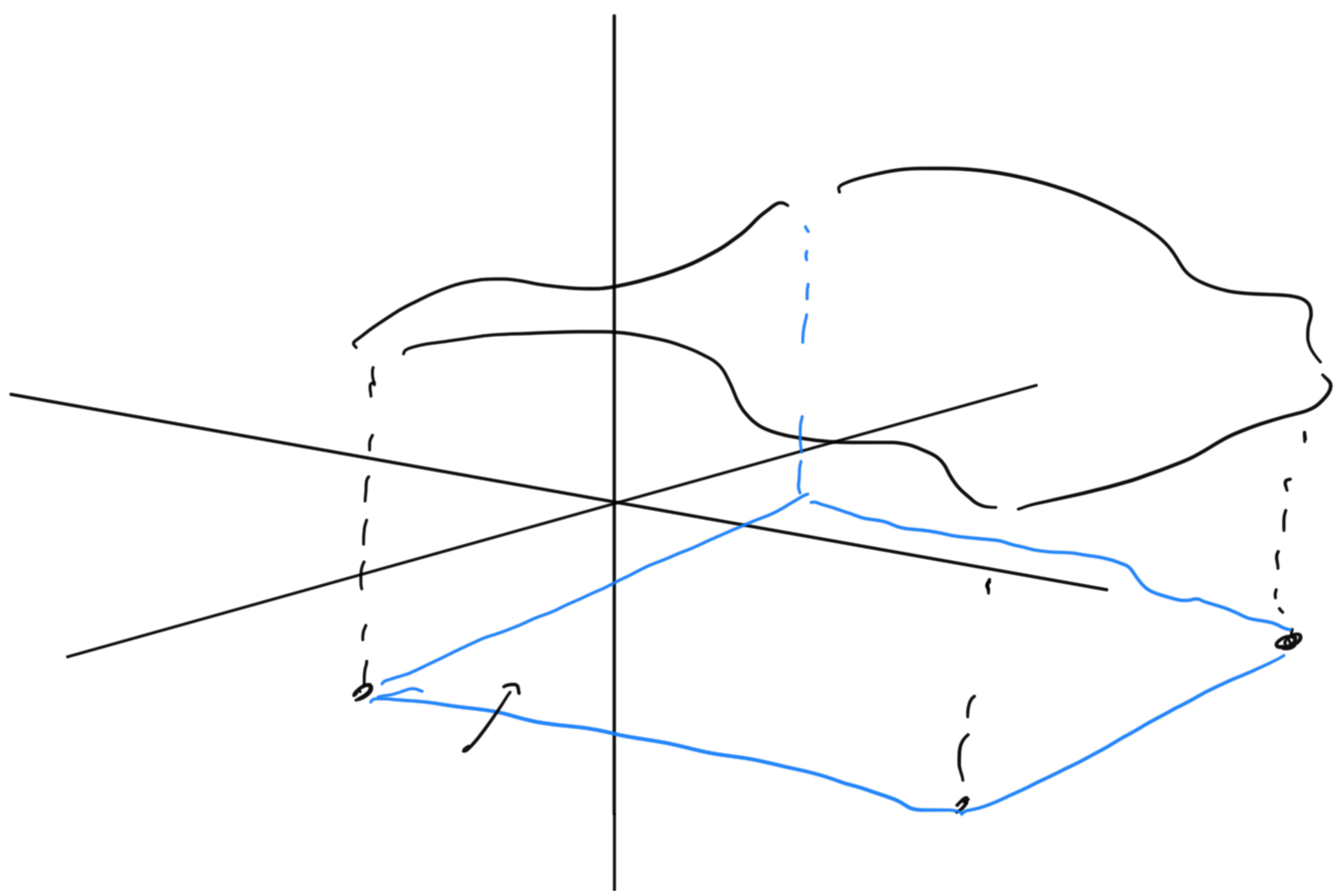
16.2 - Line Integrals

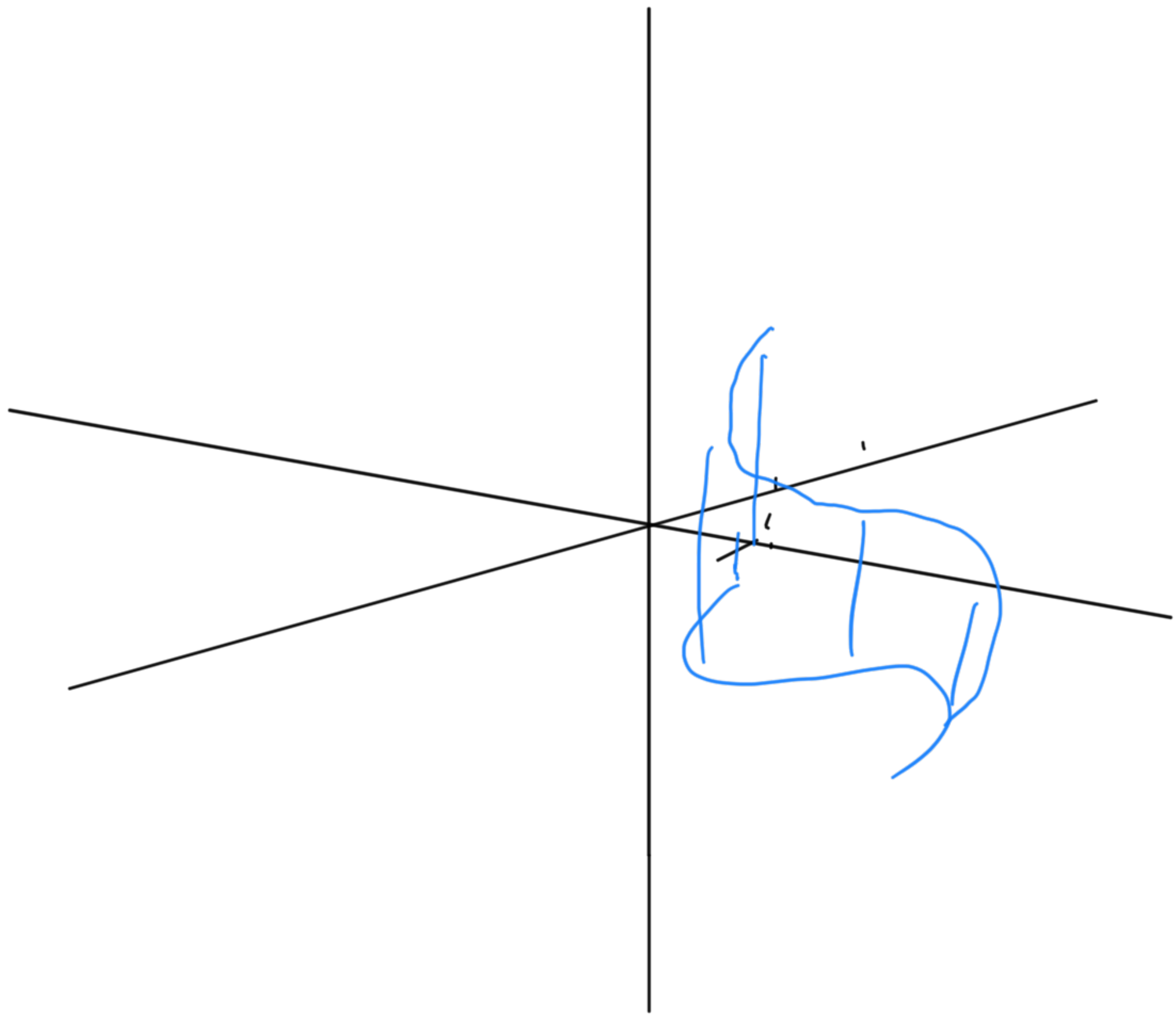
Going to seem very similar to arc length from section 13.3, and not very connected to 16.1. But will make a connection by end of section.

