Master's Project Defense

Infinite Levels of Complexity in a Family of One-Dimensional Singular Dynamical Systems

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Introduction

- The goal of Dynamical Systems is to develop a characterization of the behavior of all points under some map of interest
- A great amount of effort has been dedicated to studying rational maps of the plane such that $f : \mathbb{R}^2 \to \mathbb{R}^2$
- One well known rational map is the quadratic map

$$z_n \mapsto Q_c(z_n) = z_n^2 + c = z_{n+1}$$

- In this talk we will discuss the perturbed system

$$z_n \mapsto f_{c,\beta}(z_n) = z_n^2 + c + \frac{\beta}{\overline{z_n}^2} = z_{n+1}$$

Orbits and Long Term Behavior

- Suppose we want to describe the dynamics of some map $f_{\alpha_1,\alpha_2,\cdots,\alpha_n} : S \to S(\mathbb{R} \text{ or } \mathbb{R}^2)$
- More specifically, our goal is to characterize the long term behavior of the orbit of every point in the domain as the set of parameters vary

Definition

Forward Orbit of a Discrete System[1]: The forward orbit of some $x \in S$ is the set of points $x, f(x), f^2(x), \ldots$ and is denoted by $O^+(x)$.

- Thus we will attempt to determine some way of describing the limiting behavior of all points in the domain

Boundedness

- Generally this is impossible so we start with a more general dichotomy: bounded vs. unbounded
- Unbounded orbits grow arbitrarily large and limit to ∞
- Bounded orbits remain within some ball of finite radius
- There are many ways that a point may stay bounded

Fixed and Periodic Points

Definition

Fixed and Periodic Orbits[1]: The point x is a fixed point for f if $f(x) = x$. The point x is a periodic point of period n if $f^n(x) = x$. The least positive n such that $f^n(x) = x$ is called the prime period of x. The set of points in the orbit of a periodic point form a periodic orbit.

Definition

Attracting and Repelling Fixed Points[2]: Suppose x_0 is a fixed point for f where f is a one dimensional map. Then x_0 is an attracting fixed point if $|f'(x_0)| < 1$. The point x_0 is a repelling fixed point if $|f'(x_0)| > 1$. Finally, if $|f'(x_0)| = 1$, the fixed point is called (linearly)neutral or indifferent.

Critical Points

- Points may also be eventually fixed/periodic or in some cases remain bounded while densely filling an interval(such as in the case of a chaotic system)
- Another way to simplify the problem is to look at points affect the behavior of the entire system, such as critical points

Definition

Critical Point [1]: A point x is a critical point of a one dimensional map f if $f'(x) = 0$. The critical point is degenerate if $f''(x) = 0$ or non-degenerate otherwise.

Visualization Techniques

- Graphical Iteration: A method of displaying a small number of iterates of a one dimensional system in (x_n, x_{n+1}) space
- Orbit Diagram: A method of showing approximate long term behavior over a range of parameter values by plotting hundreds of points. Orbit Diagrams exist in phase$\times$parameter space.
- Escape Diagrams: A method for two dimensional systems where we have a plane of initial conditions(each represented by a pixel) such that each pixel is then colored by the relative escape "time" of that initial condition. Escape pictures can exist in phase space($x = (x_1, x_2)$) or parameter space($c = (c_1, c_2)$)

