12.3 - Dot Product

In previous section learned about vectors

Discovered the operations

vector addition

☆ scalar multiplication

But recall we said there are a few ways to multiply

Dot Product (aka inner product, scalar product)

Mechanics first, then interpretation

Again, works for any dimension, not just $\mathbb{R}^3$

$$\left\{ \begin{array}{l} \vec{v} = \langle a_1, a_2, \ldots, a_n \rangle \\ \vec{w} = \langle b_1, b_2, \ldots, b_n \rangle \end{array} \right. \qquad \mathbb{R}^n$$

Dot product of $\vec{v}$ and $\vec{w}$ is

$$\vec{v} \cdot \vec{w} = \underline{a_1 b_1} + \underline{a_2 b_2} + \underline{a_3 b_3} + \ldots \underline{a_n b_n}$$

For $\mathbb{R}^3$

$$\left.\begin{array}{l}\vec{v} = \langle a_1, a_2, a_3 \rangle \\ \vec{w} = \langle b_1, b_2, b_3 \rangle\end{array}\right\}$$

$$\vec{v} \cdot \vec{w} = \underbrace{a_1 b_1}_{\mathbb{R}} + \underbrace{a_2 b_2}_{\mathbb{R}} + \underbrace{a_3 b_3}_{\mathbb{R}} = \mathbb{R}$$

Take two vectors, result is just a number in $\mathbb{R}$.

Ex:

$$\vec{x} = \langle 1, -1, 7 \rangle$$
$$\vec{y} = \langle 0, -4, -2 \rangle$$

$$\vec{x} \cdot \vec{y} = 1 \cdot 0 + (-1)(-4) + 7(-2)$$
$$= 0 + 4 - 14$$
$$= -10$$

Ex:

$$(3i - 2j + 6k) \cdot (1i - 2k)$$

$$\langle 3, -2, 6 \rangle \cdot \langle 1, 0, -2 \rangle$$

$$= 3(1) + (-2)(0) + 6(-2)$$

$$= 3 - 12$$

$$= -9$$

Properties

① $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ ("length of a")^2

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$$\boxed{\vec{a} \cdot \vec{a}} = \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= a_1^2 + a_2^2 + a_3^2$$

OTGH

$$\boxed{|\vec{a}|^2} = \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2$$

$$= a_1^2 + a_2^2 + a_3^2$$

Whenever you see $\vec{a} \cdot \vec{a}$ you should think of $|\vec{a}|^2$.

$\vec{a} = \langle a_1, a_2 \rangle$

② $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

③ $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

④ $(c\vec{a}) \cdot \vec{b} = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$

⑤ $\vec{0} \cdot \vec{a} = 0$

Check the rest of properties on your own

Interpretation of Dot Product

Can think of dot product $\vec{v} \cdot \vec{w}$ as

"a <u>weighted</u> measure of angle between $\vec{v}$ and $\vec{w}$"

<u>or</u>

"a weighted measure of how perpendicular (orthogonal) $\vec{v}$ and $\vec{w}$ are"

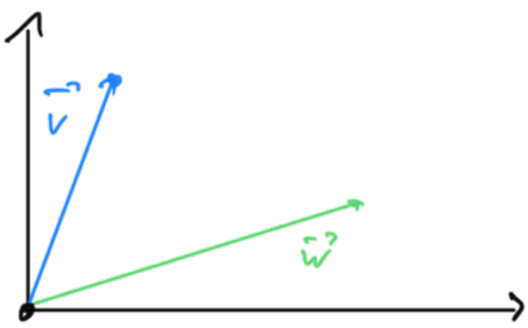
These ideas come from next result

Theorem: If θ is the angle between $\vec{a}$ and $\vec{b}$ then

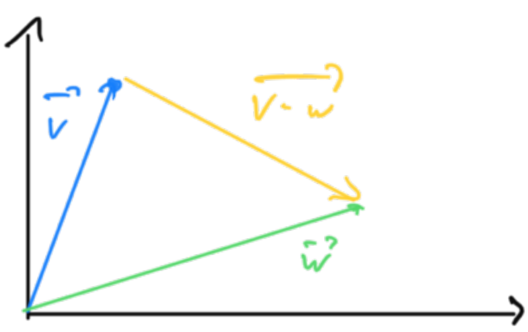
$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos(\theta)$$

