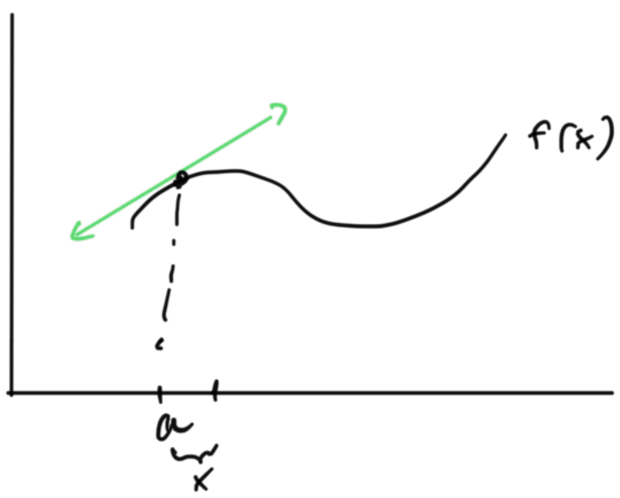
14.4 - Tangent Planes

Continue developing our ideas of derivatives and tangent

Consider $\underline{f: \mathbb{R} \to \mathbb{R}}$

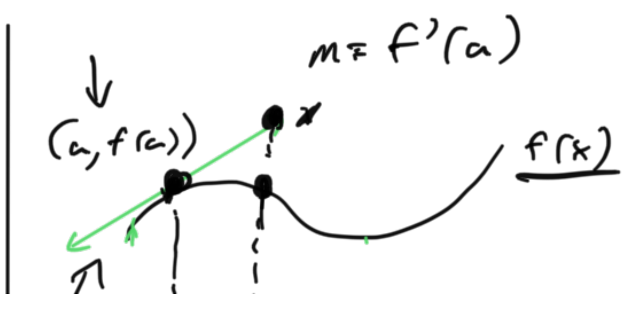


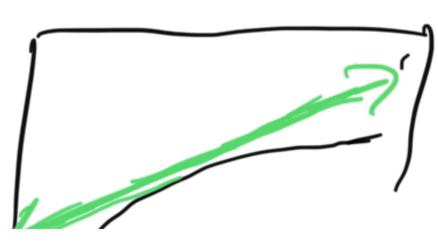
$f'(x)$ gives <u>slope</u> of tangent line at each point x

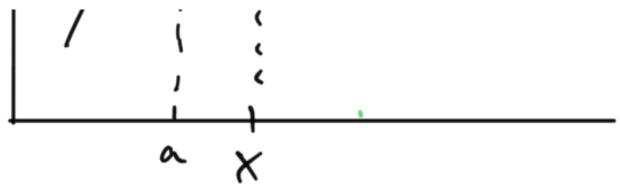
The derivative, slope of the tangent, can be thought of as roughly the rate of change

What about the tangent line itself? How can we think of it?

Consider line tangent to curve at a.







The tangent line is an approximation to the function.

It has the added benefit of being very simple compared to f, just a line. Linear approximation

What is equation of the line?

Tangent line at $x=a$

Slope : $f'(a)$

Through : $(a, f(a))$

Equation of line
$$\begin{cases} (y - f(a)) = f'(a)(x-a) \\ y = f(a) + f'(a)(x-a) \end{cases}$$

Approximation to $f(x)$.
$$\left\{ \; f(x) \approx f(a) + f'(a)(x-a) \right.$$

Could say

$$f(x) = f(a) + f'(a)(x-a) + \varepsilon(x)$$

where $\varepsilon(x) \to 0$ as $x \to a$.