Graphical Iteration

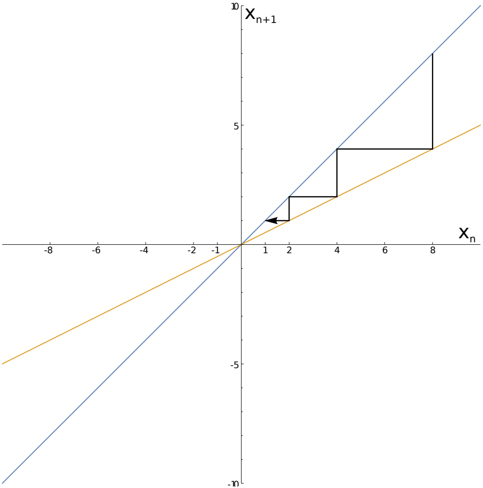


Figure: Graphical iteration of $x_0 = 8$ on $x_{n+1} = \frac{1}{2}x_n$

Orbit Diagram

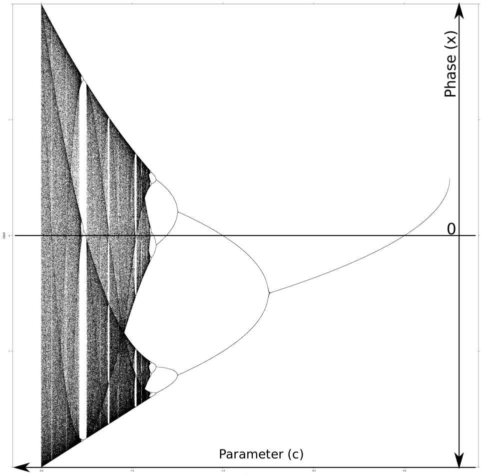


Figure: Orbit Diagram for $Q_c(x) = x^2 + c$

Orbit Diagram + Graphical Iteration

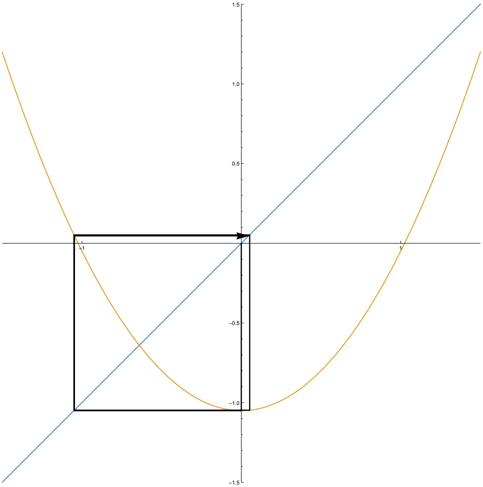


Figure: Graphical iteration of the critical point on $x_{n+1} = x_n^2 - 1.05$

Escape Picture

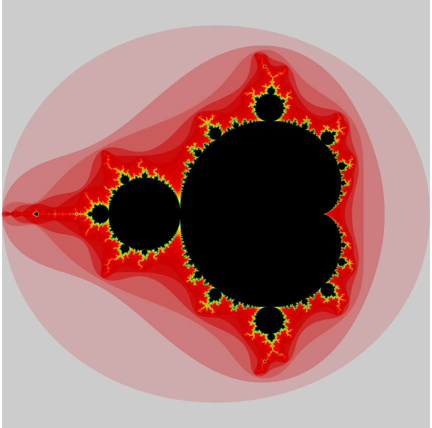


Figure: Parameter Escape Picture for $Q_c(z) = z^2 + c$

Perturbed Map

- We can now move on to a discussion our system:

$$f_{c,\beta}(x) = z^2 + c + \frac{\beta}{\overline{z}^2} \text{ for } z, c, \beta \in \mathbb{C}$$

$$f_{c,\beta}(x) = x^2 + c + \frac{\beta}{x^2} \text{ for } x, c, \beta \in \mathbb{R}$$

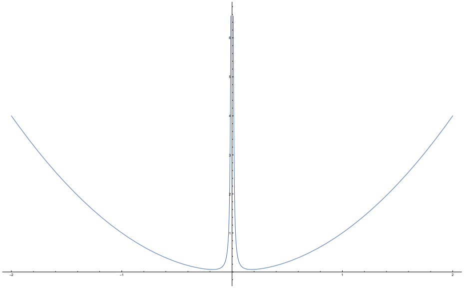
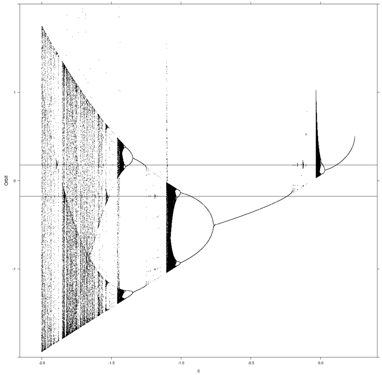
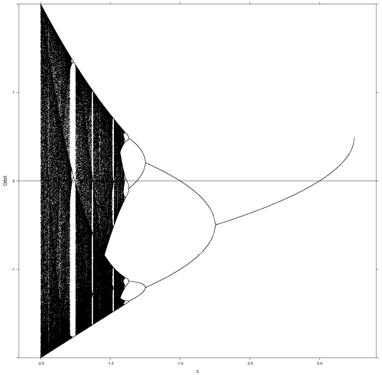


