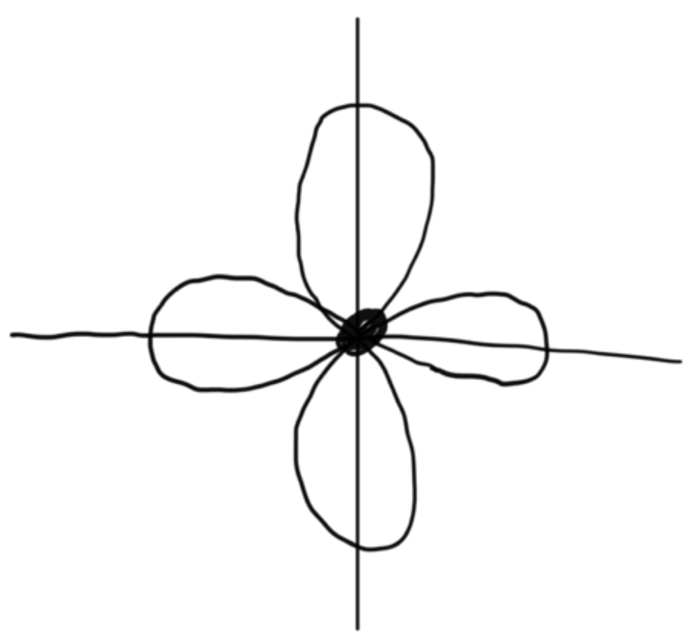
15.3 - Polar Integrals

Continue developing integration over domains in $\mathbb{R}^2$.

- Have integrated on rectangles
- Used functions in bounds to integrate on more unusual domains

But in both cases above we were integrating in Cartesian coordinates (xy-coords)



$r = \cos(2\theta)$

PDEs

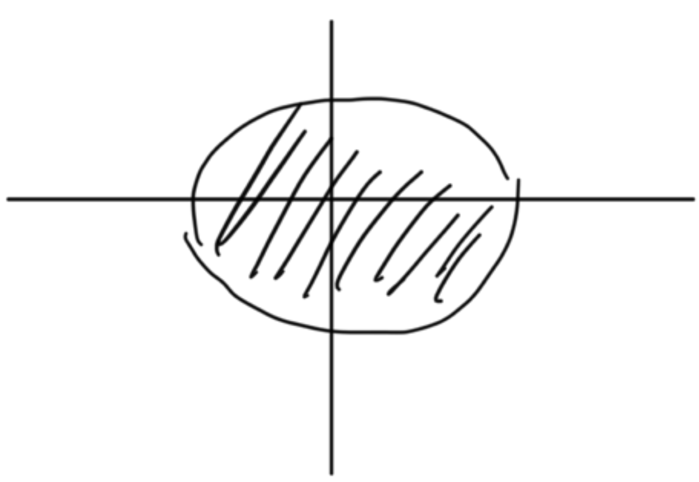
Wave Eq

Heat Eq

Cartesian coordinates are familiar/intuitive but a little limiting

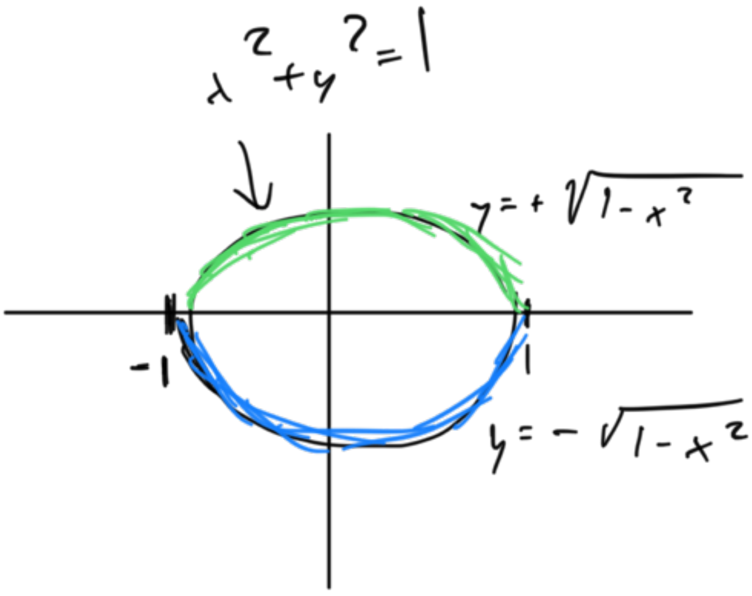
Ex: Let $R = \{ (x,y) : x^2+y^2 \leq 1 \}$

Consider $\iint_R e^{\sqrt{x^2+y^2}} \, dA$



Use techniques from 15.2 to write

iterated integral



$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}}\,dy\,dx$$

... ?

