RESEARCH ARTICLE | MARCH 01 1995

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Comput. Phys. 9, 164–169 (1995)
https://doi.org/10.1063/1.168541

SIMPLE MODELS OF SELF-ORGANIZED CRITICALITY

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Systems with correlation or distribution functions that decay as power laws are said to be scale invariant. This terminology reflects the familiar fact that power laws look the same on all scales. For example, the replacement $x \rightarrow ax$ in the function $f(x)=Ax^{-\eta}$ yields, for any value of the exponent η, a function $g(x)$ that is indistinguishable from $f(x)$, except for a change in the amplitude A by the factor $a^{-\eta}$. This invariance does not hold for functions that decay exponentially, since making the same replacement in the function $e^{-x/\xi}$ changes the correlation length ξ (the characteristic scale for the decay) by the factor a.

An enormous variety of systems in physics, chemistry, and biology (and very likely in economics, urban development, and other aspects of societal organization) seem to exhibit scale invariance[1] in some form or another.[2] Examples include the probability distribution of earthquake magnitudes, which is found empirically to decay according to a ("Gutenberg–Richter") power law;[3] many objects in nature (one illustration being the human lung), whose structures are found to be scale invariant or fractal[4] over a significant range of sizes; the temporal fluctuations in financial markets, whose power spectra have been argued (though not undisputedly),[1,5] to grow as a power of frequency for small frequencies, that is, to show $1/f$ noise;[6,7] and fully developed turbulence, which is characterized by the power-law growth of the velocity structure function,[8] $\langle|\mathbf{u}(\mathbf{r})-\mathbf{u}(\mathbf{0})|^2\rangle \sim |\mathbf{r}|^{\zeta}$, with $\zeta \approx 2/3$.

The origin of the ubiquitous occurrence of scale invariance in systems with short-ranged interactions remains largely mysterious. This is in part because the systems that have been most studied and are best understood theoretically are those in thermodynamic equilibrium. Straightforward arguments[9] show that in the absence of special symmetries[10] or long-range interactions,[11] scale invariance in equilibrium systems can occur only at isolated critical points or surfaces in parameter space; for generic choices of parameters, exponential behavior will obtain. Being loathe to accept the existence of invisible gremlins traveling the universe and tuning parameters to their critical values, one must therefore seek an explanation for the widespread occurrence of scale invariance in the realm of *nonequilibrium* systems—those driven externally and thus prevented from achieving thermodynamic equilibrium.

The general arguments prohibiting generic scale invariance in equilibrium do not carry over to the nonequilibrium domain. Simple nonequilibrium examples where scale invariance occurs without the parameter tuning required in equilibrium are known.[12] For example, the correlations in many nonequilibrium systems with a local conservation law have been shown[13,9] theoretically to decay algebraically, essentially because fluctuations in conserved quantities can only disappear via diffusive processes, which are of course described by power laws. For fluids in external temperature gradients, such algebraic correlations have been observed experimentally.[14] Although the set of nonequilibrium models known to exhibit power laws still seems too narrow to account for all the scale invariance in nature, one can reasonably hope that further examples of scale-invariant behavior will emerge from the exploration of a richer variety of nonequilibrium systems.

A few years ago, Bak, Tang, and Wiesenfeld[15] made this vague hope more concrete by proposing a new class of nonequilibrium models with short-range interactions that they argued produce scale-invariant states without parameter tuning. They called this phenomenon "self-organized critically" (SOC). The novel element of the models is the extreme rapidity with which they respond to external perturbations such as extrinsic random noise kicks. Their response times τ_R, are far shorter than the typical time scales τ_P on which they are driven or perturbed externally. Indeed, SOC occurs only in the limit where the ratio $\kappa = \tau_R/\tau_P$ is strictly zero. This limit clearly distinguishes systems that exhibit SOC from conventional nonequilibrium systems, where κ is always greater than zero and is often much bigger than unity.[16]

The extreme separation of time scales embodied by $\kappa=0$ captures an essential aspect of many natural phenomena. Perhaps the most graphic illustration of this point occurs in the seismic zone near a fault, where the slow relative motion of tectonic plates may take years or even decades to

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produce an instability that results in an earthquake. Once the earthquake begins, however, it can terminate in tens of seconds, making the effective κ extremely small.