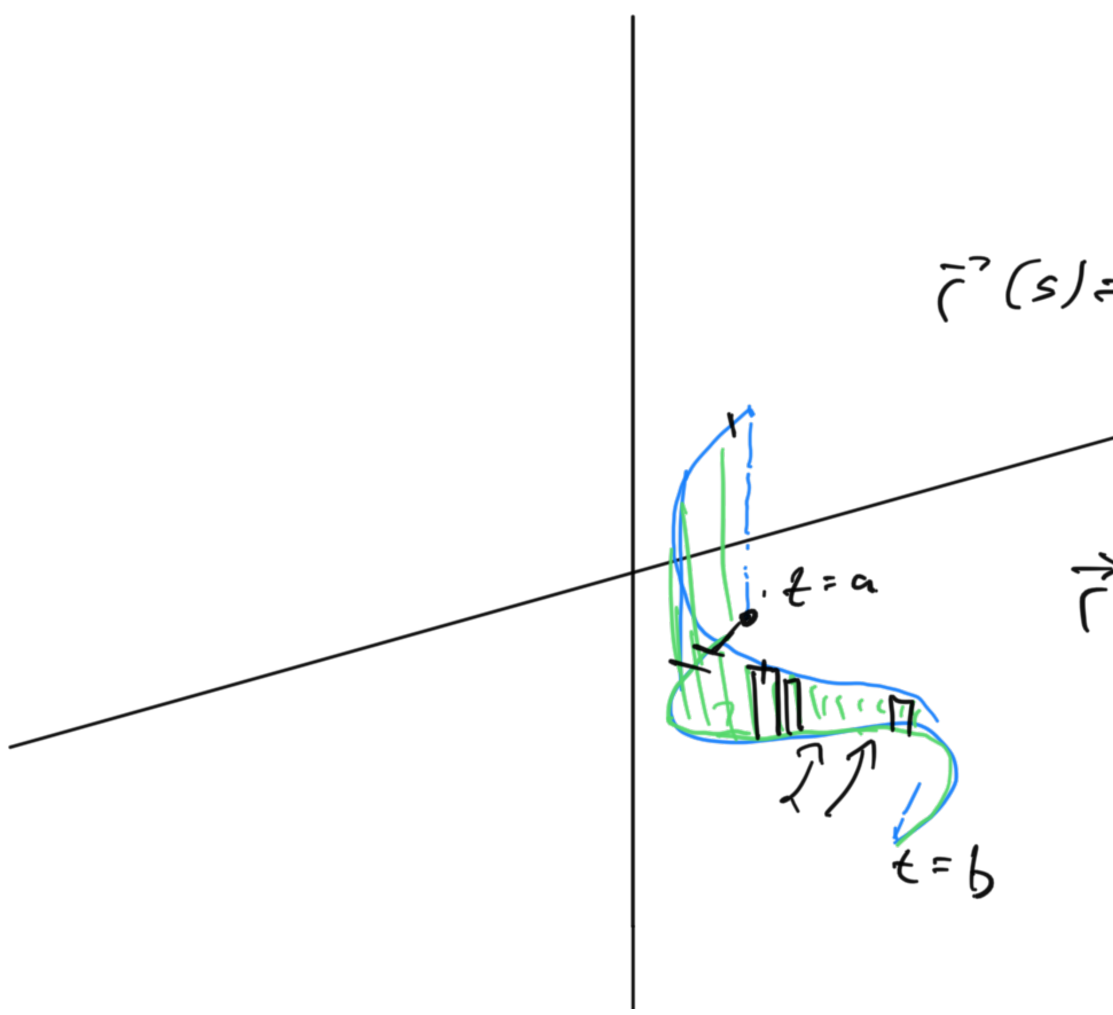
In Chapter 15 we learned how to integrate $f: \mathbb{R}^2 \to \mathbb{R}$.

Integrate on region D in $\mathbb{R}^2$, get "volume" under curve on that region



What if we don't integrate on entire region? What if we only integrate on a thin piece?