Problem is that cartesian coordinates are not great at dealing with functions that have this "circular orientation"

But as we saw in Calc 2, a switch to Polar coordinates may simplify things.

Lets develop theory of integration

over domain in polar coordinates.

Start $f(x,y)$, and a region that is hard to integrate

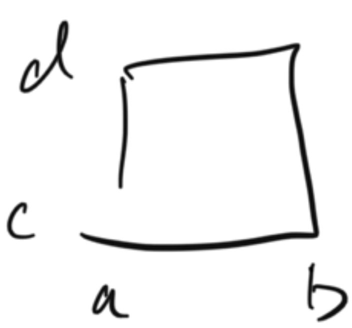
Convert function and region into polar coordinates

☆☆ Integrate ☆☆

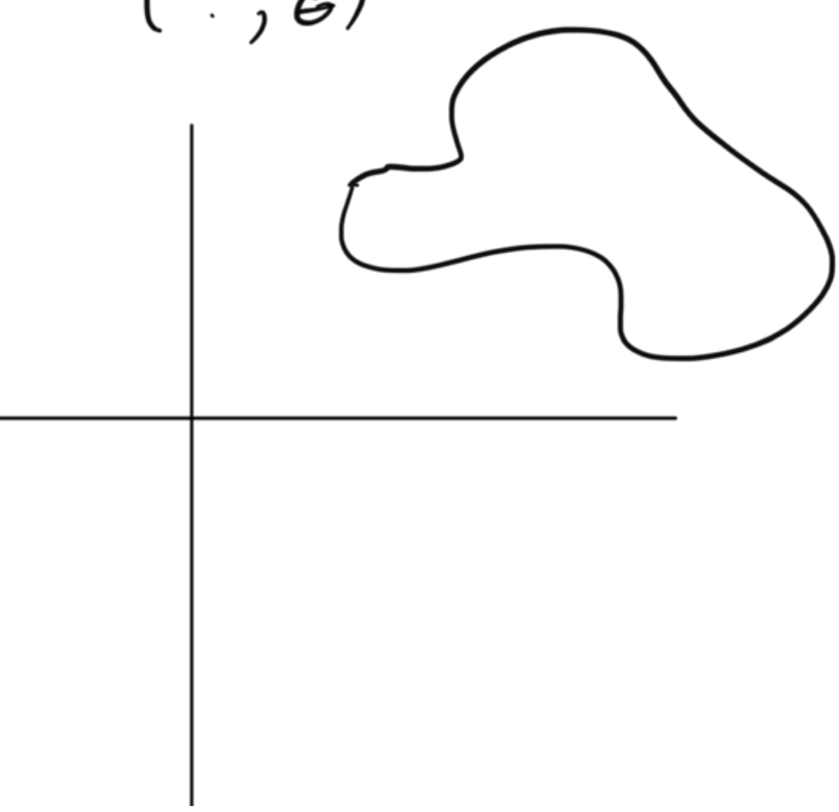
Integrating on Polar Regions

A rectangle in xy-coordinates given by

$a \leq x \leq b \qquad c \leq y \leq d$



$(\cdot , \theta)$



$$(r, \theta)$$

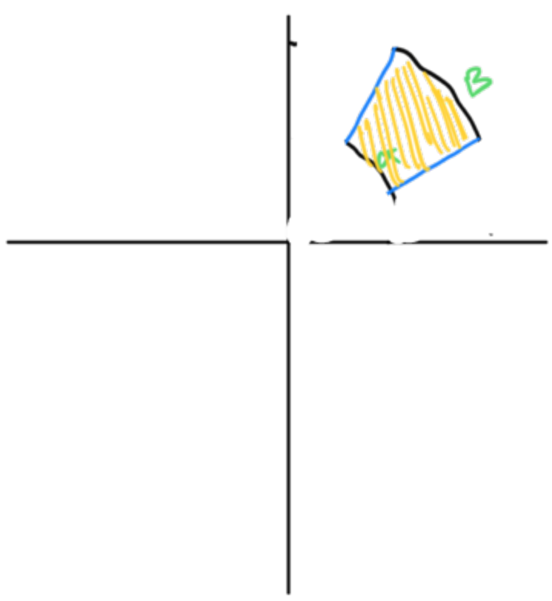
radius

angle

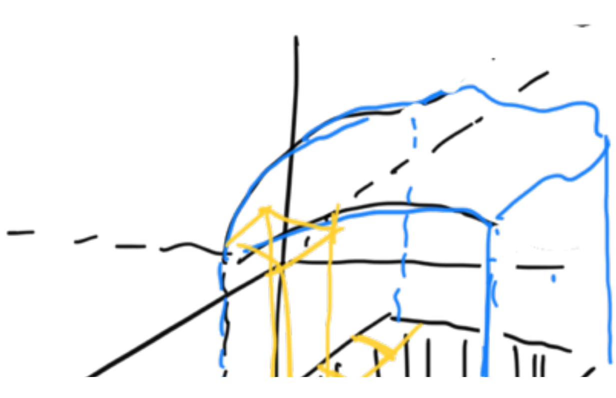
So what does this give:

☆ $\alpha \le r \le \beta$ $\quad \gamma \le \theta \le \delta$

Call this a polar rectangle, region looks like one of these



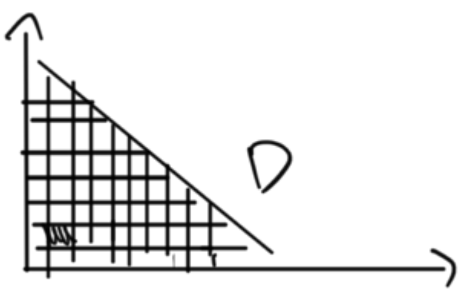
Recall in 15.1 used area of rectangles to develop Riemann sums which became integrals



function/height

Area Rectangle

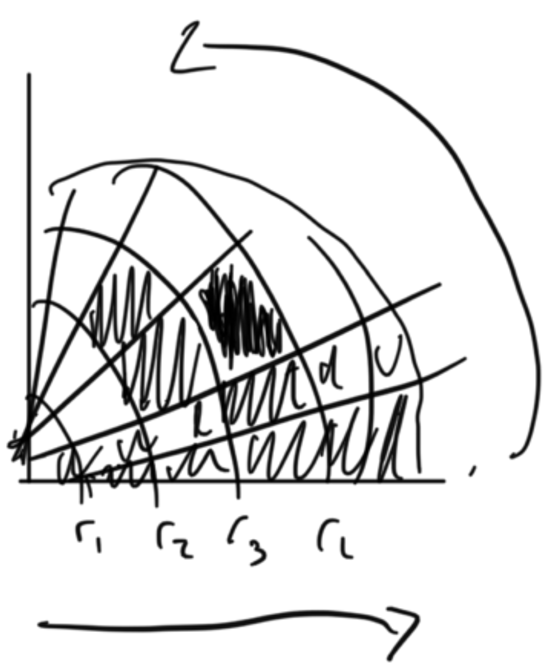
$$\sum \underbrace{f(x_i^*, y_j^*)} \underbrace{\Delta A(x_i, y_j)}$$



