Limit Order Book Dynamics and Order Size Modelling Using Compound Hawkes Process

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Hawkes Process has been used to model Limit Order Book (LOB) dynamics in several ways in the literature however the focus has been limited to capturing the inter-event times while the order size is usually assumed to be constant. We propose a novel methodology of using Compound Hawkes Process for the LOB where each event has an order size sampled from a calibrated distribution. The process is formulated in a novel way such that the spread of the process always remains positive. Further, we condition the model parameters on time of day to support empirical observations. We make use of an enhanced non-parametric method to calibrate the Hawkes kernels and allow for inhibitory cross-excitation kernels. We showcase the results and quality of fits for an equity stock's LOB in the NASDAQ exchange.

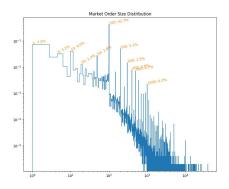
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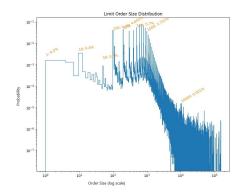
1. Introduction

The Hawkes process, known for its high adaptability, offers a more comprehensive point process methodology for modeling order book arrivals than the Poisson process and its variants, without the need to explicitly model individual traders' behaviors in the market. Its capability to replicate microstructural details such as volatility clustering and the Epps effect makes it a suitable candidate for Limit Order Book (LOB) models. It is important to highlight that these point process models are mathematically descriptive, providing full transparency in their nature and thus are suitable for applications where black-box solutions are not preferred. In their comprehensive review and tutorial, Bacry, Mastromatteo, and J.-F. Muzy 2015 outlay the major ideas of the Hawkes Process, its mathematical theory, some of its crucial properties and finally applications including a detailed review over the Order Book models. Recently, state dependent Hawkes Process have been quite popular (Morariu-Patrichi and Pakkanen 2022, Kirchner and Vetter 2022, Mucciante and Sancetta 2023, Wu et al. 2019). However, as noted in Rambaldi, Bacry, and Lillo 2017 and Lu and Abergel 2018b, individual order's size is an important of aspect of the LOB which the Hawkes model alone is unable to capture.

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- (a) Market Orders' Size Distribution (log-log scale) (b) Limit Orders' Size Distribution (log-log scale)
 - Figure 1.: Empirical Distribution of Order Sizes

1.1. Empirical Distributions of Order Sizes:

We show here in Figure 1 two trading weeks' of data (10 dates) for Apple and plot the empirical histograms on a log-log scale. We only focus on top of the book cancels and limit orders. As Figures 1aMarket Orders' Size Distribution (log-log scale)figure.caption.1 and 1bLimit Orders' Size Distribution (log-log scale)figure.caption.1 show, the market orders and limit orders sizes have several spikes at round numbers indicating the preference of the traders. For example, we see more than 40% of the market orders and 60% of limit orders are of 100 size. Naturally, Cancel Orders' sizes are capped at the size of an individual order. In fact we observe that outright cancels (i.e. full quantity of an order is cancelled) constitute 99.3% of all cancels.

1.2. Properties of the Order Book Dynamics:

There have been several variations to the Hawkes model to accommodate for certain properties of the order book in exchanges.

Prop 1: Bid-Ask Spread is always non-negative: The bid-ask spread can never be less than zero being an important one was tackled by using order arrival intensities dependent on spread-in-ticks by Lee and Seo 2022 however their model is limited to low spread-in-tick (i.e. large-tick) securities where the spread is quite tight (one to two tick wide). Small-tick securities can have very high number of price levels in the spread which makes the model which calibrates order arrivals at each price level away from mid quite high in number of parameters. Previously, Zheng, Roueff, and Abergel 2014 have used constrained Hawkes Process to control negative spreads.

Prop 2: Order intensities are dependent on time-of-day: It is well known that trading volumes follow an intraday seasonality which is observered to be stationary across multiple days. Naturally, we observe that the order intensities too exhibit this intraday seasonality (more in Section 2). Hawkes models have been adapted to account for the same by Mucciante and Sancetta 2023, Prenzel et al. 2022, and Kirchner and Vetter 2022.

Prop 3: Cross-excitations can be inhibitory: As noted in Lu and Abergel 2018a, the cross-excitation of events need not necessarily always be catalyzing. For example, we observe that cancels at opposing sides of the book have inhibitory effects on each other. Generally modelling for this can introduce negative intensities in the point process. To avoid such complications, Lu and Abergel 2018a propose to floor the total intensity of any element of the Hawkes process to zero.