$\vec{r}(s) = \langle x(s), y(s) \rangle$

$\vec{r}(t) = \underline{\langle x(t), y(t) \rangle}$

$$\sum f(x^*, y^*) \, \Delta s$$

Compute:

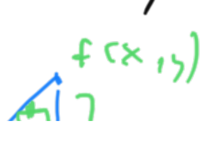
$$\bigstar \quad \bigstar \int_C f(x,y) \, ds$$

$$\int_a^b f(x(t), y(t)) \, |r'(t)| \, dt$$

See right away this won't give a volume. Should give an "area"

Once again this integral is developed via a Riemann sum.

Approximate area by rectangles.



$\vec{r}(t) = \langle f(t), g(t) \rangle$

Base of rectangles: depends on the curve we are on, depends on arc length. But for now denote as Δs

Height of rectangles: Values of f along curve, $f(x_i^*, y_i^*)$

Riemann Sum:

$$\sum_{i=1}^{n} \underbrace{f(x_i^*, y_i^*)}_{\text{Area}} \underline{\Delta s_i}$$

Integral $\Big\downarrow \quad \Big\downarrow \quad \Big\downarrow$

$$\underbrace{\int_C f(x,y)}\ \underbrace{ds}_{?}$$

In order to evaluate integral, we would like to rewrite ds in terms of x and y

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Recall from section on arc length (13.3) that if curve is $\langle x(t), y(t) \rangle$ then

$\underline{s} = \int \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$

- $$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

So if our curve $C = \langle x(t), y(t) \rangle$

the integral

☆ $\int_C f(x, y)\, \underbrace{ds}$

becomes

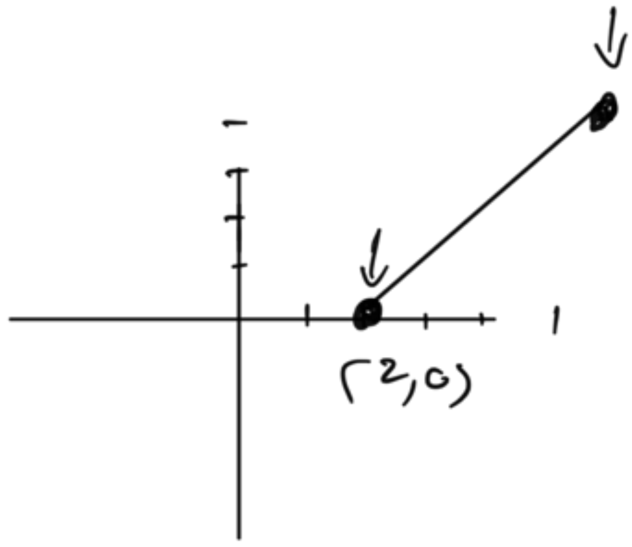
$$\int_a^b f(x(t), y(t)) \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

$a =$ t value that gives starting point of the curve

$b =$ t value that gives endpoint

Ex: $\int_C x e^y \, ds$

where C is line segment from $(2, 0)$ to $\underline{(5, 4)}$



$\vec{r}(t) = \langle 2 + 3t, 4t \rangle$

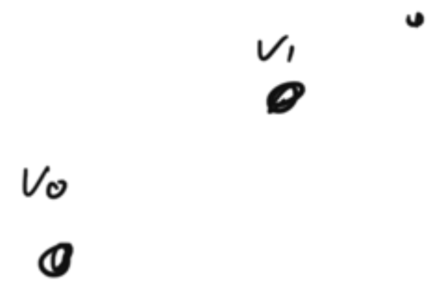
$t =$

Specifically:

$\vec{v_0}$ $\vec{v_1}$

$\vec{r}(t) = \vec{v_0} + (1 - t)(\vec{v_1} - \vec{v_0})$

$1 \le t \le 4$



$\vec{r}(t) = \langle \quad , \quad \rangle$

$\vec{r}(t) = \underline{\vec{r}_0} + t\vec{v}$

$\vec{v} = \langle 3, 4 \rangle$

$\vec{r}(t) = \langle 2, 0 \rangle + t\langle 3, 4 \rangle$

$0 \le t \le 1$

$$\boxed{\begin{array}{c}\vec{r}(t) = \langle 2+3t, 4t \rangle \\ 0 \le t \le 1\end{array}}$$

$r'(t) = \langle 3, 4 \rangle \qquad |r'(t)| = 5$

$$\int_0^1 (2+3t)e^{4t} \; 5 \, dt$$

$$5\int_0^1 2e^{4t} + 3te^{4t} \, dt$$

$$10\int_0 e^{4t}\,dt$$

$$\boxed{\frac{10}{4}\left(e^4-1\right)}$$

$$15\int_0^1 t e^{4t}\,dt$$

$$15\left(t\frac{e^{4t}}{4}\Big] - \int\frac{e^{4t}}{4}\,dt\right)$$

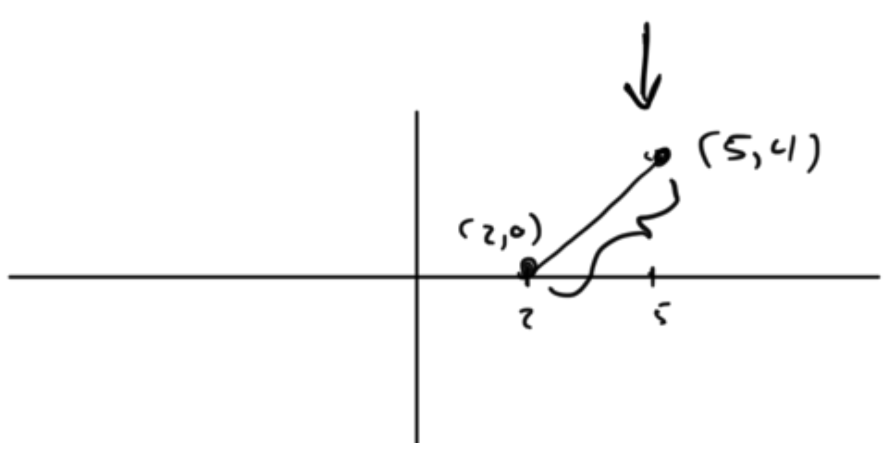
$$15\left(t\frac{e^{4t}}{4} - \frac{e^{4t}}{16}\Big]_0^1\right)$$

$$15\left(\left(\frac{e^4}{4} - \frac{e^4}{16}\right) - \left(0 - \frac{1}{16}\right)\right)$$

$$\boxed{15\left(\frac{3e^4}{16} + \frac{1}{16}\right)}$$

$$\boxed{\frac{16}{4}\left(e^4-1\right) + 15\left(\frac{3}{16}e^4 + \frac{1}{16}\right)}$$

