Real Analysis I

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1 The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

Let's begin with some familiar number systems.

$$\mathbb{N} = \{1, 2, 3, ...\} \tag{1}$$
$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\} \tag{2}$$
$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ q \neq 0 \right\} \tag{3}$$
$$\mathbb{R} = \ ? \tag{4}$$

These are the natural, integer, rational, and real number systems respectively. As we can see, it is not terribly difficult to fully characterize the natural, integer, and rational numbers. In the case of the first two, we can simply write out the elements that belong to each set one-by-one. It is harder to write out the rationals in such a way (though possible as we will see), but nevertheless we are able to produce a simple rule in set-builder notation whereby each rational is defined. The real numbers prove to be insidious in this regard. How exactly do we define the real numbers? And why do we need numbers that are not rational?

The Pythagorean theorem provides a relationship between the lengths of the bases of a right triangle and the length of the hypotenuse. If the base sides have lengths a and b, then the hypotenuse has length $c = \sqrt{a^2 + b^2}$. Consider a right triangle with base sides of length 1. Then, by the Pythagorean theorem, the hypotenuse has side length

$$c = \sqrt{1^2 + 1^2} = \sqrt{2}. \tag{5}$$

But as we will soon see, the square root of 2 is irrational. What are we to make of this? We would expect that the length of every measurable distance would be an actual number. The fact that the length of the hypotenuse is a non-rational number illustrates that there are a "larger" class of numbers for us to define. These are the so-called real numbers and they are the purpose of these notes. The following theorem was known to the Greeks.

Theorem 1. *There is no rational number whose square is 2.*

Remark. This is often stated as $\sqrt{2}$ is irrational, but this assumes $\sqrt{2}$ exists, which we also need to prove.

Proof. (by contradiction) Suppose that there were two integers p and q, $q \neq 0$, such that $(\frac{p}{q})^2 = 2$. We may assume p and q have no common factors. Then $p^2/q^2 = 2$, or equivalently,

$$p^2 = 2q^2. \tag{6}$$

The following Lemma will be used to show that both p and q are even, contradicting the premise.

Lemma. *Let n be an integer.*

(a) If n is even, then n^2 is even.

(b) If n is odd, then n^2 is odd.

Proof. The proof for each case depends on what it means for an integer to be even or odd.

(a) Suppose $n \in \mathbb{Z}$ and n is even. Then $n = 2k$ where $k \in \mathbb{Z}$. This implies that $n^2 = 4k^2 = 2(2k^2)$ is even.

(b) Suppose $n \in \mathbb{Z}$ and n is odd. Then $n = 2k + 1$ where $k \in \mathbb{Z}$. This implies that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is odd.

□

Let's use our new Lemma to finish the proof. We know that p^2 is even because $p^2 = 2q^2$. Therefore, by the Lemma, p is even. Writing $p = 2m$, $m \in \mathbb{Z}$, we find that $p^2 = 4m^2 = 2q^2$. But this means that $2m^2 = q^2$, from which it follows that q is even as well. This contradicts the assumption that p and q have no common factors. □

1.2 Some Preliminaries

Let $A, B, C, ...$ denote statements. Two statements can be joined through implication, written $A \Rightarrow B$, and read "A implies B" (alt. "if A, then B"). The implication $A \Rightarrow B$ means one of the following:

(i) A is true and B is true, or

(ii) A is false.

Example. Suppose $n \in \mathbb{N}$, and let A be the statement "n ends with 4 when written in base 10" and B the statement "n is even." Clearly $A \Rightarrow B$ because all numbers ending in 4 are even. Does $B \Rightarrow A$? The answer is no, because there exists even numbers that don't end in 4, such as 12.

When an implication is bi-directional, we write $A \Leftrightarrow B$, read "A is equivalent to B" (alt. "A if and only if (iff) B"). This is used to show that two statements are logically equivalent. The equivalence $A \Leftrightarrow B$ means one of the following:

(i) A and B are both true, or

(ii) A and B are both false.

Example. Suppose $n \in \mathbb{N}$, and let A be the statement "n ends with a 0, 2, 4, 6, or 8 when written in base 10" and B the statement "n is even." Then $A \Leftrightarrow B$.

Another way of stating the implication $A \Rightarrow B$ is to say that A only occurs if B occurs. The truth of A is contingent on B, in other words. The implication may then be read "A only if B."

Example. Suppose $n \in \mathbb{N}$, and let A be the statement "n is a multiple of 6" and B the statement "n is a multiple of 3." Then $A \Rightarrow B$. Here the phrase "if A, then B" means that if a number if a multiple of 6, then it is also a multiple of 3. The equivalent phrase "A only if B" means that a number is a multiple of 6 **only if** it is also a multiple of 3.

The introduction of certain quantifiers will aid in the mathematical discussions that follow. The symbol $\forall$ is read "for all," and means that a statement holds for all elements in the domain of inquiry. For example, the statement "$\forall n \in \mathbb{N}\ ,\ \ n < n + 1$" means that every natural number is less than the number after it. It is read "for all natural numbers n, n is less than $n + 1$." This can be verified by adding n to both sides of the inequality $0 < 1$. The symbol $\exists$ is read "there exists," and means that a statement holds for at least one element in the domain of inquiry. For example, the statement "$\exists x \in \mathbb{Q}$, $4.1 < x < 4.2$" means that there exists a rational number that is greater than 4.1 and less than 4.2. This can be verified by simply providing an x that satisfies this property, namely $x = 4.15$.

1.3 The Axiom of Completeness

We are ready to begin our discussion of the real numbers. The real number system $\mathbb{R}$ is an ordered field containing the rational numbers equipped with addition and multiplication. Moreover, the field $\mathbb{R} \supset \mathbb{Q}$ satisfies the Axiom of Completeness. What does it mean for $\mathbb{R}$ to be a field? A number system is a field if and only if it satisfies the following field axioms:

1.4 Consequences of Completeness

1.5 Cardinality

2 Sequences and Series

3 Basic Topology of $\mathbb{R}$

4 Functional Limits and Continuity

5 The Derivative

6 Sequences and Series of Functions