The basic premise of SOC is that the limit $\kappa=0$ can produce scale invariance even when there is none for $\kappa>0$. More precisely, the finite correlation length ξ, beyond which correlation or distribution functions cease to be algebraic and begin to fall off exponentially, diverges as $\kappa \to 0$ in systems exhibiting SOC. For example, correlations might decay like $x^{-\eta}e^{-x/\xi}$ for some exponent η, where $\xi \to \infty$ as $\kappa \to 0$.

The rest of this column is devoted to examining some of the evidence supporting the SOC hypothesis in three simple models and the extent to which computer simulations constitute a reliable component of this evidence. We consider only nonconserving models, because as mentioned above, scale invariance in locally conserving nonequilibrium systems is common, well understood, and does not require the $\kappa=0$ limit. Nonconserving cases, where power laws do not occur without a special symmetry or parameter tuning in conventional systems with $\kappa>0$, are therefore more exciting.

The toy models we discuss are typical of those in the growing SOC literature.[17] Each model consists of bounded, nonnegative, real variables at every site of a regular lattice—say a square lattice, for definiteness. Each is perturbed or "driven" (meaning that the variables are changed in time) slowly, until a stability threshold is exceeded at some site. The system then experiences an "avalanche" or relaxation event, that restores stability, that is, reduces the values of all the variables below threshold, whereupon a new instability is generated by the slow driving, and so on. Every avalanche is assumed to occur with infinite speed relative to the rate of driving, thereby ensuring that the defining characteristic, $\kappa=0$, of SOC holds. In practice, this means that the identity of each avalanche is well defined. If κ were greater than zero, then the driving might produce a second avalanche before the first had terminated, and avalanches might collide or otherwise interfere. Such a situation is typical of conventional nonequilibrium systems.

The behavior of conventional systems is usually monitored by means of correlation functions. In systems exhibiting SOC, however, the uniqueness of individual avalanches implies that one also can keep statistics for the sizes s and durations T of avalanches. The occurrence of scale invariance in such systems manifests itself most conveniently as the power-law decay of the probability-distribution functions, $P(s)$, for avalanches of size s.[18] Of course correlation functions remain a legitimate diagnostic of scale invariance even in the SOC limit (see problem 1).

We choose three models to span the range of possibilities: one (denoted FF1) does not, ultimately, exhibit SOC, promising early indictions notwithstanding; the second (FF2) clearly does; and the third (SSM) remains somewhat unclear, though it shows strong numerical evidence of SOC. FF1 and FF2 were proposed as schematic models for the spread of forest fires (or disease), and SSM was constructed to represent earthquakes or, more generally, the stick-slip dynamics of two frictional surfaces moving slowly over each other (see below).

FF1, a coupled-map model,[19] consists of variables $u_i(n)$ that assume real values between 0 and 4; $u_i(n)$ denotes the height of the tree on site i of an $L\times L$ lattice at discrete time n. Periodic boundary conditions are employed. The trees grow at a low constant rate, p. Trees with heights $u\geqslant 2$ catch fire spontaneously and burn down or "topple," that is, reduce their height by 2, in a single time step. The fire spreads in one step from any burning tree to all its nearest neighbors j for which $1\leqslant u_j<2$; this spreading capability allows the propagation of large fires.

Initial conditions with the $u_i(0)$ distributed randomly between 0 and 2 are used. Otherwise the model is completely deterministic. Several variants of the algorithm for the spreading of fires have been investigated, all with synchronous updating. We now summarize the original set of rules.

(1) Burning: If $u_i(n)\geqslant 2$, then $u_i(n+1)=u_i(n)-2$.
(2) Spreading: If $1\leqslant u_i(n)<2$ and $u_j(n)\geqslant 2$, where j is a nearest neighbor of i, then $u_i(n+1)=2u_i(n)$.
(3) Growth: If $u_i(n)$ satisfies neither (1) nor (2), then $u_i(n+1)=u_i(n)+p$.

