Section 2.3

Characterizations of Invertible Matrices

A matrix being invertible requires a very special set of circumstances. There is a lot of information packed into the word "invertible", a lot of which has to do with <u>one-to-one</u> and <u>onto</u>

"The following are equivalent"

Theorem: A is $n \times n$ matrix. Tfae:

a. A is an invertible matrix. row reduce $A \Rightarrow I_{n \times n}$

b. A is row equivalent to $I_{n \times n}$

- Have seen first two already when we learned general method to calculate inverses

c. A has n pivot positions

- Think of reduced echelon form of $n \times n$ matrix. If it is identity, n pivots. If n pivots, it's identity.

d. $A\vec{x} = 0$ has only trivial solution. $n \times n$ $\begin{bmatrix} & & \\ & & \end{bmatrix}$

- $A x = \vec{0}$ only trivial solution means no free variables, so pivot every column. (n pivots). OTOH n pivots means pivot every column, means no free variables

e. Columns of A are linearly independent.

coeff

- Again, no free variables.

f. $\vec{x} \mapsto A\vec{x}$ is one-to-one

- Again, no free variables

g. $A\vec{x} = \vec{b}$ has at least one solution for all $\vec{b} \in \mathbb{R}^n$

- Always consistent, so A has pivot in every row, so pivot in every column.

⑧ Columns of A span $\mathbb{R}^n$

- Same as saying $A\vec{x}=\vec{b}$ consistent for all $\vec{b}\in\mathbb{R}^n$

⑨ $x \mapsto A\vec{x}$ is onto $\mathbb{R}^n$.

- Same as previous

⑩ There is an $n\times n$ matrix C s.t. $CA = I_{n\times n}$

- If A invertible, this is obvious. If $CA = I_{n\times n}$ and everything square matrix, must be that $C = A^{-1}$

⑪ There is an $n\times n$ matrix D s.t. $AD = I_{n\times n}$

- Same idea as above

⑫ A^T is an invertible matrix.

- Saw this in last section.

Corollary: Theorem, Lemma, Proposition, Corollary

Let A and B be square matrices. If $AB = I$ then A, B both invertible with $B = A^{-1}$ and $A = B^{-1}$.

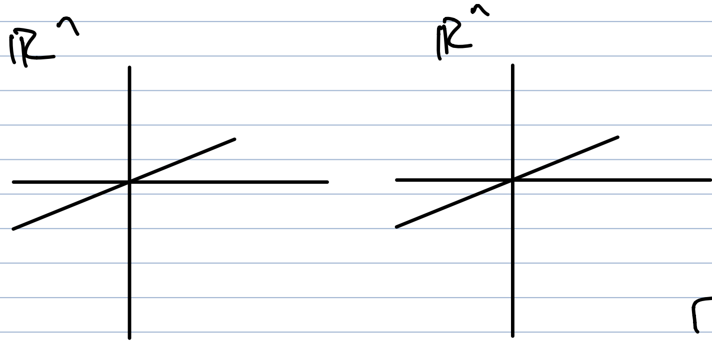
Some pictures:

Picture invertible matrices as transformations.

Usually think of transformations in terms of how they affect a space

"T rotates points by $\pi/2$" etc.

