12.2 – Vectors

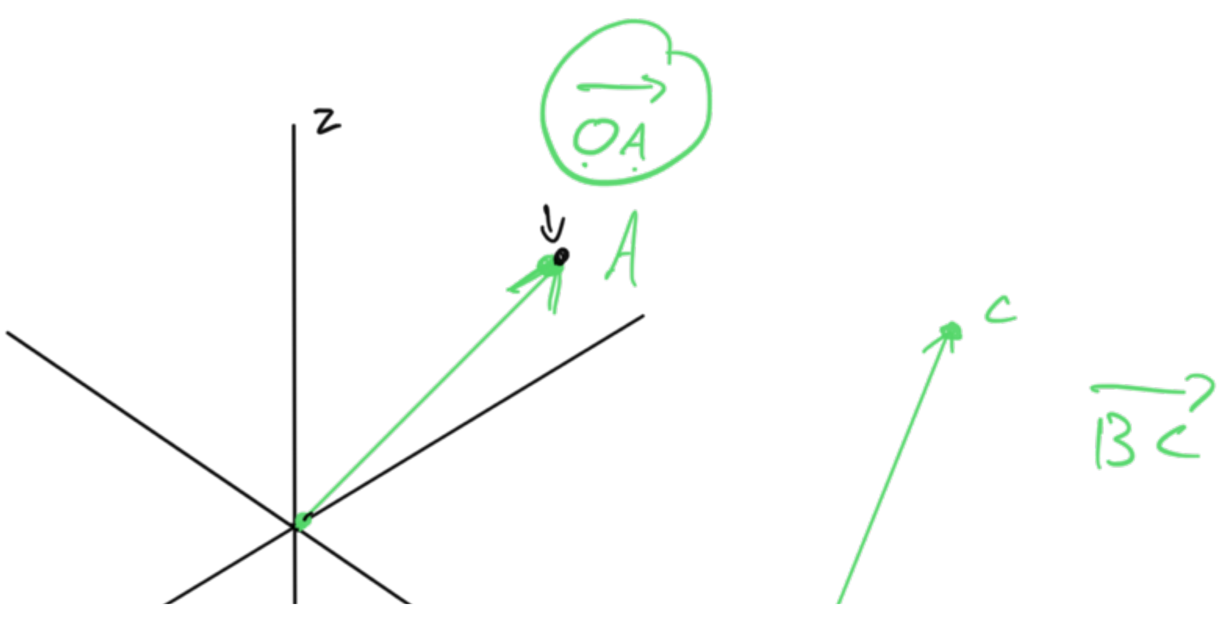
In last section dealt with points in $\mathbb{R}^3$.

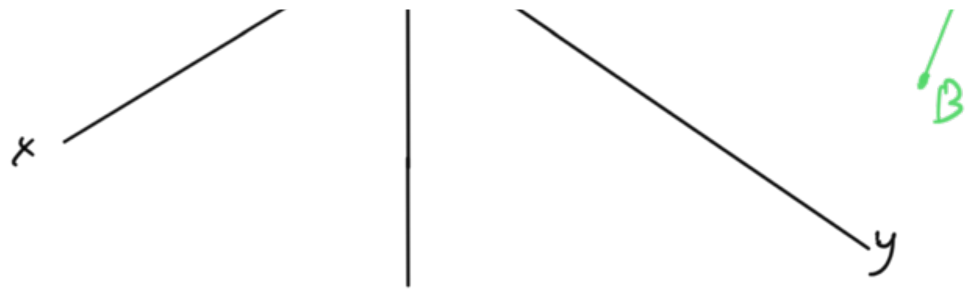
Now introduce object called a vector.

Textbook describes them thusly:

"quantity that has both magnitude and direction"

Consider our points in $\mathbb{R}^3$





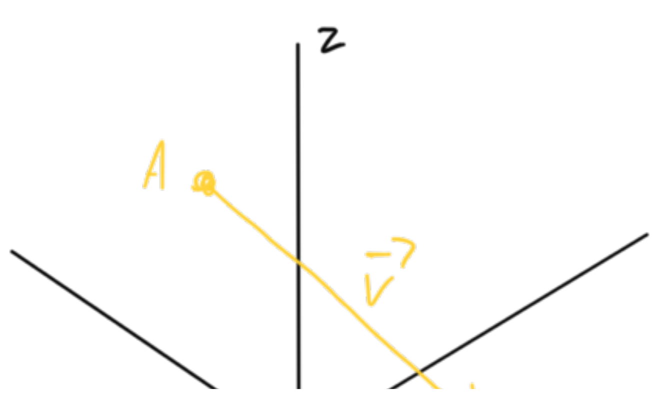
So can think of arrow from origin to point A as a vector

Denote vectors by bold letters or letters with arrows over them

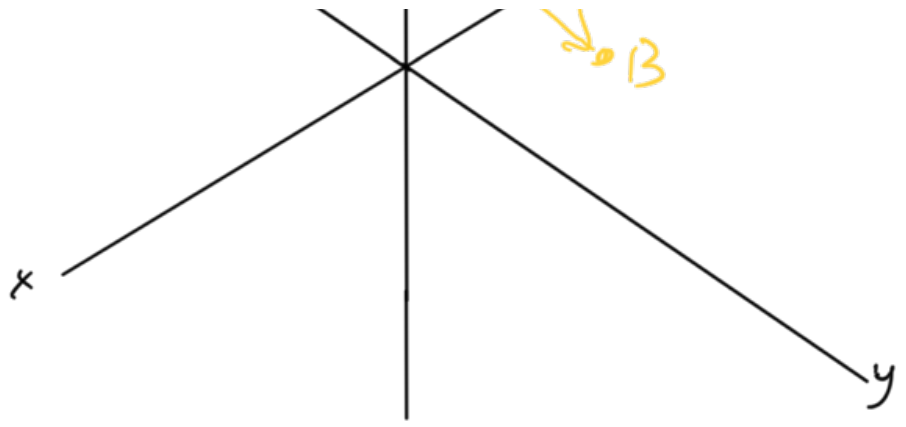
Ex: $\mathbf{v}$

$\vec{v}$

Vectors dont have to start at the origin.