Note that only in the limit $p=0$ do "relaxation" (that is, burning) processes proceed infinitely fast relative to the growth process. In this limit, no growth occurs while fires are actually burning, so that the infinite separation of time scales intrinsic to SOC is incorporated. We shall see in our study of the SSM that there is a way to perform calculations directly at $p=0$. For now we take the historical route of approaching $p=0$ by considering progressively smaller values of p.

To achieve a statistical steady state of FF1 requires many iterations of the algorithm above. Indeed, the required transient time grows as $1/p$ with decreasing p.[20] Hence as $p\to 0$, it becomes progressively more costly even to achieve a steady state, let alone to acquire reliable data on correlation or distribution functions. This constraint limited the range of p over which early calculations[19] could produce trustworthy results to approximately $p>0.01$.

For $p>0$, the steady state achieved by the system has a constant, nonzero density of burning trees. Individual forest fires are therefore not well defined, and correlation functions must be used as a diagnostic for scale invariance. Chen *et al.*[19] observed in simulations that fires form strikingly one-dimensional fronts (see Fig. 1), burning through groups of "mature" (large) trees on the lattice and leaving "burned" (small) ones in their wake. Correspondingly, they found that the fractal dimension[4] D of the set of fires is very nearly 1 for distances x less than a typical correlation length ξ, which seems to grow as $1/p$ as p decreases to the numerical limit of $p\sim 0.01$. For $x>\xi$, D was found to be 2, as required by the nonvanishing density of fires. These numerical findings strongly suggested a divergent ξ and concomitant scale invariance in the $p=0$ limit for infinite systems.

However, subsequent larger-scale computations[21] down

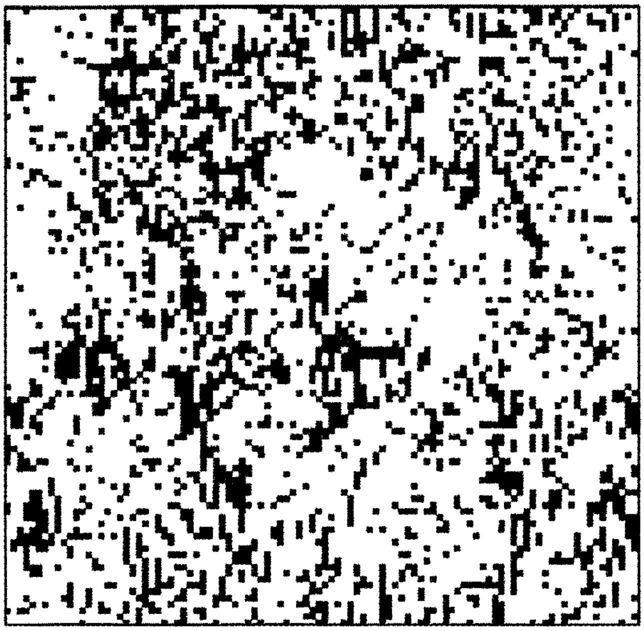

Figure 1. An instantaneous configuration of the FF1 model with $p=0.01$ and $L=100$. Small trees ($u_i<1$), large trees ($1\leq u_i<2$), and burning trees ($u_i\geq 2$) are shown as white, shades of green, and red, respectively.

to $p=0.0001$ indicate that the increase of ξ with decreasing p ceases around $p=0.003$; ξ actually *decreases* with further reductions of p (see Fig. 2 and Ref. 22), suggesting that the system is not in fact becoming scale invariant at $p=0$. Calculations of the probability distribution of forest-fire sizes (see Problem 1) directly at $p=0$ confirm this conclusion. Hence FF1 is apparently not scale invariant. There is not yet a satisfactory answer to the subtle question of why ξ is as large as 50–100 for certain values of p, though the long correlation lengths may have to do with incipient temporally periodic oscillations of the average tree height.[21] In any event, the behavior of FF1 teaches us to be alert to the possibility of large, unanticipated correlation lengths that can produce a false impression of scale invariance in numerical (and, presumably, real) experiments.

