Physics 174 - Spring 2021: Homework 1

Due: Monday, March 29, 2:30pm

Question 1: A familiar experience from swimming at finite Reynolds' numbers is the ability to coast. We can push off from the side of the pool and travel a finite distance without having to expend additional energy. Here we will explore what "coasting" is like at very low Reynolds' number.

In class we wrote the Navier-Stokes equation as:

$$\rho\vec{g} - \vec{\nabla}P + \text{ add'n driving terms } = \rho(\vec{u}\cdot\vec{\nabla})\vec{u} + \eta\nabla^2\vec{u}$$

The Navier-Stokes equation governs the motion of fluids and through Newton's 3rd law this also governs the motion of swimmers in the fluid. The organisms we are interested in consist primarily of water and are swimming in water so $\rho\vec{g}$ is largely cancelled by buoyancy.

In fluid mechanics $\vec{\nabla}P$ is often written as a driving force as would be the case for studying pressure driven flow through pipes or channels. However, when we talk about an organism swimming, the driving force comes from the organism's self-propulsion and $\vec{\nabla}P$ and $\eta\nabla^2\vec{u}$ together describe the reactionary/drag force exerted by the liquid on the organism where the pressure term describes the normal force per unit area from the fluid on the organism and the viscous term describes the viscous shear stress or tangential force.

For flow at low Reynold's number, we can get a pretty good estimate of $\vec{F}_{drag}$ from dimensional analysis. At low Re the inertial term is not significant, meaning that ρ does not play a significant role in the equation of motion. The physics of the problem is therefore captured by our characteristic velocity, u; our characteristic length scale, L; and the viscosity, η.

a) using dimensional analysis, what is the drag force (defined up to a constant)

$u : m/s$

$L : m$

$\eta : [Pa \cdot s] = [N s/m^2] = [Kg\, m^{-1} s^{-1}]$

$F_D : [\eta u L] = [Kg\, m^{-1} s^{-1} \cdot m s^{-1} \cdot m] = [N]$

b) For a bacterium moving through water what would you use for the characteristic length scale and velocity. Numbers are not expected, just a connection between the equations and the motion/physical properties of the bacterium (which you can assume is approximately spherical)

Characteristic Length: bacterium's diameter

Characteristic Velocity: baterium's speed

* The a in Stokes' formula is the radius not the diameter. However, for this order of magnitude estimate it doesn't matter, bacteria are ~1-2 μm so 1 μm is a good estimate for a either way.

Question 2: The drag force for a sphere at low Reynolds' number was calculated by Stokes and is given by $F_D = -6\pi\eta a\vec{v}$, where a is the diameter / radius of the sphere and $\vec{v}$ is the velocity of the sphere relative to the fluid. You can find a derivation of Stokes formula for the drag force here:
https://www.math.nyu.edu/faculty/childres/chpseven.PDF
Bacteria have flagella that they use to propel themselves. Imagine a bacterium is moving at velocity, $\vec{u}$ when it stops propelling itself.

Approximately how far can the bacterium coast (travel without propulsion)? You can use the approximation we made in class that u 10 BL/sec (body lengths per second) and approximate the bacterium as a sphere of 1μm diameter. (Hint: you will have to reintroduce the inertial term into the equation of motion to answer this question.)

The only force acting on the bacterium is the drag force, F_D

$$m\frac{d\vec{u}}{dt} = \vec{F}_D = -6\pi\eta a\vec{u}$$

$$m\frac{du}{dx}\cancel{\frac{dx}{dt}} = -6\pi\eta a\cancel{\frac{dx}{dt}}$$

$$m\,du = -6\pi\eta a\,dx$$

$$m\int_{u_0}^{0} du = -6\pi\eta a\int_{x_0}^{x_f} dx$$

$$m u_0 = 6\pi\eta a\Delta x$$

$$\Delta x = \frac{m u_0}{6\pi\eta a} = \frac{\frac{4}{3}\pi a^2 \rho_B u_0}{6\pi\eta}$$

$$a \sim 1\,\mu m$$

$$u_0 \sim 10\,BL/s \sim 10\,\mu m/s$$

$$\rho_B \sim \rho_{H_2O} \quad \text{so} \quad \frac{\eta}{\rho_B} \sim \frac{\eta}{\rho_{H_2O}} = \nu$$

Kinematic viscosity

$$\nu_{H_2O} \sim 1\,cSt = 10^{-6}\,m^2/s$$

centiStoke

$$\therefore \Delta x \sim \frac{2}{9}\,\frac{(1\times10^{-6}m)^2\,(10^{-5}\,m/s)}{10^{-6}\,m^2/s}$$

$$\Delta x \sim \frac{2}{9} \cdot 10^{-12}\ m \sim 2\ pm$$

less than 1 atom!

Question 3: a) A 1 μm gold sphere and a 2 μm silver sphere sediment (fall under gravity) in a viscous fluid (you can assume it is in the low Reynolds' number regime). Which one sediments faster?

The motion is governed by

$$\vec{F}_g - \vec{F}_b - \vec{F}_D = m\vec{a}$$

gravity, buoyancy, drag

Because of their small size we will be in a low Reynolds # regime and can ignore $m\vec{a}$. The drift velocity at which the particles settle is therefore determined by $\vec{F}_D = 6\pi\eta a\vec{u}$. Since gold + silver both have much greater density than water the buoyancy force is also insignificant so $u = \frac{4}{3}\pi a^3\rho / 6\pi\eta a = C a^2 \rho$

$$\therefore \frac{u_{s\,Ag}}{u_{s\,Au}} = \frac{(2\mu m)^2 \rho_{Ag}}{(1\mu m)^2 \rho_{Au}} = 4 \cdot \frac{(10.5\ g/cm^3)}{(19.3\ g/cm^3)} \sim 2$$

The silver spheres will sediment faster due to their larger size.

b) Molecular biologists routinely make use of centrifugation (spinning solutions containing protein, at rates as high as 10^5 rpm) to separate proteins and other macromolecules by size. Find an expression for the drift velocity of a protein, in a solvent of viscosity η and density, ρ undergoing centrifugation, with an angular velocity ω at a distance R from the axis of rotation. Treat the protein as a sphere of radius, R a; density, ρ_P. Hint: to solve this problem you will need to set up the equation of motion equating viscous drag to a term proportional to the centriptal acceleration. It is important to note in this equation that the relevant mass that is being accelerated is not the total mass but the relative mass from the greater density of the protein to water (this is analogous to Archimedes principle).

$$F_D = (\rho_P - \rho_{H_2O})\frac{4}{3}\pi a^3 \omega^2 R = 6\pi\eta a v_D$$

$$v_D = \frac{(\rho_P - \rho_{H_2O})\frac{4}{3}\pi a^3 \omega^2 R}{6\pi\eta a}$$

$$v_D = \frac{2(\rho_p - \rho_{H_2O})\, a^2 \omega^2 R}{9\eta}$$

You were marked correct if you did this problem as written or if you recognized that there should have been different variables. I apologize for the typo.