In this work we show some lack of support for the hypothesis that the order arrival intensities are impacted by the past order sizes. Thus we conclude that the Compound Hawkes Process is a suitable candidate for the model. We then create a stationary distribution of the order sizes for each type of order which closely mimics the empirical distribution. This distribution is used to sample the order sizes in the compound Hawkes Process. We create a novel formulation of the Hawkes

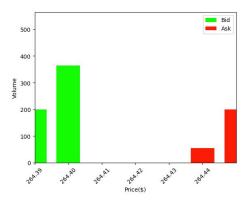


Figure 2.: Order Book Snapshot

intensities which satisfies Properties 1, 2 and 3. In Section 3, we show the calibration results for the Apple Inc. stock by using Level 2 data in the NASDAQ exchange. We also provide an analysis for the quality of fit.

2. Methodology

We consider the task of modelling LOB's current best bid and ask (i.e. Level 1 LOB). We note that since placing order in the spread is a common practice in equities since the spread usually has many empty price levels to attract market participants to place a more aggressive order, the Hawkes formulation should contain modelling the intensity of the nearby empty levels as well. Hence we formulate the order book as a queueing system with six different queues which are $\{ask_{+1}, ask_0, ask_{-1}, bid_{+1}, bid_0, bid_{-1}\}$.

Here ask_0 and bid_0 are the respective best ask and bid prices. Where the subscripts denote the price level distance in ticks from the best ask/bid. For eg. ask_{+1} is the price level which is 1 tick more than ask. The 2×6 vector of $((q_d^s, s_d)_{s \in \{ask, bid\}, d \in \{-1, 0, +1\}})$, where q_d^s denotes the queue size, is what we call the state of the order book. The motivation to model just these six levels and not the entire order book is two-fold. Firstly, the question of parsimony becomes important when we model more levels than just these six. Secondly, we observe that the LOB state changes are generally events which move the prices by 1 tick. Indeed we observe, in our dataset, 97.4% of all in-spread orders occur 1-tick away from the best resp. quote with mean being 1.035 ± 0.285 ticks for 2 million samples and 97.5% of all price changes have the next non-empty price level at 1-tick distance as well with the mean being 1.033 ± 0.283 ticks.

Mechanical Constraints: Naturally, since ask_0 and bid_0 are the best ask and bid prices, the queue size at ask_{-1}/bid_{+1} is zero. Therefore any incoming limit order at ask_{-1}/bid_{+1} (i.e. an in-spread (IS) order) creates a new best ask/bid and therefore the LOB state transforms from $S := \{(q_{+1}^{ask}, ask_{+1}), (q_0^{ask}, ask_0), (0, ask_{-1}), (0, bid_{+1}), (q_0^{bid}, bid_0), (q_{-1}^{bid}, bid_{-1})\}$ to $S_{IS} = \{(q_0^{ask}, ask_0), (q_{-1}^{ask}, ask_{-1}), (0, ask_{-2}), (0, bid_{+1}), (q_0^{bid}, bid_0), (q_{-1}^{bid}, bid_{-1})\}$ for an in-spread ask side limit order. Note that it is possible that the spread is only 1-tick wide which would mean ask_{-1} and bid_{+1} coincides. Hence the intensities for this price level will be exactly equal between these two queues. There is also the possibility of 0 spread in which case ask_{-1} and bid_{+1} do not exist. We will put some constraints on the Hawkes process intensities for these levels further in this section to account for these possibilities. On the other hand, if a queue-depletion (QD) event at best bid/ask happens (for eg, a large enough market order), the spread widens and therefore the state moves from S to $S_{QD} = \{(q_{+2}^{ask}, ask_{+2}), (q_{+1}^{ask}, ask_{+1}), (0, ask_0), (0, bid_{+1}), (q_0^{bid}, bid_0), (q_{-1}^{bid}, bid_{-1})\}$ for a queue-depleting ask side market order/cancel order. Here we observe an unknown quantity q_{+2}^{ask} . We sample this unknown quantity from a stationary distribution calibrated from empirical

data. Finally, we note that Market Orders can only occur at ask_0/bid_0 and ask_{-1}/bid_{+1} cannot have Cancel Orders since by definition the quantity there is zero.

