16.5 - Curl / Divergence

Curl / Divergence are two properties of vector fields. Often used to describe "flow" of fluids/currents.

Very widely used in physics. See Maxwell's equations.

Will try to give some intuition for these, but may have to think of it in abstract terms.

Curl Describes rotation of vector field around a point.

Big ugly formula: (for $F = \langle P, Q, R \rangle$)

☆ Curl (F) =

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

These terms remind us of cross product, or determinant of a 3×3 matrix

How can we write as a cross product?
Of what 2 vectors?

Let's make a new "vector" ∇

Recall for function f, $\nabla f = \langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \rangle$

Define ∇ by itself as

$$\nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$$

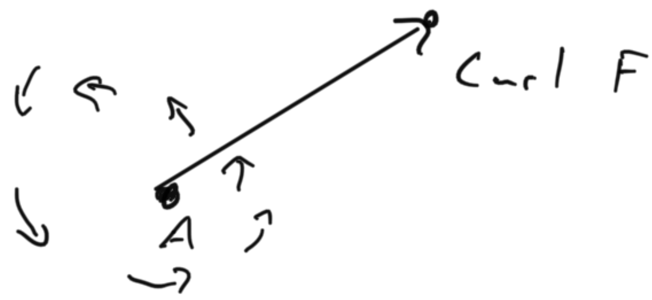
Now consider "cross product" $\nabla \times F$
($\nabla \times F$ will be notation for curl (F))

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} - \underbrace{\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j}} + \left(\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\right)\vec{k}$$

Can re-arrange this term to
get $\nabla \times F$ to match the big
ugly formula.

Interpretation: Curl (F) gives a vector.



Particles <u>tend</u> to rotate around this vector and $|\text{curl}(F)|$ a measure of how quickly they rotate

☆ Note: Curl (F) defined at a point. So curl (F) at (a,b,c) may not be same as at (d,e,f)

☆ Note: If curl $(F) = \vec{0}$ at (a,b,c) say <u>F is irrotational at the point</u>

Connection with conservative vector fields:

<u>Theorem:</u> If $F = \langle P, Q, R \rangle$ is conservative and P, Q, R have continuous first order partials then curl $(F) = \vec{0}$ (at all points).

$\Downarrow$

$\square$ F conservative so $F = \nabla f$ for

for some T,

i.e. $\langle P, Q, R \rangle = \langle f_x, f_y, f_z \rangle$. Since we have continuous derivatives, mixed partials are equal (Clairaut's Theorem). Meaning

$$f_{xy} = f_{yx} \quad \text{etc.}$$

So: $F = \langle P, Q, R \rangle \qquad F = \langle f_x, f_y, f_z \rangle$

$$\text{curl } F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= \left(f_{zy} - f_{yz}\right)\vec{i} + \left(f_{xz} - f_{zx}\right)\vec{j} + \left(f_{yx} - f_{xy}\right)\vec{k}$$

$$= \vec{0}$$

☆ Thus, if $\text{curl}(F) \neq \vec{0}$ then F not conservative.

Theorem: If $F = \langle P, Q, R \rangle$ is vector field defined on all of $\mathbb{R}^3$, P, Q, R have continuous partials, and $\text{curl}(F) = \vec{0}$ (everywhere) then F is conservative.

Divergence

For $F = \langle P, Q, R \rangle$

$$\operatorname{div} F = \partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$$

or in our new notation

$$\operatorname{div} F = \nabla \cdot F$$

$$\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle P, Q, R \rangle$$

Interpretation: Divergence gives a scalar, not a vector. So, an amount.

Divergence is "amount" of vector field flowing from/to a point.

- Diverging from
- Compressing to

If $\operatorname{div} F = 0$ say F is incompressible.

Theorem: If $F = \langle P, Q, R \rangle$ is a vector field over $\mathbb{R}^3$ and P, Q, R have continuous second-order partials then

$$\operatorname{div}(\operatorname{curl}(F)) = 0$$

$\square$ Obvious if think of it as

$$\nabla \cdot (\nabla \times F)$$

$\blacksquare$

<u>Notation difference</u>:

Consider $\operatorname{div}(\nabla f)$ for some f

$$= \frac{\partial}{\partial x}(f_x) + \frac{\partial}{\partial y}(f_y) + \frac{\partial}{\partial z}(f_z)$$

$$0 = f_{xx} + f_{yy} + f_{zz}$$

☆ <u>Laplace Equation</u>

Since <u>div</u> $(\underline{\nabla f})$ can be written as $\nabla \cdot (\nabla f)$, book denotes Laplace equation as $\nabla^2 f$

We used ∇^2 for 2^{nd} derivative matrix.

Better notation for Laplace equation is

$$\Delta f = f_{xx} + f_{yy} + f_{zz}$$

"Laplacian"

Rewriting Green's Theorem

Assume $F = \langle P, Q \rangle$ vector field over $\mathbb{R}^2$

$$\oint_C F \cdot dr = \iint_D \left(\partial Q/\partial x - \partial P/\partial y \right) dA$$

But lets rewrite in terms of div/curl

First, turn F into 3-D vector field by adding a 0 in z-component

$$F = \langle P, Q, 0 \rangle$$

Now note curl $F = \nabla \times F$ gives

$$\left(\partial Q/\partial x - \partial P/\partial y \right) \vec{k}$$

To get rid of $\vec{k}$, take dot product with $\vec{k} = \langle 0, 0, 1 \rangle$

$$\left(\partial Q/\partial x - \partial P/\partial y \right) \vec{k} \cdot \vec{k} = \left(\partial P/\partial x - \partial P/\partial y \right)$$

So

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \vec{k} \cdot \vec{k} \, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

So can rewrite Green's theorem as

$$\int_C F \cdot dr = \int (\text{curl } F) \cdot \vec{k} \, dA$$

$\int_C F \cdot dr$ calculates / adds up the F along boundary. May also be interested in F crossing boundary

$$\int_C F \cdot \vec{n} \, ds = \iint_D \text{div}(F) \, dA$$