Figure: Plot of $f_{0,.001} = x^2 + \frac{.001}{x^2}$

Comparison of Orbit Diagrams



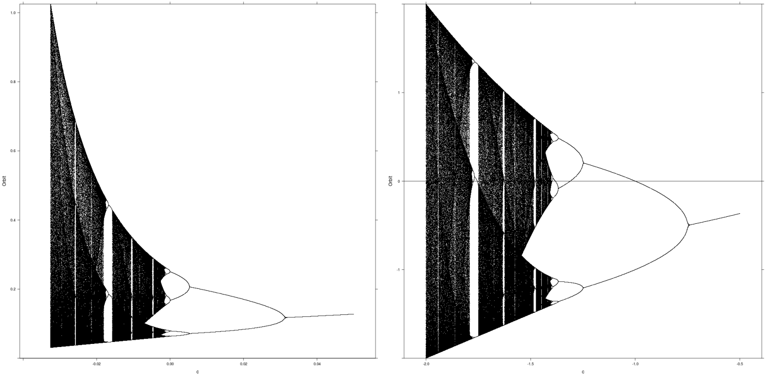
(a) Orbit diagram of $x^2 + c + \frac{.001}{x^2}$



(b) Orbit diagram of $x^2 + c$

Figure: Orbit diagrams of the original and perturbed systems

Other Interesting Intervals

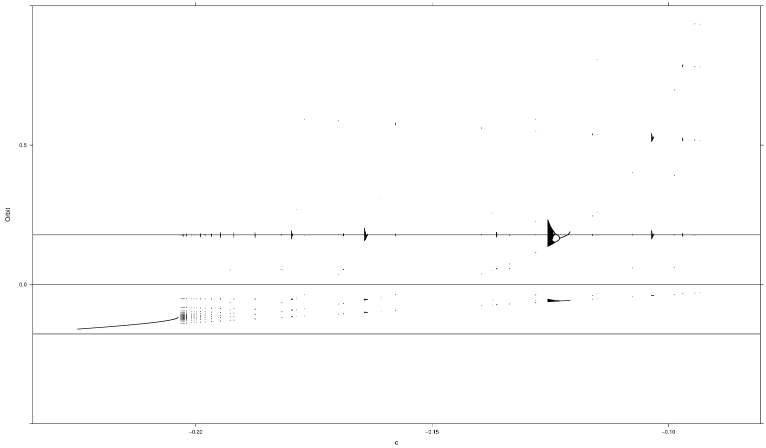


(a) Orbit Diagram for $Q_c(x)$ where $c \in (-.035, .05)$

(b) Orbit Diagram for $Q_c(x)$ where $c \in (-2, -.5)$

Figure: Orbit diagrams of the original and perturbed systems

Zoom on Important Interval



Orbit Codings

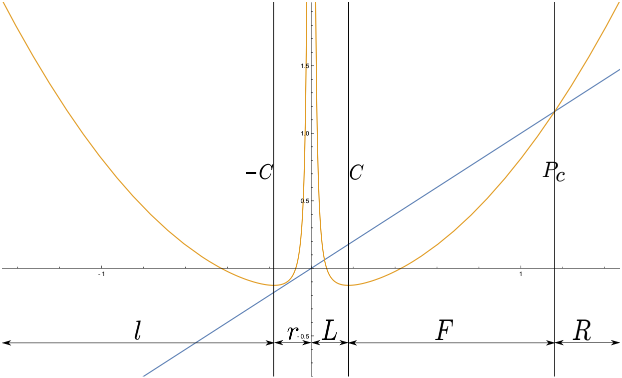


Figure: A partitioning of $\mathbb{R}$ into the intervals l, r, L, F, R

Graphical Iteration

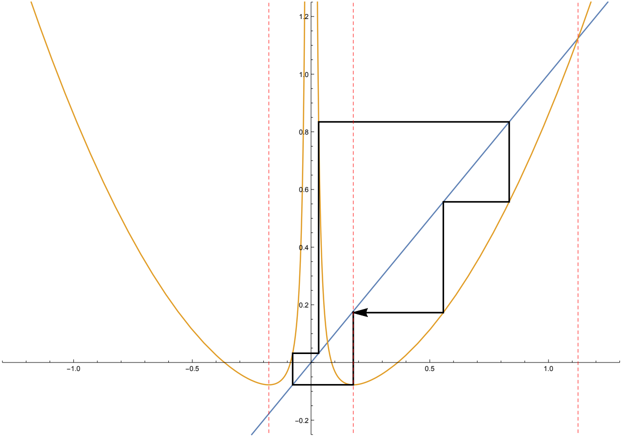


Figure: Graphical Iteration an periodic orbit with coding *CrLFFC*

Special Notation

- Let C be the positive critical point
- Let P_c be the right most fixed point(when it exists)
- Let h_n, p_n, z_n be parameter values such that $f^n_{h_n}(C) = P_c$(roughly corresponding to a homoclinic parameter value), $f^n_{p_n}(C) = C$(corresponding to a superattracting periodic orbit parameter value), and $f^n_{z_n}(C) = 0$(corresponding to a prezero parameter value)