$$\sum \sum f(x_i^*, y_j^*) \, \Delta A(x_i, y_j)$$

$$\iint f(x, y) \, dA$$

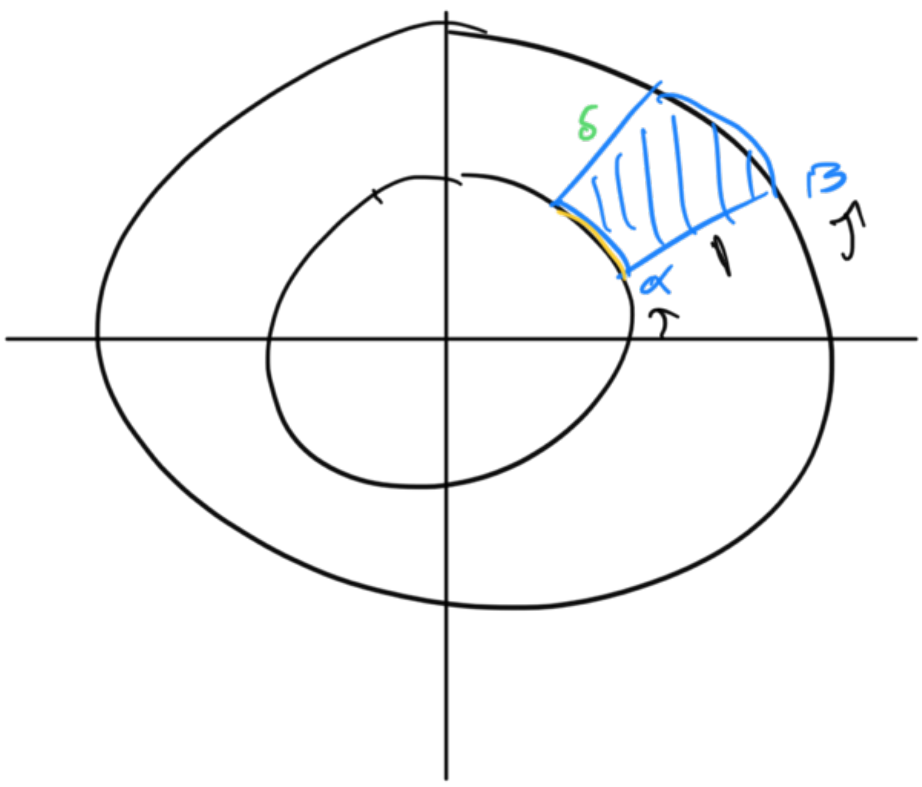
Want to develop an integral for polar coordinates in the same way using "polar rectangles"



$$f(r_{ij}^{*}, \theta_j^{*}) \underline{\Delta A_{ij}}$$

Step 1: Compute $f(r_i^{*}, \theta_j^{*}) \Delta A_{ij}$ on a single polar rectangle

What is area ΔA_{ij} ?



