Section 6.1

This section mostly material from calc 3.

$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

Dot product $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 + \dots u_n v_n$

Note $\vec{u} \cdot \vec{v}$ is same as:

$$\vec{u}^T \vec{v} = [u_1 \dots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Dot product sometimes referred to as the **scalar product** or inner product

(Note: Inner product is actually a much more general idea, the dot product above is just a special case of inner products)

Properties: $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, c a scalar

① $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

② $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u}\vec{w} + \vec{v}\vec{w}$

③ $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u}(c\vec{v})$

④ $\vec{u} \cdot \vec{u} \geq 0$, equals 0 iff $\vec{u} = \vec{0}$

Length

Recall "length" of vector $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is

$\sqrt{v_1^2 + v_2^2 + \dots v_n^2}$

Examine term inside square root

$$v_1^2 + v_2^2 + \dots v_n^2$$
$$= v_1 v_1 + v_2 v_2 + \dots v_n v_n$$
$$= \vec{v} \cdot \vec{v}$$

Thus, can rewrite "length" of $\vec{v}$ as

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

Note: $\|c\vec{v}\| = |c| \, \|\vec{v}\|$

Distance:

Distance between vectors $\vec{u}, \vec{v}$ is just the length of $\vec{u} - \vec{v}$

So

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$
$$= \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}$$

Orthogonality

Definition: $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ are **orthogonal** to each other if $\vec{u} \cdot \vec{v} = 0$

Recall from calc 3 that orthogonal is same as perpendicular for our familiar spaces $\mathbb{R}^2, \mathbb{R}^3$
The definition above gives convenient way to think about this concept in higher dimensions

Theorem: (Pythagorean theorem)
Two vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal iff
$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

$\square$ First note that:

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v})$$
$$= (\vec{u} \cdot \vec{u}) + (\vec{u} \cdot \vec{v}) + (\vec{v} \cdot \vec{u}) + (\vec{v} \cdot \vec{v})$$
$$= \|\vec{u}\|^2 + 2\,\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$$

Above is always true.
Now assume that $\vec{u}, \vec{v}$ are orthogonal.
Then $2\,\vec{u} \cdot \vec{v} = 0$. So

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\,\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$$
$$= \|\vec{u}\|^2 + 0 + \|\vec{v}\|^2$$
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Now other direction. Assume $\|\vec{u}+\vec{v}\|^2 = \|u\|^2 + \|\vec{v}\|^2$
Since $\|\vec{u}+\vec{v}\|$ also equals $\|u\|^2 + 2\,\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$
have

$$\|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

$$\|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Subtract $\|\vec{u}\|^2$, $\|\vec{v}\|^2$ from both sides, get:

$$2\vec{u}\cdot\vec{v} = 0$$

so:

$$\vec{u}\cdot\vec{v} = 0$$

Thus $\vec{u}, \vec{v}$ are orthogonal ∎

All of above should be review from calc 3. Now some new stuff.

Have seen two vectors orthogonal to each other. Begin to expand idea a bit.

Terminology

Let W be a subspace of some vector space V and let $\vec{v} \in V$. Say $\vec{v}$ is orthogonal to W if $\vec{v}$ is orthogonal to <u>every</u> vector $\vec{w}$ in W, i.e. $\vec{v}\cdot\vec{w} = 0$ for all $\vec{w} \in W$.

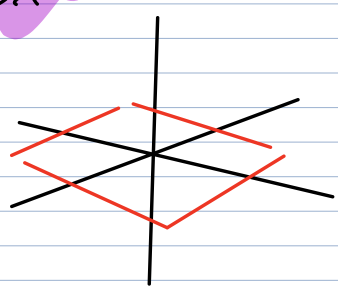
Expand a bit further.

There may be many vectors that are orthogonal to all W. Can group them all together in one "object".

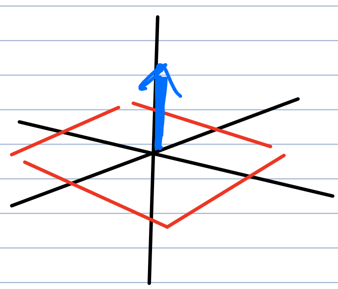
Definition:
Set of all vectors $\vec{v} \in V$ that are orthogonal to W is called the orthogonal complement of W, denoted $W^{\perp}$

$$W^{\perp} = \{ \vec{v} \in V : \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W \}$$

Ex



$V = \mathbb{R}^3$
$W =$ horizontal plane



Any vector on vertical axis is orthogonal to W. So should have vertical axis is $W^{\perp}$

Know that vertical axis is a subspace of $\mathbb{R}^n$.

Question: Is $W^{\perp}$ always a subspace of the larger vector space V? Yes.

Theorem: Let W be a subspace of the vector space V. Then $W^\perp$ is also a subspace of V.

$\square$ $\vec{0}$ is orthogonal to all vectors in V (and thus W) so can say for certain $\vec{0} \in W^\perp$.

Assume $\vec{x}_1, \vec{x}_2 \in W^\perp$, $c \in \mathbb{R}$. Let $\vec{\omega}$ be an arbitrary vector in W.

$$(c\vec{x}_1) \cdot \omega = c(\vec{x}_1 \cdot \vec{\omega}) = 0$$

So $W^\perp$ closed under scalar multiplication.

$$(\vec{x}_1 + \vec{x}_2) \cdot \vec{\omega} = \vec{x}_1 \cdot \vec{\omega} + \vec{x}_2 \cdot \vec{\omega} = 0 + 0 = 0$$

So $W^\perp$ closed under vector addition.

Thus $W^\perp$ is a subspace. $\blacksquare$

Theorem: $\vec{v} \in W^\perp$ iff $\vec{v}$ is orthogonal to the basis vectors of W.

$\square$ Assume $\vec{v} \in W^\perp$. Then $\vec{v}$ is orthogonal to all vectors in W, including any and all of its basis vectors.

Let $B = \{\vec{\omega}_1, \dots \vec{\omega}_k\}$ be basis for W and assume $\vec{v} \cdot \vec{\omega}_i = 0$ for $1 \leq i \leq k$. Now pick arbitrary vector $\vec{w}$ in W.

Since β basis for W, $\vec{w} = c_1 \vec{w_1} + \dots c_k \vec{w_k}$

Then

$$\vec{v} \cdot \vec{w} = \vec{v} \cdot (c_1 \vec{w_1} + \dots c_k \vec{w_k})$$
$$= c_1 (\vec{v} \cdot \vec{w_1}) + \dots + c_k (\vec{v} \cdot \vec{w_k})$$
$$= c_1 (0) + \dots c_k (0)$$
$$= 0.$$

Thus $\vec{v} \cdot \vec{w} = 0$ for all $\vec{w} \in W$. So $\vec{v} \in W^{\perp}$

$\blacksquare$