1 GRADIENT FLOW IN HILBERT SPACE $\mathcal{H}$

Let $\varphi : \mathcal{H} \to \mathbb{R}$ be of class C^1. With $x_0 \in \mathcal{H}$ given, suppose we have a way to solve the ODE $\left\{ \begin{array}{l} x(0) = x_0 \\ \dot{x}(t) = -\nabla\varphi(x(t)) \end{array} \right\}$.

Then: $\frac{d}{dt}\varphi(x(t)) = \nabla\varphi(x(t)) \cdot \dot{x}(t) = -|\nabla\varphi(x(t))|^2 \leq 0$, so

- φ decreases along $x(\cdot)$
- $\frac{d}{dt}\varphi(x(t)) = 0$ holds iff the motion is at a critical point of φ.

In particular, if φ has a unique stationary point which is also a global minimizer, then one expects $x(\cdot)$ to converge to it.

HOW DOES ONE CONSTRUCT A SOLUTION? Well, one can appeal to the familiar PICARD–LINDELÖF iterations, which require smoothness. A rough-and-ready way is via numerical schemes due to EULER, to wit:

Explicit: Find $x(t+\tau)$ that satisfies $\frac{x(t+\tau) - x(t)}{\tau} = -\nabla\varphi(x(t))$

Implicit: .... $\frac{x(t+\tau) - x(t)}{\tau} = -\nabla\varphi(x(t+\tau))$.

We shall stick with the latter: set $x_0^\tau = x_0$; then, given $k \geq 0$, x_k^τ, find x_{k+1}^τ that satisfies

$$\frac{x_{k+1}^\tau - x_k^\tau}{\tau} = -\nabla\varphi(x_{k+1}^\tau), \text{ or}$$

$$\nabla_x \underbrace{\left(\frac{\|x - x_k^\tau\|^2}{2\tau} + \varphi(x) \right)}_{\psi_k^\tau(x)} \Bigg|_{x = x_{k+1}^\tau} = \frac{x_{k+1}^\tau - x_k^\tau}{\tau} + \nabla\varphi(x_{k+1}^\tau) = 0; \text{ i.e.,}$$

x_{k+1}^τ is a $\underbrace{\text{critical point}}_{\text{say, global minimizer}}$ of $\psi_k^\tau(\cdot)$.

Definition: We call a curve $x : [0,\infty) \to \mathcal{H}$ "absolutely continuous", if it is differentiable a.e., has locally integrable derivative $\dot{x}(\cdot)$, and the fundamental thm. of calculus $x(t) - x(s) = \int_s^t \dot{x}(u)\,du$, $s<t$ holds.

- Such a curve $x(\cdot)$ we call gradient flow for the convex, l.s.c. $\varphi : \mathcal{H} \to \mathbb{R}$, with initial point $x_0 \in \mathcal{H}$, if: $x(0) = x_0$

$$\dot{x}(t) \in -\partial\varphi(x(t)), \quad \text{a.e. } t>0.$$

- We construct an implicit Euler scheme for this by finding, given $\tau > 0$ and x_k^τ, a minimizer x_{k+1}^τ of $\psi_k^\tau(\cdot)$. This exists, and is actually unique.

And we set $x^\tau(t) := x_k^\tau$, for $(k-1)\tau < t \leq k\tau$.

Then, we let $\tau \downarrow 0$ and show the existence of a limit curve $\dot{x}(\cdot)$.....

Details in Ambrosio/Gigli.

Remark: Solutions of gradient flows are unique, and stable: if $x(\cdot), y(\cdot)$ start at x_0, y_0 and $\varphi \in C^1$ is convex, then

$$\frac{d}{dt}\frac{\|x(t)-y(t)\|^2}{2} = \langle x(t)-y(t), \dot{x}(t)-\dot{y}(t)\rangle = -\langle x(t)-y(t), \nabla\varphi(x(t)) - \nabla\varphi(y(t))\rangle \leq 0.$$

And if x_0, y_0 are close, then $x(\cdot), y(\cdot)$ stay close for all times.

Example: If $\mathcal{H} = \mathbb{L}^2(\mathbb{R}^d)$ and $\varphi(u) \triangleq \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x)|^2 dx$ for $u \in W^{1,2}(\mathbb{R}^d)$ (and $\varphi(u) = +\infty$ otherwise), then $\partial\varphi(u) \neq \emptyset \Leftrightarrow \Delta u \in \mathbb{L}^2(\mathbb{R}^d)$

and in this case $\partial\varphi(u) = \{-\Delta u\}$, and then the gradient flow of φ with respect to the $\mathbb{L}^2$-scalar product is the heat equation: i.e.,

$$\partial_t u(t) \in -\partial\varphi(u(t)) \Longleftrightarrow \partial_t u(t,x) = \Delta u(t,x).$$

<u>In words:</u> The Heat Equation is the $\mathbb{L}^2$-gradient flow of the Dirichlet Energy Functional
$$\varphi(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx \, ;$$
and can be solved by the implicit EULER Scheme
$$u_{k+1}^\tau \text{ is the minimizer in } \mathbb{L}^2 \text{ of } u \mapsto \frac{\|u - u_k^\tau\|^2_{\mathbb{L}^2(\mathbb{R}^d)}}{2\tau} + \varphi(u)$$
and then sending $\tau \downarrow 0$.

JORDAN–KINDERLEHRER–OTTO (1998): The Heat Equation is the Quadratic-Wasserstein-Distance flow of the Entropy $\int \rho \log(\rho)$, and can be solved by the implicit EULER Scheme
$$\rho_{k+1}^\tau \text{ is the minimizer in } \mathcal{P}(\mathbb{R}^d) \text{ of } \rho \mapsto \frac{W_2^2(\rho, \rho_k^\tau)}{2\tau} + \int_{\mathbb{R}^d} \rho \log \rho$$
and then sending $\tau \downarrow 0$.