15.7 - Cylindrical Coordinates

Cylindrical coordinates are one possible way to extend polar coordinates to 3 dimensional domains.

For cylindrical coordinates, write two dimensions and polar, third in cartesian.

(x, y, z) $(1, 2, 4)$

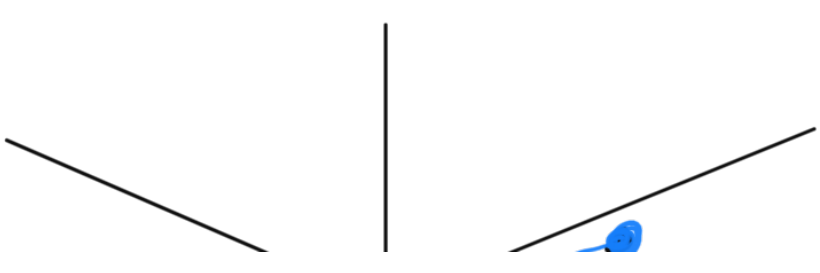
☆ (r, θ, z) $\left(2, \frac{\pi}{4}, 4\right)$

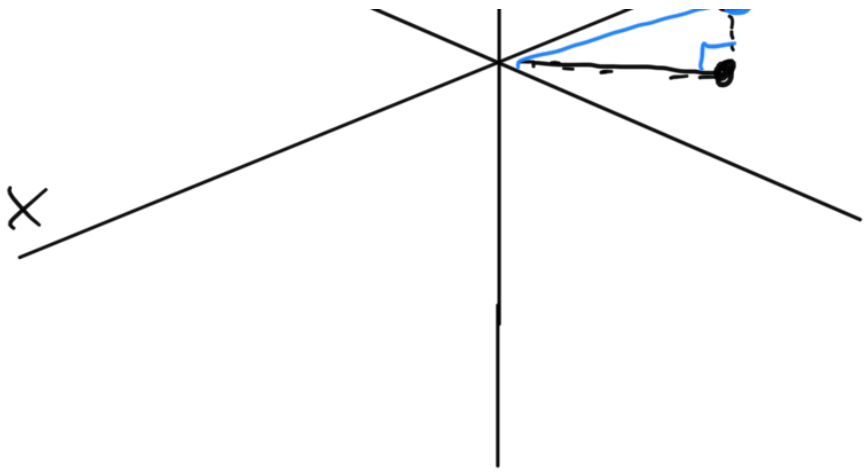
$(x, \underbrace{r, \theta})$ $\left(3, 4, \frac{\pi}{6}\right)$

etc.

Book seems to focus on (r, θ, z) form.

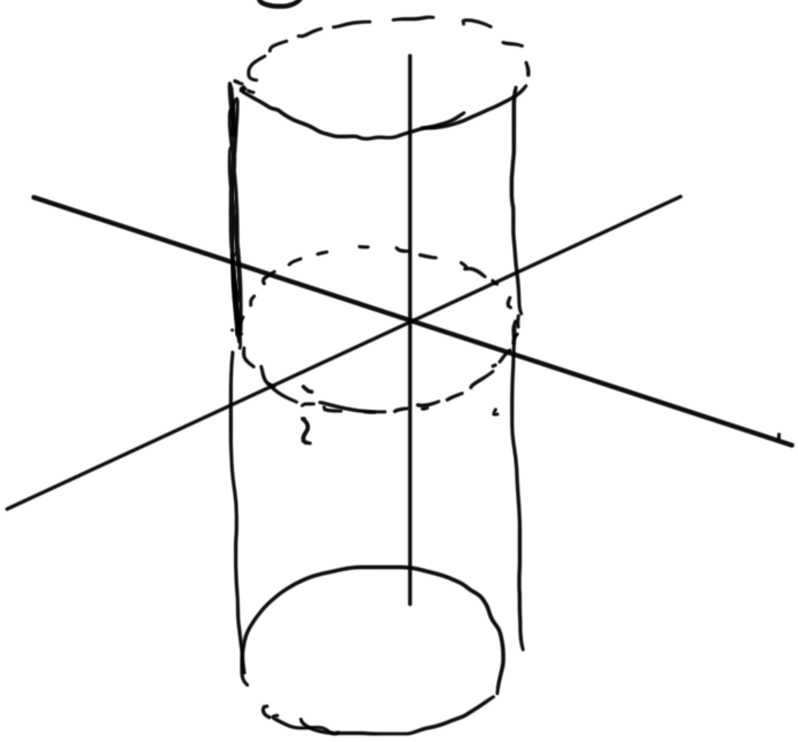
Ex: Plot point $\left(2, \frac{2\pi}{3}, 1\right)$



