12.4 - Cross Product

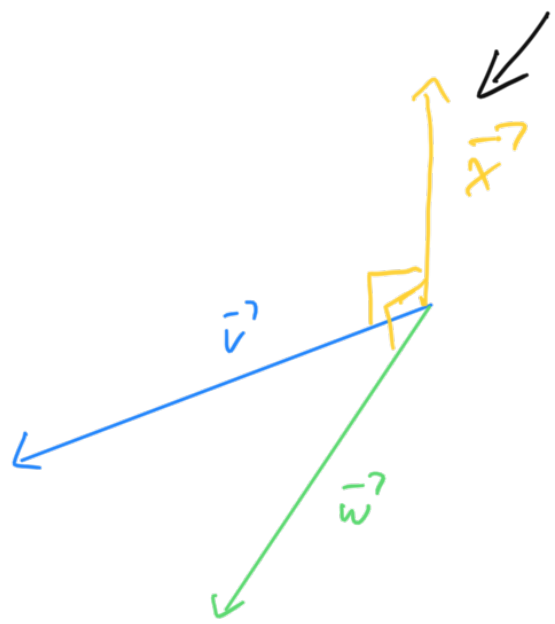
Another way to "multiply" two vectors

Dot product $\vec{v} \cdot \vec{w} =$ number

☆ Cross product $\vec{v} \times \vec{w} =$ $\boxed{\text{vector}}$

The vector we get from cross product will be orthogonal to $\vec{v}$ and $\vec{w}$

So :



Remember, if $\vec{x}$ is orthogonal to $\vec{v}$ must have

$$\vec{v} \cdot \vec{x} = 0$$ ☆

Also want

$$\vec{w} \cdot \vec{x} = 0$$ ☆

These requirements tell us how to

compute cross product

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$
$$\vec{w} = \langle w_1, w_2, w_3 \rangle$$
$$\vec{x} = \langle x_1, x_2, x_3 \rangle$$

System of equations:

$$\begin{cases} v_1 x_1 + v_2 x_2 + v_3 x_3 = 0 \\ w_1 x_1 + w_2 x_2 + w_3 x_3 = 0 \end{cases}$$

$$\begin{cases} w_3(v_1 x_1 + v_2 x_2 + v_3 x_3) = 0 \\ v_3(w_1 x_1 + w_2 x_2 + w_3 x_3) = 0 \end{cases}$$

$$\begin{array}{rl} & w_3 v_1 x_1 + w_3 v_2 x_2 + w_3 v_3 x_3 = 0 \\ - & (v_3 w_1 x_1 + v_3 w_2 x_2 + v_3 w_3 x_3) = 0 \\ \hline \star = & (w_3 v_1 - v_3 w_1) x_1 + (w_3 v_2 - v_3 w_2) x_2 = 0 \end{array}$$

If
$$\left.\begin{array}{l} x_1 = (v_2 w_3 - v_3 w_2) \star \\ x_2 = (v_3 w_1 - v_1 w_3) \star \end{array}\right]$$
equation satisfied

These values imply $x_3 = (\underbrace{v_1 w_2 - v_2 w_1})$

So for vectors $\vec{v}, \vec{w}$ their cross product $\vec{v} \times \vec{w}$ is:

$$\vec{v} \times \vec{w} = \vec{x}$$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$
$\vec{w} = \langle w_1, w_2, w_3 \rangle$

where

$$\vec{x} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle$$

Don't worry!

Theres easy method to remember/calculate cross product

First need something called a determinant

Determinants

This is a number we get from a matrix

2 x 2

☆ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad \det = \underline{ad - bc}$

$\underline{3 \times 3}$

☆ $\begin{bmatrix} i & j & k \\ a & b & c \\ d & e & f \end{bmatrix}$ $\boxed{i(bf - ec) - j(af - dc) + k(ae - bd)}$

$$\det = i(bf - ce) - j(af - cd) + k(ae - bd)$$

How does this help us?

$\vec{v} \times \vec{w}$

$\vec{v} = \langle v_1, v_2, v_3 \rangle$

$\vec{w} = \langle w_1, w_2, w_3 \rangle$

$\rightarrow \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$

$$\det = \vec{i}\,(v_2 w_3 - v_3 w_2) - \vec{j}\,(v_1 w_3 - v_3 w_1)$$
$$+ \vec{k}\,(v_1 w_2 - v_2 w_1)$$

$$= \langle v_2 w_3 - v_3 w_2,\ v_3 w_1 - v_1 w_3,\ v_1 w_2 - v_2 w_1 \rangle$$

So "det" $= \vec{v} \times \vec{w}$!!

