Section 6.2

Definition: A set of vectors $\{\vec{u}_1 \dots \vec{u}_p\}$ is an orthogonal set if for each pair $\vec{u}_i, \vec{u}_j$ where $i \neq j$ we have $\vec{u}_i \cdot \vec{u}_j = 0$.

OTOH $\vec{u}_i \cdot \vec{u}_i = \|\vec{u}_i\|^2 \neq 0$ unless $\vec{u}_i = \vec{0}$

Orthogonal sets are very convenient to work with because there will typically be a lot of cancellations leaving very simple result. Will see specific examples later.

Another convenient property of orthogonal sets given in next result.

Theorem: If $S = \{\vec{u}_1 \dots \vec{u}_p\}$ is an orthogonal set of nonzero vectors in V, then S is a linearly independent set and so S is a basis for $\text{Span}\{\vec{u}_1 \dots \vec{u}_p\}$.

Pf. Assume $c_1\vec{u}_1 + \dots c_p\vec{u}_p = \vec{0}$. To show lin. ind, need to show $c_1 = c_2 = \dots = c_p = 0$.

$\star$ $c_1\vec{u}_1 + \dots c_p\vec{u}_p = \vec{0}$ Left / right side are vectors

$\vec{u}_1 \cdot (c_1\vec{u}_1 + \dots c_p\vec{u}_p) = \vec{u}_1 \cdot \vec{0}$ Do dot product with $\vec{u}_1$ on both sides

$\downarrow$

$\vec{u}_1 \cdot (c_1\vec{u}_1 + \dots c_p\vec{u}_p) = 0$ Dot product with $\vec{0}$ always just 0.

$c_1(\vec{u}_1 \cdot \vec{u}_1) + c_2(u_1 \cdot u_2) + \dots c_p(\vec{u}_1 \cdot \vec{u}_p) = 0$ Expand left side

$c_1(\vec{u}_1 \cdot \vec{u}_1) + 0 + \dots + 0 = 0$ Since S orthogonal set $u_i \cdot u_j = 0$ when $i \neq j$. Thus all terms except $u_1 \cdot u_1$ cancel

$c_1(\vec{u}_1 \cdot \vec{u}_1) = 0$

Since $\vec{u}_1$ non-zero vector, $\vec{u}_1 \cdot \vec{u}_1 = \|\vec{u}\|^2 > 0$.

Thus, must be that $c_1 = 0$.

Repeat process for other vectors $\vec{u}_i$. Then $c_i = 0$ for all i. Thus S linearly independent.

Clearly S span$\{S\}$ Thus S is basis for span$\{S\}$.

$\blacksquare$

Definition: If S is an orthogonal set <u>and</u> a basis for some subspace, call it an **orthogonal basis**.

If $S = \{\vec{u}_1 \dots \vec{u}_n\}$ is a (orthogonal) basis for W, can write any $\vec{w}$ as lin. comb of $\vec{u}_1 \dots \vec{u}_n$

$$\star \quad \vec{w} = c_1\vec{u}_1 + \dots + c_n\vec{u}_n \quad \star$$

Have a nice way to figure out what each c_i should be.

Theorem: $c_i = \dfrac{\vec{u}_i \cdot \vec{w}}{\vec{u}_i \cdot \vec{u}_i}$ scalar projection $\text{proj}_{\vec{u}_i}\vec{w}$

☐ Have $\vec{w} = c_1\vec{u_1} + \dots c_n\vec{u_n}$.

Do dot product with $\vec{u_i}$ similar to previous theorem.

Trying to find c_i.

$$\vec{w} = c_1\vec{u_1} + \dots c_n\vec{u_n}$$

$$\vec{u_i}\cdot\vec{w} = \vec{u_i}\cdot(c_1\vec{u_1} + \dots c_n\vec{u_n})$$

$$= c_1(u_i\cdot u_1) + \dots c_n(\vec{u_i}\cdot\vec{u_n})$$

$$= c_i(\vec{u_i}\cdot\vec{u_i})$$

Thus $\vec{u_i}\cdot\vec{w} = c_i(\vec{u_i}\cdot\vec{u_i})$

Can divide both sides safely by $u_i\cdot u_i$. Since $\vec{u_i}$ nonzero vector, $\vec{u_i}\cdot\vec{u_i} \neq 0$. So

$$\frac{\vec{u_i}\cdot\vec{w}}{\vec{u_i}\cdot\vec{u_i}} = c_i \quad ☆$$

■

So, orthogonal sets are nice to work with. Don't have nice formula for other bases. Would just have to solve augmented matrix

$$\left[\begin{array}{c|c}\vec{u_1}\dots\vec{u_n} & \vec{w}\end{array}\right]$$

If matrix small, not bad. But for larger matrices row reduction very difficult. Easier to use formula for c_i if possible.