□ Setup: $\boxed{\mathbb{R}^2}$

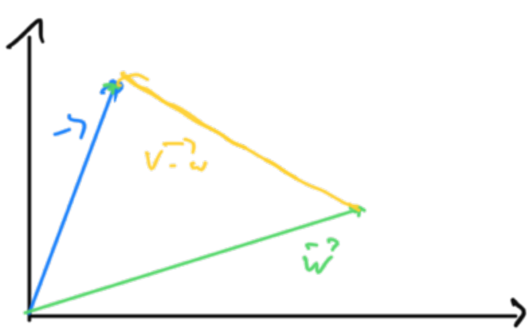
Consider $\vec{a}$ and $\vec{b}$ with bases at origin



Dimension doesn't matter



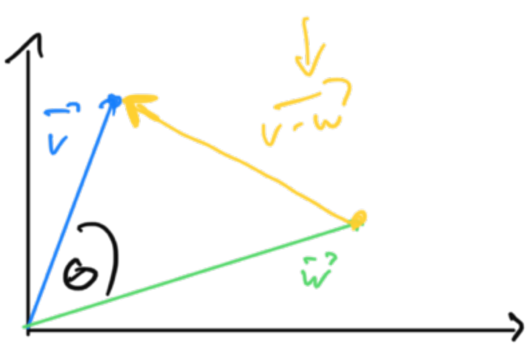
The vector that connects arrows is actually $\overrightarrow{v-w}$ (or $\overrightarrow{w-v}$)



Assume we know length of $\vec{v}$, length of $\vec{w}$. Can we find length of $\overrightarrow{v-w}$?

Pythagorean Theorem

Law of cosines



$$|v-w|^2 = |v|^2 + |w|^2 - 2|v||w|\cos(\theta)$$

OTOH $|v-w|^2 = (\overrightarrow{v-w}) \cdot (\overrightarrow{v-w})$

$$= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{w} - \vec{w}\cdot\vec{v} + \vec{w}\cdot\vec{w}$$

$$= \underline{|\vec{v}|^2} + \underline{|w|^2} - \underline{2\,\vec{v}\cdot\vec{w}}$$

So

$$|v-w|^2 = |v|^2 + |w|^2 - 2|v||w|\cos(\theta)$$

$$\downarrow$$

$$\underline{|v|^2} + \underline{|w|^2} \boxed{- 2\,\vec{v}\cdot\vec{w}} = \underline{|v|^2} + \underline{|w|^2} \boxed{- 2|\vec{v}|\,|\vec{w}|\cos\theta}$$

$$\downarrow$$

$$-2\,\vec{v}\cdot\vec{w} = -2\,|\vec{v}|\,|\vec{w}|\cos\theta$$

<u>so</u>: $\vec{v}\cdot\vec{w} = |\vec{v}|\,|\vec{w}|\cos\theta$ ∎

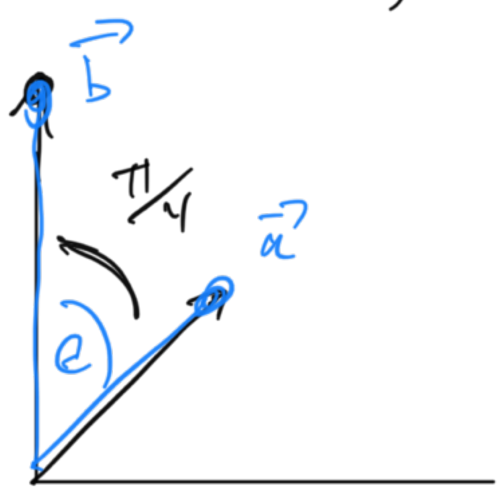
☆ $\underline{\vec{v}\cdot\vec{w}} = \underbrace{|\vec{v}|\,|\vec{w}|}_{\text{"weights"}}\ \underbrace{\cos\theta}_{\text{Gives some idea about angle between } \vec{v} \text{ and } \vec{w}}$