Compound Hawkes Process: A compound point process (CPP) is defined as

$$Z_t = \sum_{i=1}^{N(t)} Y_i$$

where N(t) is the counting process associated with a point process Y and Y_i is a random variable in some sample space. The Compound Hawkes Process (CHP) is a special case of the CPP where Y is a Hawkes Process. For a d-dimensional Hawkes process the intensity of the process λ_t^i for $i = 1, \ldots, d$:

$$\lambda_t^i = \mu_t^i + \sum_{j=1}^d \sum_{\{T_i\}} \phi_{t-T_j}^{ij}$$

where $\{T_j\}$ denotes the set of event times in the j dimension of the Hawkes Process. Here, μ_t is the exogenous intensity and $\phi_{t-T_j}^{ij}$ is the excitation term from j-th dimension to i-th dimension. Generally the excitation terms are a function of the time since event (generally a decaying function in time like exponential decay or power law decay). An alternate but equivalent formulation is the following:

$$\lambda_t^i = \mu_t^i + \sum_{j=1}^d \int_{-\infty}^t \phi_{t-s}^{ij} dN_s^j$$

Limit Order Book at Compound Hawkes Process: We define a 12D Compound Hawkes Process (in accordance with the mechanical constraints) for the 6 queues in the order book state where each dimension corresponds to the following event types: $\{LO_{ask_{+1}}, CO_{ask_{+1}}, LO_{ask_{0}}, CO_{ask_{0}}, MO_{ask_{0}}, LO_{ask_{-1}}, LO_{bid_{+1}}, LO_{bid_{0}}, CO_{bid_{0}}, MO_{bid_{0}}, LO_{bid_{-1}}, CO_{bid_{-1}}\}$ where LO is Limit Order, CO is Cancel Order and MO is Market Order. Here each of the 12 event types' order size (κ^{e} for event e) is sample from their own specific calibrated distribution ($\Pi^{e}(\Theta)$). We postulate that the order intensities themselves are not impacted by the past order sizes but only the past order event-times as is the general assumption in Hawkes models applied to LOB data. We provide some weak evidence for this claim in the Appendix A. We note that our work on the distribution of order size closely follows the work by Lu and Abergel 2018b however they make use of a Poisson model. Let us also mention that Abergel et al. 2016 too use parametric distributions for order sizes in the Hawkes Process model. For Cancel Order quantities, since we observe almost all cancels in empirical data are outright cancels, we draw randomly, with equal probability, from the available limit orders' quantities in the queue.

$$\kappa_i^e \sim \Pi^e(\Theta); i = 1, \dots, N^e(t)$$

$$q_d^s(t) = \sum_{e \in E_d^s} \sum_i^{N^e(t)} \kappa_i^e; E_d^s := \text{set of eligible events for } s \text{ and } d$$

To control the spread of the process (Prop 1), we formulate the intensities of $LO_{ask_{-1}}$ and $LO_{bid_{+1}}$ in the following manner as a function of the current spread-in-ticks (s). If $\psi_t^{(.)}$ represents

the intensity (either exogenous or excitation), then,

$$\psi_t^{(.)}(s) = (\psi_t^{(.)})s^{\beta}; \beta > 0, s \in \mathbb{N}$$

In this formulation we enforce the two mechanical constraint i.e. if $s=1, \lambda_t^{LO_{ask_{-1}}}=\lambda_t^{LO_{bid_{+1}}}$, and, if $s=0, \lambda_t^{LO_{ask_{-1}}}=\lambda_t^{LO_{bid_{+1}}}=0$ with an extra parameter β . We motivate the choice of a power law dependence of order intensity over spread in Appendix B.

We observe a strong dependence between the exogenous intensities $\mu_t^{(.)}$ on the time of day of the trading day (Prop 2). There is a "U"-shape of order intensities with respect to the time of day i.e. right after the open auction and right before the close auction, the activity in the market is much higher than the middle of the day. This effect is strongly observed for trading volumes in equities with a number of probable causes but the most commonly accepted is because auctions cause a halt in trading leading to increased activity just before or after the auction. To maintain model parsimony, we bin the 6.5 hour trading day of NASDAQ into thirteen 30-minute bins with each bin having its own calibrated exogenous intensity. We assume here that any intraday seasonality in the order intensities are captured by the exogenous intensities. For a binning operator $Q(t): \mathbb{R} \to \{1, \ldots, 13\}$,

$$\mu_t^e = \mu^e(Q(t))$$

Finally, to make sure the inhibitory kernels do not create negative intensities, we floor the total excitation λ_t^i to zero.

