BASIC NOTIONS OF RIEMANNIAN GEOMETRY

Consider a compact, d-dimensional manifold M, embedded in $\mathbb{R}^d$ (topological space, each point of which has a neighborhood homeomorphic to an open subset of $\mathbb{R}^d$).

Tangent Space: Given $p \in M$, we define the tangent space $T_pM \subset \mathbb{R}^d$ of M at p as

$$T_pM := \{ \dot{\gamma}(0) \mid \gamma : (-1,1) \to M, \gamma(\cdot) \in C^1, \gamma(0) = p \}.$$

Intuitively: this contains "all directions tangent to M at p".
Here and in what follows, all "curves" $\gamma(\cdot)$ we consider are of class C^1.

Gradient: Let the function $F: M \to \mathbb{R}$ be smooth. We define its gradient $\nabla F : M \to \mathbb{R}^d$ as the unique tangent vector field $\nabla F(x) \in T_xM$, $\forall\, x \in M$ with the property

$$\frac{d}{dt} F(\gamma(t)) \Big|_{t=0} = \langle \nabla F(\gamma(0)), \dot{\gamma}(0) \rangle$$

for every curve $\gamma : (-1,1) \to M$.
(We are using here the fact, that the Euclidean scalar product endows the tangent spaces also with a scalar product $\langle \cdot, \cdot \rangle$.)

Length: Given a curve $\gamma : [a,b] \to M$, its length is $\int_a^b |\dot{\gamma}(t)|\, dt$.

(Invariant under reparametrization; to compute it, we need only compute the Euclidean norm only of vectors tangent to the manifold M.)

Riemannian Distance: Given any two points x, y in M, their distance

$$d_M(x,y) \triangleq \inf \left\{ \int_a^b |\dot{\gamma}(t)|\, dt \;\middle|\; \gamma : [0,1] \to M, \gamma(0) = x, \gamma(1) = y \right\}.$$

The smallest length among all curves on the manifold that connect x with y. Because M is compact, this infimum is attained.

Minimizing Geodesic: A curve $\gamma : [a,b] \to M$ with $\gamma(a)=x$, $\gamma(b)=y$ and constant speed $|\dot{\gamma}(\cdot)|$, whose length attains the infimum:

$$d_M(x,y) = (b-a) \cdot |\dot{\gamma}| .$$

Minimizing geodesics are "straight lines in a curved space". Apart from the distortion induced by M, "they go as straight as possible":

(✱) $$\ddot{\gamma}(t) \perp T_{\gamma(t)} M , \quad \forall\, t \in [a,b].$$

In other words, there is no "skewness" in the acceleration (it lies entirely in the tangent space, like a purely centrifugal force). This is a consequence of minimality, and follows from a variational argument.

GEODESIC: A curve $\gamma : [a,b] \to M$ satisfying (✱) right above.

- Every Geodesic has constant speed: this is because

$$\frac{d}{dt} |\dot{\gamma}(t)|^2 = 2 \langle \dot{\gamma}(t), \ddot{\gamma}(t) \rangle = 0 , \quad \text{as} \quad \ddot{\gamma}(t) \perp T_{\gamma(t)} M \ni \dot{\gamma}(t) .$$

- Every Geodesic is Locally Minimizing: if a curve $\gamma : [a,b] \to M$ satisfies (✱), then for any given $t_0 \in (a,b)$ there exists $\varepsilon > 0$ such that the restriction of $\gamma(\cdot)$ to $[t_0 - \varepsilon, t_0 + \varepsilon]$ is a minimizing geodesic.

Riemannian Metric on M: A symmetric, positive definite scalar product $g_x : T_xM \times T_xM \to \mathbb{R}$, defined on each tangent space T_xM, $x \in M$, with $x \mapsto g_x$ continuous. With the collection $g = (g_x)_{x \in M}$, we say that (M, g) is a RIEMANNian Manifold.

In such a setting, the notions of gradient, length, Riemannian distance, and minimizing geodesic, make perfect sense. For instance, the length of a curve becomes

$$\int_a^b \left(g_{\gamma(t)}\left(\dot{\gamma}(t), \dot{\gamma}(t)\right)\right)^{1/2} dt$$

It is a little more delicate to generalize $(*)$, and thus to define the notion of (not necessarily minimizing) Geodesic.