_2 — \\ — \vec{u}_n — \end{bmatrix}}_{U^T}\underbrace{\begin{bmatrix} \vec{y} \end{bmatrix}}_{\vec{y}} = \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}}_{U}\begin{bmatrix} \vec{u}_1\cdot\vec{y} \\ \vdots \\ \vec{u}_n\cdot\vec{y} \end{bmatrix}$$

$$= (\vec{u}_1\cdot\vec{y})\begin{bmatrix} \vec{u}_1 \end{bmatrix} + (\vec{u}_2\cdot\vec{y})\begin{bmatrix} \vec{u}_2 \end{bmatrix} + \dots$$

$$= (\vec{u}_1 \cdot \vec{y})\vec{u} + (\vec{u}_2 \cdot \vec{y})\vec{u}_2 + \dots$$

18 b)

$$\vec{y} = \begin{bmatrix} 7 \\ 9 \end{bmatrix} \qquad \vec{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$$

$\vec{u}_1$ orthonormal basis for $W = \text{span}\{\vec{u}_1\}$

$$\text{proj}_{\vec{u}}\,\vec{y} = \underline{U}^T y$$

$$\begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$2 \times 1 \qquad 1 \times 2$

$$\begin{bmatrix} 1/10 & -3/10 \\ -3/10 & 9/10 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

UU^T

$$= \begin{bmatrix} -20/10 \\ 60/10 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$