Calibration: We follow the non-parametric method of calibration in Kirchner 2017 and take inspiration from Bacry, Jaisson, and J.-.-F. Muzy 2016 to create a time grid at both linear and log scales to account for the slowness of the decay kernels. This method has the advantage of not having any priors on the shape of the kernels themselves (popular choices in the literature include exponential decay and power-law decay) since it is non-parametric. We fit parametric functions on each of these kernels on these point estimates and here we make use of power-law functions and exponential functions as the candidates. Unlike Kirchner 2017, however, we do not calibrate the parameters in 30-minute windows and average over a day's 13 windows, but rather we use the entire day's data to calibrate the parameters. We also modify the calibration methodology to account for the time-of-day dependence of the exogenous intensities as well as the spread dependent formulation we developed for in-spread queues. We calibrate the data for several days individually and test the stationarity (by day) of the calibrated parameters. Since we do see stationarity in the kernel parameters over multiple days, we conclude by using the average parameters over multiple days as our final calibrated parameters. We use a heuristic method to calibrate the order size distribution. A key observation is the preference of traders of round numbers in their order quantity. This is an important stylized fact of the order book dynamics so we make use of Dirac delta functions to add spikes in the distribution function to account for this.

Finally, we use the thinning algorithm to simulate the order book from the calibrated parameters and provide some visualizations of the calibrated parameters in the following section along with the quality of fit results.

3. Results

3.1. Calibration of Hawkes Process:

We now show the observed intensities conditioned by time of day for a sample dimension (Limit Order at Top) in Figure 3Intensities conditioned by Time of Day: we report the average number of events per 10 min bin in a 6 month periodfigure.caption.3. We observe the common U-shape across all 12 dimensions. In Figure 4aNorms of Kernelfigure.caption.4, we show the the norm of

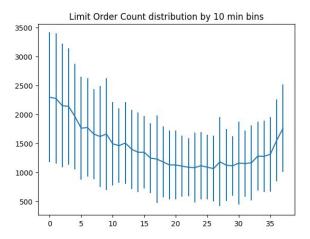


Figure 3.: Intensities conditioned by Time of Day : we report the average number of events per 10 min bin in a 6 month period

kernels. The x axis shows the excitor and y axis is the excitee. We also show a sample fitted kernel shape in Figure 4bKernel Shapefigure.caption.4. As we can see here in the translucent blue lines, the kernel shapes fitted over multiple days are stable across days.

Note that these estimated kernels are estimated in a non-parametric manner and hence need to be further fitted on a parametric function. We choose between the power-law kernel and the exponential kernel to do the parametric fit by comparing the fit's Akaike Information Criteria (AIC). We see that the power-law kernel is selected 100% of the time for this dataset. We show the fitted line in 4bKernel Shapefigure.caption.4 in red. As we can see, the power law line fits the point estimates quite well. Indeed we see an average mean square fit error to be $\sim 10^{-3}$. Another noteworthy aspect of the fitting results is the presence of inhibitory kernels.

3.2. Calibration of Order Size Distributions

Following Lu and Abergel 2018a, we use Dirac delta at round numbers to account for the stylized facts we observe in Figure 1aMarket Orders' Size Distribution (log-log scale)figure.caption.1 and 1bLimit Orders' Size Distribution (log-log scale)figure.caption.1. We choose the set 1, 10, 50, 100, 200, 500 as the set of round numbers we wish to put spikes in the PDF at. The remainder of the PDF is modelled by a Geometric distribution. We fit this distribution using the maximum likelihood method. We show the final calibrated PDF in Figure 5Fitted Distribution of Order Sizesfigure.caption.5.

3.3. Quality of fit metrics:

We present the quality of fit metrics in 6Resultsfigure.caption.6 by using the Q-Q plot on the residuals of the Hawkes process against the exponential distribution as recommended by Lu and Abergel 2018a. In the future we plan on performing a realism quality of fit by comparing some stylized facts of the simulated data to the empirical data. We plan to make use of inter-event durations' Q-Q plots, signature plots, distribution of spread and returns, average shape of the book, autocorrelation of returns and order flow, and sample price paths as our set of stylized facts.