Can also get a length and direction as



we move from A to B.

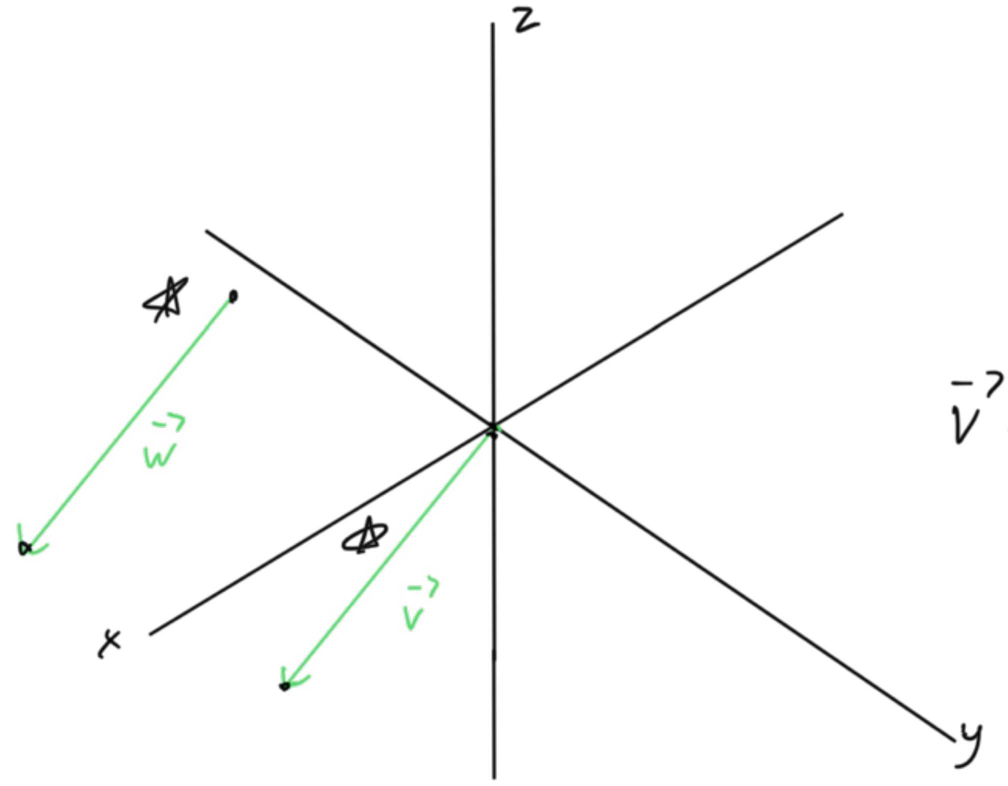
May denote as

$$\vec{v} = \overrightarrow{AB}$$

or

$$\mathbf{v} = \overrightarrow{AB}$$

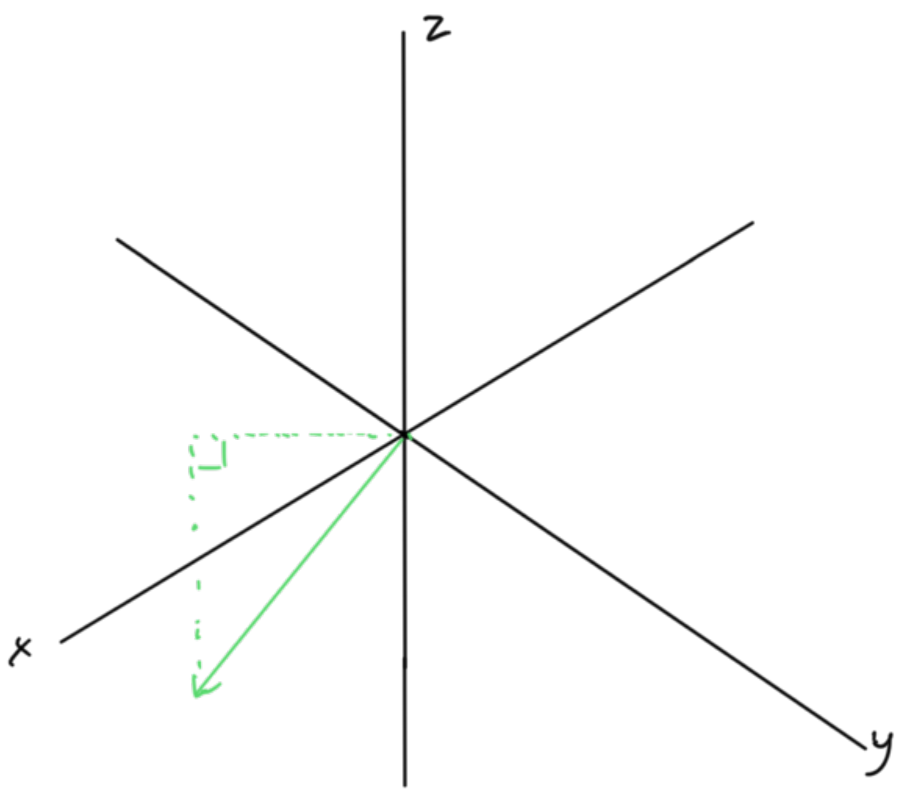
A vector is independent of its initial point, endpoint



$\vec{v} = \vec{w}$

$$\vec{v} = \vec{w}$$

A vector also independent of how we represent it.