13 First, consider C.



$$\vec{r}(t) = \langle \underline{x(t)}, \underline{y(t)} \rangle$$

$$\vec{r}(t) = \langle 2,0 \rangle + t\langle 3,4 \rangle$$

for $0 \le t \le 1$

$$\vec{r}(t) = \langle 2+3t, 4t \rangle$$

$0 \le t \le 1$

$$\int_0^1 f(x,y) \sqrt{(3)^2 + (4)^2}\, dt$$

$\sqrt{(dx/dt)^2 + (dy/dt)^2}$

$$\int_0^1 f(\underline{2+3t}, \underline{4t}) \sqrt{3^2+4^2}\, dt$$

$$\int_0^1 (2+3t)e^{4t}\sqrt{25}\, dt$$

$$= 5\int_0^1 (2e^{4t} + 3te^{4t})\, dt$$

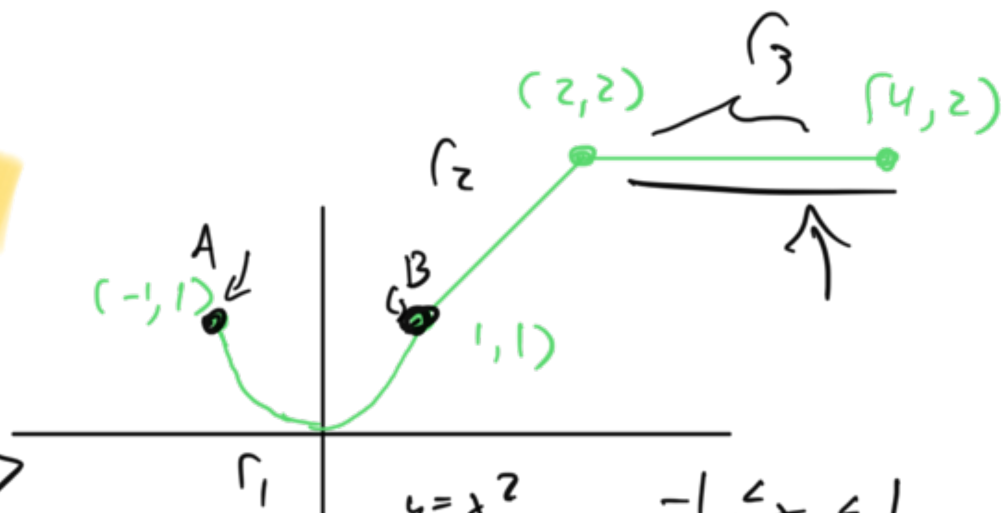
$$= 5\left(\frac{2}{4}e^{4t} + \frac{3}{4}te^{4t} - \frac{3}{16}e^{4t}\right)\Big]_0^1$$

$$= \left(\frac{10}{4}e^4 + \frac{15}{4}e^4 - \frac{15}{16}e^4\right) - \left(\frac{10}{4}\right)$$

These problems become more complicated as curve C we integrate along becomes more complicated.

C may be <u>piecewise smooth</u>

Ex:



$r_1(t) = \langle \quad , \quad \rangle$

<u>$0 \le t \le 1$</u>

$r_2(t)$

$3 \le t \le 4$

$y = x^2 \qquad -1 \le x \le 1$

$y = x \qquad 1 \le x \le 2$

$y = 2 \qquad 2 \le x \le 4$

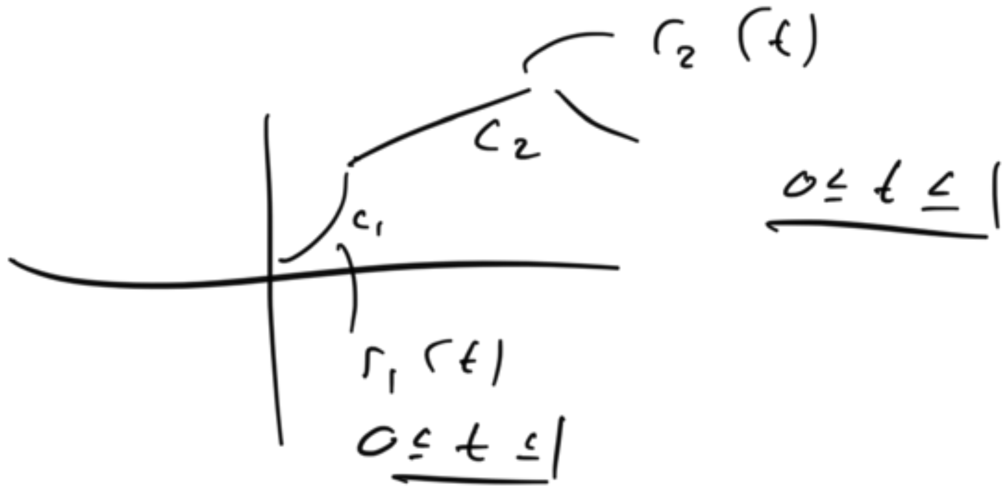
But in general can follow these steps:

