Analysis I

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Contents

1 The Real Numbers

1.1 Discussion: The Irrationality of $\sqrt{2}$

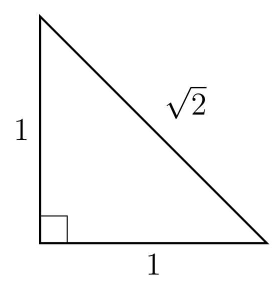
We begin with some familiar number systems.

$$\begin{aligned}
\mathbb{N} &= \{1, 2, 3, ...\} \\
\mathbb{Z} &= \{..., -3, -2, -1, 0, 1, 2, 3, ...\} \\
\mathbb{Q} &= \left\{ \frac{p}{q} \;\middle|\; p, q \in \mathbb{Z},\ q \neq 0 \right\} \\
\mathbb{R} &= \ ?
\end{aligned}$$

These are the natural, integer, rational, and real number systems, respectively. As we can see, it is not terribly difficult to fully characterize the natural, integer, and rational numbers. In the case of the first two, we can simply write out the elements that belong to each set one-by-one. It is harder to write out the rationals in such a way (though possible as we will see), but nevertheless we are able to produce a simple rule in set-builder notation whereby each rational is defined. The real numbers prove to be insidious in this regard. How exactly do we define the real numbers? And why do we need numbers that are not rational?

The Pythagorean theorem provides a relationship between the lengths of the bases of a right triangle and the length of the hypotenuse. If the base sides have lengths a and b, then the hypotenuse has length $c = \sqrt{a^2 + b^2}$. Consider a right triangle with base sides of length 1. Then, by the Pythagorean theorem, the hypotenuse has length

$$c = \sqrt{1^2 + 1^2} = \sqrt{2}.$$



But as we will soon see, the square root of 2 is irrational. What are we to make of this? We would expect that the length of every measurable distance would be an actual number. The fact that the length of the hypotenuse is a non-rational number illustrates that there are a "larger" class of numbers for us to define. These are the so-called real numbers and they are the purpose of these notes. The following theorem was known to the Greeks.

Theorem 1. *There is no rational number whose square is 2.*

Remark. This is often stated as $\sqrt{2}$ is irrational, but this assumes $\sqrt{2}$ exists, which we also need to prove.

Proof (by contradiction). Suppose that there were two integers p and q, $q \neq 0$, such that $(\frac{p}{q})^2 = 2$. We may assume p and q have no common factors. Then $p^2/q^2 = 2$, or equivalently,

$$p^2 = 2q^2.$$

The following lemma will be used to show that p and q are both even, contradicting the assumption that they have no common factors.

Lemma. *Let n be an integer.*

(a) If n is even, then n^2 is even.

(b) If n is odd, then n^2 is odd.

Proof of lemma. The proof for each case depends on what it means for an integer to be even or odd.

(a) Suppose $n \in \mathbb{Z}$ and n is even. Then $n = 2k$ where $k \in \mathbb{Z}$. This implies that $n^2 = 4k^2 = 2(2k^2)$ is even.

(b) Suppose $n \in \mathbb{Z}$ and n is odd. Then $n = 2k + 1$ where $k \in \mathbb{Z}$. This implies that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ is odd.

□

Let's use our new lemma to finish the proof of Theorem 1. We know that p^2 is even because $p^2 = 2q^2$. Therefore by the lemma, p is even. Writing $p = 2m$, $m \in \mathbb{Z}$, we find that $p^2 = 4m^2 = 2q^2$. But this means that $2m^2 = q^2$, from which it follows that q is even as well. This contradicts the assumption that p and q have no common factors. □

Since $\sqrt{2}$ can be the length of a line segment (e.g. the hypotenuse above), there is a "hole" in the rational number line precisely where this length is. The real number system remedies this by filling in the holes of the rational numbers.

Example. We will show that there is no rational number whose square is 3. Like the previous proof, it will be helpful to establish an intermediate lemma.