We now consider FF2, also conceived[23] as a forest-fire model, though a probabilistic one. Here the $u_i(n)$ assume only two values, 0 (holes) and 1(trees). The discrete-time, synchronously updated version of the model we discuss here is therefore a cellular automaton. Again, periodic boundary conditions and random initial conditions are used. The rules are simple. At each time step, every hole grows into a tree with (small) probability p, while every tree catches fire and burns ("is hit by lightning") with (smaller) probability f. The fire produced by each lightning strike spreads instantaneously through the entire cluster of trees connected to the stricken tree by nearest neighbor bonds. The cluster is thereby reduced to holes in a single time step. We now summarize the algorithm.

(1) Growth: If $u_i(n)=0$, then $u_i(n+1)=1$ with probability p.

(2) Lightning: If $u_i(n)=1$, then $u_i(n+1)=0$ with probability f.

(3) Spreading: A tree on site j at time n, that is, $u_j(n)=1$, which is not struck by lightning as in rule (2), but belongs to a cluster of trees connected through nearest neighbor bonds to a tree that *is* stricken at time n, burns down, that is, $u_j(n+1)=0$, with probability one.

Note that in FF2 the instantaneous spreading of fire through a cluster ensures the infinite separation of time scales required by the definition of SOC. Each connected cluster of s trees that burns down constitutes a single fire of size s. Therefore FF2 has many independent fires (separate burning clusters) at any given time.

The balance between the average rate of growth and burning of trees in steady state suggests a value for the free parameter f/p at which this model is likely to be scale invariant. If the average steady-state density of trees is ρ, then at each time step the average number of new trees appearing is $pN(1-\rho)$, while for small f the average number destroyed is $fN\rho\langle s\rangle$. Here N is the total number of sites, and $\langle s\rangle$ is the average number of trees in a cluster. Equating these two rates yields $\langle s\rangle\sim[(1-\rho)/\rho](p/f)$. Since $0<\rho<1$, it follows that $\langle s\rangle\rightarrow\infty$ in the limit $f/p\rightarrow 0$. From the relation $\langle s\rangle=\Sigma_{s=1}^{\infty}sP(s)$, one infers that in this limit the distribution function $P(s)$ must decay more slowly than exponentially with s. One therefore expects that $P(s)\sim s^{-\tau}$, with $\tau<2$ as required by the divergence of $\langle s\rangle$. This expectation has been demonstrated convincingly to hold in extensive numerical simulations[23] of the model. There are, moreover, exact results[24] confirming the occurrence of power laws in the one-dimensional version of the model.

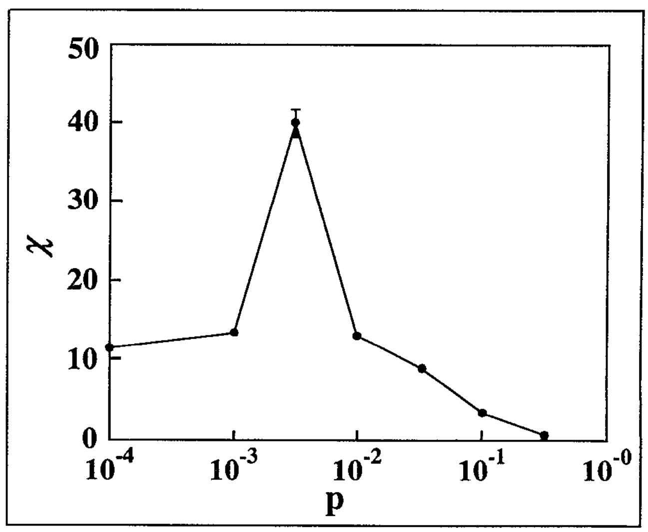

Figure 2. The susceptibility χ vs p for the FF1 model. The points indicate saturation values of χ at large L. The value at $p=0.003$ is somewhat uncertain, as indicated, since clear saturation did not occur for L up to 128, and extrapolation of data from smaller L was necessary. Lines are guides to the eye. This figure is the same as Fig. 7 of Ref. 21.