$f(x)$ $\qquad$ $f'(x)$

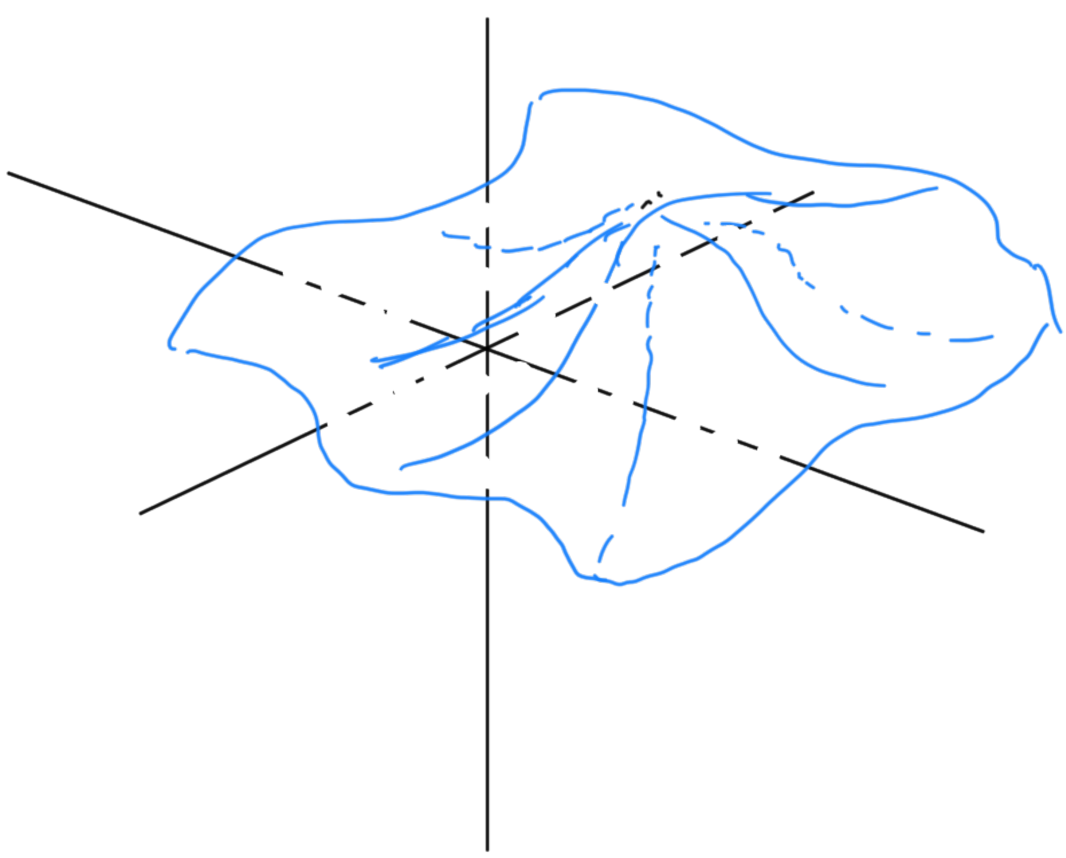
slope of tangent

Really, the idea of "<u>differentiable</u>" is about the existence of these linear approximations.

Straightforward for $\boxed{f : \mathbb{R} \to \mathbb{R}}$

Develop similar idea for $f : \mathbb{R}^n \to \mathbb{R}$

For $f : \mathbb{R}^2 \to \mathbb{R}$ have derivatives in many directions, many tangent lines