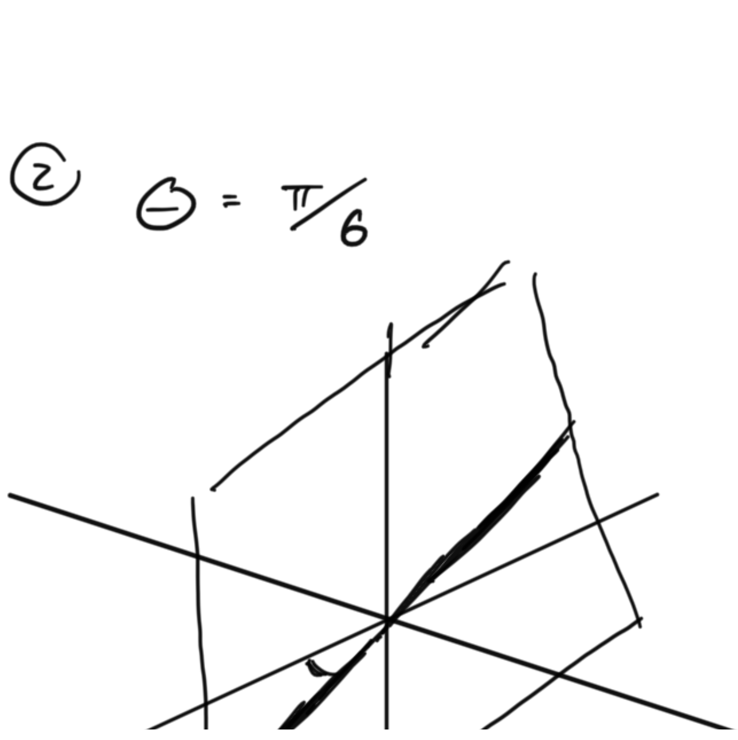


Ex: Describe surfaces given by

① $r = 2$ $\qquad (r, \theta, z)$



② $\theta = \pi/6$

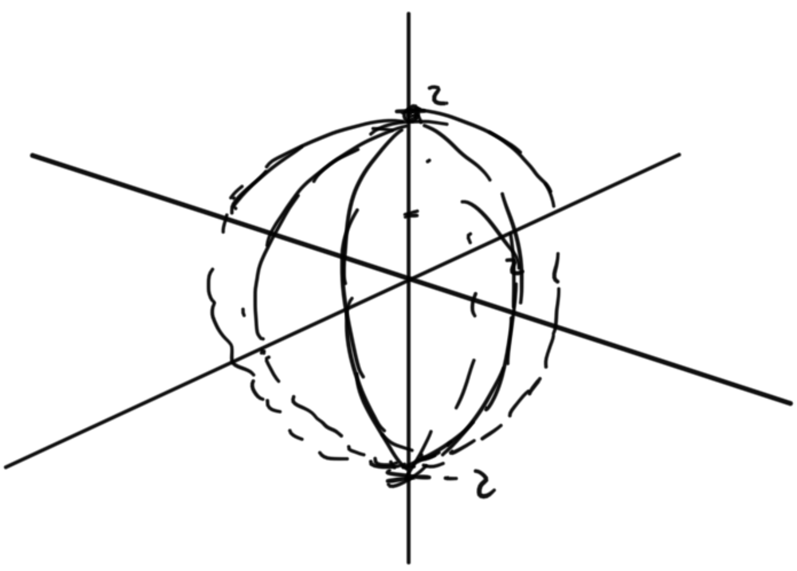


(3) $r^2 + z^2 = 4$

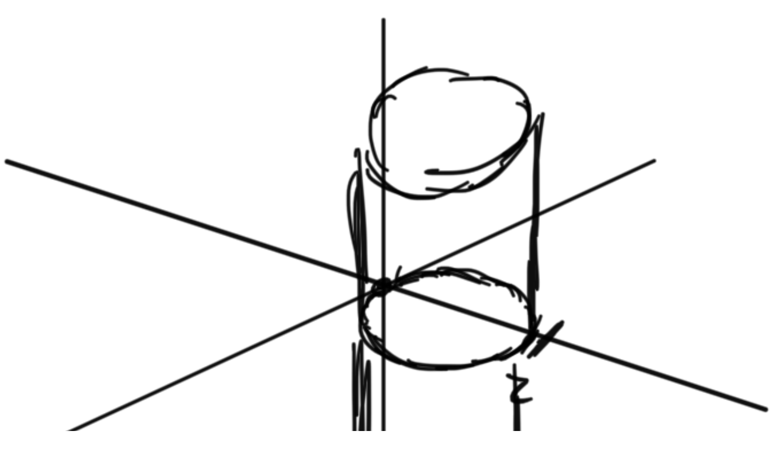
$r = \sqrt{x^2+y^2}$

$$\left(\sqrt{x^2+y^2}\right)^2 + z^2 = 4$$