- Let s_n be a parameter value where the n^{th} iterate goes through a saddle node bifurcation

Fixed Points

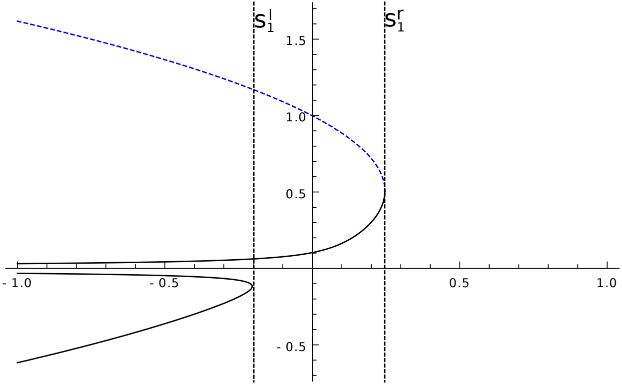


Figure: Plot of all of the fixed points of our system in parameter $\times$ phase space. The curve for P_c is in blue/dashed

Statement of Results I

Proposition

On the interval $(p_1^{-C}, h_2^{CrP_c})$, there is an accumulation of parameter values p_n and z_n which give critical orbit coding CrF^{n-1} for any integer $n \geq 2$

$$z_n < p_n < z_{n+1} < p_{n+1} < \cdots < h_2^{CrP_c}$$

The numerics suggest that the accumulation will limit to the point $h_2^{CrP_c}$.

Statement of Results II

Proposition

On the interval $(p_1^{-C}, h_2^{CrP_c})$, there is an accumulation of parameter values p_n, z_n, and h_n which give critical orbit coding Cr^{n-2} for any integer $n \geq 2$ such that

$$s_1^l < \cdots < z_{n+1} < p_{n+1} < h_{n+1} < z_n < p_n < h_n$$

The numerics suggest that the accumulation will limit to the point s_1^l.