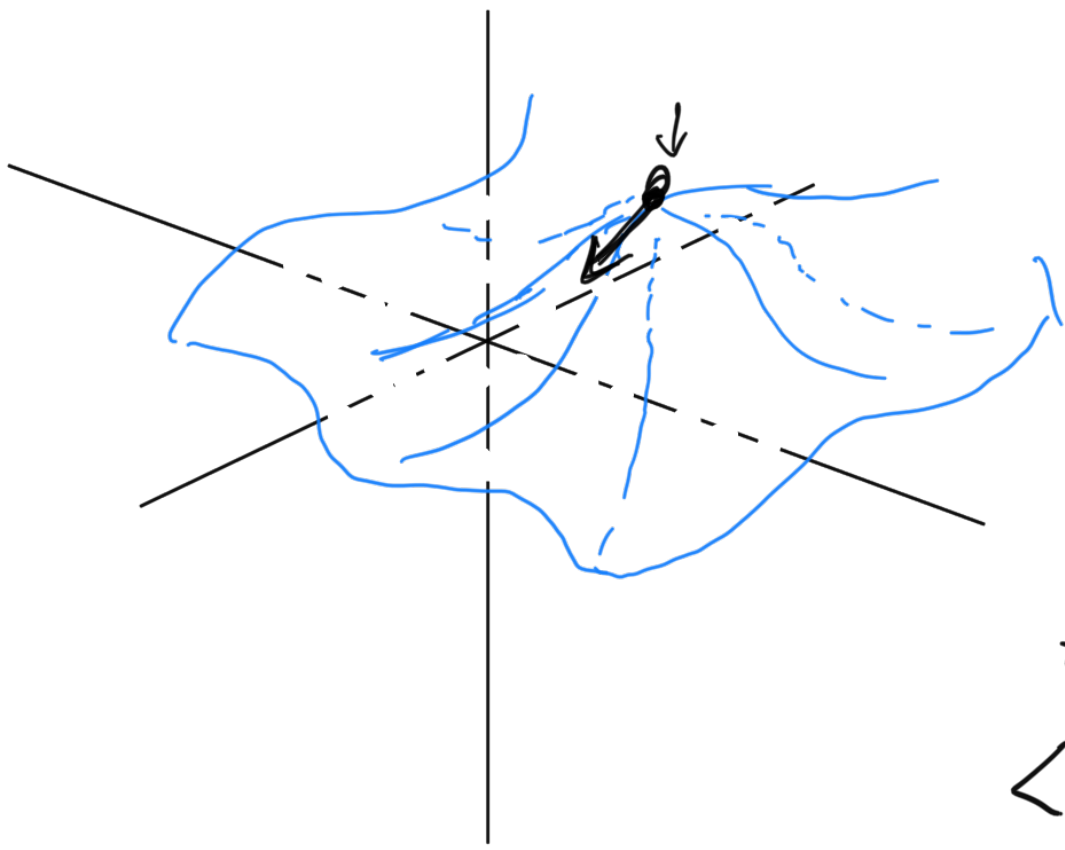
In particular have partial derivatives f_x and f_y and those tangent lines

Can we create a tangent object from all the tangent lines?

Yes! (**Sometimes**)

Instead of tangent lines, form a

tangent plane $f: \mathbb{R}^2 \to \mathbb{R}$



$f(x,y)$

$f_x = -2$

$\langle 1, 0, -2 \rangle$

In 12.5 saw how to find equations of planes.

$$f: \mathbb{R}^n \to \mathbb{R}$$

For two vectors and a point in plane

Take cross product of vectors to get normal vector

Ex: Find equation of tangent plane for $z = 2x^2 + y^2$ at point $\boxed{(1,\underline{1},3)}$

$$f_x = 4x = 4$$

$$f_y = 2y = 2$$

$$\vec{v}_1 \quad \langle 1, 0, 4 \rangle$$

$$\vec{v}_2 \quad \langle 0, 1, 2 \rangle$$

$$\begin{array}{ccc} i & j & k \\ \hline 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}$$

$$(0-4)i - (2-0)j + (1-0)k$$

$$\vec{n} = \langle -4, -2, 1 \rangle$$

$$\vec{n} \left(\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle \right) = 0$$

$$\langle -4, -2, 1 \rangle \cdot \langle x-1, y-1, z-3 \rangle = 0$$

$$-4(x-1) - 2(y-1) + 1(z-3) = 0$$

$$\boxed{z - 3 = 4(x-1) + 2(y-1)}$$

Formula for Tangent Plane

For $\boxed{z=f(x,y)}$ equation of tangent plane at $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

Similar to the equation of tangent line, the tangent plane provides an approximation to our function $f(x,y)$.

☆ See textbook.

Again, approximation has the benefit of being simple compared to function.

Linear

Ex: $\boxed{ax + by = 0}$ in 2 variables

$\boxed{ax + by + cz = 0}$ Plane

"Hyperplane" $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0$

Rearrange equation of tangent plane to see our approximating function

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$\Rightarrow \quad z = z_0 + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

or

$$f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

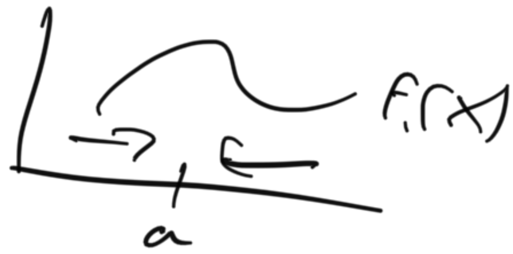
$$\mathbb{R}^2$$

Said that tangent plane / linear approximation

exists sometimes

exists sometimes. Where's the problem?