However will see a standard way to represent them later

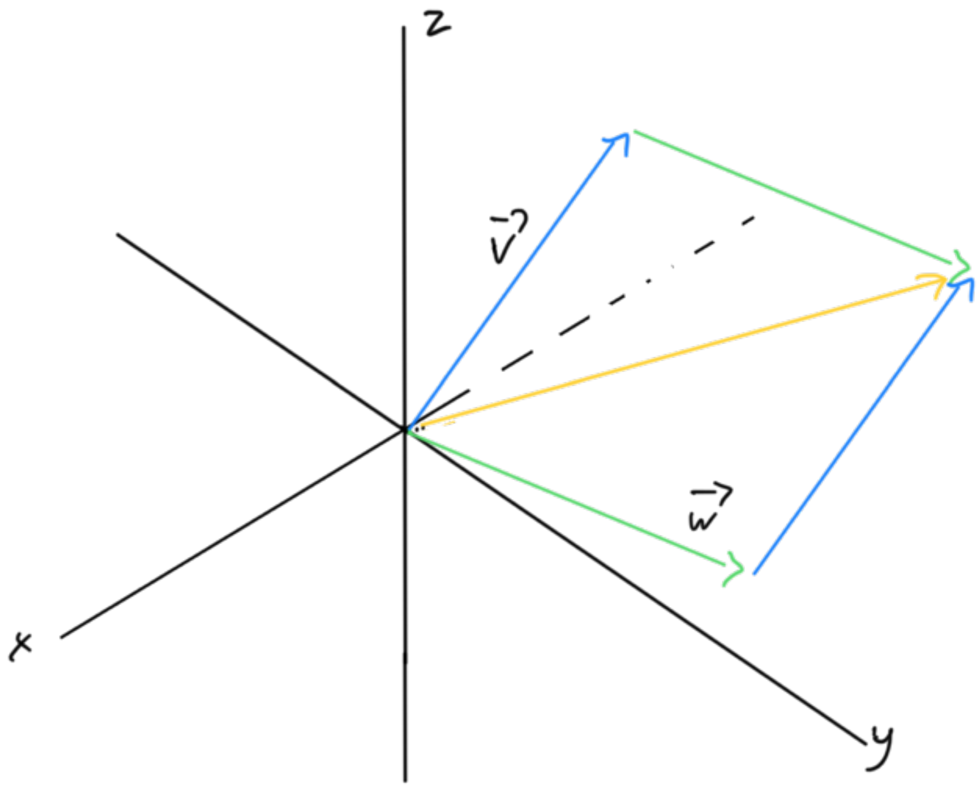
If given two numbers in $\mathbb{R}$, have a couple standard operations

$a + b$ addition

$a \cdot b$ multiplication

Would like to develop similar operations for vectors

$\vec{v} + \vec{w}$



Can we add $\vec{v}$ and $\vec{\omega}$ somehow?

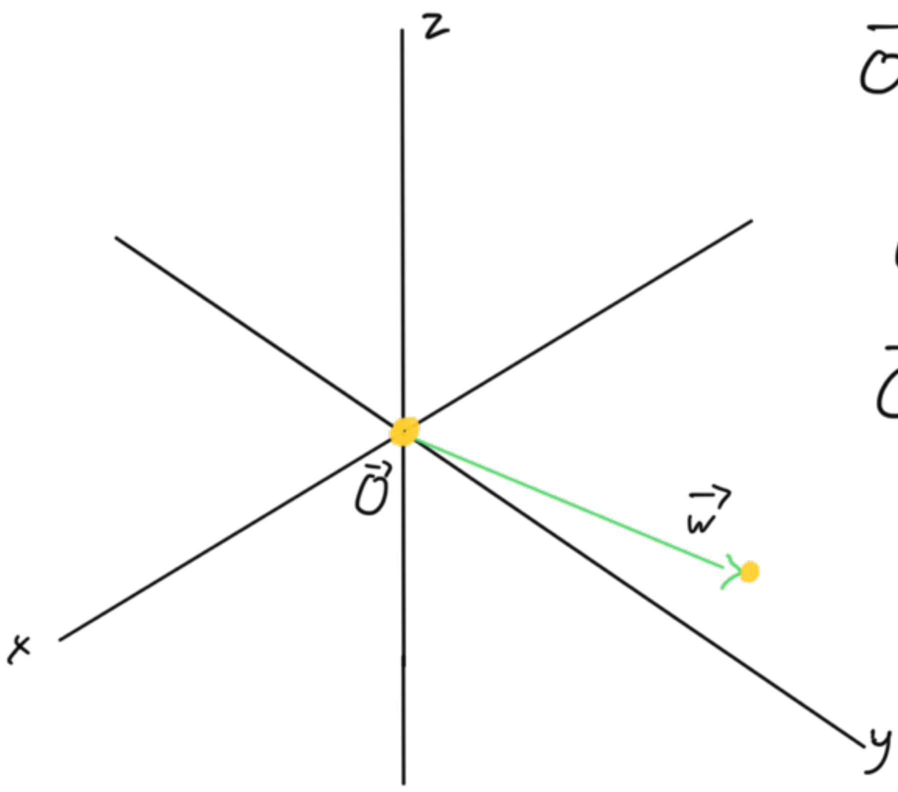
- Triangle Law
- Parallelogram Law
- $\vec{v} + \vec{w} = \vec{\omega} + \vec{v}$