$$\frac{\theta}{2} r^2$$

$$\frac{\delta - \gamma}{2} \alpha^2$$

$$\frac{\delta - \gamma}{2} \beta^2$$

$$\frac{\delta - \gamma}{2} \beta^2 - \frac{\delta - \gamma}{2} \alpha^2$$

$$\theta \,(\beta^2 - \alpha^2)$$

$$\frac{\theta}{2}(\beta-\alpha)(\beta+\alpha)$$

$$\frac{\theta}{2}(\beta-\alpha)\boxed{\frac{(\beta+\alpha)}{2}}$$

The polar rectangle is difference of two sectors of a circle

$\frac{\theta}{2} r_1^2$

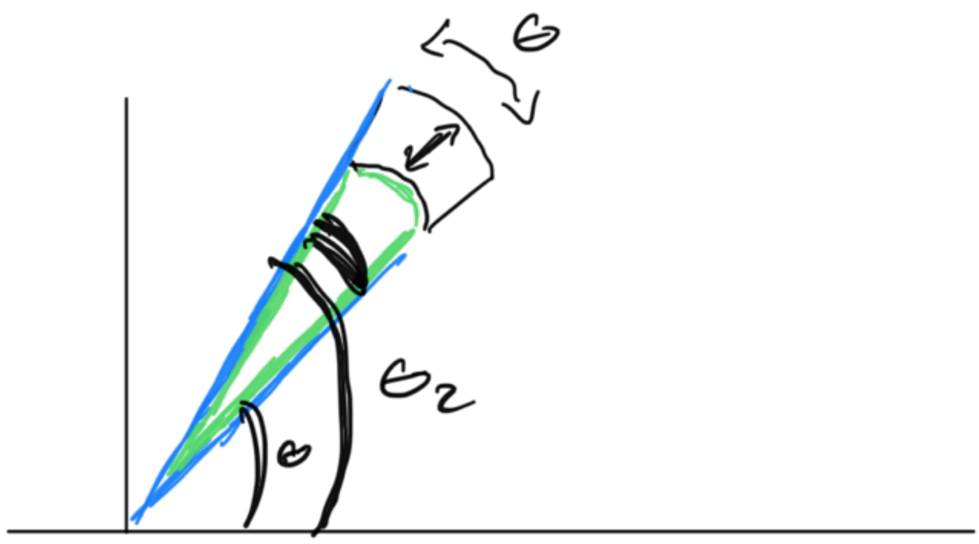
$\frac{\theta}{2} r_2^2$

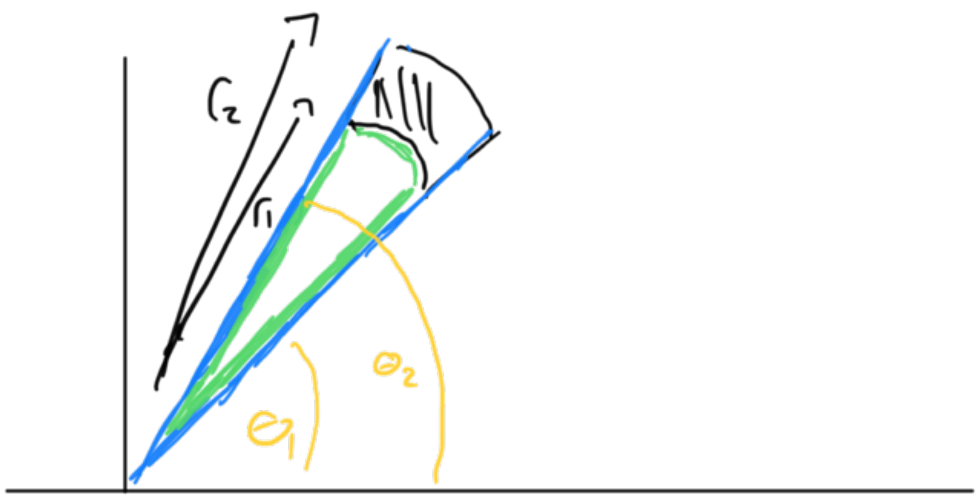
$$\text{Area} = \frac{\theta}{2}\left(r_2^2 - r_1^2\right)$$

$$= \frac{\theta}{2}(r_2 - r_1)(r_2 + r_1)$$

$$= \boxed{\theta}\,\underline{(r_2 - r_1)}\,\underline{\frac{(r_2 + r_1)}{}}$$

2





$$= \underbrace{(\theta_2 - \theta_1)}\,\underbrace{(r_2 - r_1)}\,\underbrace{\left(\frac{r_2 + r_1}{2}\right)}$$

$$A_{ij} = \underbrace{\Delta\theta\ \Delta r\ (r^*)}$$

Step 2: Add all polar rectangles up to get a Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(r_{ij}^*, \theta_{ij}^*)\, \Delta A_{ij}$$

$$\star = \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_{ij}^{*}, \theta_{ij}^{*}) \underline{\Delta r_{ij} \Delta \theta_{ij} (r_{ij}^{*})}$$

$$\star = \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_{ij}^{*}, \theta_{ij}^{*})\, r_{ij}^{*}\, \Delta r_{ij}\, \Delta \theta_{ij}$$