Plot of Critical Point

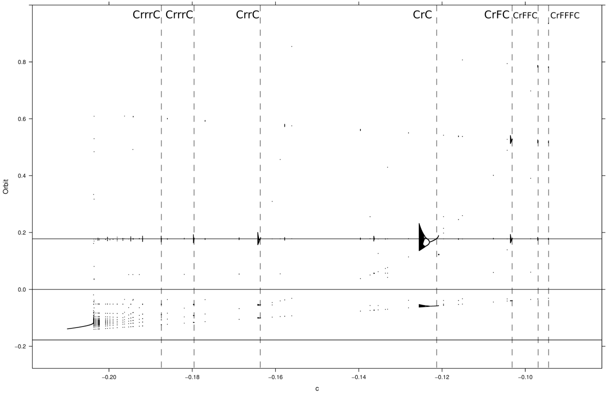


Figure: Orbit Diagram for $x^2 + c + \frac{.001}{x^2}$

Plot of Critical Point

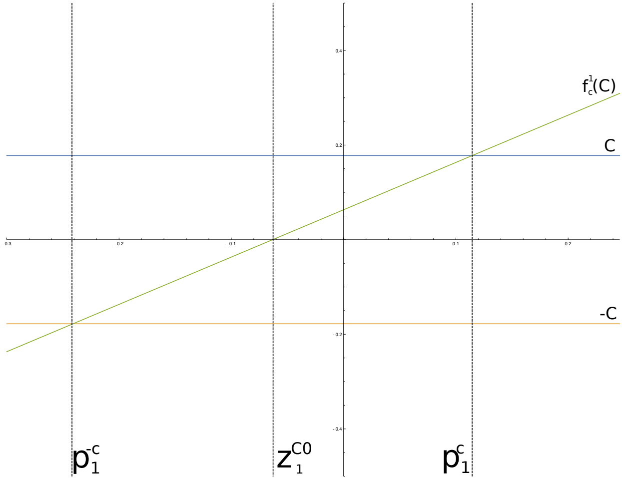
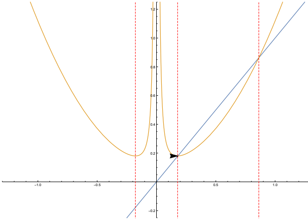
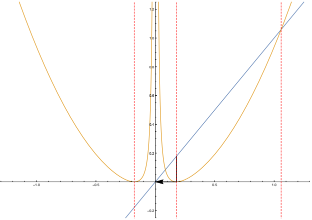


Figure: Plot of $f_c^1(C)$ for $c \in (-.3, .246)$

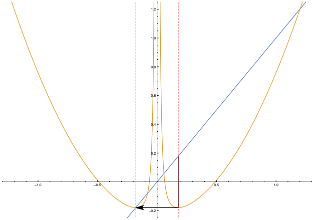
Graphical Iteration



(a) $c \approx p_1^C$



(b) $c \approx z_1^{C0}$



(c) $c \approx p_1^{-C}$

Plot of Critical Point

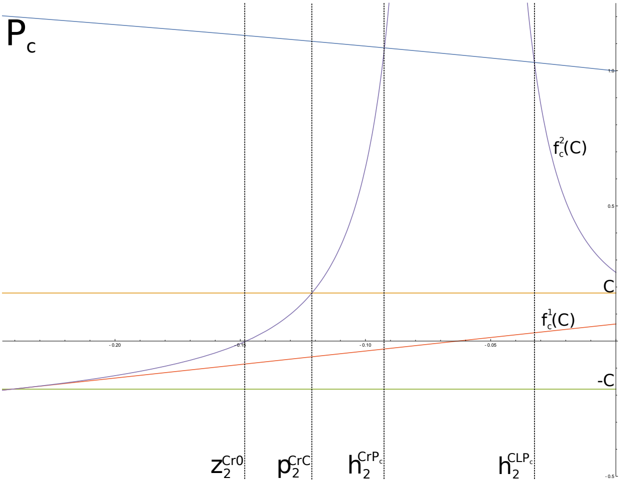
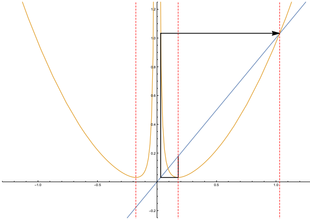
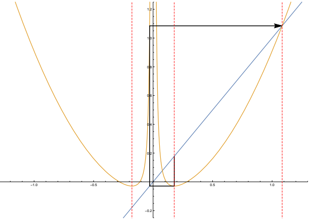