$$\int_C f(x,y)\, ds$$

0. Draw C

1. Break curve up into recognizable pieces (functions), integral for each piece

2. Parameterize each piece of curve



$\boxed{8}$

$$\int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds \ldots$$

$$\downarrow$$

$$\int f(\quad)\,|r_1'(t)|\,dt + \int f(\,)\,|r_2'(t)|\,dt$$

$$\vec{r}(t) = \langle \underline{x(t)}, \underline{y(t)} \rangle$$

③ Convert each integral to parameter

Ex: Curve: $\langle x(t), y(t) \rangle$ for $a \le t \le b$

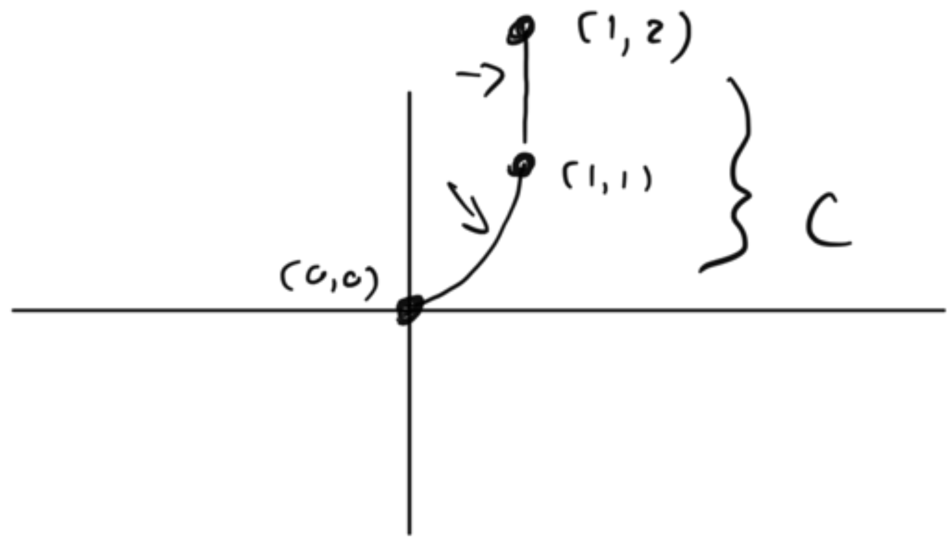
$$\int_C \underline{f(x,y)}\,ds \rightarrow \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

④ Integrate each piece, then add results together

Ex: (Example 2)

$\int_C 2x\,\underline{ds}$ where $\underline{C}$ is curve $y = x^2$ from $(0,0)$ to $(1,1)$

then vertical line from $(1,1)$ to $(1,2)$



① $\begin{cases} y = x^2 & 0 \le x \le 1 \\ x = 1 & 1 \le y \le 2 \end{cases}$

② $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\boxed{\begin{array}{l} x(t) = t \qquad y(t) = t^2 \\ \vec{r}(t) = \langle t, t^2 \rangle \end{array}} \qquad \boxed{0 \le t \le 1}$$

★ $$\boxed{\begin{aligned} \vec{r}(t) &= (1,1) + t(0,1) \\ &= \langle 1, 1+t \rangle \end{aligned}} \qquad \boxed{0 \le t \le 1}$$

③ $\int_C f(x,y)\,ds$

★ $\int_0^1 2t\sqrt{1 + 4t^2}\,dt$

$\int_0^1 2(1)\sqrt{0^2 + (1)^2}\,dt$

★ $\int_0^1 2 \, dt$

(4) $\int_0^1 2t\sqrt{1+4t^2}\, dt + \int_0^1 2\, dt$

$u = 1+4t^2$
$du = 8t\, dt$

$2t \Big]_0^1$

$\frac{1}{4} \int u^{\frac{1}{2}} \, du$

$\frac{1}{4} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big]$

$= \frac{1}{6} (1+4t^2)^{\frac{3}{2}} \Big]_0^1$

$$\boxed{\frac{1}{6}\left(5^{\frac{3}{2}}-1\right) + 2}$$

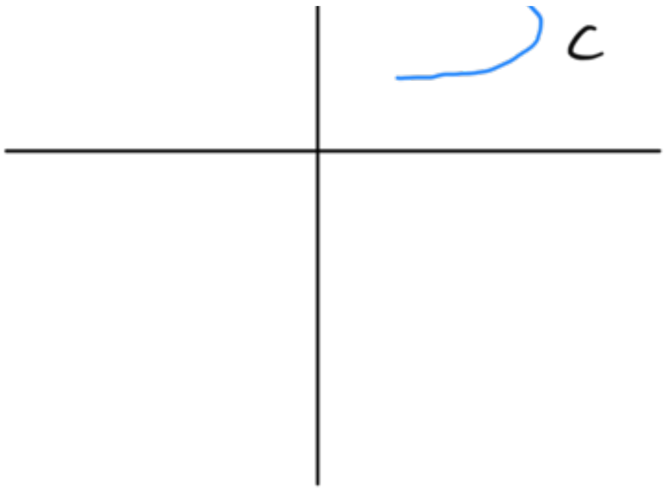
$$\boxed{\frac{5\sqrt{5}-1}{6} + 2}$$

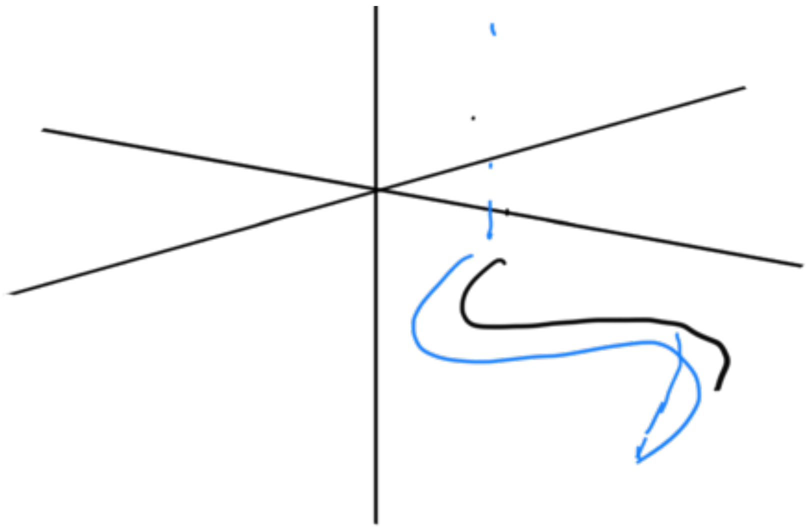
Line Integrals in Space (Domain is $\mathbb{R}^3$)