$$\underbrace{x^2 + y^2 + z^2 = 4}$$



(4) $r = 2 \sin\theta$

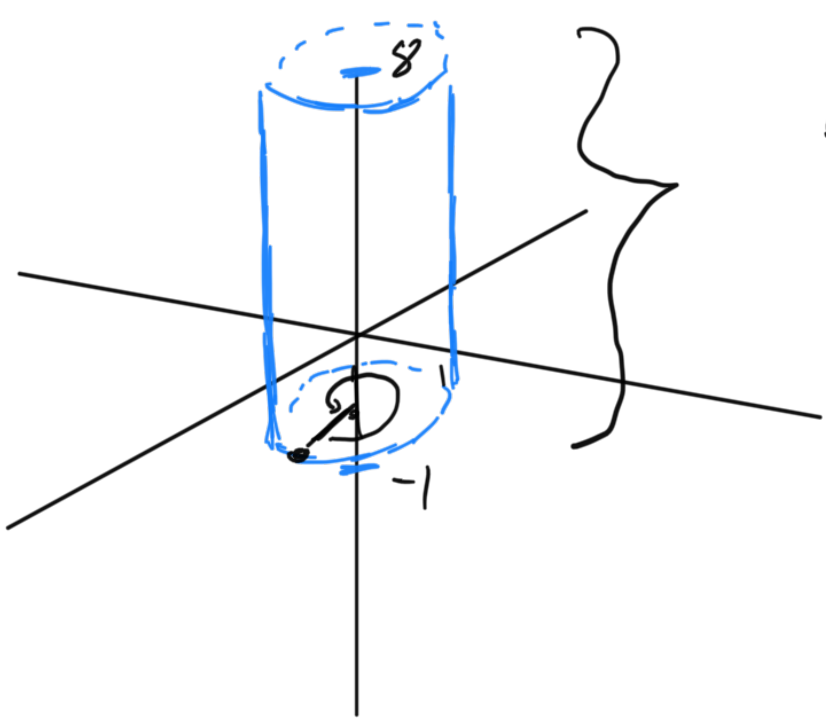


How to integrate using cylindrical:

Simply change to cylindrical coordinates (i.e. change two dimensions to polar and leave third alone) and integrate normally.