Figure: Plot of $f_c^1(C)$ and $f_c^2(C)$ for $c \in (-.245, 0)$

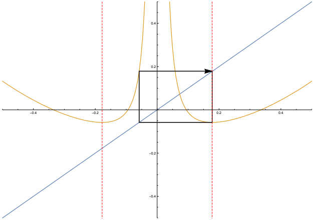
Graphical Iteration



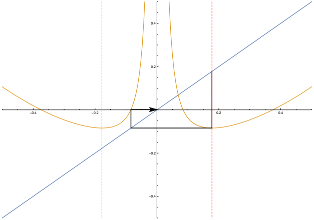
(a) $c \approx h_2^{CLP_c}$



(b) $c \approx h_2^{CrP_c}$



(c) $c \approx p_2^{CrC}$



(d) $c \approx z_2^{Cr0}$

Plot of Critical Point

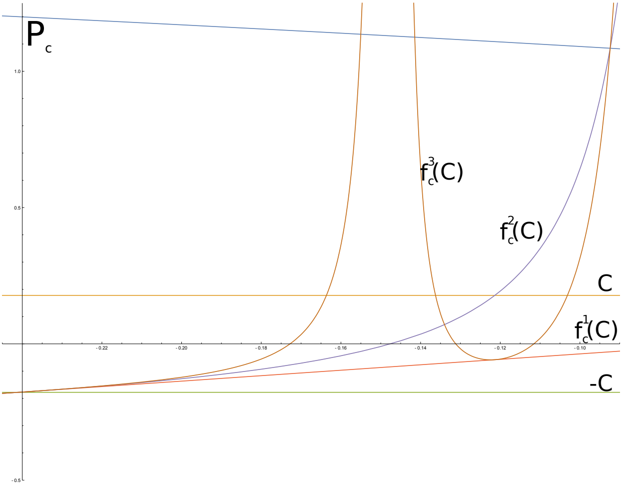


Figure: Plot of $f_c^1(C), f_c^2(C), f_c^3(C)$ for $c \in (-.245, -0.09) \approx (p_1^{-C}, h_2^{CrP_c})$

Right Hand Accumulation

Proposition

On the interval $(p_1^{-C}, h_2^{CrP_c})$*, there is an accumulation of parameter values* p_n *and* z_n *which give critical orbit coding* Cr^{n-2} *for any integer* $n \geq 2$

$$z_n < p_n < z_{n+1} < p_{n+1}$$

The numerics suggest that the accumulation will limit to the point $h_2^{CrP_c}$.

Plot of Critical Point

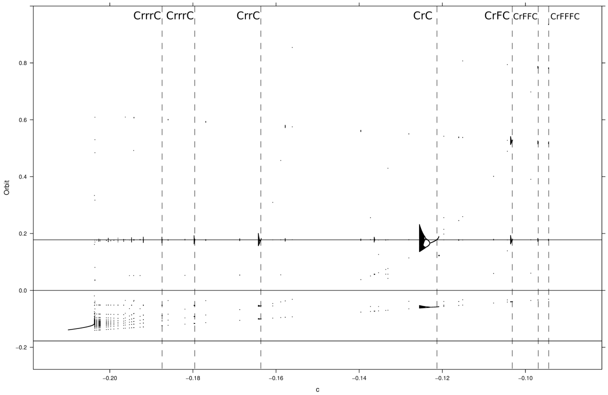
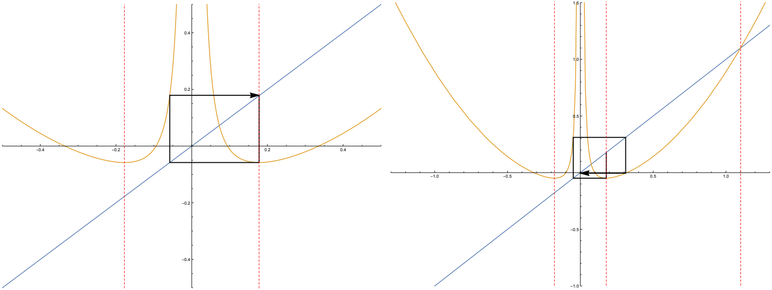


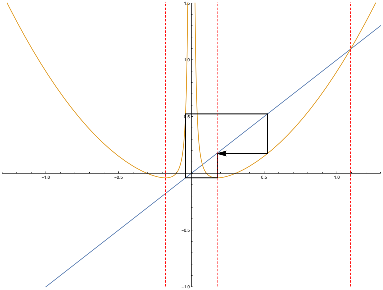
Figure: Orbit Diagram for $x^2 + c + \frac{.001}{x^2}$

Graphical Iteration I

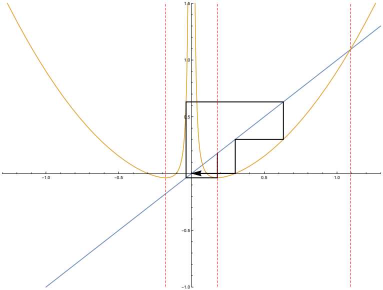


(a) $c \approx -.12125 \approx p_2^{Cr}$ (b) $c \approx -.112 \approx z_3^{CrF0}$

