Real Analysis II

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2025

These notes will prove, among other things, the fundamental theorem of calculus, the theorems of Arzela-Ascoli and Dini, the existence of an everywhere-continuous nowhere-differentiable function, the Stone-Weierstrass theorem for general algebras of functions, the summation of the Bessel series $\sum 1/n^2 = \pi^2/6$ using Fourier analysis, and the equivalence of the inverse and implicit function theorems in $\mathbb{R}^n$.

1 The Riemann-Stieltjes Integral

1.1 Definition and Existence of the Integral

Before embarking on a serious treatment of the theorems of integral calculus, it is prudent to construct a flexible integral defined on interval subsets of the real line that can handle the pathologies not present in "well-behaved" functions. This integral should be defined for any bounded function with countably many discontinuities. In particular, the presence of a removable discontinuity should not change the value of the integral, that is to say, the integral of a discontinuous function should be equal to that of its continuous extension. These requirements are lax enough to allow us to integrate a whole host of misbehaved functions, such as the infinitely oscillatory $x \sin(x)$ over the interval $[0, 1]$.

Definition. A partition P of the interval $[a, b]$ is a set of finitely many points $\{x_n\}$ with

$$a = x_0 < x_1 < x_2 < ... < x_n = b.$$

Assume f is a bounded function on the interval $[a, b]$, i.e. $m \leq f(x) \leq M$ on $[a, b]$ for some numbers m and M. We will define $L(f, P)$, the lower (Riemann) sum associated with f and P, to be

$$L(f, P) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \inf_{x_i \leq x \leq x_{i+1}} f(x),$$

where

$$\inf_{x_i \le x \le x_{i+1}} f(x) = \inf\{f(x) : x_i \le x \le x_{i+1}\}.$$

This infinum is guaranteed to exist by the inequality $m \le f$. Similarly, the upper sum of f on the interval $[a, b]$ is defined to be

$$U(f, P) = \sum_{i=0}^{n-1} (x_{i+1} - x_i) \sup_{[x_i, x_{i+1}]} f.$$

It is easy to see that $L(f, P) \le U(f, P)$ for any given partition P of $[a, b]$.

Definition. We say that a partition P^* refines (or is a refinement of) P if P^* contains all the points in P.

Lemma. *If P^* refines P, then $L(P, f) \le L(P^*, f)$ and $U(P^*, f) \le U(P, f)$.*

Proof. It suffices to consider one interval within the partition P and see what happens if we add an extra point. If P is given by $a = x_0 < x_1 < x_2 < ...x_n = b$, then let $I = [x_0, x_1]$ denote the first interval of the partition. Suppose that the refinement P^* contains a point x^* situated between x_0 and x_1. The interval I in P^* is then broken into two intervals: $I_1 = [x_0, x^*]$ and $I_2 = [x^*, x_1]$. Then

$$\begin{aligned} L(f, P) &= (\text{length of } I) \times \inf_I f + ... \\ &\le (\text{length of } I_1) \times \inf_{I_1} f + (\text{length of } I_2) \times \inf_{I_2} f + ... = L(f, P^*). \end{aligned}$$

The case for the upper sums is analogous. □

Corollary. *For every two partitions P_1 and P_2, we have $L(f, P_1) \le U(f, P_2)$.*

Proof. Let P^* be the partition that contains all points of P_1 and P_2. Then P^* is a refinement of both P_1 and P_2. Now

$$L(f, P) \le L(f, P^*) \le U(f, P^*) \le U(f, P_2).$$

□

Definition. Assume f is bounded on the interval $[a, b]$. Let the lower integral of f be

$$\underline{\int_a^b} f = \sup\{L(f, P) : P \text{ is any partition of } [a, b]\}.$$

Similarly, the upper integral of f is

$$\overline{\int_a^b} f = \inf\{U(f, P) : P \text{ is any partition of } [a, b]\}.$$

These integrals always exist (recall $m \le f \le M$) and $\underline{\int_a^b} f \le \overline{\int_a^b} f$.

Definition. A function f is called Riemann integrable ($f \in \mathfrak{R}$) on $[a,b]$ if $\underline{\int_a^b} f = \overline{\int_a^b} f$, with the integral being the common value.

When asking whether a given function is Riemann integrable, it is almost never practical to appeal to this definition as the construction of both the upper and lower integrals can be quite laborious. We will instead opt for a much more powerful test for integrability.

Theorem 1 (Criterion for Integrability). *A function f is Riemann integrable if and only if for every $\epsilon > 0$, there exists a partition P_ϵ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.*

Proof. "$\Leftarrow$": To prove $\underline{\int} = \overline{\int}$, it suffices to prove that $\overline{\int} - \underline{\int} < \epsilon$ for every $\epsilon > 0$. We are done by noting that

$$L(f, P_\epsilon) \leq \underline{\int} \leq \overline{\int} \leq U(f, P_\epsilon).$$

"$\Rightarrow$": Let the partitions $P_{1\epsilon}$ and $P_{2\epsilon}$ be such that $\int - L(f, P_{1\epsilon}) < \epsilon/2$ and $U(f, P_{2\epsilon}) - \int < \epsilon/2$. Consider P_ϵ the common refinement of $P_{1\epsilon}$ and $P_{2\epsilon}$. Then

$$\begin{aligned} \int - \epsilon/2 \leq L(f, P_{1\epsilon}) \leq L(f, P_\epsilon) \leq \\ \leq U(f, P_\epsilon) \leq U(f, P_{2\epsilon}) \leq \int + \epsilon/2. \end{aligned}$$

From this, it follows that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. $\square$

We are ready to prove some basic facts about integrable functions, making use of our test above. The theorem below, for instance, will ensure the existence of $\int_0^1 x \sin(x) \mathrm{d}x$.