In previous cases integrated along curve in $\mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = z$





Domain of function

Graph

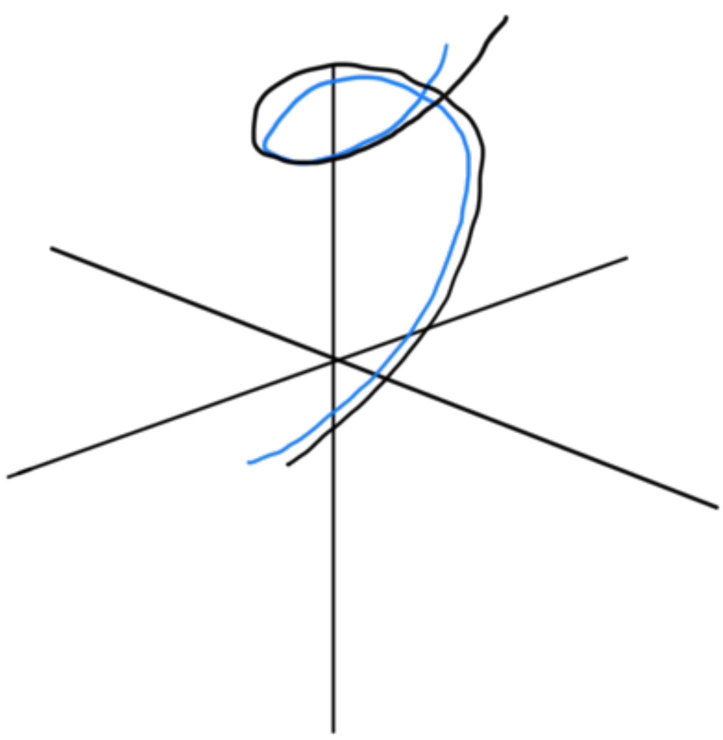
$$\int_C f(x,y)\, \underline{ds}$$

But can do same process in higher dimensions such as $\mathbb{R}^3$.

$f: \mathbb{R}^3 \to \mathbb{R}$

C in $\mathbb{R}^3$

?



Domain of function

Graph

Steps we laid out previously remain same

Integral only slightly different

$C = \langle x(t), y(t) \rangle$

$C = \langle x(t), y(t), z(t) \rangle$

$$\int_c f(x,y,z)\,ds \longrightarrow$$

$$\int_a^b f(x(t), y(t), z(t)) \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2 + \frac{dz}{dt}^2}\, dt$$

12 $\int_C (x^2 + y^2 + z^2)\, ds$

$C: \quad x = t \quad y = \cos(2t) \quad z = \sin(2t)$

$0 \le t \le 2\pi$

1
2 $C = \langle t,\ \cos(2t),\ \sin(2t) \rangle$

$0 \le t \le 2\pi$

3 $\int_0^{2\pi} t^2 + (\cos(2t))^2 + (\sin(2t))^2$

$\sqrt{(1)^2 + (-2\sin(2t))^2 + (2\cos(2t))^2}\, dt$

$$\int_0^{2\pi} (t^2 + 1)\sqrt{1 + 4\sin^2(2t) + 4\cos^2(2t)}\, dt$$

$$= \int_0^{2\pi} (t^2 + 1)\sqrt{1+4}\, dt$$

$$\sqrt{5}\int_0^{2\pi} (t^2 + 1)\, dt$$

$$\sqrt{5}\left(\frac{t^3}{3} + t \Big]_0^{2\pi}\right)$$

$$\sqrt{5}\left(\frac{8\pi^3}{3} + 2\pi\right)$$

An annoying variation

So far, line integrals make intuitive sense.
To be clear, we have been doing line integrals with respect to <u>arc length</u> along curve C

Could also consider line integral along C with <u>respect to x</u> (or y)

Makes less sense. Sorry. But computation not much different.

Was $\int_{\underline{C}} f(x,y)\, \underset{\smile}{ds}$

Now $\int_{\underline{C}} f(x,y)\, \underline{dx} \longrightarrow \int_a^b f(x(t),y(t))\, x'(t)\, dt$

Indicates w.r.t x

Steps:

1. Break curve up

2. Find parameterization each piece

☆ 3. Convert each integral

$$\int_c f(x,y)\,dx \to \int_a^b f(x(t),y(t))\,\overset{dx/dt}{x'(t)}\,\overset{dt}{dt}$$

$$\int_c f(x,y)\,dy \to \int_c^b f(x(t),y(t))\,y'(t)\,dt$$

4. Integrate

Usually see integrals w.r.t x, y together:

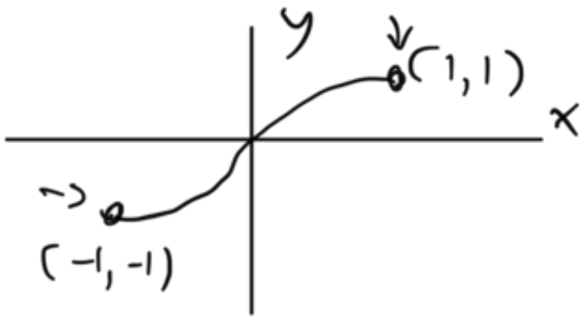
$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

In order to avoid writing integral sign twice, use following notation (standard)

$$\boxed{\int_c P(x,y)\,dx + Q(x,y)\,dy}$$

Ex: $\int_c e^x\,dx$

where C is curve $\underline{x=y^3}$ from $(-1,-1)$ to $(1,1)$



① ✓

②

$$y(t) = t$$

$$x(t) = (y(t))^3$$

$$x(t) = (t)^3$$

$$\vec{r}(t) = \langle t^3, t \rangle$$

③ $\int_{-1}^{1} e^{t^3}\, 3t^2 \, dt$

$$u = t^3 \quad du = 3t^2$$

$$\int e^u \, du$$

$$e^{t^3}\Big]_{-1}^{1} = \boxed{e - \tfrac{1}{e}}$$

By itself, not too valuable, but will see a connection to vector fields next

Line Integrals for Vector Fields

Now connect this section to 16.1.

