# FOCS HW3

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**Problem 5.10 (j)**

3 divides $n^3+5n+6$

Let P(n): 3 divides $n^3+5n+6$

P(1): $n^3+5n+6 = 1+5+6 = 12$ which is dividible by 3. P(1) is True.

Assume P(n) is True, Let's prove P(n+1)

P(n+1): $(n+1)^3+5(n+1)+6 = n^3+3n^2+3n+1+5n+5+6 = n^3+3n^2+8n+12$

$= (n^3+5n+6)+(3n^2+3n+6) = (n^3+5n+6)+3(n^2+n+2)$

From the assumption P(n) we can know that $(n^3+5n+6)$ is dividible by 3. Also $3(n^2+n+2)$ is divisible by 3. As the result P(n+1) is divisible by 3.

Since P(1) is True and P(n) -> p(n+1) is True. By Induction P(n) is True for all $n \geq 1$.

**Problem 5.12 (i)**

$n! \geq n^n e^{-n}$

Let P(n): $n! \geq n^n e^{-n}$

P(1): $1 \geq 1e^{-1}$, $1 \geq 1/e$, since $e>1$, so $1/e<1$. P(1) is True

Assume P(n) is True, Let's prove P(n+1):$(n+1)^{n+1}e^{-n-1}$

$$(n+1)! = \frac{e}{e}(n+1)n! \geq \frac{(1+\frac{1}{n})^n}{e}(n+1)n! = \frac{(\frac{n+1}{n})^n}{e}(n+1)n!$$

$$= \frac{(\frac{n+1}{n})^n}{e}(n+1)n! = \frac{(n+1)^{n+1}}{n^n}\ \frac{e^n}{e^{n+1}}\, n! = \frac{(\frac{n+1}{e})^{n+1}}{(\frac{n}{e})^n}\, n!$$

$$\geq \frac{(\frac{n+1}{e})^{n+1}}{(\frac{n}{e})^n}\, n^n e^{-n} = \frac{(\frac{n+1}{e})^{n+1}}{(\frac{n}{e})^n}\ (\frac{n}{e})^n = (\frac{n+1}{e})^{n+1} = (n+1)^{n+1}e^{-n-1}$$

P(n+1) is True

By Induction, P(n) is True for all $n \geq 1$.

**Problem 5.18 (a)**

$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + 1/n$, for $n \geq 1$.

Prove: $H_1 + H_2 + \ldots + H_n = (n+1)H_n - n$

Let $P(n)$: $H_1 + H_2 + \ldots + H_n = (n+1)H_n - n$

$P(1)$: $H_1 = 1/1 = 1$; $(1+1)H_1 - 1 = 1$

$P(1)$ is True

Assume $P(n)$ is True

Let's prove $P(n+1)$ is True:

$H_1 + H_2 + \ldots + H_n + H_{n+1}$

$= (n+1)H_n - n + H_{n+1}$

$= (n+1)(H_n - \frac{1}{n+1} + \frac{1}{n+1}) - n + H_{n+1}$

$= (n+1)(H_{n+1} - \frac{1}{n+1}) - n + H_{n+1}$

$= (n+1)H_{n+1} - (n+1)\frac{1}{n+1} - n + H_{n+1}$

$= (n+2)H_{n+1} - n - 1$

$P(n+1)$ is True

By Induction, $P(n)$ is True for all $n \geq 1$

**Problem 5.60**

(a) 42

(b) P(n): The perimeter is even for all n≥1

P(1): The perimeter for one square is 4 which is even.

P(1) is True

Let's assume P(n) is True and prove P(n+1) is True

When adding a new square, there can be five possibilities:

(1) No adjacent

Adding another 4 to the perimeter which will still be even number.

(2) 1 adjacent

Perimeter minus one and add 3 which equivalent to add 2 which is also even number.

(3) 2 adjacent

Perimeter minus 2 and add 2 which does not change; even number.

(4) 3 adjacent

Perimeter minus 3 add 1 which equivalent to minus 2; even number.

(5) 4 adjacent

Perimeter minus 4; even number.

In all 5 circumstances, the perimeter is all even. Therefore, P(n+1) is True.

By Induction, P(n) is True for all n≥1.