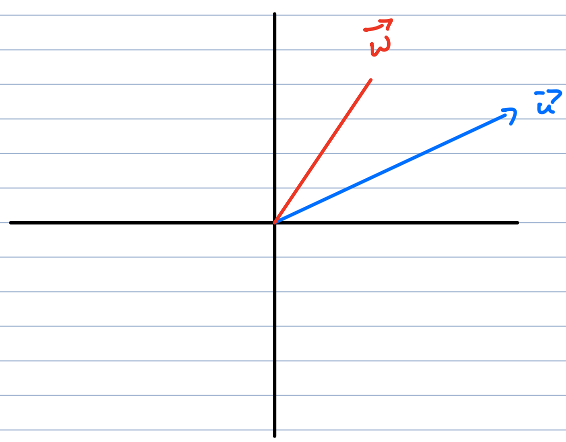
Orthogonal Projections

Consider some vector $\vec{u}$ in $\mathbb{R}^n$

(picture in $\mathbb{R}^2$ for now)

$\text{proj}_{\vec{u}}\,\vec{w}$

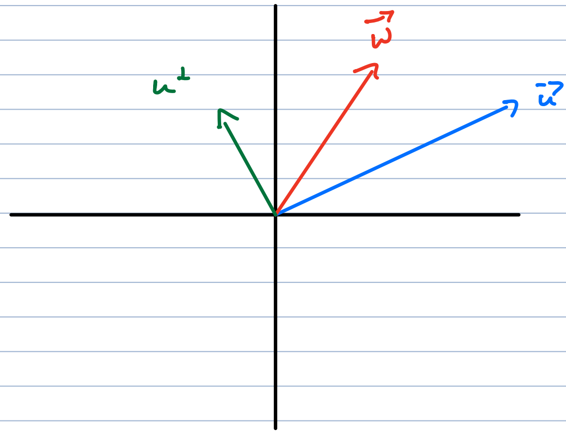
$\vec{w} = c\,\vec{u} + \_\_\_$



Now consider vector $\vec{w}$ and try to write $\vec{w}$ as multiple of $\vec{u}$ and something orthogonal to $\vec{u}$ ($\vec{u}^{\perp}$)

$\vec{w} = \alpha\,\vec{u} + \vec{u}^{\perp}$

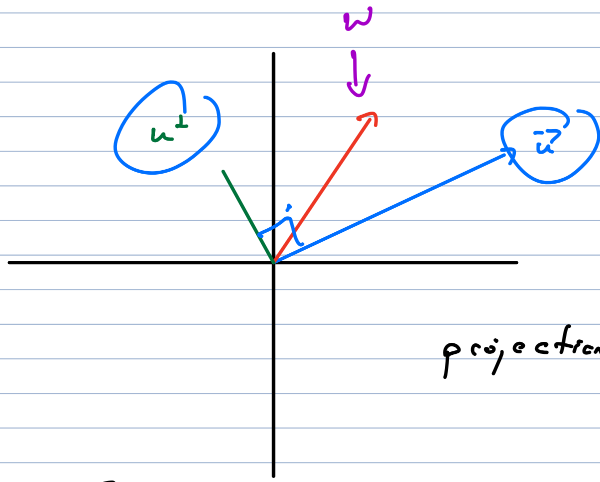
$\vec{w} = \underline{\alpha\,\vec{u}} + \_\_\_$



How do we do this?

In particular, what should $\alpha \vec{u}$ be?

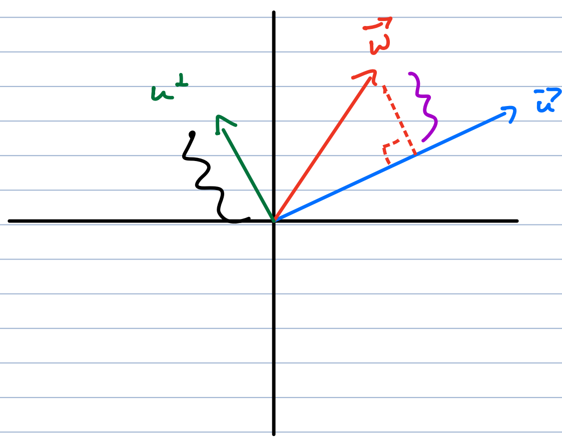
Ask, how much of $\vec{w}$ "points in same direction" as $\vec{u}$?