Note:

$$\vec{v} \cdot \vec{w} = \underline{|\vec{v}|\,|\vec{w}|} \cos\theta$$

$$\Rightarrow \frac{\vec{v}\cdot\vec{w}}{|\vec{v}|\,|\vec{w}|} = \cos\theta$$

Ex: Find angle between $\vec{a} = \langle 4, 4 \rangle$ and $\vec{b} \langle 0, 7 \rangle$.



$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

$$\underline{\langle 4,4\rangle \cdot \langle 0,7\rangle} = 4 \cdot 7 \cdot \sqrt{2} \cos(\theta)$$

$$28 = 28\sqrt{2} \cos\theta$$

$$\frac{1}{\sqrt{2}} = \cos\theta$$

$$\frac{\sqrt{2}}{2} = \underline{\cos\theta}$$

$\pi/4$

Direction angles $\mathbb{R}^3$

Apparently

α is angle vector makes w/ x-axis

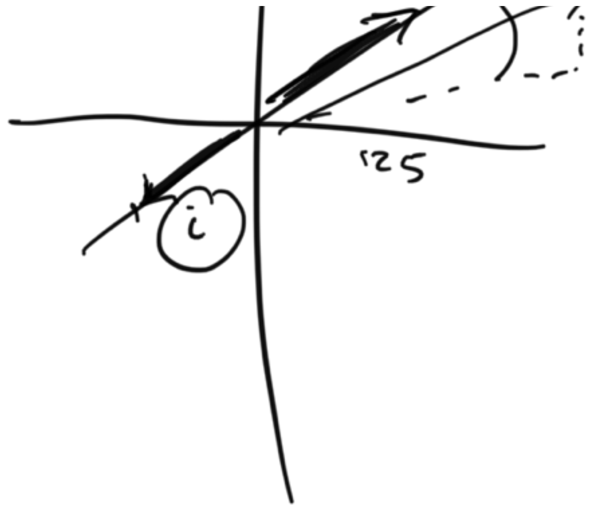
β w/ y-axis

γ w/ z-axis

Given vector, how to find α, β, γ ?

$\vec{v} = \langle -2, 2, 2 \rangle$

$\vec{v} \cdot \vec{i} = |v|\, \underline{|i|} \cos(\alpha)$



$$\langle -2,2,2\rangle \cdot \langle 1,0,0\rangle$$

$$-2 = \sqrt{12}\,|\cos(\alpha)$$

$$-2 = 2\sqrt{3}\cos(\alpha)$$

$$\frac{-1}{\sqrt{3}} = \cos(\alpha)$$

$$-\frac{\sqrt{3}}{3} = \cos(\alpha)$$

$$\alpha = \cos^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

How to interpret the numerical values

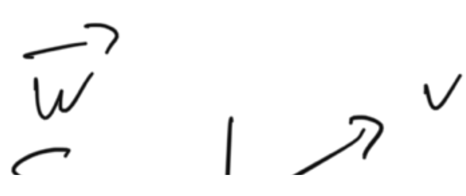
$\boxed{\vec{v}\cdot\vec{w}}$ easier to compute than $|\vec{v}|\,|\vec{w}|\cos\theta$ ☆ so...