Step 3: Take limit as $\Delta A_{ij} \to 0$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{f(r_{ij}^{*}, \theta_{ij}^{*})}\, \underline{r_{ij}^{*}}\, \underline{\Delta r_{ij}}\, \underline{\Delta \theta_{ij}}$$

$$\boxed{\iint_D f(r, \theta)\, \boxed{r}\, dr\, d\theta}$$

Notice the extra r term! $\star$

$x = r\cos\theta$
$y = r\sin\theta$

$$\star \iint_R f(x, y)\, dx\, dy \rightarrow \iint \underline{f(r\cos\theta, r\sin\theta)}\, r\, \underline{dr\, d\theta}$$

Ex: $f(r,\theta) = \boxed{r\cos(\theta)}$

Find volume below this surface on region $D = \{ (r,\theta) : 0 \le r \le 4,\ 0 \le \theta \le \frac{\pi}{2} \}$

$$\int_0^{\frac{\pi}{2}}\int_0^4 r\cos(\theta)\, r\, dr\, d\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^4 r^2 \cos(\theta)\, dr\, d\theta$$

$$\int_0^{\frac{\pi}{2}}\left(\frac{r^3}{3}\Big]_0^4\right)\cos(\theta)\, d\theta$$

$$\int_0^{\frac{\pi}{2}}\left(\frac{64}{3}\right)\cos(\theta)\, d\theta$$

$$\frac{64}{3}\left(\sin\theta\Big]_0^{\frac{\pi}{2}}\right)$$

$$= \frac{64}{3}$$

$f(r,\theta)$ $\qquad R$

$$\iint f(r,\theta)\, r\, dr\, d\theta$$

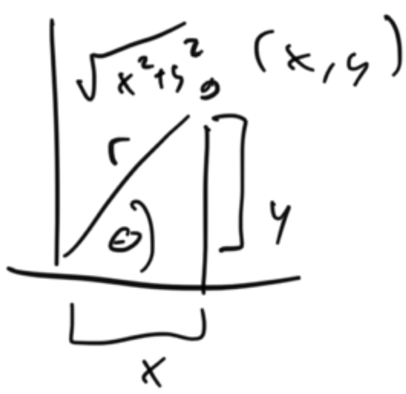
$$f(r,\theta) = r^2\sin\theta$$

$$\iint r^2 \sin\theta\ r\, dr\, d\theta$$

Converting Cartesian to Polar

Forget integrals for a second. Recall how to convert cartesian to polar coordinates.