The combination of analytic arguments and numerics provides a compelling case for SOC in FF2. Note, however, the *two* infinitely separated sets of time scales are required to produce the power laws: The rate of lightning strikes is infinitesimally small compared to the growth rate of trees $(f/p \to 0)$, which in turn is infinitesimally small relative to the instantaneous spreading of fire through a connected cluster. The relaxation of either of these two infinite separations produces scale invariance only out to some finite cutoff size,[23] even for $L=\infty$. Thus power laws occur only in a more restricted set of circumstances than envisaged in the original conception of SOC. Nonetheless, FF2 is clearly a success.

Finally, we consider the stick-slip model SSM,[25] which shows great promise of producing scale invariance with only a single infinite separation of time scales. This deterministic, synchronously updated, coupled-map model seeks to capture the essence of earthquake dynamics, wherein the relative motion of two tectonic plates at a fault produces slowly but steadily increasing stresses at various points (imagined to lie on a regular lattice) along the fault. The stresses $u_i(n)$ are nonnegative real numbers. When the stress at any point exceeds a threshold value (arbitrarily taken as 4), that point is imagined to slip, reducing its stress to zero, but transferring some of the lost stress to nearest neighbor points, whose own stresses may consequently be increased above threshold, causing them to slip and transfer stress. The resulting cascade of slips is an earthquake, the size of which is measured by the total number of slips s.

Random initial conditions with $0 \leq u_i(0) < 4$ for all i are typically used in the SSM. It is important, however (see problem 2), that *open* rather than periodic boundary conditions be used. We now summarize the algorithm, which is similar in spirit to FF1 in the strict $p=0$ limit, where the condition $\kappa=0$ is fulfilled explicitly.

(1) Stick dynamics: If $u_i(n)<4$ for *all* i, identify the site, $i=I$, for which $u_i(n)$ achieves its maximum value. Then set $u_i(n+1)=u_i(n)+4-u_I(n)$ for each i, thereby making site I unstable and precipitating an earthquake.
(2) Slip ("earthquake") dynamics: For each i with $u_i(n) \geq 4$, set $u_i(n+1)=0$, and set $u_j(n+1)=u_j(n)+\alpha u_i(n)$ for each nearest neighbor j of i with $u_j(n)<4$. If a given j has more than one nearest neighbor i, with $u_i(n) \geq 4$, then j receives transferred stress from *each* of them at time $n+1$.

At present, the SSM is more like FF1 than FF2, in that one is largely at the mercy of the computer.

No purely analytic argument to explain the occurrence of scale invariance exists.[26] However, the numerical results for $P(s)$, obtained with values of the free parameter in the range $0.05<\alpha<0.25$, do look very encouraging. For $\alpha=0.2$ and square lattices of linear size L up to 200, the data indicate that $P(s) \sim s^{-(B+1)}$, with $B \approx 0.9$.[27] Interestingly, the exponent B seems to vary continuously with the parameter α. (Note that for $\alpha=0.25$, the redistribution rule for slips conserves the total stress in the system. As mentioned above, there are sound arguments for expecting scale invariance in conserving cases.)

In the absence of a satisfactory explanation for the power laws observed numerically in the SSM with $\alpha<0.25$, it is probably wise to be cautious about declaring it scale invariant. It is possible that there is a finite correlation length (larger than the one that eventually manifested itself in FF1), whose origins we simply do not understand at present.[28] There also is some question as to how generic the SSM is. It has been argued,[21] that the model contains a parameter γ that has been implicitly tuned to unity, thereby placing the SSM on the very edge of chaotic behavior, that is, giving it a maximum Lyapunov exponent[29] of zero. The parameter is introduced by generalizing the stress-transfer condition, $u_j(n+1)=u_j(n)+\alpha u_i(n)$, in rule 2 of the above algorithm to $u_j(n+1)=\gamma u_j(n)+\alpha u_i(n)$. It is conceivable that the SSM is scale invariant only for the special value $\gamma=1$.