Lemma. *Let n be an integer.*

(a) If n is divisible by 3, then n^2 is divisible by 3.

(b) If n is not divisible by 3 (has remainder 1 or 2), then n^2 is not divisible by 3.

Proof of lemma. We will proceed, again, by cases.

(a) If n is divisible by 3, then $n = 3k$ for some integer k. Then $n^2 = 9k^2 = 3(3k^2)$ is divisible by 3.

(b) Case 1: If $n = 3k + 1$ for some integer k, then $n^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ is not divisible by 3.
Case 2: If $n = 3k+2$ for some integer k, then $n^2 = 9k^2+12k+4 = 9k^2+12k+3+1 = 3(3k^2 + 4k + 1) + 1$ is not divisible by 3.

$\square$

We can restate the lemma as follows: for an integer n, n^2 is divisible by 3 if and only if n is divisible by 3. Now assume that $(\frac{p}{q})^2 = 3$ where p and $q \neq 0$ are integers with no common factor. Then $p^2 = 3q^2$, which implies that p^2 is divisible by 3. The lemma shows us that p is also divisible by 3, for which we can write $p = 3k$ for some integer k. Now,

$$p^2 = 3q^2 \implies (3k)^2 = 3q^2 \implies 9k^2 = 3q^2 \implies 3k^2 = q^2.$$

But since q^2 is divisible by 3, it follows that q is itself divisible by 3. This contradicts the assumption that p and q have no common factors and we are forced to conclude that there is no rational number whose square is 3.

1.2 Some Preliminaries

1.2.1 Notions from logic

Let A and B denote statements. Two statements can be joined through implication, written $A \implies B$, and read "A implies B" (alt. "if A, then B"). The implication $A \implies B$ means one of the following:

(i) A is true and B is true, or

(ii) A is false.

Example. Suppose $n \in \mathbb{N}$, and let A be the statement "n ends with 4 when written in base 10" and B the statement "n is even." Clearly $A \implies B$ because all numbers ending in 4 are even. Does $B \implies A$? The answer is no, because there exist even numbers that don't end in 4, such as 12. This lack of implication may be written $B \not\Longrightarrow A$.

When an implication is bi-directional, we write $A \iff B$, read "A is equivalent to B" (alt. "A if and only if (iff) B"). This is used to show that two statements are logically equivalent. The equivalence $A \iff B$ means one of the following:

(i) A and B are both true, or

(ii) A and B are both false.

Example. Suppose $n \in \mathbb{N}$, and let A be the statement "n ends with a 0, 2, 4, 6, or 8 when written in base 10" and B the statement "n is even." Then $A \iff B$.

Another way of stating the implication $A \implies B$ is to say that A only occurs if B occurs. The truth of A is contingent on B, in other words. The implication may then be read "A only if B." This is not to say that *every* time B occurs, A must as well (that would be "A if B"). This is the subtle difference between "A if B" ($B \implies A$) and "A only if B" ($A \implies B$).

Example. Suppose $n \in \mathbb{N}$, and let A be the statement "n is a multiple of 6" and B the statement "n is a multiple of 3." Then $A \implies B$. Here the phrase "if A, then B" means that if a number n is a multiple of 6, then n is also a multiple of 3. The equivalent phrase "A only if B" means that a number n is a multiple of 6 **only if** n is also a multiple of 3.

1.2.2 Convention regarding definitions

Nearly all mathematical definitions are stated with "if" when "iff" is meant. For example, the definition below really means that a number is prime if and only if it is not 1 and has no factors other than 1 and itself.

Definition. A natural number $n \neq 1$ is **prime** if n has no factors other than 1 and n.

