Section 5.2

Recall in previous section we learned how to find eigenbasis if we knew eigenvalue. Now we will see how to find eigenvalues.

When we knew eigenvalue c, we looked at Null Space of $n \times n$ matrix $(A - cI)$, i.e solutions for homogeneous system

A) $$(A - cI)\vec{x} = \vec{0}$$

Similar starting point for finding eigenvalues.

Recall λ is eigenvalue of A if there is nonzero vector s.t. $A\vec{x} = \lambda\vec{x}$, i.e. nontrivial solution to:

B) $$(A - \underline{\lambda} I)\vec{x} = \vec{0}$$

Difference between A) and B):

In A), <u>know</u> value of c, want to find vectors $\vec{x}$ that solve system

In B), λ unknown. Don't care about $\vec{x}$, just want to know for what values of λ there <u>will</u> be a nontrivial solution

Finding Eigenvalues Step 1: Find matrix $A - \lambda I$. Leave λ as a variable.

Ex $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix}$

How do we figure out B) ?

- Existence of nontrivial solution iff $(A - \lambda I)$ is <u>not</u> one-to-one.
- Not one-to-one iff $(A - \lambda I)$ is <u>not</u> invertible ☆

Do we have a simple test to determine if $(A - \lambda I)$ invertible? <u>Yes</u>.

Determinant

Finding Eigenvalues Step 2: Try to calculate determinant of $A - \lambda I$. Should end up with polynomial/equation, λ is variable. Called characteristic equation, characteristic polynomial

Ex: $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{bmatrix}$ ~~A~~

$$\det(A - \lambda I) = (1-\lambda)(1-\lambda) - 2(0)$$
$$= (1-\lambda)^2$$

Want to know what values of λ are eigenvalues. So want to know for what values of λ determinant $= 0$.

Finding Eigenvalues Step 3: Solve (factor) determinant

Ex
$$1 - 2\lambda + \lambda^2 = 0$$
$$(1-\lambda)(1-\lambda) = 0$$
$$\lambda = 1$$

How do we interpret results? The factors we find are:

- values that make characteristic polynomial 0
- which makes determinant 0
- which makes matrix singular (non-invertible)
- which means not one-to-one
- which means there is nontrivial solution to $(A - \lambda I)\vec{x} = 0$ for our values of λ

$$A\vec{x} - \lambda\vec{x} = \vec{0} \qquad A\vec{x} = \lambda\vec{x}$$

- which means $A\vec{x} = \lambda\vec{x}$ for these λ

Conclusion: • which means these λ are eigenvalues!

Ex Eigenvalues of $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ just 1

Let A be $n \times n$ matrix. How many <u>distinct</u> eigenvalues are possible?

- May have 0 eigenvalues

What if follow steps and get characteristic polynomial $\lambda^2 + 1 = 0$?

<u>No solutions</u>! No eigenvalues

Ex $A = \begin{bmatrix} \cos \pi/3 & -\sin \pi/3 \\ \sin \pi/3 & \cos \pi/3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\sqrt{3}/2 \\ \sqrt{3}/2 & \frac{1}{2} \end{bmatrix}$

- May have between 0 and n eigenvalues
- <u>At most</u> n distinct eigenvalues

Recall if λ_1 and λ_2 are distinct, their eigenvectors are linearly independent. In n-dimensional vector space at most n linearly independent vectors. So, at most n distinct eigenvalues.

A is $n \times n$,

$0 \leq \#$ eigenvalues $\leq n$

Ex:

4×4 matrix

$$p_A = (\lambda - 3)^1 (\lambda - 2)^1 (\lambda - 1)^2$$

Eigenvalues : 3, 2, 1

alg mult 1

alg mult 2

Power of each factor / value of λ is called its algebraic multiplicity.

Now for a seemingly separate concept.

Definition: Let A, B be square $n \times n$ matrices. Say that A is similar to B if there is invertible matrix P such that $P^{-1}AP = B$

- $A \mapsto P^{-1}AP$ is similarity transformation
- If $A \sim B$ then $B \sim A$

Theorem: If A, B are similar then they have same characteristic equation and so same eigenvalues.

$\square$ Characteristic equation B given by determinant of $(B - \lambda I)$.

- • Now recall $\det(GH) = \det(G)\det(H)$
- • $\det(G^{-1}) = 1/\det(G)$
- • Note if G of form cI, it commutes with other square matrices, i.e. $GH = HG$

So

$$B - \lambda I$$

$= P^{-1}AP - \lambda I$	Because $A \sim B$
$= P^{-1}AP - P^{-1}P(\lambda I)$	$P^{-1}P = I$ so can add anywhere
$= P^{-1}AP - P^{-1}(\lambda I)P$	λI commutes, so can rearrange
$= P^{-1}(AP - (\lambda I)P)$	Factor out P^{-1}
$= P^{-1}(A - \lambda I)P$	Factor out P

Thus

char poly B $= \underline{\det(B - \lambda I)} = \det\left(P^{-1}(A - \lambda I)P\right)$

$$= \det(P^{-1})\det(A - \lambda I)\det(P) \quad \star$$

$$\star = \cancel{\frac{1}{\det(P)}}\ \cancel{\det(P)}\ \underline{\det(A - \lambda I)}$$

$$= \det(A - \lambda I) = \text{char poly } A$$

So $\det(A - \lambda I) = \det(B - \lambda I)$. Since characteristic equation is this determinant, A B have same char. eq.

Since eigenvalues are roots of char poly, A and B have same eigenvalues.

Why do we care? Want to try and rewrite A in terms of eigenvalues/eigenvectors. Can't do it directly but can say A similar to B where B is determined by eigenvectors, eigenvalues.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \qquad B = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 5 \\ -1 & 2 & 4 \end{bmatrix}}_{\text{pain}}$$

A^2

$$\begin{bmatrix} 1^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & (-2)^2 \end{bmatrix}$$

$$A^{15} \begin{bmatrix} 1^{15} & 0 & 0 \\ 0 & 3^{15} & 0 \\ 0 & 0 & (-2)^{15} \end{bmatrix}$$