Call this <u>vector addition</u>

The zero

The zero vector. The vector with a length of zero

Denoted $\mathbf{0}$, or $\vec{0}$



$\vec{0} + \vec{w} = \vec{w}$

$0 + 7 = 7$

$\vec{0} + \vec{v} = \vec{v}$

So far, have done vector addition graphically. Will also see how to do it numerically once we introduce components

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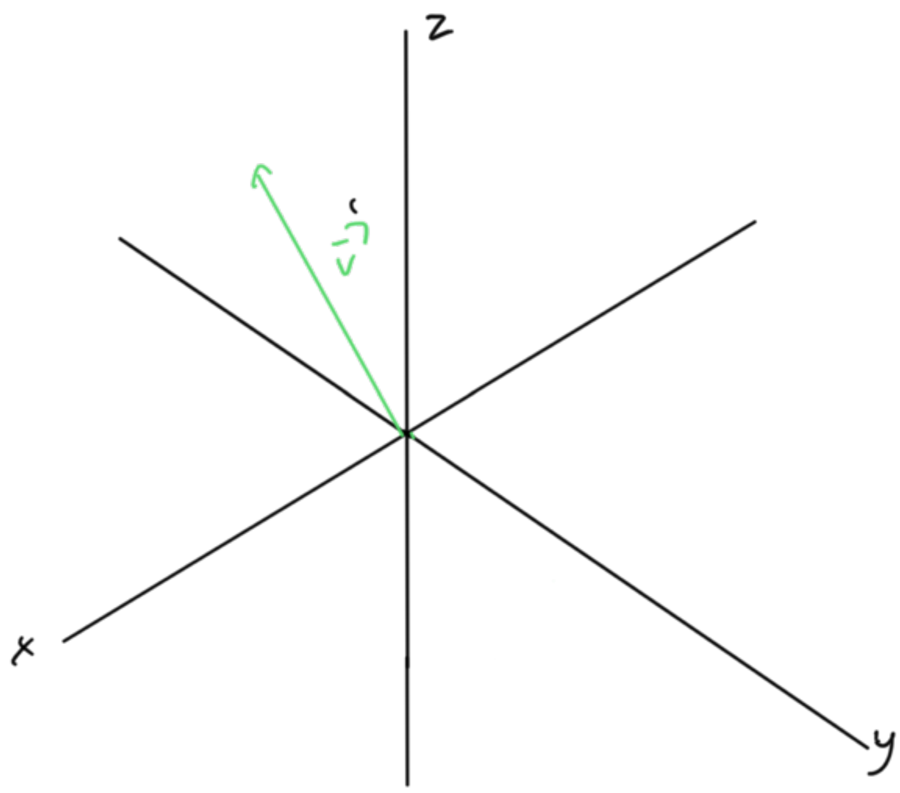
There are a couple ways to

multiply two vectors but a bit

complicated for now.