Ex: Integrate $f(x,y,z) = 2x + 2y$ on cylinder

$-1 \leq z \leq 8 \qquad x^2 + y^2 \leq 4$



r, θ, z

$$\iiint 2x + zy \; dx\,dy\,dz$$

$$r = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{y}{x} = \tan\theta$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\int_{-1}^{8}\int_{0}^{2\pi}\int_{0}^{2} \Big(2\, r\cos(\theta) + z\, r\sin(\theta)\Big) r\,dr\,d\theta\,dz$$

$$\iiint 2r^2\cos\theta + z r^2 \sin\theta \; dr\,d\theta\,dz$$

$$\iint \frac{2}{3} r^3\cos\theta + z\frac{r^3}{3}\sin\theta \Big]_0^2 d\theta\,dz$$

$$\iint r^3\Big]_0^2\left(\frac{2}{3}\cos\theta + \frac{z}{3}\sin\theta\right) d\theta\,dz$$

$$= \int_{-1}^{8}\int_{0}^{2\pi} 8\left(\frac{2}{3}\cos\theta + \frac{z}{3}\sin\theta\right) d\theta\,dz$$

$$\int_{-1}^{8} 8\left(\frac{2}{3}\sin\theta - \frac{z}{3}\cos\theta\right)\Big]_0^{2\pi} dz$$

$$= \int_{-1}^{8} 8\left[\left(0 - \frac{z}{3}(1)\right) - \left(0 - \frac{z}{3}(1)\right)\right] dz$$