The "shadow" of $\vec{w}$ on $\vec{u}$

projection of $\vec{w}$ onto $\vec{u}$, $\text{proj}_{\vec{u}} \vec{w}$

$\vec{w} = \underline{\alpha \vec{u}} + \_\_$



This portion is orthogonal to $\vec{u}$ and clear that

Note that this is $\vec{w} - \text{proj}_{\vec{u}} \vec{w}$, same vector as $\vec{u}^\perp$

Thus $u^\perp = \vec{w} - \text{proj}_{\vec{u}} \vec{w}$ and so

☆ $\vec{w} = \text{proj}_{\vec{u}} \vec{w} + (\vec{w} - \text{proj}_{\vec{u}} \vec{w})$

So what is actual vector $\text{proj}_{\vec{u}} \vec{w}$?

$$\text{proj}_{\vec{u}} \vec{w} = \left(\frac{\vec{u} \cdot \vec{w}}{\vec{u} \cdot \vec{u}} \right) \vec{u}$$

Note for any other vector $\vec{x}$ on same line as $\vec{u}$, result of $\text{proj}_{\vec{x}} \vec{w}$ is the same

Orthogonal Projections on Subspaces

By note above, $\text{proj}_{\vec{u}} \vec{w}$ is same for any multiple of $\vec{u}$. Can consider this to be projection of $\vec{w}$ onto "subspace spanned by $\vec{u}$"

Can consider projections onto larger subspaces.

Let Y be subspace of V, $\mathcal{B} = (\vec{y}_1, \dots, \vec{y}_k)$ an orthogonal basis for Y.

For vector $\vec{v} \in V$, projection of $\vec{v}$ onto Y

$\text{proj}_Y \vec{v} = c_1 \vec{y}_1 + \dots c_k \vec{y}_k$

where $c_i = \dfrac{\vec{v} \cdot y_i}{y_i \cdot y_i}$, just like in a prev theorem.

Orthonormal Sets :

A slight refinement of orthogonal vectors.

Definition: An orthonormal set of vectors is an orthogonal set of unit vectors (so orthogonal to each other and all have length one).

Orthonormal basis if orthonormal set and a basis.

Standard basis is simplest example.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots$$

Take dot product and see they are orthogonal. And clearly all have length one.

- Can easily make an orthogonal set / basis into orthonormal. Just divide each vector by its own length.

Orthogonal $\{\vec{v}_1, \ldots \vec{v}_k\}$

Orthonormal $\left\{\frac{\vec{v}_1}{\|\vec{v}_1\|}, \ldots \frac{\vec{v}_k}{\|\vec{v}_k\|}\right\}$

Theorem:

The $m \times n$ matrix U has columns of orthonormal vectors iff $\underbrace{(U^T)}_{n \times m}\underbrace{U}_{m \times n} = \underbrace{I}_{n \times n}$

⇒ Assume U made up of orthonormal vectors.

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & & & \\ \vdots & & & \\ u_{m1} & & & \end{bmatrix}_{m \times n} \qquad U^T = \begin{bmatrix} u_{11} & u_{21} & \cdots & u_{m1} \\ u_{12} & & & \\ \vdots & & & \\ u_{1n} & & & \end{bmatrix}_{n \times m}$$

$$\underbrace{\begin{bmatrix} u_{11} & u_{21} & \cdots & u_{m1} \\ u_{12} & & & \\ \vdots & & & \\ u_{1n} & & & \end{bmatrix}}_{n \times m}^{U^T} \underbrace{\begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & & & \\ \vdots & & & \\ u_{m1} & & & \end{bmatrix}}_{m \times n}^{U} = \underbrace{\begin{bmatrix} 1 & 0 & \cdot \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{n \times n}^{I}$$

Assume $U^T U = I_{n \times n}$. Then ① $\vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j$ and ② $\vec{u}_i \cdot \vec{u}_i = 1$. ① Tells us vectors are orthogonal. By ② since $\vec{u}_i \cdot \vec{u}_i = \|\vec{u}_i\|^2$, have $\|\vec{u}_i\|^2 = 1$. So $\|\vec{u}_i\| = 1$. So orthonormal. ∎

$$U^T U = I_{n \times n}$$

Definition: An $n \times n$ matrix U whose columns are orthonormal set is called an orthogonal matrix.

- U is invertible. ☆

  By prev theorem $U^T U = I_{n \times n}$ so U invertible

- $U^T = U^{-1}$

  Since U, U^T square matrices s.t. $U^T U$ they are inverses of each other.

- Transformation $x \mapsto U\vec{x}$ preserves inner product and thus orthogonality

$$U\vec{x} \cdot U\vec{y} = \vec{x} \cdot \vec{y}$$

- Transformation preserves length

$$\|Ux\| = \|x\|$$

- The transformation $x \mapsto U\vec{x}$ is a rotation, a reflection, or combination of the two