Graphical Iteration II

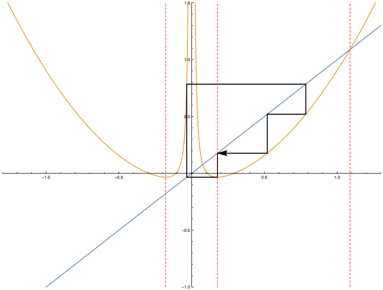


(c) $c \approx -.10325 \approx p_3^{CrF}$

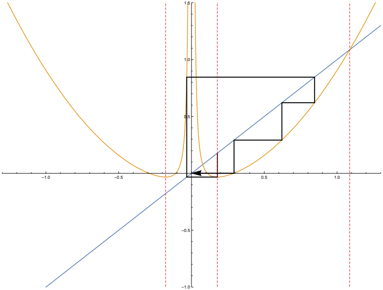


(d) $c \approx -.10025 \approx z_4^{CrFF0}$

Graphical Iteration III

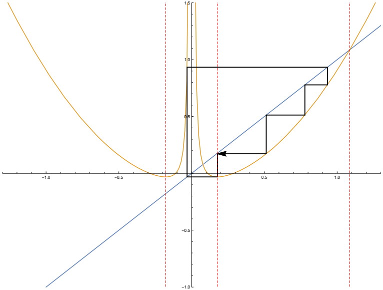


(e) $c \approx -.09695 \approx p_4^{CrFF}$

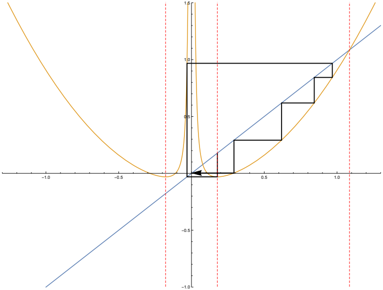


(f) $c \approx -.09585 \approx z_5^{CrFFF0}$

Graphical Iteration IV

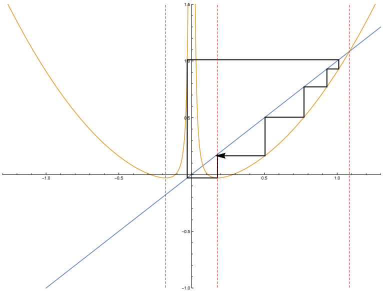


(g) $c \approx p_5^{CrFFF}$

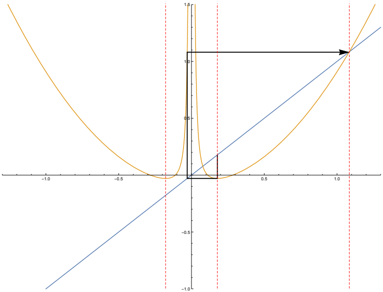


(h) $c \approx z_6^{CrFFFF0}$

Graphical Iteration V

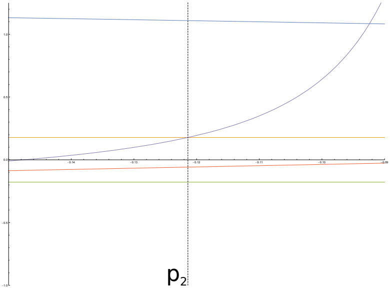


(i) $c \approx p_6^{CrFFFF}$

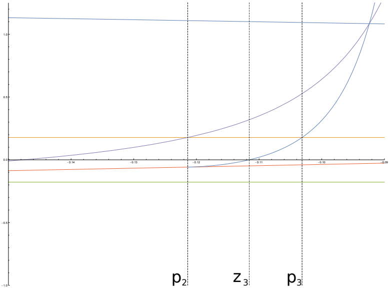


(j) $c \approx h_2^{CrP_c}$

Right Side Accumulation I

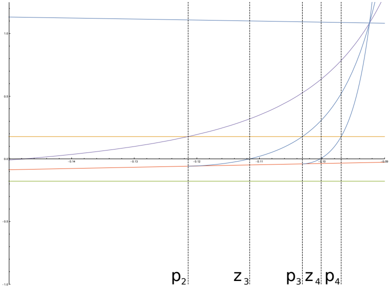


(a) $n = 2$

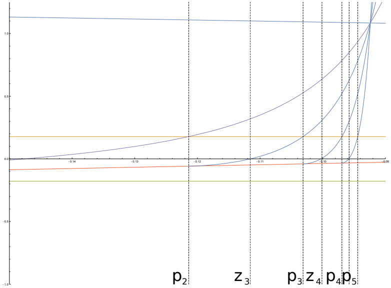


(b) $n = 3$

Right Side Accumulation II

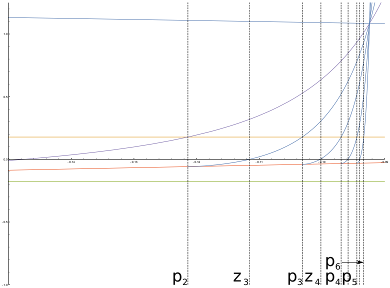


(c) $n = 4$

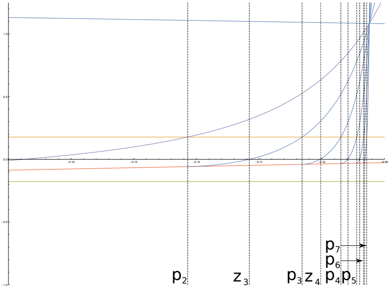