Given (x, y), $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ (check this against your quadrant).



$$\frac{x}{r} = \cos\theta$$
$$x = r\cos\theta$$

$$\frac{y}{r} = \sin\theta$$
$$y = r\sin\theta$$

More importantly:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

So $f(x,y) \Rightarrow f(r\cos\theta, r\sin\theta)$

Ex: Convert $f(x,y)=x^2y$ to polar.

$$f(r,\theta)=\underline{(r\cos(\theta))^2}\,\underline{(r\sin\theta)}$$
$$=r^3\cos^2\theta\sin\theta$$

$$\iint f(x,y)\,dx\,dy$$
$$\iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Ex: Let $R=\{(x,y): x^2+y^2\le 1\}$

Consider $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{\sqrt{x^2+y^2}}\,dy\,dx$

$$\boxed{\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}}$$

$$\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}$$
$$=\sqrt{r^2\cos^2\theta+r^2\sin\theta}$$
$$=r\sqrt{\cos^2\theta+\sin^2\theta}$$
$$=r$$

$$\boxed{\star\ r=\sqrt{x^2+y^2}}\qquad \theta=\tan^{-1}\left(\frac{y}{x}\right)\star$$

$$\int^{2\pi}\int^{1} r \ldots$$

$$\int_0 \int_0 e \quad r\,dr\,d\theta$$

$$\int_0^{2\pi} \left(re^r - e^r \Big]_0^1 \right) d\theta$$

$$= \int_0^{2\pi} e^r(r-1) \Big]_0^1 \, d\theta$$

$$= \int_0^{2\pi} \left(e(0) - (-e^1) \right)$$

$$= \int_0^{2\pi} e \, d\theta$$

$$e \cdot \theta \Big]_0^{2\pi}$$

$$\boxed{2\pi \cdot e}$$

Ex: 8

$$\iint_R (2x-y)\,\underset{dx\,dy}{dA}$$

$x^2+y^2=4$ $\qquad$ $x=0$

$y=x$

$$\iint (2r\cos\theta - r\sin\theta)\, r\,dr\,d\theta$$

$$\iint r^2 (2\cos\theta - \sin\theta)\,dr\,d\theta$$

$$0 \le r \le 2$$

$\pi/4 \le \theta \le \pi/2$

$$\int_{\pi/4}^{\pi/2}\int_0^2 r^2 (2\cos\theta - \sin\theta)\, dr\, d\theta$$

$$= \int_{\pi/4}^{\pi/2} \frac{8}{3} (2\cos\theta - \sin\theta)\, d\theta$$

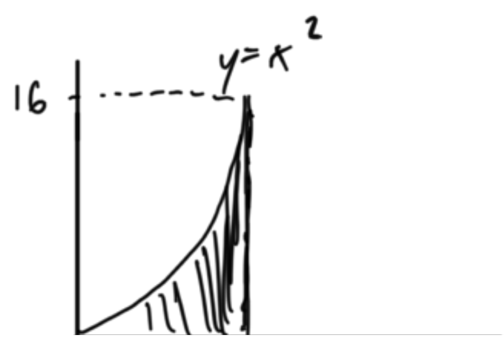
$$\frac{8}{3}\left(2\sin\theta + \cos\theta \Big]_{\pi/4}^{\pi/2} \right)$$

$=$

Polar Regions Bounded by Functions

For cartesian, started integrating on rectangles then considered regions bounded by functions

Ex:



$$\int_0^4 \int_0^{x^2} dy\, dx$$

or

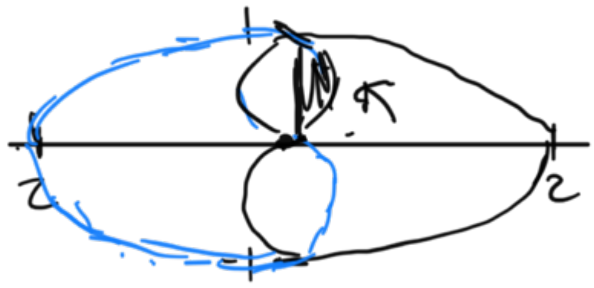
$$\int_0^{16} \int_{\sqrt{y}}^{4} dx\, dy$$

