Section 4.1

Chapter 4 is about vector spaces, which make up the some of most fundamental structure in mathematics.

Vector spaces are the setting we work in.
Range from very basic/familiar

$\mathbb{R}$, $\mathbb{R}^2$, etc.

to fancy/complicated

Normed spaces, Hilbert spaces, Sobolev spaces

Can be used to make sense of everything from simple operation like addition to the solutions to systems of differential equations.

Definition: A <u>vector space</u> is a nonempty set V of objects (vectors) along with well defined operations of <u>vector addition</u>, <u>scalar multiplication</u> such that following properties hold for all $\vec{u}, \vec{v}, \vec{w} \in \underline{V}$ and all $\underline{a, b} \in \mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$)

① If $\vec{u}, \vec{v} \in V$ then $\vec{u} + \vec{v} \in V$ (closed under vec. addition)

② $\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (Commutativity)

3. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ (Associativity)

4. $\exists\ \vec{0} \in V$ s.t. $\vec{u}+\vec{0}=\vec{u}$ for all $\vec{u}\in V$

5. For each $\vec{u}\in V$, $\exists\ -\vec{u}\in V$ s.t. $\vec{u}+(-\vec{u})=\vec{0}$

6. If $\vec{u}\in V$, $c\in\mathbb{F}$ then $c\vec{u}\in V$ (closed under scalar mult.)

7. $c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

8. $(c+d)\vec{u} = c\vec{u}+d\vec{u}$

9. $c(d\vec{u}) = (cd)\vec{u}$

10. $1\vec{u} = \vec{u}$

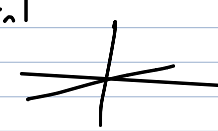
$\mathbb{R}^2$

Vector spaces can be familiar and easily represented.

Ex Consider set $\mathbb{R}^n$

$V=$ Elements/vectors in $\mathbb{R}^n$ of form $\vec{x}=\begin{vmatrix} x_1 \\ \vdots \\ x_n \end{vmatrix}$

Vector addition $\vec{x}+\vec{y} = \begin{vmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{vmatrix}$



Scalar mult. $c\vec{x} = \begin{vmatrix} cx_1 \\ \vdots \\ cx_n \end{vmatrix}$

Used to all these operations, pretty familiar

Ex Set of continuous functions on $[0,1]$

$V=$ Vectors are functions. No convenient way to write as a list of components

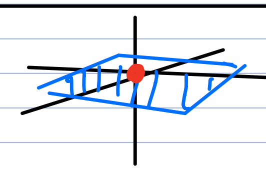
$$\left\{\begin{array}{l} \text{Vector addition: } f+g \text{ is continuous function s.t.} \\ \quad \underline{f(x)}+\underline{g(x)} = \underline{(f+g)(x)} \text{ for all } x\in[0,1] \\ \text{Scalar mult: } cf \text{ is cont. function s.t.} \\ \quad cf(x) = c\cdot f(x) \text{ for all } x\in[0,1] \end{array}\right.$$

Often want to consider a part of a larger vector space.

Definition: A subspace, H, of a vector space V is a subset of V that satisfies following:

1. Zero vector of V is in H
2. H is closed under vector addition (i.e. $\vec{u}, \vec{v} \in H$ means $\vec{u} + \vec{v} \in H$)
3. H is closed under scalar multiplication ($c \in \mathbb{F}$, $\vec{u} \in H$ then $c\vec{u} \in H$)

Ex Vector space $\mathbb{R}^3$



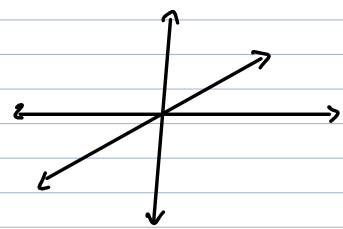
xy-plane of form $\begin{vmatrix} x \\ y \\ 0 \end{vmatrix}$ is subspace

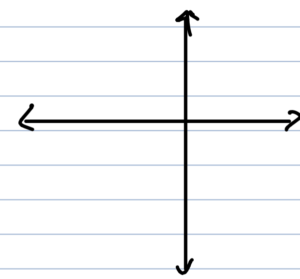
Ex Polynomials on $[0,1]$ form subspace of $C([0,1])$

Every vector space has some subspace

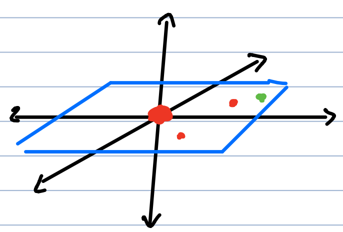
- V a vector space. V is a subspace of itself
- V a vector space with zero vector $\vec{0}$. $H = \{\vec{0}\}$ is subspace of V. (Trivial/zero subspace)

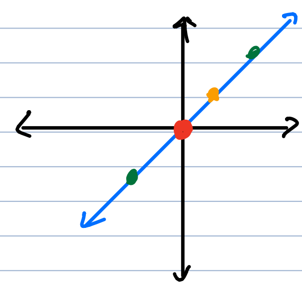
Vector spaces may be abstract but may be useful to imagine them having some geometry as $\mathbb{R}^n$ $\quad C([0,1])$





Have similar ways to picture subspaces





May recall these pictures similar to when we introduced idea of span. Indeed, will see that if $\vec{v}_1, \dots \vec{v}_k \in V$ then $\text{Span}\{\vec{v}_1, \dots \vec{v}_k\}$ always subspace of V

Theorem: If $\vec{v}_1, \dots \vec{v}_k$ are in V then $H = \text{Span}\{\vec{v}_1, \dots \vec{v}_k\}$ is a subspace of V.

$\square$ $\quad \vec{y} \in \text{Span}\{\vec{v}_1 \dots \vec{v}_k\}$

$\underline{\vec{y}_1} = \underline{a_1 \vec{v}_1 + \dots a_k \vec{v}_k}$

$$\underline{\vec{y_2}} = \underline{b_1 \vec{v_1} + \dots b_k \vec{v_k}}$$

$\vec{y_1} + \vec{y_2}$ in Span ?

$$\underline{(a_1+b_1)}\,\underline{\vec{v_1}} + \dots (a_k+b_k)\,\underline{\vec{v_k}} \in \text{Span}\{v_1, \dots v_k\} = H$$

$$c\,\vec{y_1} = c\,(a_1 \vec{v_1} + \dots c_k \vec{v_k})$$

$$= \underline{c a_1}\,\underline{\vec{v_1}} + \dots \underline{c a_k}\,\underline{\vec{v_k}}$$

$$\in \text{span}\{v_1 \dots v_k\} = H$$

Is $\vec{0} \in \text{Span}\{v_1, \dots v_k\}$?

$$\underbrace{0\vec{v_1} + \dots 0\vec{v_k}}_{} = \vec{0} \in \text{Span}\{v_1 \dots v_k\}$$