Easy and intuitive way to multiply

by real numbers



$2\vec{v}$

$(.5)\vec{v}$

$-1\vec{v}$

Since multiplying by real number has

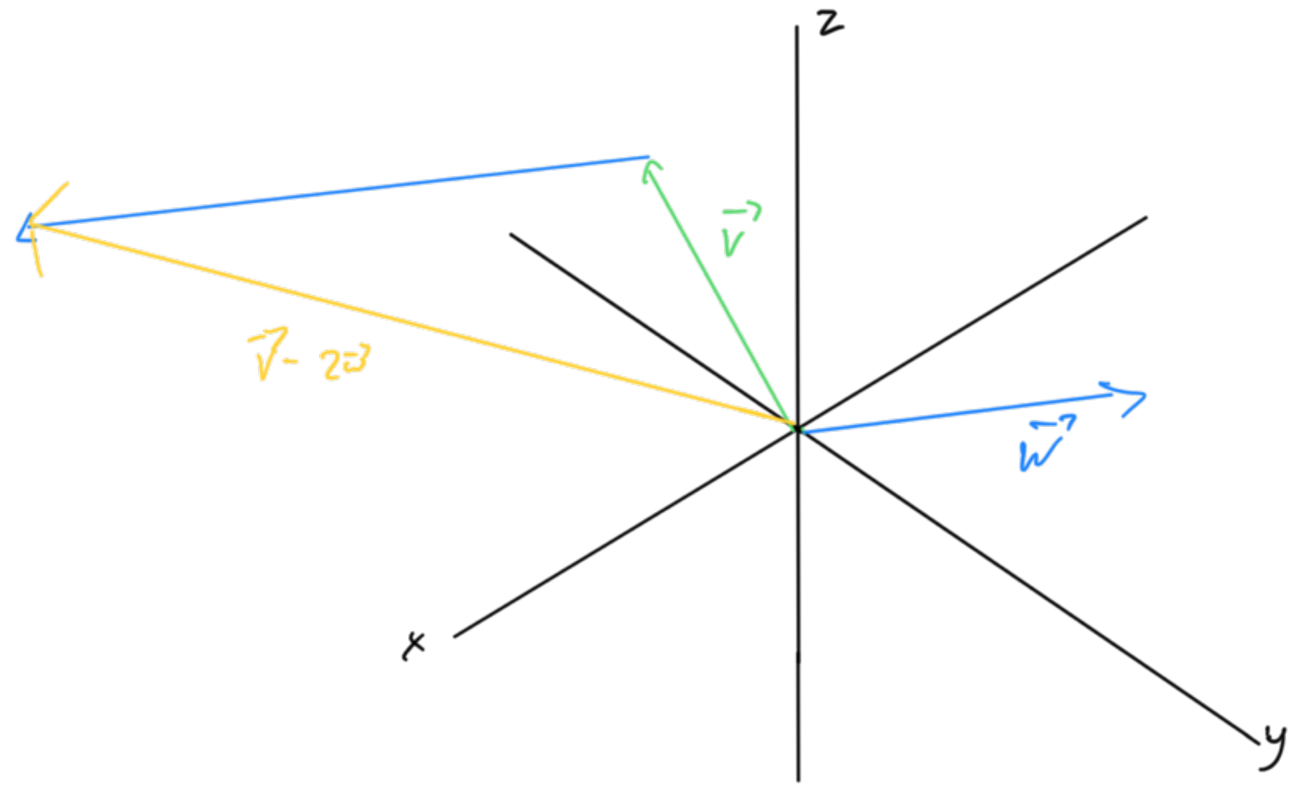
effect of "scaling" the vector

we can call the real number

a <u>scalar</u> and process scalar multiplication

Ex: Find $\vec{v} - 2\vec{w}$.

$\vec{v} + (-2)\vec{w}$



Representations/Coordinates

Have seen how to work with vectors graphically.

Numerically?

Start with direction. How can

we write a vector down that

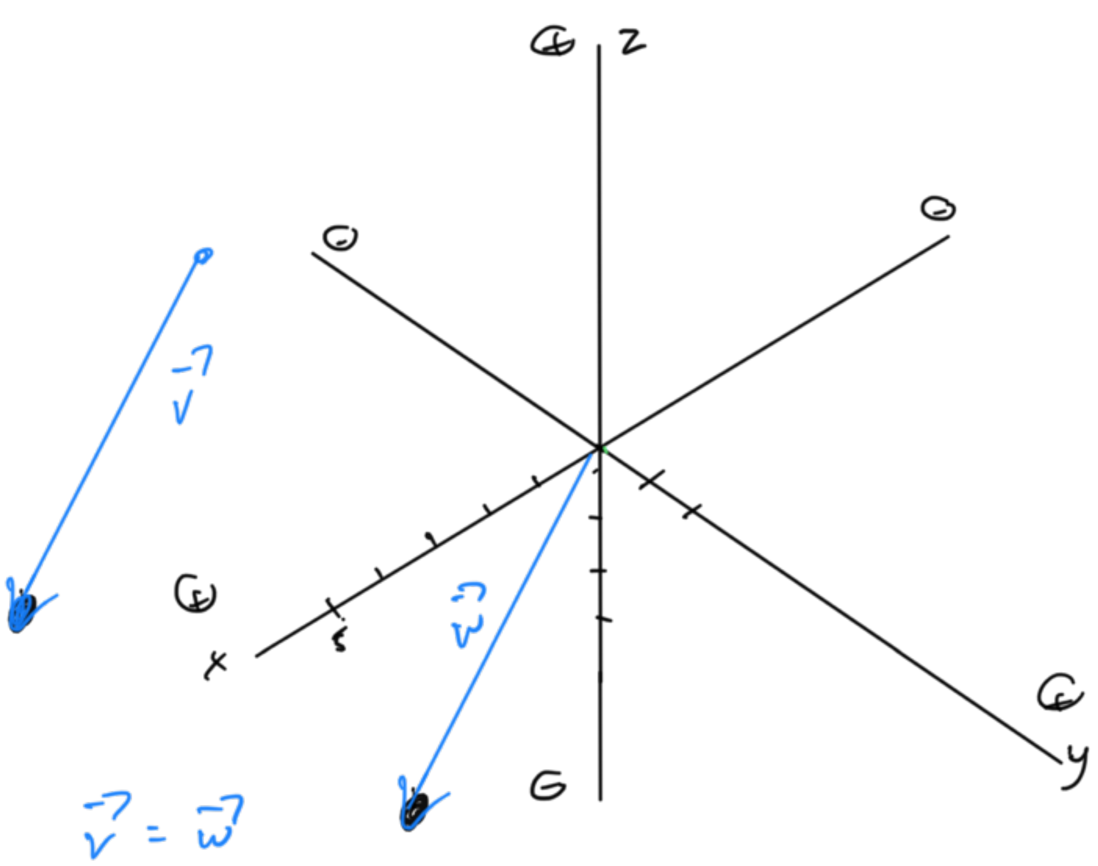
moves

5 units in x direction
2 units in y direction
-3 units in z direction

$$\begin{array}{ccc} x & y & z \\ \downarrow & \downarrow & \downarrow \end{array}$$
$$\langle 5, 2, -3 \rangle$$