Similarly, may have regions that are not polar rectangles

$\theta_2 = g(r)$

$\theta_1 = \pi/6$

Ex: Find area of region enclosed by $r = 1 + \cos(\theta)$ and $r = 1 - \cos(\theta)$.



$$4\int_0^{\pi/2}\int_0^{1-\cos(\theta)} 1 \quad r\, dr\, d\theta$$

$$4\int_0^{\pi/2}\int_0^{1-\cos\theta} r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{r^2}{2}\Big]_0^{1-\cos\theta}\right)d\theta$$

$$= 4\int_0^{\pi/2}\frac{(1-\cos\theta)^2}{2}\,d\theta$$

$$= \frac{4}{2}\int_0^{\pi/2}\left(1-2\cos\theta+\cos^2\theta\right)d\theta$$

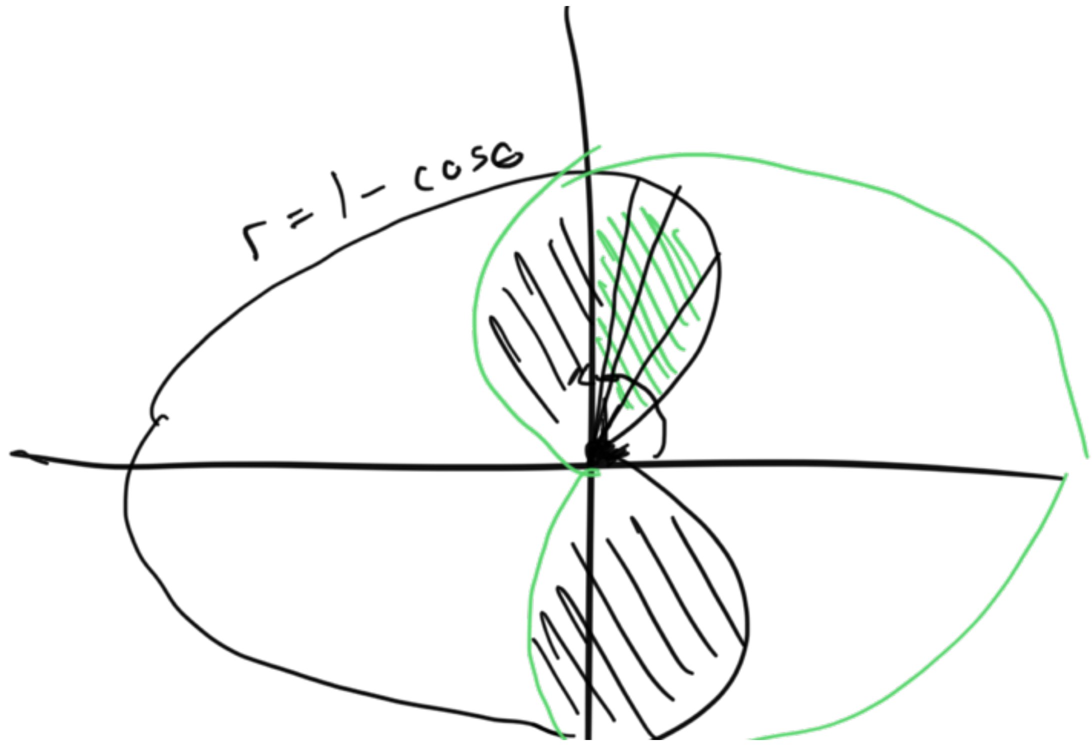
$$= 2\int_0^{\pi/2} 1 - 2\cos\theta + \tfrac{1}{2}\left(1+\cos(2\theta)\right)d\theta$$

$$= 2\left(\theta - 2\sin\theta + \tfrac{1}{2}\theta + \frac{\sin(2\theta)}{4}\Big]\right)$$

$$= 2\left(\left(\tfrac{\pi}{2} - 2 + \tfrac{\pi}{4} + \tfrac{0}{4}\right)\right)$$

$$= \pi - 4 + \frac{\pi}{2}$$

$$= \boxed{\tfrac{3\pi}{2} - 4}$$


r = 1 - cosθ