Theorem 2. *If f is continuous on $[a,b]$, then $f \in \mathfrak{R}$.*

Proof. Fix $\epsilon > 0$. We need to find a partition P_ϵ with $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. We will appeal to the definition of continuity: f is continuous at x if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, x)$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$. Further, f is uniformly continuous (UC) on $[a,b]$ if for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ for all $x, y \in [a,b]$. All continuous functions are uniformly continuous on an interval, so we can pick δ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{b-a}$. Choose any partition P_ϵ containing exclusively intervals that are less than δ. Then

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum \Delta x_i \Big(\sup_{[x_i, x_{i+1}]} f - \inf_{[x_i, x_{i+1}]} f \Big) < \Big(\sum \Delta x_i \Big) \frac{\epsilon}{b-a} = \epsilon.$$

$\square$

Theorem 3. *If f is monotonic on $[a, b]$, then $f \in \mathfrak{R}$.*

Proof. To start, notice that if we choose P such that $\Delta x_i = \frac{b-a}{n}$ is uniform for each i, then

$$\sum \Delta x_i(\sup_{[x_i,x_{i+1}]} f - \inf_{[x_i,x_{i+1}]} f) = \sum \Delta x_i(f(x_{i+1}) - f(x_i)) = \frac{b-a}{n}(f(b) - f(a)).$$

Without loss of generality, assume f is monotonically increasing. We need to prove that for every $\epsilon > 0$, there exists a P_ϵ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. Given our scratch work above, it is clear that we would like $\frac{b-a}{n}(f(b) - f(a)) < \epsilon$. Choose n such that $\frac{b-a}{n}(f(b) - f(a)) < \epsilon$, and pick P_ϵ to be the uniform partition with n (equal sized) intervals. Then note that

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) = \sum \Delta x_i(f(x_{i+1}) - f(x_i)) &= \frac{b-a}{n}\sum_{i=0}^{n-1}(f(x_{i+1}) - f(x_i)) \\ &= \frac{b-a}{n}(f(b) - f(a)) < \epsilon. \end{aligned}$$

□

It is natural to ask ourselves how many discontinuities a function can have in its domain and still be integrable. We saw above that there are no restrictions in this regard if the function in question is monotonic. In general, however, the integral thus-defined will not be able to treat functions that are discontinuous "almost everywhere" (this language will become more meaningful later). For example, the everywhere-discontinuous Dirichlet function $f : [0, 1] \to \mathbb{R}$ defined by

$$f = \begin{cases} 0, \text{if } x \in \mathbb{Q}, \\ 1, \text{if } x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable. Functions that have a finite number of discontinuities, however, can still be treated.

Theorem 4. *Assume $f : [a, b] \to \mathbb{R}$ is bounded ($|f(x)| \leq M$) and the set of discontinuities $\mathrm{Disc}(f)$ is finite. Then $f \in \mathfrak{R}$.*

Proof. Let ϵ. We need to find the P_ϵ.
Step I: □

1.2 Properties of the Integral

Corollary. *If $f, g \in \mathfrak{R}$, then*

1. $f + g \in \mathfrak{R}$ *and*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

2. $fg \in \mathfrak{R}$.

Corollary (Triangle Inequality). 1. *If* $f_1(x) \le f_2(x)$, *then* $\int_a^b f_1 \le \int_a^b f_2$.

2. $f \in \mathfrak{R}$ *implies that* $|f| \in \mathfrak{R}$, *and* $|\int_a^b f| \le \int_a^b |f|$.

1.3 Integration and Differentiation

This section provides the link between differential and integral calculus.

Theorem 5. *Assume* $f \in \mathfrak{R}([a,b])$. *Let* $F(x) = \int_a^x f$ *for some* $x \in [a,b]$. *Then*

1. F *is uniformly continuous on* $[a,b]$.

2. *If* f *is continuous at* x_0, *then* F *is differentiable at* x_0 *and* $F'(x_0) = f(x_0)$.

The following theorem is occasionally useful.

Theorem 6 (The Fundamental Theorem of Calculus). *Let* $f \in \mathfrak{R}([a,b])$. *Assume* f *has an antiderivative* F, *i.e.* F *is differentiable and* $F'(x) = f(x)$. *Then*

$$\int_a^b f = F(b) - F(a).$$

Theorem 7 (Integration by Parts). *Let* $F, G : [a,b] \to \mathbb{R}$ *be differentiable functions. Assume* $F' = f \in \mathfrak{R}$, $G' = g \in \mathfrak{R}$. *Then*

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fg.$$

2 Sequences and Series of Functions

2.1 Discussion of the Main Problem