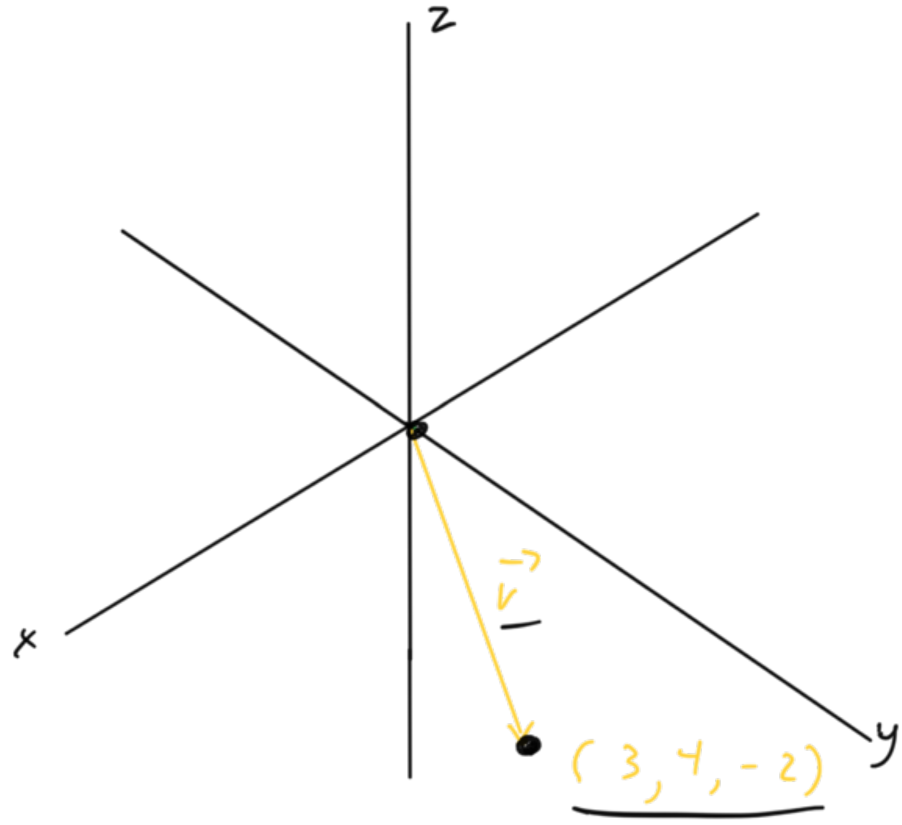
Entries are the <u>components</u> of the vector

Where could we draw this vector?



OTOH

Consider vector $\vec{v}$ with base at origin, tip at some point.



$\langle 3, 4, -2 \rangle$

$\vec{v} = \langle 3, 4, -2 \rangle$

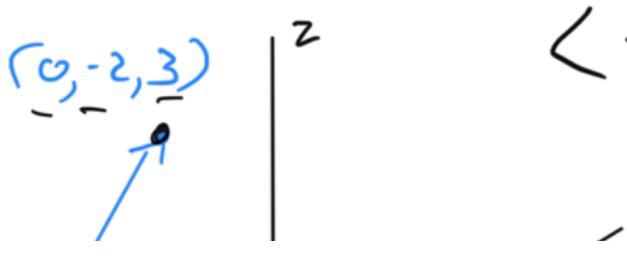
(, ,) point

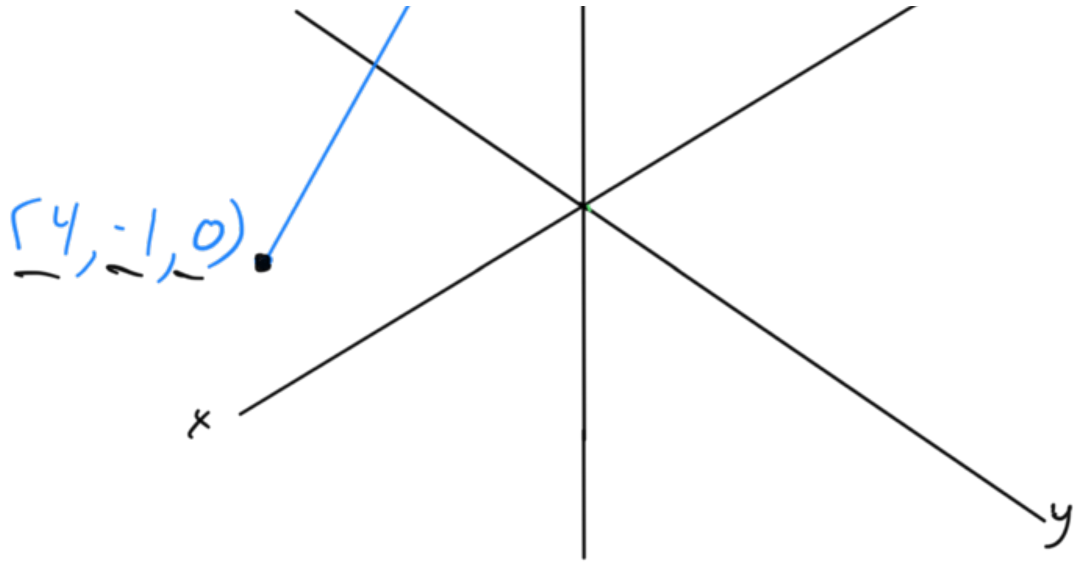
$\langle\ ,\ ,\ \rangle$ vector

What about vectors with base <u>not</u> at origin?

$\langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

$\langle -4, -1, 3 \rangle$





What is the movement in x-direction?

$$x_2 - x_1$$

$$0 - 4 = -4$$

What is the movement in y-direction?

$$y_2 - y_1$$

$$-2 - (-1) = -1$$

What is the movement in z-direction?

$$z_2 - z_1$$

$$3 - 0 = 3$$

So the coordinates of this vector are

$$\langle 0-4, -2-(-1), 3-0 \rangle$$

$$= \langle -4, -1, 3 \rangle$$

What are components of zero-vector?

$$\vec{0} = \langle 0, 0, 0 \rangle$$

These components of a vector can be thought of as the "direction" of the vector

Remember, a vector has a direction <u>and</u> a magnitude (length).