$$\underline{f : \mathbb{R} \to \mathbb{R}}$$



$\underline{f'(x)}$ exists $\Rightarrow$ tangent exists

$$\underline{f : \mathbb{R}^n \to \mathbb{R}}$$

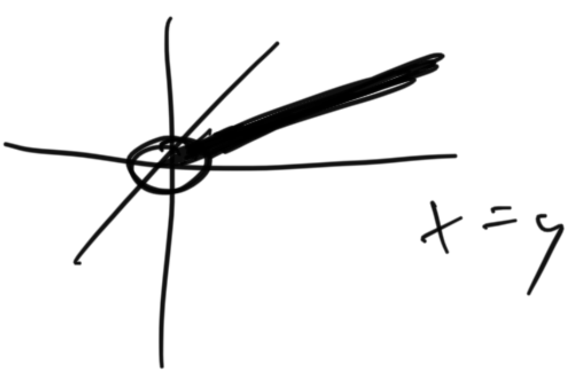
Even if partial derivatives f_x, f_y etc. exist, may not have tangent plane

☆ f may not even be continuous!

Problem once again stems from idea of "direction of approach" for limits.

Ex:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$



$$\frac{x^2}{x^2+x^2} = \frac{1}{2}$$

f_x close to $(0,0)$

f_y

$$f_x(0,0) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

0

$f_x(0,0) = 0$

$f_y(0,0) = 0$

$$f(x,y) \approx \underbrace{\cancel{f(0,0)}}_{=0} + \cancel{0(x-0)} + \cancel{0(y-0)}$$

$f(x,y) \approx 0$ in neighborhood around $(0,0)$

To get around this, develop stronger notion of derivative for $f: \mathbb{R}^n \to \mathbb{R}$

<u>Mine</u> (14.6 preview)

Say $\underline{f: \mathbb{R}^2 \to \mathbb{R}}$ is **differentiable** at $\underline{(a,b)}$ if

$$\lim_{|\langle h,j\rangle|\to 0} \frac{|f(a+h,b+j)-f(a,b)|}{|\langle h,j\rangle|}$$

exists.

Book

Say $z=f(x,y)$ is differentiable at (a,b) if Δz can be expressed in form

small change in z

$$\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y)\to(0,0)$

What is this saying? Just a definition.

Call $f: \mathbb{R}^n \to \mathbb{R}$ differentiable if it has a linear approximation.

Next question: When is a function $f: \mathbb{R}^n \to \mathbb{R}$ differentiable? (How can we tell?)

<u>Theorem</u>: If partial derivatives of $f: \mathbb{R}^n \to \mathbb{R}$ exist <u>and</u> are continuous at point $\vec{x}_0$, <u>then</u> f is differentiable at $\vec{x}_0$.

Differentials

Notation closely related to derivatives that indicate small changes in values in relation to one another

<u>Ex</u>: $f: \mathbb{R} \to \mathbb{R}$ where

$$y = f(x)$$

(a small change in y)

$$dy = f'(x)\,dx$$

(change in y in response to change in x) (small change in x)

For $f: \mathbb{R}^n \to \mathbb{R}$

, have more than one possible differential. Must be specific.

$$z = f(x, y)$$

$$dz = f_x\, dx \qquad ?$$

$$dz = f_y\, dy \qquad ?$$

<u>Total Differential</u>

$$\underline{dz} = f_x\, dx + f_y\, dy$$

Seems confusing but just think of it as the same as linear approximation

Ex: $f(x,y) = \underline{x^2 + 3xy - y^2}$

$$dz = f_x(x,y)\, dx + f_y(x,y)\, dy$$

$$dz = \underline{(2x + 3y)}\, dx + \underline{(3x - 2y)}\, dy$$

If x changes from $\underline{2}$ to $\underline{2.05}$ and y changes from 3 to 2.96

$$(2, 3) \to (2.05, 2.96)$$

$$dz = (13)\,dx + \cancel{(0)\,dy}$$

$$dz = (13)(.05)$$

$$= .65$$

$$\Delta z =$$

z_1 at $(2, 3)$

z_2 at $(2.05, 2.96)$

$$\boxed{\Delta z = z_2 - z_1}$$