**Problem 6.6**

(a) P(n): $H_1/1+H_2/2+H_3/3+...+H_n/n \le (1/2)H_n^2+1$

Prove P(1):

$(1/1)/1 \le 1/2(1/1)^2+1 \qquad 1 \le 3/2$

P(1) is True

Let's Assume P(n) and Prove P(n+1)

$H_1/1+H_2/2+H_3/3+...+H_n/n+H_{n+1}/(n+1) \le (1/2)H_n^2+1+H_{n+1}/(n+1)$

$RHS = ½\ H_n^2+1+H_{n+1} \cdot \frac{1}{n+1} = ½\ H_n^2+1+ \frac{2Hn+1}{2} \cdot (H_{n+1}-H_n)$

$\ge ½\ H_n^2+1+ \frac{Hn+1+Hn}{2} \cdot (H_{n+1}-H_n) = ½\ H_n^2+1+½(H_{n+1}+H_n) \cdot (H_{n+1}-H_n)$

$= ½\ H_n^2+1+½(H_{n+1}^2-H_n^2) = ½\ H^2_{n+1}+1$

The problem is that $\le$ should be $\ge$

(b) P(n): $H_1/1+H_2/2+...+H_n/n \le ½\ H_n^2+½\ (1/1^2+1/2^2+...+1/n^2)$

Prove P(1): $1/1 \le ½\ (1)^2+½\ (1/1^2) \qquad 1 \le 1$

P(1) is True

Let's Assume $H_1/1+H_2/2+...+H_n/n \le ½\ H_n^2+½\ (1/1^2+1/2^2+...+1/n^2)$ is True

Prove P(n+1):

$H_1/1+H_2/2+...+H_n/n+H_{n+1}/(n+1) \le ½\ H_n^2+½\ (1/1^2+1/2^2+...+1/n^2)+H_{n+1}/(n+1)$

$RHS = ½\ H_n^2+½\ (1/1^2+1/2^2+...+1/n^2)+H_{n+1} \cdot (1/(n+1))$

$= ½\ H_n^2+½\ (1/1^2+1/2^2+...+1/n^2)+(2H_{n+1}/2) \cdot (H_{n+1}-H_n)$

$= ½\ H_n^2+½\ (1/1^2+1/2^2+...+1/n^2)+(2H_{n+1}/2) \cdot (H_{n+1}-H_n)+½\ (1/(n+1)^2-1/(n+1)^2)$

$= ½\ H_n^2+½\ (1/1^2+1/2^2+...+1/n^2)+(2H_{n+1}/2) \cdot (H_{n+1}-H_n)+½\ (1/(n+1)^2-(1/(n+1))(H_{n+1}-H_n))$

$= ½\ H_n^2+½\ (1/1^2+1/2^2+...+1/n^2)+(2H_{n+1}/2) \cdot (H_{n+1}-H_n)+½\ (1/(n+1)^2)-½((1/(n+1))(H_{n+1}-H_n))$

$= ½\ H_n^2+½\ (1/1^2+1/2^2+...+1/n^2)+(2H_{n+1}-1/(n+1)) \cdot (H_{n+1}-H_n)+½\ (1/(n+1)^2)$

$= ½\ H_n^2+½\ (H_{n+1}+H_n)(H_{n+1}-H_n)+½\ (1/1^2+1/2^2+...+1/n^{2+}1/(n+1)^2)$

$= ½\ H_n^2+½\ (H^2_{n+1}-H^2_n)+½\ (1/1^2+1/2^2+...+1/n^{2+}1/(n+1)^2)$

$= ½\ H_{n+1}^2+½\ (1/1^2+1/2^2+...+1/n^{2+}1/(n+1)^2)$

P(n+1) is True, By Induction P(n) is True for all $n \ge 1$

The claim the stronger because on the RHS it increases faster than the first one

**Problem 6.45 (a)**

There is a one-way flight between every pair of cities. Prove that there is at least one special city that can be reached from every other city either directly or via one-stop.

P(n): there is at least one special city that can be reached from every other city either directly or via one-stop.

P(2): Only two cities, each city can be reached by the other. P(2) is True

Assume P(n), let's prove P(n+1)

By assumption, we can say that there are 3 kinds of cities:

(1) The special city(we assume only one)
(2) The city that can directly fly to special city directly
(3) The city that can fly to the second type of city directly

We use contradiction: Assume adding a new will result in no special city

However, adding a new city into the system will result in 2 results

(a) The special city remains special
(b) The new city become the new special city

The reason is if the new one has a flight to special city, the special city remains special and the new city will become the second type. If the new one has a flight to the second type of city, the new one will become the third type of city and the special city remains special. On the other hand, if the new city has a flight from the previous special city and flights from previous type 2 cities. The new city will become the new special city. The previous special city will become a type 2 city and the previous type 2 cities remain in type 2 and the type 3 cities remain in type 3 since it can reach the special city from type 2 cities.

By contradiction, P(n+1) is True

By Induction, P(n) is True for all $n \geq 2$