① $\vec{v}\cdot\vec{w} > 0$ $\qquad 0 < \theta < \pi/2$

"Same direction-ish"

② $\vec{v}\cdot\vec{w} < 0$

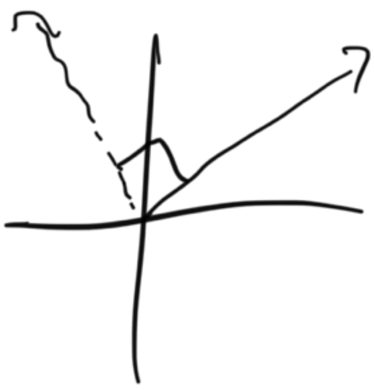
$$\overset{-}{\vec{v}\cdot\vec{w}} = \overset{+}{|\vec{v}|}\,\overset{+}{|\vec{w}|}\,\overset{-}{\cos\theta}$$



$\frac{\pi}{2} < \quad \leq \pi$

(3) $\vec{v} \cdot \vec{w} = 0$

$$\underbrace{\vec{v} \cdot \vec{w}}_{0} = \overset{+}{|\vec{v}|} \, \overset{+}{|\vec{w}|} \, \underbrace{\cos(\theta)}_{0}$$



$\vec{v}$ and $\vec{w}$ are perpendicular

$\Updownarrow$

$\vec{v}$ and $\vec{w}$ are orthogonal

[$\vec{v} \cdot \vec{w}$ are orthogonal (perpendicular)
if and only if $\vec{v} \cdot \vec{w} = 0$

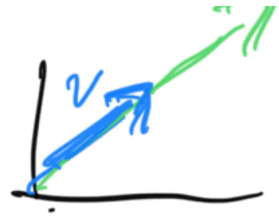
$\vec{v} \cdot \vec{w} = 0$ implies $\vec{v}$ and $\vec{w}$ are orthogonal

$\vec{v}$ and $\vec{w}$ orthogonal implies $\vec{v} \cdot \vec{w} = 0$

How $\vec{v} \cdot \vec{w}$ and $|\vec{v}| \, |\vec{w}| \cos\theta$ relate to each other

W

④ $\boxed{\vec{v} \cdot \vec{w} = |\vec{v}|\,|\vec{w}|}$ ☆



$\Rightarrow \cos(\theta) = 1$

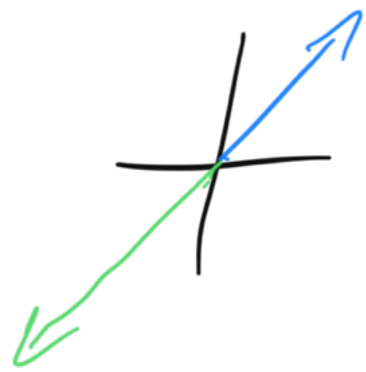
$\Rightarrow \theta = 0$

So $\vec{v}$, $\vec{w}$ parallel and point same direction

⑤ $\vec{v} \cdot \vec{w} = \left(|\vec{v}|\,|\vec{w}| \right) \cancelto{-1}{\cos\theta}$