So let's build some intuition/pictures of invertible matrices

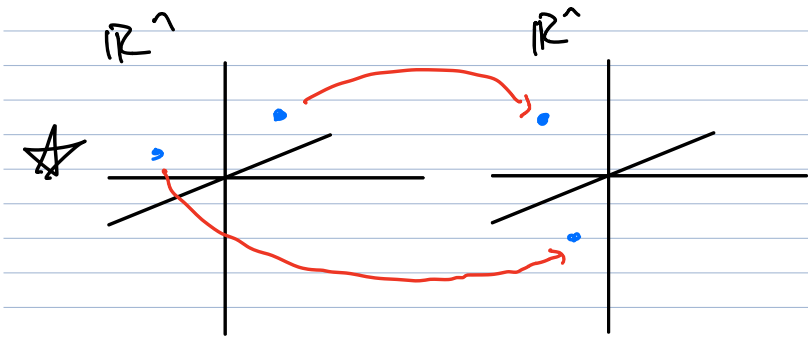


$n \times n$

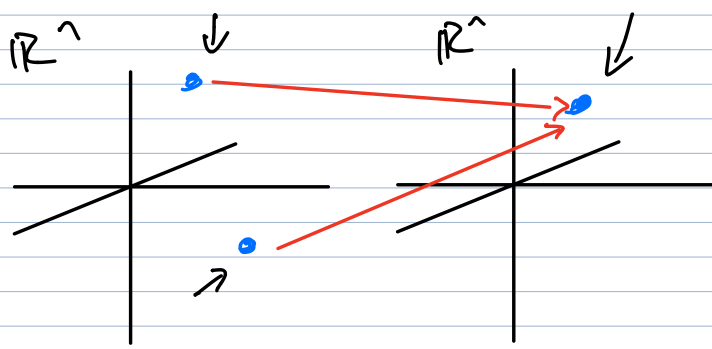
Invertible matrices are $n \times n$

So can think of them as just having some effect on $\mathbb{R}^n$

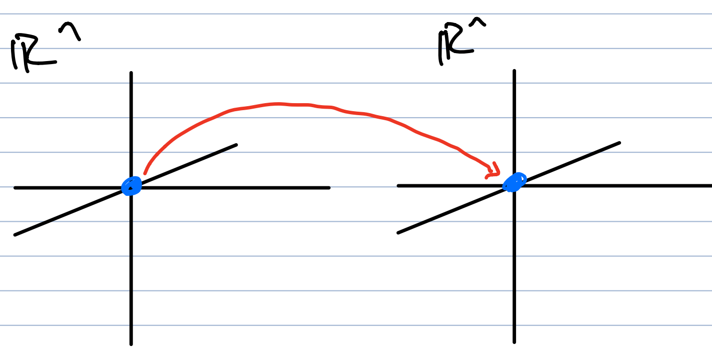
(rotation, reflection, etc)



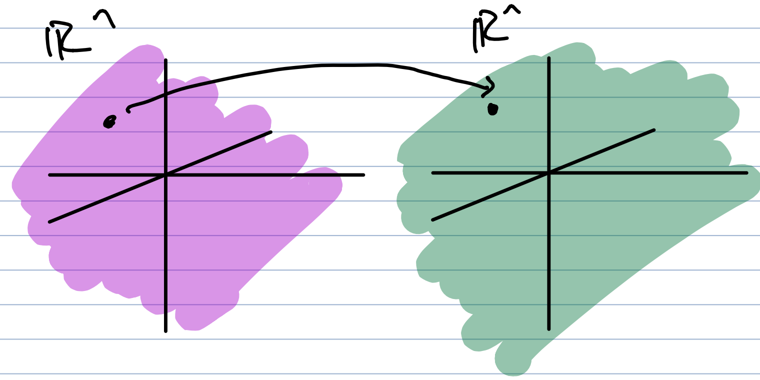
One-to-one



<u>Can't</u> have this



Recall one-to-one ☆ same as saying only solution to $A\vec{x}=\vec{0}$ is $\vec{0}$



Onto: every point is image of something

Solution to every $A\vec{x}=\vec{b}$

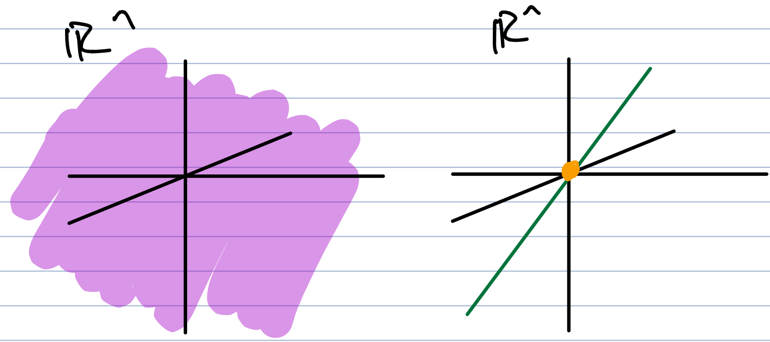


Image can't be proper subspace – can't be "smaller"

Transformation Viewpoint:

Remember matrices can be thought of as transformations (functions). Focusing on square matrices:

$$T: \mathbb{R}^n \to \mathbb{R}^n \qquad \text{where} \qquad T(\vec{x}) = A\vec{x}$$

Recall definition of <u>inverse functions</u>.

$$f^{-1}(f(x)) = x$$
$$f(f^{-1}(x)) = x$$

for all x in domain

Same applies for linear transformations

$$T^{-1}(T(\vec{x})) = \vec{x} = T(T^{-1}(\vec{x}))$$

or in matrix notation:

$$\underbrace{A^{-1}A}\,\vec{x} = \vec{x} = \underbrace{AA^{-1}}\,\vec{x}$$
$$I\vec{x} = \vec{x} = I\vec{x}$$

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be standard matrix for T. Then $\underline{T}$ is ☆ invertible if and only if A is an invertible matrix. In that case, linear transformation S given by $\underline{T^{-1}}(\vec{x}) = \underline{A}^{-1}\vec{x}$ is unique function satisfying

$$☆)\quad T^{-1}(T(\vec{x})) = \vec{x} = T(T^{-1}(x)) \quad \text{for all } \vec{x} \in \mathbb{R}^n$$