So what is magnitude of vector with components $\vec{v} = \langle a, b, c \rangle$?

$$|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$$

$$\vec{v} = \langle a, b \rangle$$

$$|\vec{v}| = \sqrt{a^2 + b^2}$$

(or $\|\vec{v}\|$)

Ex: $\vec{v} = \langle 4, 0, -3 \rangle$

$$|\vec{v}| = \sqrt{4^2 + 0^2 + (-3)^2}$$

$$= \sqrt{16 + 0 + 9}$$

$$= \sqrt{25}$$

$= 5$

Now that can represent vectors by components, can use components to deal with vector addition and scalar multiplication

- $\vec{v} = \langle a, b, c \rangle$

  $\vec{w} = \langle d, e, f \rangle$

  $\vec{v} + \vec{w} = \langle a+d, b+e, c+f \rangle$

- $s \in \mathbb{R}$

  $s\vec{v} = s\langle a, b, c \rangle$

  $= \langle s \cdot a, s \cdot b, s \cdot c \rangle$

Ex:

$\vec{v} = \langle 2, 0, 1 \rangle$

$\vec{w} = \langle 0, -3, -3 \rangle$

$\vec{v} + 2\vec{w} = \, ?$

Properties of vectors

1. $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$
3. $\vec{a}+0=\vec{a}$
4. $\vec{a}+(-\vec{a})=0$
5. $c(\vec{a}+\vec{b}) = c\vec{a}+c\vec{b}$
6. $(c+d)\vec{a} = c\vec{a}+d\vec{a}$
7. $(cd)\vec{a} = c(d\vec{a})$
8. $1\vec{a} = \vec{a}$

Check these your self.

Let
$$\vec{a} = \langle x_1, y_1, z_1\rangle$$
$$\vec{b} = \langle x_2, y_2, z_2\rangle$$
$$c, d \in \mathbb{R}$$

Ex:

1. $$\begin{aligned}\vec{a}+\vec{b} &= \langle x_1, y_1, z_1\rangle + \langle x_2, y_2, z_2\rangle \\ &= \langle x_1+x_2, y_1+y_2, z_1+z_2\rangle \\ &= \langle x_2+x_1, y_2+y_1, z_2+z_1\rangle \\ &= \langle x_2, y_2, z_2\rangle + \langle x_1, y_1, z_1\rangle \\ &= \vec{b}+\vec{a}\end{aligned}$$

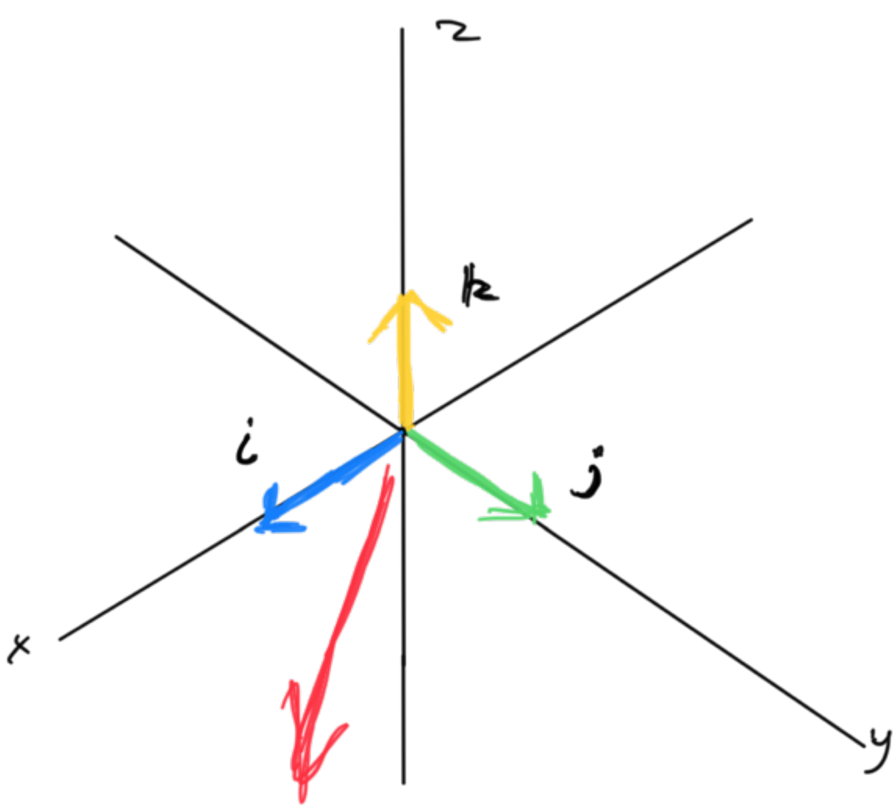
Basis Vectors

Consider these simple vectors:

$\mathbf{i} = \langle 1, 0, 0 \rangle$

$\mathbf{j} = \langle 0, 1, 0 \rangle$

$\mathbf{k} = \langle 0, 0, 1 \rangle$



Can make <u>any</u> vector in $\mathbb{R}^3$

by combinations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$

Ex: $\vec{v} = \langle 7, -.6, 137 \rangle$

$\vec{v} = 7\mathbf{i} - .6\mathbf{j} + 137\mathbf{k}$

$$7\langle 1,0,0\rangle - 6\langle 0,1,0\rangle + 13\langle 0,0,1\rangle$$

__Unit Vectors__ - A unit vector is any vector whose magnitude is 1

Ex: $i = \langle 1,0,0\rangle \qquad j \qquad k$

$$\vec{v} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$|\vec{v}| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}$$