Should it transpire that the SSM is indeed scale invariant, the mechanism responsible[26] for the algebraic decays is likely to be one with which we have no prior familiarity. Potentially, therefore, this mechanism is an important piece of the puzzle posed by the ubiquitous occurrence of scale invariance in our universe.

Understanding simple models such as those described here is obviously just one step in the complex task of elucidating the origins of this scale invariance. Before more realistic descriptions of scale-invariant phenomena are constructed and tested, one cannot reasonably claim to have unraveled the mystery. It is of course also possible that very different classes of nonequilibrium systems capable of producing scale invariance with minimal parameter tuning remain to be discovered.

Suggestions for further study

1. Write a computer program to investigate the probability distribution, $P(s)$, of forest-fire sizes s in the FF1 model in the $p=0$ limit. The strategy for working right at $p=0$ involves destabilizing stable states of the system by increasing all the variables uniformly until the largest one reaches the instability threshold and catches fire. The general method is described above, in the context of the SSM.

Start the system from random initial conditions, and let it evolve until it achieves steady state. (Sadly, there is no automatic prescription for inferring the waiting times required to emerge from the transient regime and into steady state. In general, one must monitor quantities of interest after one thinks steady state has been achieved, and check that waiting longer does not alter the results. For scale-invariant systems, one usually expects transient times to grow algebraically with the linear system size.) Once steady state is reached, measure s for each forest fire. Recall that s is the total number of trees that burn in a given fire; if a particular tree burns m times, then it contributes m to the total. Obtain enough statistics to develop a reliable estimate of the function $P(s)$. [Again there is no magic formula for divining when your statistics are adequate. You must simply increase the running times to verify that $P(s)$ does not change significantly. Luckily, the computer performs most of this drudgery, not you.] Make log–log and log–linear plots of $P(s)$ versus s to see whether $P(s)$ falls off algebraically or exponentially. You will soon encounter some of the great "joys" of numerical work: It takes a long time to accumulate acceptable statistics, even for modest system sizes such as $L=32$; and once you do, it is difficult to tell what the best fit is, in part because the rapid decay of $P(s)$ at large s due to the system's finite size is difficult to distinguish from a truly finite correlation length. Only by studying systems of progressively larger size, for which the power-law regime should extend to progressively larger s values if the model is indeed scale invariant, can one infer the true large-s behavior of $P(s)$ with reasonable confidence. Unfortunately, the analysis requires looking at values of L of up to 100 or more, which requires a powerful computer or a *very* long wait.

Given your data, can you decide objectively whether FF1 is scale invariant? Is your conclusion consistent with that of Ref. 21 and the above discussion?

2. Compute the distribution function, $P(s)$, for avalanche sizes s in the SSM as described in the text, but with periodic boundary conditions. Use small L values in the range 4 to 16. Is there a difference between your results and those obtained with open boundaries? Is the system with periodic boundary conditions scale invariant? Can you understand the state that this system finds and that is responsible for the observed $P(s)$. (Running a very small system such as $L=4$ and monitoring the average variable value for 100 or so time steps should provide a strong hint.) Does the system with open boundaries find the same kinds of states? Your conclusions can be compared with those of Refs. 21, 26, and 27.

3. In models like the SSM, for which understanding is limited and one is to a large extent at the mercy of simulations, efficient algorithms that increase the size of the systems one can handle numerically are obviously valuable. Models such as the SSM and FF1 often have rather flagrant potential sources of inefficiency. For example, the site that first topples to begin an avalanche is the one with the highest value of u_i. There are more efficient ways of locating this site than performing a global search each time. As a second example, only the sites neighboring a site that topples at time n can possibly topple at time $n+1$. An efficient way of keeping track of the sites ripe for toppling at any given time is therefore a great advantage. By dealing with such issues cleverly, Grassberger[27] has constructed an algorithm based on linked lists and pointers that allows the collection of considerably better statistics than brute force approaches. Compare the speed of this algorithm with that of the one you concocted for problem 2.

Acknowledgments

We are grateful for the many insights provided by our collaborator Joshua Socolar.

From the editors. Please send us your comments, suggestions, and manuscripts for submission to this column to `jant@kzoo.edu` or `hgould@clarku.edu`.

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