$\Rightarrow \cos(\theta) = -1$ ☆

$\Rightarrow \theta = \pi$



So $\vec{v}$, $\vec{w}$ parallel but point in opposite directions

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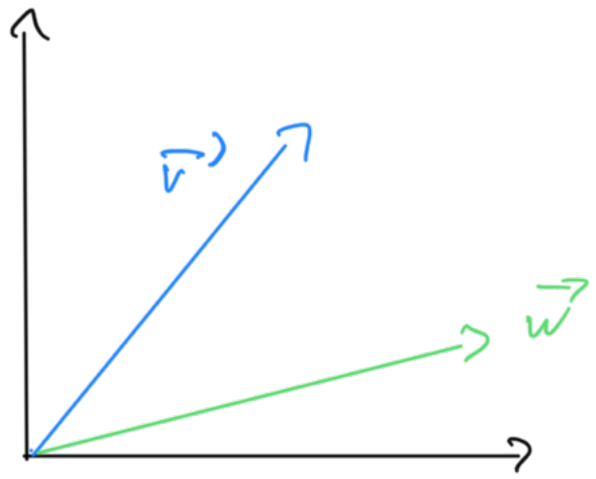
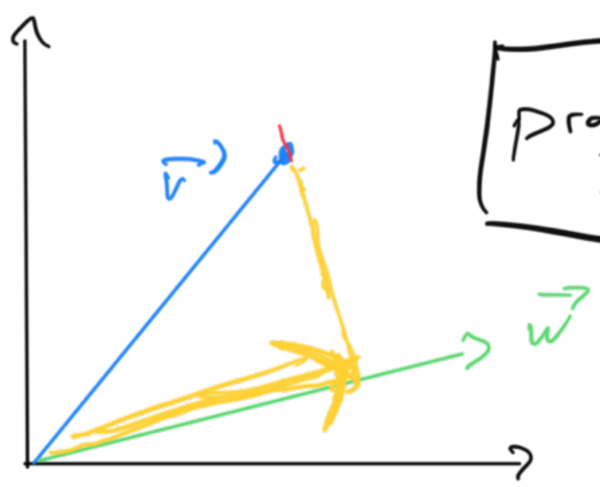
Projections

Hugely important. Used in things like data analysis, machine learning, many more branches of applied math.

<u>What is a projection?</u>



Lets find projection of $\vec{v}$ onto $\vec{w}$.



$$\boxed{\text{proj}_{w}\vec{v}} = \frac{\vec{v}\cdot\vec{w}}{|\vec{w}|} \frac{\vec{w}}{|\vec{w}|}$$

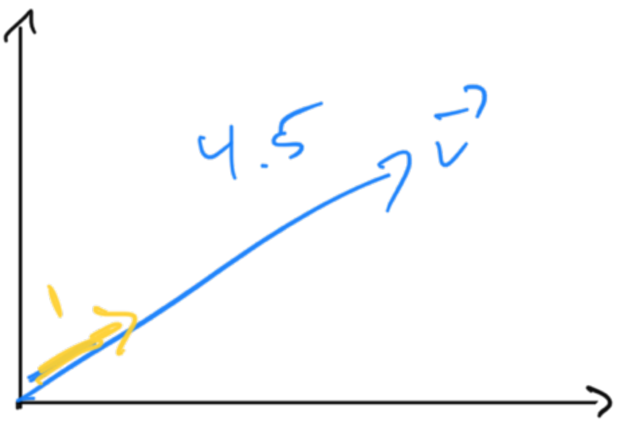
$$= \frac{\vec{v}\cdot\vec{w}}{|w|^2}\vec{w}$$

$$= \boxed{\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\underline{\vec{w}}}$$

vector projection of v onto $\vec{w}$

How to find projection

Unit vectors : any vector of length one



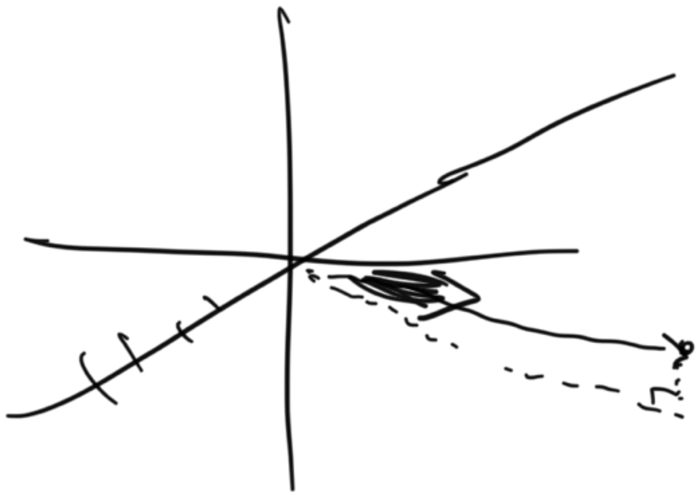
$\vec{v} = \langle \underline{v_1}, \underline{v_2} \rangle$