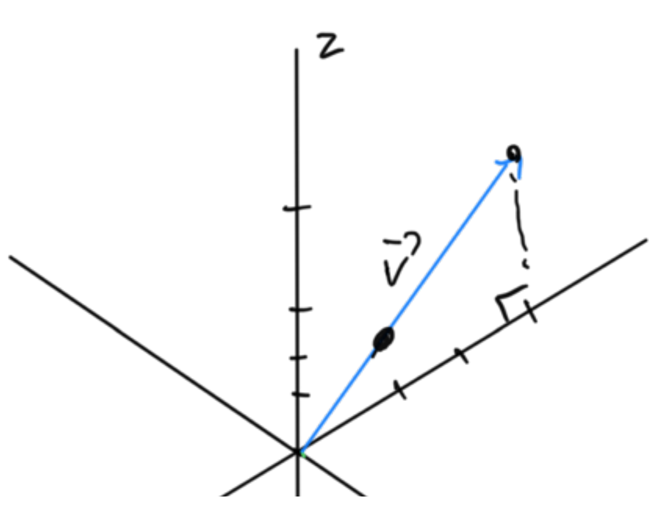
$$\sqrt{1} = 1$$

Given vector $\vec{v}$, may want to find a unit vector pointing in same direction as $\vec{v}$

Ex $\vec{v} = \langle -3, 0, 4\rangle$ ✓

$$|\vec{v}| = \sqrt{(-3)^2 + 0^2 + 4^2} = \sqrt{25}$$

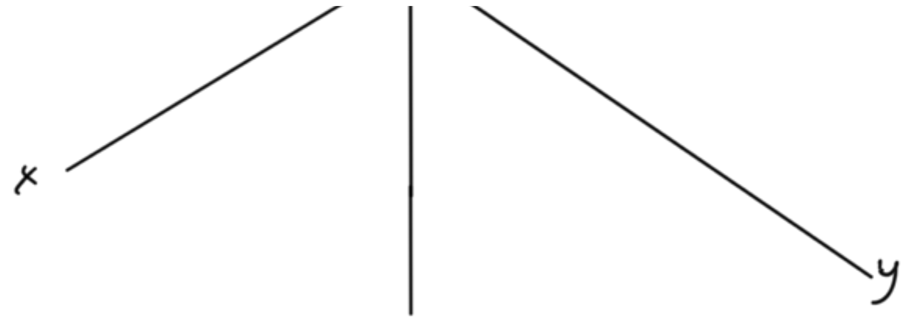
$$= 5$$



$$\left\| \frac{1}{5}\vec{v} \right\|$$

$$\frac{1}{5}\vec{v} = \left\langle \frac{-3}{5}, 0, \frac{4}{5} \right\rangle$$

$$= \sqrt{\left(\frac{-3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$



$$= \sqrt{\left(\frac{25}{25}\right)}$$

$$= 1$$

Take $\vec{v}$, "divide" by $|\vec{v}|$

So, if $|\vec{v}| \neq 0$, $\frac{1}{|\vec{v}|}\vec{v}$ is a unit vector in same direction as $\vec{v}$

Applications

Force as a vector - Magnitude and angle

Resultant Force

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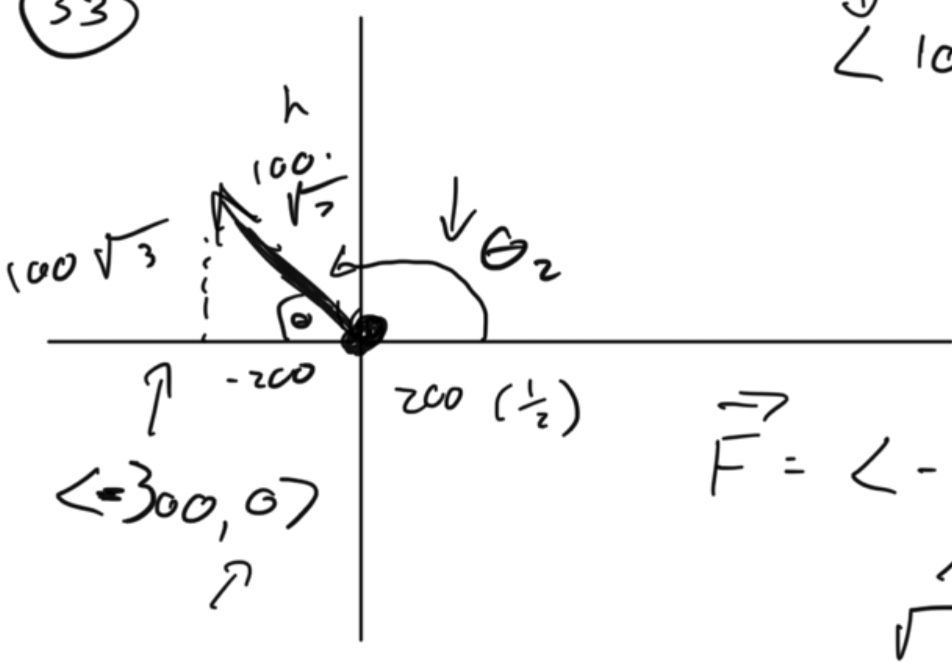

$$\langle 100, 100 \cdot \sqrt{3} \rangle$$

$$\sin \theta = \frac{100\sqrt{3}}{100\sqrt{7}}$$

$$\theta = \sin^{-1}\left(\sqrt{\frac{3}{7}}\right)$$

$$\vec{F} = \langle -200, 100\sqrt{3} \rangle$$

$$\sqrt{40{,}000 + 30{,}000}$$

$$= \sqrt{7} \cdot 100$$

Decomposition of Force