Ex: Find a vector orthogonal to ~~cross product of~~

$\vec{v} = \langle 1, 1, 1 \rangle$ and $\vec{w} = \langle 0, -3, 4 \rangle$

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 0 & -3 & 4 \end{bmatrix}$$

$$\vec{i}\,(4+3) - \vec{j}\,(4-0) + \vec{k}\,(-3-0)$$

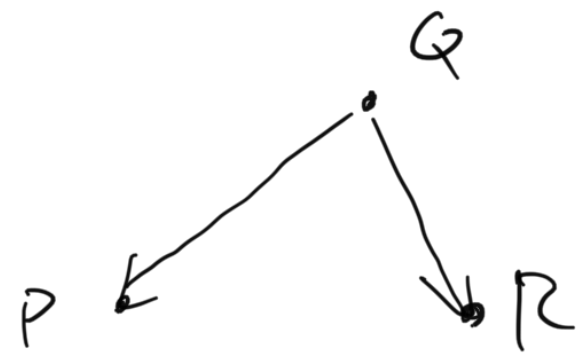
$$= \vec{i}\,7 - \vec{j}\,4 + \vec{k}\,(-3)$$

$$= 7\vec{i} - 4\vec{j} - 3\vec{k}$$

$$= \langle 7, -4, -3 \rangle$$

Ex: Find vector perpendicular to plane that passes through

$P(1,4,6)$ $Q(-2,5,1)$ $R(1,-1,1)$



$\overrightarrow{QP} = \langle 1-(-2),\ 4-5,\ 6-1 \rangle = \langle 3, -1, 5 \rangle$

$\overrightarrow{QR} = \langle 1-(-2),\ -1-5,\ 1-1 \rangle = \langle 3, -6, 0 \rangle$

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 5 \\ 3 & -6 & 0 \end{bmatrix}$$

$$\vec{i}\,(0+30) - \vec{j}\,(0-15) + \vec{k}\,(-18+3)$$

$$= 30\vec{i} + 15\vec{j} - 15\vec{k}$$

$$= \langle 30, 15, -15 \rangle$$

Notes:

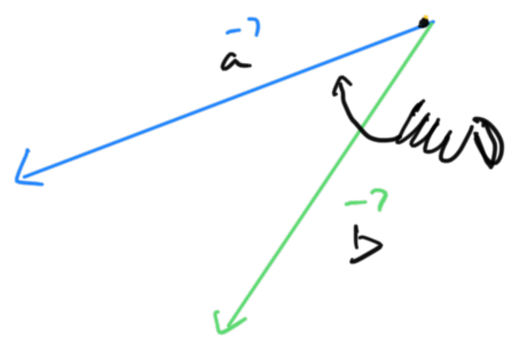
As expected, if $\vec{a} \times \vec{b} = \vec{c}$

then $\vec{a}$ and $\vec{c}$ orthogonal and

$\vec{c}$ and $\vec{b}$ orthogonal.

So $$\vec{a} \cdot \vec{c} = 0$$
$$\vec{b} \cdot \vec{c} = 0$$

$\vec{c}$ is a vector orthogonal to $\vec{a}$ and $\vec{b}$, but <u>not</u> unique



$\vec{b} \times \vec{a}$

<u>Right Hand Rule</u>

Properties

1. $(\vec{a} \times \vec{b}) = -(\vec{b} \times \vec{a})$

2. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b}) = \vec{a} \times (c\vec{b})$

3. $\vec{a} \times (\underline{\vec{b} + \vec{c}}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$ ☆

6. $a \times (b \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

☆ ☆

$\vec{c}$

$(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})$

$((\vec{a} \times \vec{b}) \times \vec{a}) \cdot \vec{b}$ ☆

$= (-\vec{a} \times (\vec{a} \times \vec{b})) \cdot \vec{b}$

$= (-((\vec{a} \cdot \vec{b})\vec{a} - (a \cdot a)\vec{b})) \cdot \vec{b}$

☆

$\vec{c} \cdot (\vec{a} \times \vec{b})$

$= (\vec{c} \times \vec{a}) \cdot \vec{b}$

$= ((\vec{a} \times \vec{b}) \times \vec{a}) \cdot \vec{b}$

Surprisingly

$$\boxed{\vec{a} \times \vec{a} = \vec{0}}$$

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$$\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{bmatrix}$$