$|\vec{v}| = \sqrt{v_1^2 + v^2}$

$$\boxed{\frac{\vec{v}}{|\vec{v}|}} = \left\langle \frac{v_1}{|\vec{v}|}, \frac{v_2}{|\vec{v}|} \right\rangle$$

Ex:

$\vec{v} = \langle 4, 7, 2 \rangle$



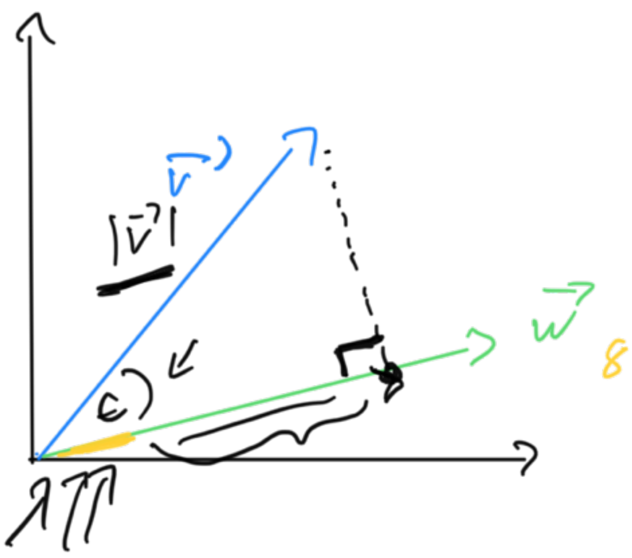
$$\frac{\vec{v}}{|\vec{v}|}$$

$$|\vec{v}| = \sqrt{4^2 + 7^2 + 2^2}$$

$$= \sqrt{16 + 49 + 4}$$

$$= \sqrt{69}$$

$$\left\langle \frac{4}{\sqrt{69}}, \frac{7}{\sqrt{69}}, \frac{2}{\sqrt{69}} \right\rangle$$



$$\cdot \frac{\vec{w}}{|\vec{w}|}$$

$$\boxed{\frac{\vec{v} \cdot \vec{w}}{|v|\,|w|}} = \frac{P}{|\vec{v}|}$$

$$P = \frac{\cancel{|\vec{v}|}\ \vec{v} \cdot \vec{w}}{\cancel{|\vec{v}|}\ |\vec{w}|}$$

$$p = \boxed{\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|}}$$

$$p \frac{w}{|\vec{w}|} = \text{projection}$$

scalar projection of $\vec{v}$ onto $\vec{w}$ $\quad \dfrac{\vec{v}\cdot\vec{w}}{|\vec{w}|}$

$$\boxed{\text{comp}_{\vec{w}}\, \vec{v}}$$

<u>Ex</u>: $\vec{v} = \langle 1, 4, 6 \rangle$

$\vec{w} = \langle -3, 3, 3 \rangle$

$$\text{proj}_w \vec{v} = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\,\vec{w}$$

$$\vec{v}\cdot\vec{w} = 1(-3) + 4(3) + 6(3)$$

$$27$$

$$\vec{w}\cdot\vec{w} = 9 + 9 + 9$$

$$= 27$$

$$\text{proj}_{\vec{w}}\vec{v} = \frac{27}{27}\langle -3, 3, 3\rangle$$

$$= \langle -3, 3, 3\rangle$$

$$\vec{v} = \langle 9, 1, 6\rangle$$

$$\vec{w} = \langle 0, 0, 3\rangle$$

$$\text{proj}_w v = \frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w}$$

$$\vec{v}\cdot\vec{w} = 9(0) + 1(0) + 6(3) = 18$$

$$\vec{w}\cdot\vec{w} = 9$$

$$\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\vec{w} = \frac{18}{9}\vec{w}$$

$$= 2\vec{w}$$

$$= 2\langle 0, 0, 3\rangle$$

$$\text{proj}_w\vec{v} = \langle 0, 0, 6\rangle$$

40N