1.2.3 Quantifiers

The introduction of certain quantifiers will aid in the mathematical discussions that follow. The symbol $\forall$ is read "for all" and means that a statement holds for all elements in the domain of inquiry. For example, the statement "$\forall n \in \mathbb{N}\ ,\ \ n < n + 1$" means that every natural number is less than the number after it. It is read "for all natural numbers n, n is less than $n+1$." This can be verified by adding n to both sides of the inequality $0 < 1$. The symbol $\exists$ is read "there exists" and means that a statement holds for at least one element in the domain of inquiry. For example, the statement "$\exists x \in \mathbb{Q}$ such that $4.1 < x < 4.2$" means that there exists a rational number that is greater than 4.1 and less than 4.2. This can be verified by simply providing an x that satisfies this property, namely $x = 4.15$.

We may speak of the negation of a proposition A, written $\neg A$, whose truth value is opposite that of A's. For instance, the negation of

$$\exists \frac{p}{q} \in \mathbb{Q} \text{ such that } \left(\frac{p}{q}\right)^2 = 2$$

is

$$\nexists \frac{p}{q} \in \mathbb{Q} \text{ such that } \left(\frac{p}{q}\right)^2 = 2,$$

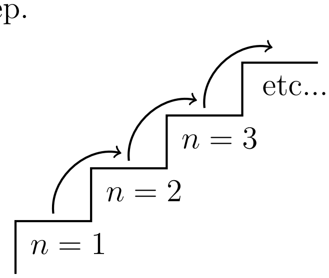
or equivalently,

$$\forall \frac{p}{q} \in \mathbb{Q}, \left(\frac{p}{q}\right)^2 \neq 2.$$

The contrapositive of an implication is equivalent to the implication itself. Thus, if $A \implies B$, then $\neg B \implies \neg A$. Why is this true? Well $A \implies B$ is true iff (A, B are both true) or (A is false). The contrapositive $\neg B \implies \neg A$ is true iff ($\neg B$, $\neg A$ are both true) or ($\neg B$ is false), which is equivalent to (B, A are both false) or (B is true). Every truth assignment that makes the original implication true makes the contrapositive true, and vice versa. Thus the two statements are logically equivalent.

1.2.4 Proof by induction

Suppose that for each natural number n, $S(n)$ is a statement involving n. The goal of proof by induction is to prove that the statements $S(n)$ are true for all $n \in \mathbb{N}$. The main idea is to begin by showing $S(1)$ is true, i.e. that the statement is true for the first natural number. This is often referred to as the base case. Next, we want to show that if the statement is true for a generic natural n, then it is also true for $n+1$, i.e. that $S(n) \implies S(n+1)$. This is referred to as the inductive step.



Example. Let $y_1 = 1$ and for each $n \in \mathbb{N}$, define $y_{n+1} = \frac{3y_n+4}{4}$. We will use induction to prove that the sequence satisfies $y_n < 4$ for all natural numbers n. More specifically, induction will be used to show that $S(n) =$ "$y_n < 4$" is true for all $n \in \mathbb{N}$.
Base case: For $n = 1$, we have $y_1 = 1 < 4$ and the statement $S(1)$ is true.
Inductive step: Assume that $S(n)$ is true, i.e. that $y_n < 4$. We must show that $S(n+1)$ is also true, i.e. that $y_{n+1} < 4$. Multiplying both sides of the given inequality by 3 yields $3y_n < 12$. A little more algebra produces

$$\frac{3y_n + 4}{4} < \frac{12+4}{4} = \frac{16}{4} = 4.$$

But the LHS of this inequality is precisely y_{n+1}, and so we have that $y_{n+1} < 4$ and the statement is true for $S(n+1)$. By the principle of induction, $y_n < 4$ for all natural numbers.

Example. Mathematical induction can be used to show that $4^n - 1$ is divisible by 3 for all $n \in \mathbb{N}$.
Base case: For $n = 1$, we have $4^n - 1 = 4 - 1 = 3$ is divisible by 3.
Inductive step: Assume $4^n - 1$ is divisible by 3, i.e. $4^n - 1 = 3k$ for some integer k. Then $4^n = 3k + 1$. Multiplying both sides by 4 produces $4^{n+1} = 12k + 4$, from which we have

$$4^{n+1} - 1 = 12k + 3 = 3(4k+1)$$

is divisible by 3.