My explanation of general formula:

Consider vector field $\underline{F}: \underline{\mathbb{R}^3 \to \mathbb{R}^3}$

To each point (x,y,z) assigns output

$\langle \underline{F_1(x,y,z)}, \underline{F_2(x,y,z)}, \underline{F_3(x,y,z)} \rangle$ ☆ ☆

$$F(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle$$

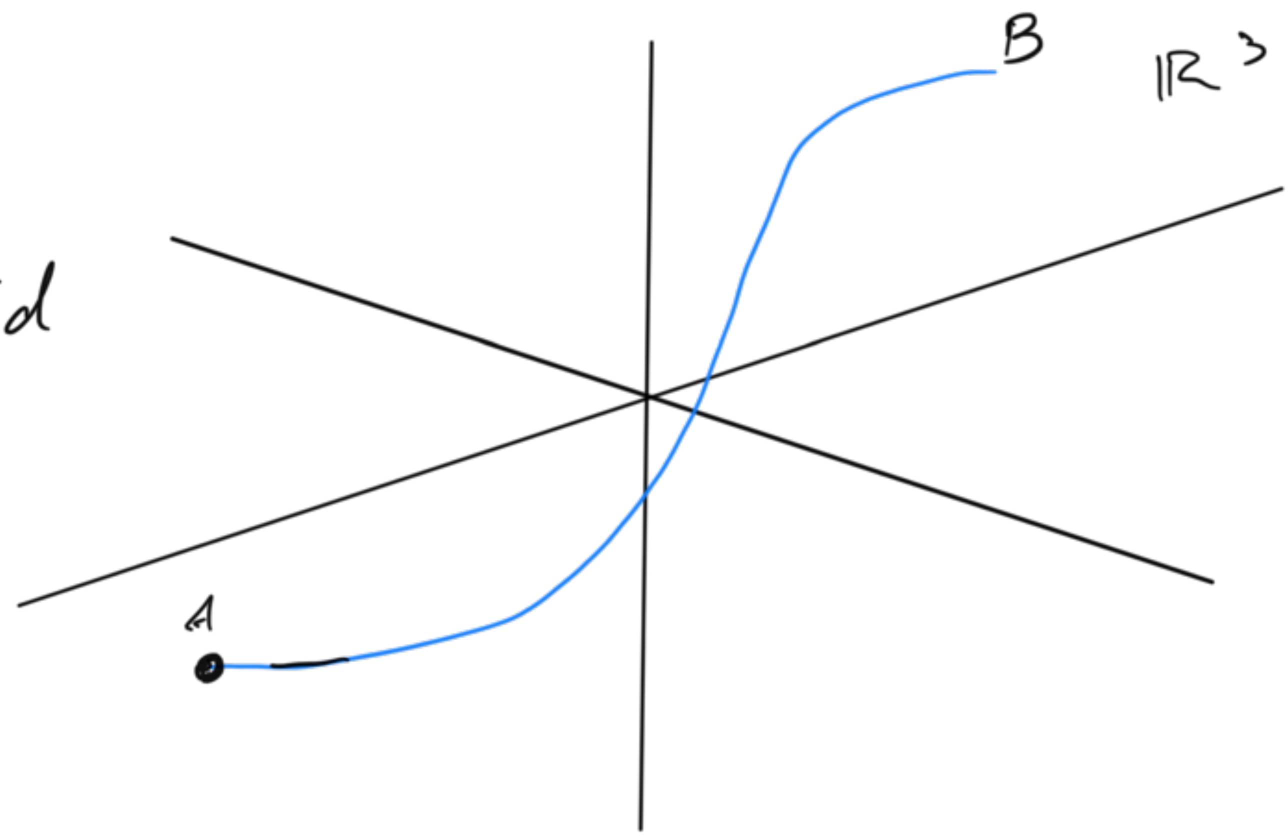
Think of $F_1(x,y,z)$ as <u>amount</u> of output pointing in x-direction, when we are at the point (x,y,z).

$F_2(x,y,z)$ in y-direction.

$F_3(x,y,z)$ in z-direction.

Now consider a curve C in $\mathbb{R}^3$

$W = Fd$



$\mathbb{R}^3$ (domain)

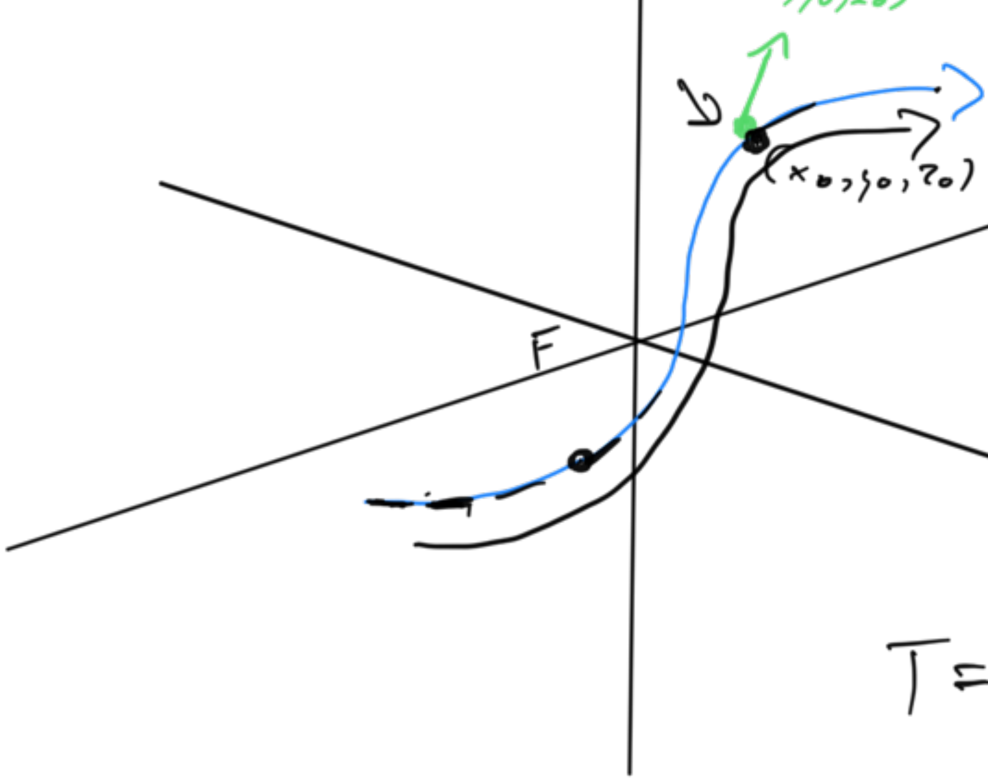
Want to try and "add up" the amount of output pointing in same direction as the curve C.