(d) $n = 5$

Right Side Accumulation III



(e) $n = 6$



(f) $n = 7$

An Infinite Hierarchy

Proposition

Suppose we have two distinct parameter values $z_{n_1}^{C\alpha 0}, z_{n_2}^{C\beta 0} \in (p_1^{-C}, h_2^{CrP_c})$ *where* α *and* β *are coding sequences such that* $\alpha_i \neq \beta_i$ *for at least one* i*. Then there must be at least two other prezero parameter values* $z_{n_3}^{\gamma}$ *and* $z_{n_4}^{\delta}$ *on the interval* $int\{z_{n_1}^{C\alpha 0}, z_{n_2}^{C\beta 0}\}$ *such that* $\gamma_i \neq \delta_i$ *for at least one* i*.*

Plot of Critical Point

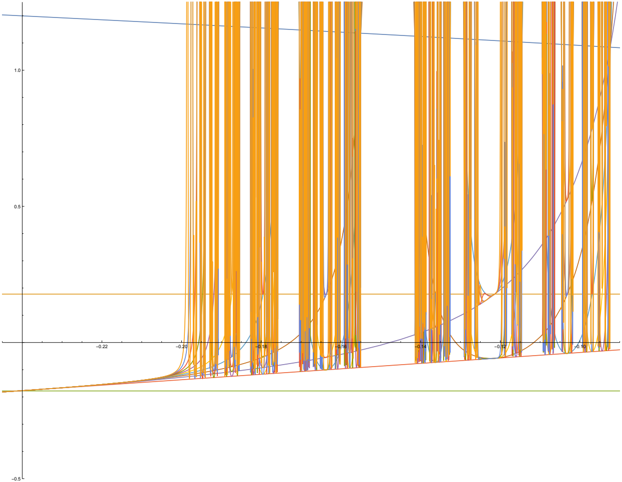
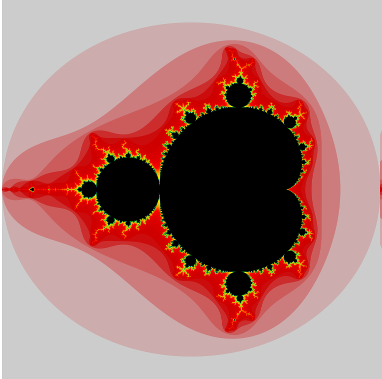
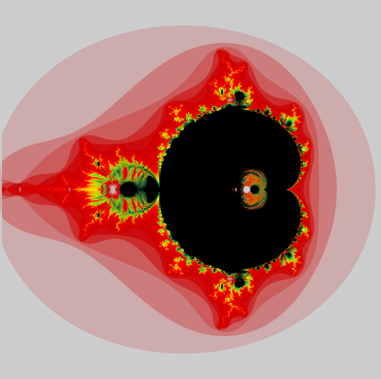


Figure: Plot of first 10 iterates of C for $c \in (-.245, -0.09) \approx (p_1^{-C}, h_2^{CrP_c})$

Back to the Plane



(a) Parameter Space Escape Picture for $z^2 + c$



(b) Parameter Space Escape Picture for $z^2 + c + \frac{.001}{z^2}$

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