Example. Let $y_1 = 1$ and for each $n \in \mathbb{N}$, define $y_{n+1} = \frac{3y_n+4}{4}$. We will use induction to prove that the sequence

$$(y_n)_{n=1}^{\infty} = (y_1, y_2, y_3, ...)$$

is strictly increasing. This amounts to showing that $y_n < y_{n+1}$ for all natural n.
Base case: For $n = 1$, we must show that $y_1 < y_2$. We have that $y_1 = 1$ and

$$y_2 = \frac{3y_1 + 4}{4} = \frac{3+4}{4} = \frac{7}{4}.$$

Since $1 < \frac{7}{4}$, the base case is true.
Inductive step: Suppose $y_n < y_{n+1}$. We must show that $y_{n+1} < y_{n+2}$. Behold:

$$y_n < y_{n+1} \implies 3y_n < 3y_{n+1} \implies 3y_n + 4 < 3y_{n+1} + 4 \implies \frac{3y_n + 4}{4} < \frac{3y_{n+1} + 4}{4}$$
$$\implies y_{n+1} < y_{n+2}.$$

1.3 The Axiom of Completeness

We are ready to begin our discussion of the real numbers. The real number system $\mathbb{R}$ is an ordered field containing the rational numbers equipped with addition and multiplication. Moreover, the field $\mathbb{R} \supset \mathbb{Q}$ satisfies the Axiom of Completeness. The fact that $\mathbb{R}$ is a field means it satisfies the following field axioms for all $a, b, c \in \mathbb{R}$:

1. $a + b = b + a$ (commutativity of addition)
2. $a + (b + c) = (a + b) + c$ (associativity of addition)
3. $a + 0 = 0 + a = a$ (additive identity)
4. $ab = ba$ (commutativity of multiplication)
5. $a(bc) = (ab)c$ (associativity of multiplication)
6. $a + (-a) = 0$ (additive inverse)
7. $a \cdot 1 = 1 \cdot a = a$ (multiplicative identity)
8. $a(b + c) = ab + ac$ (distributive property)
9. $a \cdot \frac{1}{a} = 1$ (multiplicative inverse)
10. $0 \neq 1$

Further, $\mathbb{R}$ is an ordered field meaning there is a relation called $<$ defined on pairs of real numbers. More concretely, for all $x, y \in \mathbb{R}$, either $x < y$, $y < x$, or $x = y$, with one and only one of these holding. This ordering is transitive; if $x, y, z \in \mathbb{R}$, then $x < y$ and $y < z$ implies that $x < z$.

Definition. A subset $A \subset \mathbb{R}$ is bounded above if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. Such a number b is called an upper bound of the set A.

Example. Let $A = \{1, 5\} = \{x \in \mathbb{R} : 1 < x < 5\}$. Then some upper bounds of A include 76, 10, and 5.

Example. Let $A = [1, 5] = \{x \in \mathbb{R} : 1 \leq x \leq 5\}$. Then 76, 10, and 5 are still upper bounds of A despite $5 \in A$.

Definition. If $A \subset \mathbb{R}$ is bounded above, then an upper bound c of A is a least upper bound (supremum) of A if $c \leq b$ for every upper bound b.

That this upper bound is unique is immediate, and we will write $\sup A = c$.

Example. What is the sup of $A = (1, 5)$?
Our candidate is 5 and we shall prove it is in fact the supremum.

Claim. There is no upper bound smaller than 5.

Proof. s □

1.4 Consequences of Completeness

1.5 Cardinality

2 Sequences and Series

3 Basic Topology of $\mathbb{R}$

4 Functional Limits and Continuity

5 The Derivative

6 Sequences and Series of Functions