How to calculate this?

Assume C has parameterization $\vec{r}(t)$

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

$F(x_0, y_0, z_0) = \langle F_1, F_2, F_3 \rangle$



$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$r'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$

$$T = \frac{r'(t)}{|r'(t)|}$$

What <u>is</u> direction of C at point (x_0, y_0, z_0) ?

$$\underline{\vec{r}\,'(t_0)} = \langle x'(t_0), y'(t_0), z'(t_0) \rangle$$

But want to only consider direction so make this unit tangent vector

$$\star \quad T(t_0) = \frac{r'(t_0)}{|r'(t_0)|}$$

So "amount" of F in direction of C should be

$$F(x,y,z) \cdot \underline{T(t)}$$

$$\star \quad \langle F_1(x(t), y(t), z(t)), F_2, F_3 \rangle \cdot \frac{\langle x'(t), y'(t), z'(t) \rangle}{\| r'(t) \|}$$

For this to make sense want F, T to be in same units

$$F(x(t), \ldots) \cdot T(t)$$

$F(x(t), y(t), z(t)) \cdot r'(t)$

$$\star = F(x(t), y(t), z(t)) \cdot \frac{r'(t)}{|r'(t)|}$$

Then integrate with respect to arc length

$$\int_a^b \langle F_1, F_2, F_3 \rangle \cdot \frac{\langle x', y', z' \rangle}{\cancel{|r'(t)|}} \cancel{|r'(t)|} dt$$

$$= \int_a^b \langle F_1, F_2, F_3 \rangle \cdot \langle x', y', z' \rangle dt$$

$$\int_a^b F_1 x'(t) dt + F_2 y'(t) dt + F_3 z'(t) dt$$

$$\int_a^b F \cdot \frac{r'(t)}{|r'(t)|} |r'(t)| dt$$

$$= \int_a^b F(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\, dt$$

F is a vector of form

$$\langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$$

$$\int_a^b \langle P, Q, R\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle \, dt$$

$$= \int_a^b \left[P\, x'(t) + Q\, y'(t) + R\, z'(t) \right] dt$$

$$\star \int_a^b \underline{P\, x'(t)\,dt} + \underline{Q\, y'(t)\,dt} + \underline{R\, z'(t)}\,dt$$

$$\star \int_C P\,dx + Q\,dy + R\,dz$$

Summary:

$\star$ Vector field $F = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$

Line integral of F along curve C

is

$$\int_C F_1\,dx + F_2\,dy + F_3\,dz$$

In other words

$$\left.\begin{aligned} &\int_a^b P(x(t), y(t), z(t))\, x'(t)\,dt \\ &+ \int_a^b Q(x(t), y(t), z(t))\, y'(t)\,dt \\ &+ \int_a^b R(x(t), y(t), z(t))\, z'(t)\,dt \end{aligned}\right\}$$

$$\int_a^b F(x(t), y(t), z(t)) \cdot \frac{r'(t)}{|r'(t)|} \boxed{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}$$

But note $\qquad = |r'(t)|$

$$= \boxed{\int_a^b F(x(t), y(t), z(t)) \cdot r'(t)\; dt}$$

(19) $F(x,y) = xy^2\,\vec{i} - x^2\,\vec{j}$

$= \langle xy^2, -x^2 \rangle$

$C : \vec{r}(t) = t^3\,\vec{i} + t^2\,\vec{j}$

$\langle t^3, t^2 \rangle$

$0 \le t \le 1$

$$\int_0^1 \langle t^3(t^2)^2, -(t^3)^2 \rangle \cdot \langle 3t^2, 2t \rangle\, dt$$

$\int_0 \langle t, -t \rangle \cdot \langle 3t^2, 2t \rangle \, dt$

$\int_0^1 (3t^9 - 2t^7) \, dt$

$= \frac{3}{10} t^{10} - \frac{2}{8} t^8 \Big]_0^1$

$= \frac{3}{10} - \frac{2}{8} = \frac{24}{80} - \frac{20}{80}$

$= \frac{4}{80}$

$\boxed{= \frac{1}{20} = .05}$

Notation

Line integral of vector field F along curve C

$$\int_C F \cdot dr = \boxed{\int_a^b F(r(t)) \cdot r'(t) \, dt}$$

$$= \int_C F \cdot T \, ds$$

Connect to line integrals w.r.t. x, y, z:

Recall $F: \mathbb{R}^3 \to \mathbb{R}^3$ can be written as

$$\langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z)\rangle$$

or

$$\langle \underline{P(x,y,z)}, \underline{Q(x,y,z)}, \underline{R(x,y,z)}\rangle$$

Then

$$\int_a^b \underline{F(x,y,z)} \cdot r'(t)\,dt$$

$$= \int_a^b P(x,y,z)\,x'(t) + Q(x,y,z)\,y'(t) + R(x,y,z)\,z'(t)]\,dt$$

$$= \int_a^b P(x,y,z)\,x'(t)\,dt$$

$$+ \int_a^b Q(x,y,z)\,y'(t)\,dt$$

$$+ \int_a^b R(x,y,z)\,z'(t)\,dt$$