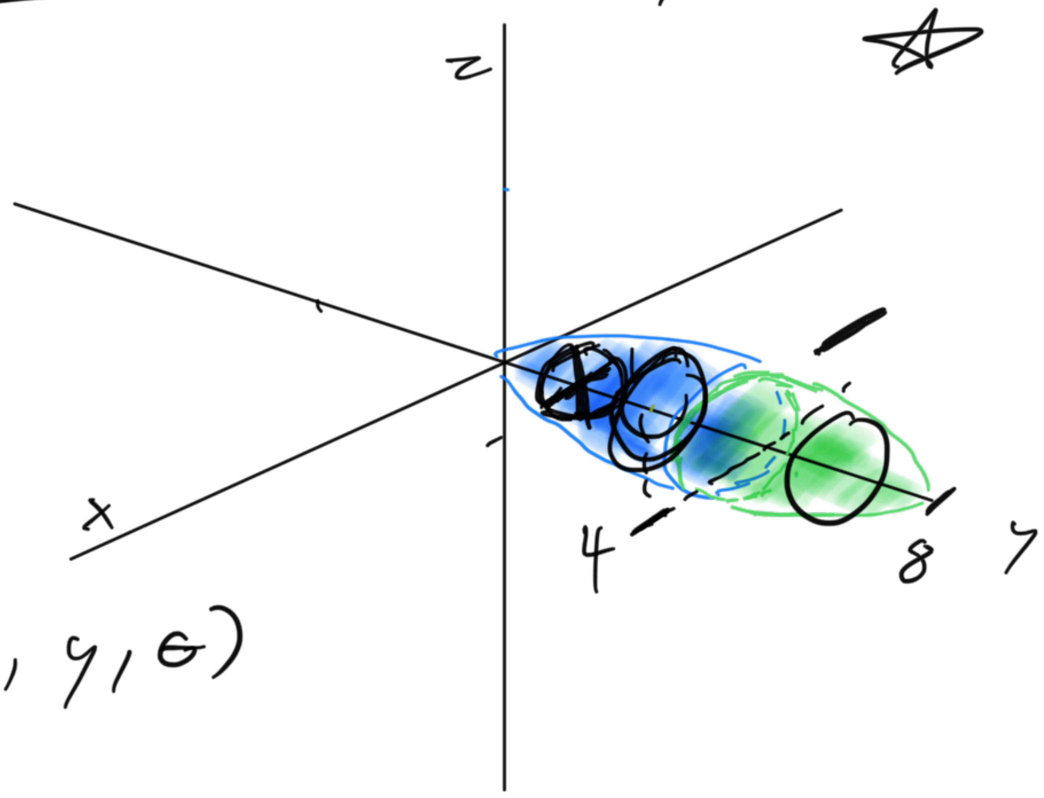
$$= \int_{-1}^{8} 8 \cdot 0 \, dz$$

$$= \int_{-1}^{8} 0 \, dz$$

$$= 0$$

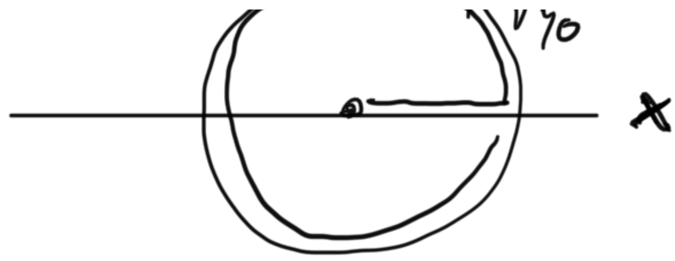
Ex: Finding volume enclosed by these curves:

$y = x^2 + z^2$ and $y = 8 - x^2 - z^2$



(r, y, θ)

$$2 \int_0^4 \int_0^{\sqrt{y}} \int_0^{2\pi} r \, d\theta \, dr \, dy$$



radius $= \sqrt{y_0}$

$$2\int_0^4\int_0^{\sqrt{y}}\int_0^{2\pi} r \, d\theta \, dr \, dy$$

$$2\int_0^4\int_0^{\sqrt{y}} 2\pi r \, dr \, dy$$

$$2\int_0^4 \pi r^2 \Big]_0^{\sqrt{y}}$$

$$\pi r^2 \Big]_0^{\sqrt{y}}$$

$$= 2\int \pi y \, dy$$

$$2\left(\pi \frac{y^2}{2}\Big]_0^4\right)$$

$$2\left(\pi 8\right)$$

$$= 16\pi$$

Ex. Solve by converting to cylindrical coordinates

$$\int_{-2}^{2}\int_{x=-\sqrt{4-y^2}}^{x=\sqrt{4-y^2}}\int_{z=\sqrt{x^2+y^2}}^{2} (xz)\, dz\, dx\, dy$$

Converting to cylindrical

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

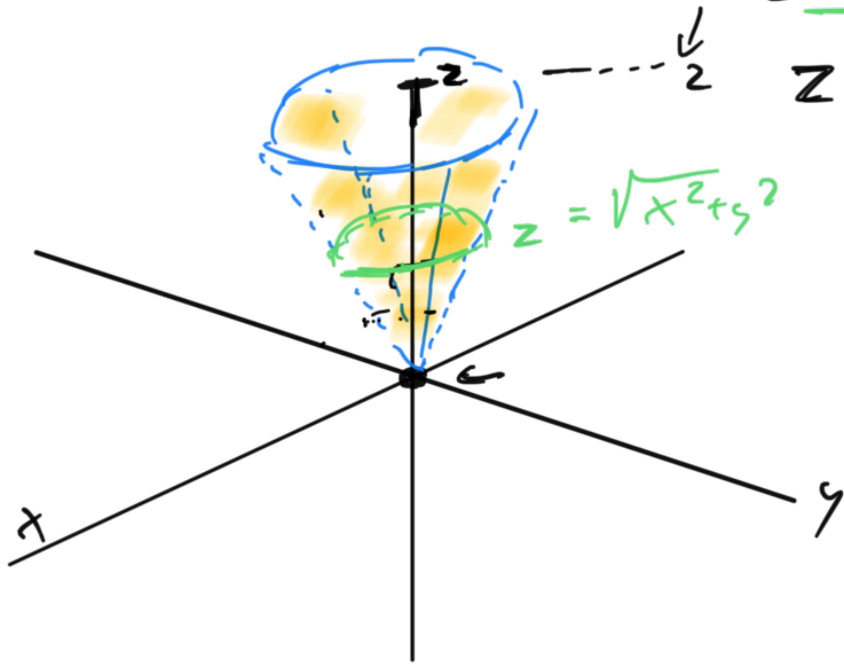
$$r^2\cos\theta \cdot z \quad dz \quad dr\,d\theta$$