$\vec{i}(\cancel{a_2 a_3 - a_2 a_3}) - j(\cancel{a_1 a_3 - a_1 a_3})$

$+ \vec{k}(\cancel{a_1 a_2 - a_1 a_2})$

In fact

$\vec{a} \times \vec{b} = \vec{0}$ if and only if

$\vec{a}$ and $\vec{b}$ are parallel

can write $\underline{\vec{b}} = \underline{r}\,\underline{\vec{a}}$

so $\vec{a} \times \vec{b} = \underline{\vec{a}} \times \underline{r\,\vec{a}}$

$= \underline{r}\,(\underline{\vec{a} \times \vec{a}})$

$= \underline{r\,\vec{0}}$

$= \underline{\vec{0}}$

$\bigstar$ $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,\sin(\theta)$

$\bigstar$ $|a \times b|^2 \overset{\bigstar}{=} (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})$

$= \left[\underbrace{(\vec{a} \times \vec{b}) \times \vec{a}}\right] \cdot \vec{b}$

$\left[(\vec{a} \cdot \vec{a})\vec{b} - (\vec{a} \cdot \vec{b})\vec{a}\right] \cdot \vec{b}$

$(\vec{a} \cdot \vec{a})\vec{b} \cdot \vec{b} - (\vec{a} \cdot b)(\vec{a} \cdot \vec{b})$

$|\vec{a}|^2\,|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$

$|\vec{a}|^2\,|b|^2 - (|\vec{a}|\,|\vec{b}|\cos\theta)^2$

$|\vec{a}|^2\,|b|^2 - |\vec{a}|^2\,|b|^2\cos^2\theta$

$|\vec{a}|^2\,|b|^2\,(1 - \cos^2\theta)$

$|a|^2\,|b|^2\,\sin^2\theta$

$$|\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 \, |\vec{b}|^2 \sin^2 \theta$$

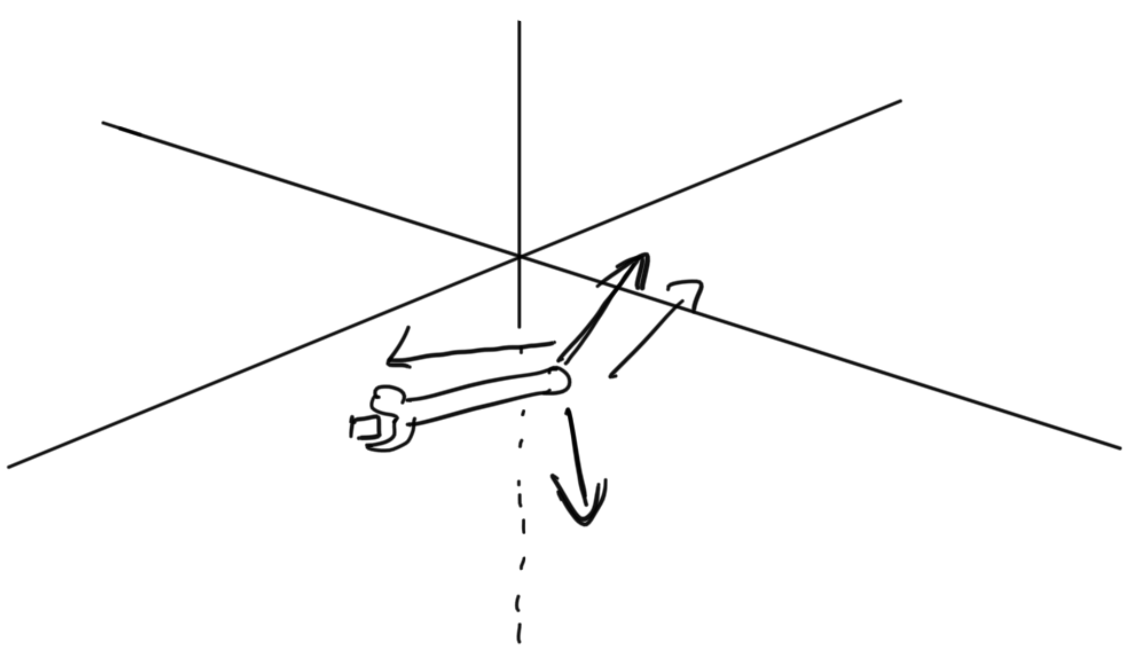
$$|\vec{a} \times \vec{b}| = |\vec{a}| \, |\vec{b}| \sin \theta$$

<u>Fact</u>: $|a \times b|$ is equal to area of parallelogram given by $\vec{a}$ and $\vec{b}$.

Example 4 in book

<u>Torque</u>

Classic example of cross product used in physics



(39) 60N, $\vec{F}$

$$|\tau| = |\vec{F} \times \vec{P}| = \boxed{60(.18)\sin(80)}$$