4. Conclusion & Future Work

We tackle the problem of simulating a realistic order book by using the Compound Hawkes Process. We particularly focus on building a simulator which is realistic in its order sizes and exhibits

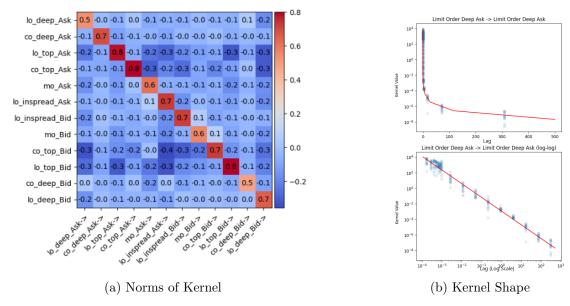


Figure 4.: Excitation Kernels

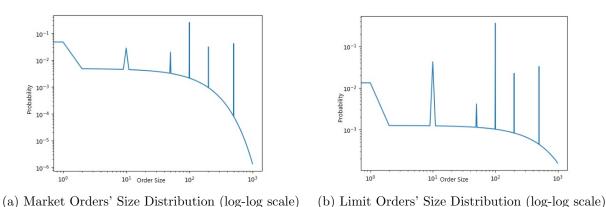
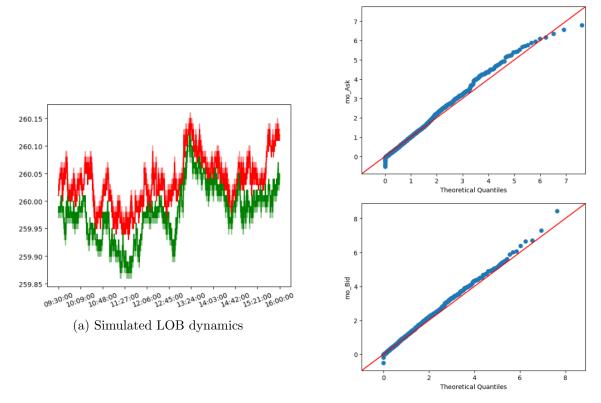


Figure 5.: Fitted Distribution of Order Sizes

the mechanical constraints that a real order book has such as non-negative spreads and positive intensities. We further condition the intensities to be dependent on time-of-day in order for the simulator to not be ignorant of the increase in trading intensities around open and close auctions. Future directions of research includes finding evidence for or against the presence of long-tailed behaviour in the order sizes of an order book i.e. is the order size distribution conditional on previous order sizes? The number of dimensions of this Hawkes Process is quite high, future research could be focused on simulating the order book with lesser number of dimensions.

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(b) Q-Q plot of residuals against the exponential distribution

Figure 6.: Results

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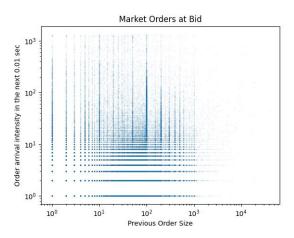


Figure A1.: Previous Order Size vs Future Arrival Rates

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Appendix A: Previous Order Sizes do not impact Future Order Arrival Rates

We calculate the next order intensity in our dataset for a past order size by count the number of future events in a window of $t=0.01\mathrm{sec}$. The joint scatter plot for Market Orders at Bid is shown in Figure A1Previous Order Size vs Future Arrival Ratesfigure.caption.8. Qualitatively the distribution looks to be quite uniform. We make use of the Hoeffding Independence Test to calculate the distance between the observed joint distribution of these two random variables and the distribution if they were independent. The test statistic is observed to be 0.00068 which is sufficiently low for us to conclude, albeit with weak evidence, that the two variables are independent. We observe similar scatter plots and Hoeffding statistics for all other events.

Appendix B: In-Spread order intensity depends on current spread

We plot the order intensity against the current spread-in-ticks in empirical data (1 month data) in Figure B1In-Spread Arrival Rates vs Current Spreadfigure.caption.9. Here we show a violin plot since the intensities are approximated by the number of in-spread orders in the next 0.01 seconds and therefore are random. We show the distribution of data points in the translucent red bars. We take the mean of the data for each spread-in-tick group as our input. We fit a linear regressor on log-log transformation of these data points (excluding 0 and 1 spread) and find the best exponent to

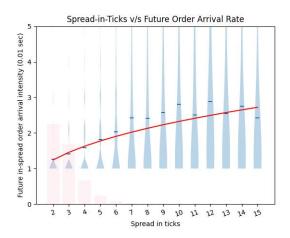


Figure B1.: In-Spread Arrival Rates vs Current Spread

be 0.41. The red line in the plot shows the fitted line. We observe an \mathbb{R}^2 of 0.85 for this regression.