$$x = \sqrt{4-y^2} \quad ☆$$

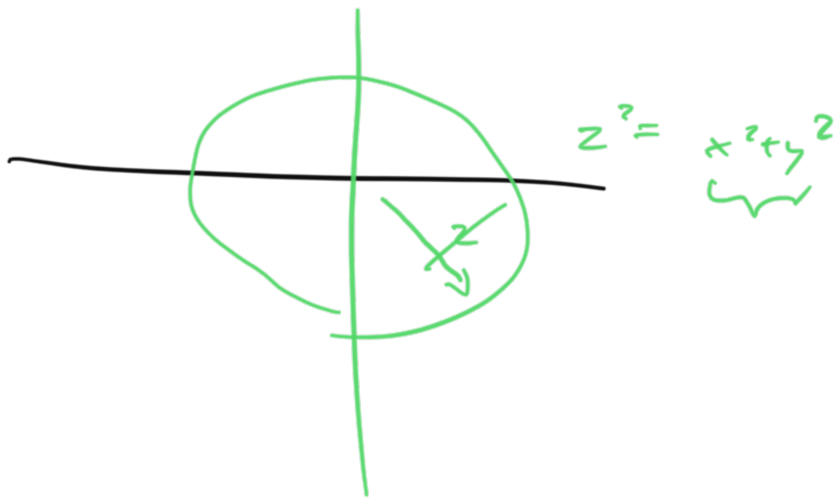
$$\int_{\sqrt{x^2+y^2}}^{2}$$

$z = \sqrt{x^2+y^2}$

$z^2 = x^2+y^2$



(r, θ, z)



$$\int_0^2 \int_0^z \int_0^{2\pi} r^2 \cos(\theta) \cdot z \, d\theta \, dr \, dz$$

$$\iint 0 \, dr \, dz$$

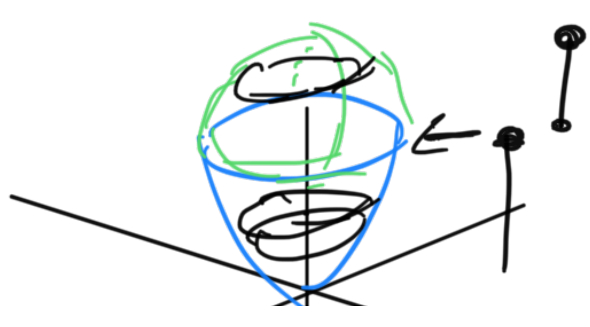
$$\iint 0$$

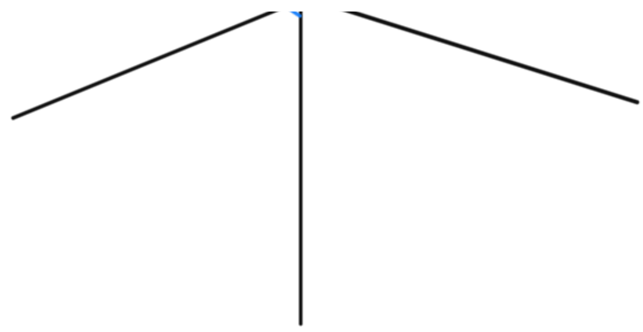
$$= 0$$

(24) Find volume between

$z = x^2 + y^2$

$x^2 + y^2 + z^2 = 2$





(r, θ, z)

$$\int \qquad dz$$