FOCS HW4

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Problem 7.4(c)

$A_0=1$; $A_1=2$; $A_n=2A_{n-1}-A_{n-2}+2$ for $n \geq 2$

$A_2=2A_1-A_0+2=5$
$A_3=2A_2-A_1+2=10$
Guess P(n): $A_n=n^2+1$, ***for*** $n \geq 0$
P(0): $A_0=0^2+1=1$, which is True
Let's Assume P(n) and prove P(n+1):

$$\begin{aligned} A_{n+1} &= 2A_n-A_{n-1}+2 \\ &= 2(n^2+1)-((n-1)^2+1)+2 \\ &= 2n^2+2-n^2+2n-1-1+2 \\ &= n^2+2n+2 \\ &= (n+1)^2+1 \end{aligned}$$

P(n+1) is True
By induction, Our guess is True

Problem 7.56

(a) $M(0,k) = 0$, don't need drop the egg from the first floor
$M(n,1) = n$, drop one time per floor
$M(n,k) = \log_2(n)$, binary relationship(try n/2, then n/4 or 3n/4 depend on if it breaks)

(b) $M(n,k) \to (x) \to$:
If it breaks, $M(n,k)=M(x,k-1)+1$
If it doesn't break, $M(n,k)=M(n-x,k)+1$
Basis: $M(0,k)=0$, $M(n,1)=n$

(worst cases)
Since we have infinite eggs, a binary search will be a good solution. So the number of egg drop will be logarithm.
When the egg breaks, the second egg drop will be in range of 0 to x
We need $\log_2 x$ of eggs to drop if this egg break
When the egg survives, we need to move up to a range of x to n
So we need $\log_2(n-x)$ of eggs if the egg survive

(c) $M(7,3)=3$, $M(8,3)=3$, $M(9,3)=4$

Problem 8.6

Recursive Definition:

1. $1 \in A$ (base case)
2. $x \in A \rightarrow 2 \cdot x \in A$ (constructor)

(a) Every element of your set is a non-negative power of 2

P(n): n is a non-negative power of 2

P(1): $1=2^0$, which is True

Let's assume P(n) and prove P(n+1):

$a_n=2^k$ for $k \geq 0$,

child of n is $2x=2^{k+1}$ for $k+1 \geq 0$

The child is also a non-negative power of two

By structural induction, every element in set A is a non-negative power of 2

(b) Every non-negative power of 2 is in your set

Let's assume $x=2^k$ and $x \notin A$

Because of $x/2 = 2^{k-1}$ that x can be constructed by x/2

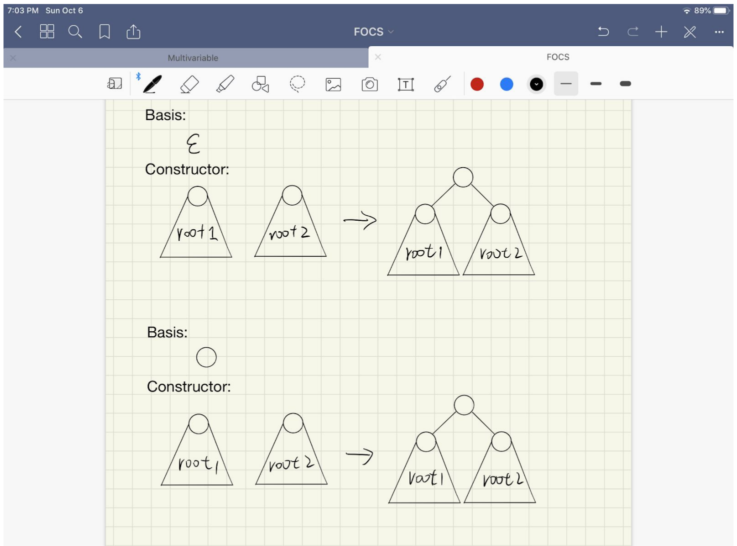
So, $x/2 \notin A$

When it goes on: $x/4=2^{k-2} \notin A$, $x/8 \notin A \ldots x/2^k=2^{k-k}=1 \notin A$

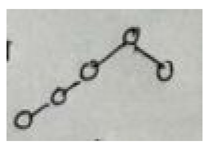
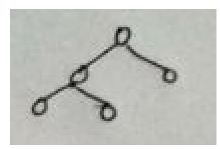
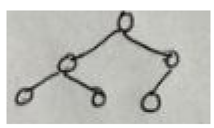
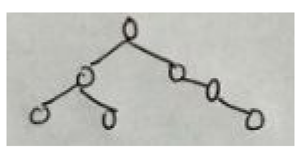
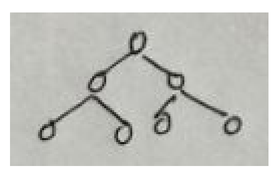
However, 1 is the base case of A

By contradiction, Every non-negative power of 2 is in set A

Problem 8.18



(a) Give examples, with derivations, of RBTs and RFBTs with 5,6 and 7 vertices

	RBT	RFBT
5 vertices		
6 vertices		Not possible for even number
7 vertices		

(b) Prove by structural induction that every RFBT has an odd number of vertices

P(n): when RFBT tree has the height of n, the number of vertices is 2^{n+1}-1 which is odd, for n≥0.

P(0): number of vertices when the height is 0: 2-1=1. Which is odd. True

Let's assume P(n) and prove P(n+1):

$2^{n+1+1}-1=2*2^{n+1}-2+1=2(2^{n+1}-1)+1$

$2(2^{n+1}-1)$ is even, so $2(2^{n+1}-1)+1$ is odd

P(n+1) is True, By induction, every RFBT has an odd number of vertices

Problem 9.3

(b)

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)$$

$$= \sum_{i=1}^{n}(\sum_{j=1}^{n} i + \sum_{j=1}^{n} j)$$

$$= \sum_{i=1}^{n}(i^2 + \tfrac{1}{2}i(i+1))$$

$$= \sum_{i=1}^{n}(\tfrac{3}{2}i^2 + \tfrac{1}{2}i)$$

$$= \sum_{i=1}^{n}\tfrac{3}{2}i^2 + \sum_{i=1}^{n}\tfrac{1}{2}i$$

$$= \tfrac{3}{2}\sum_{i=1}^{n} i^2 + \tfrac{1}{2}\sum_{i=1}^{n} i$$

$$= \tfrac{3}{2}\ \tfrac{1}{6}n(n+1)(2n+1) + \tfrac{1}{2}\tfrac{1}{2}n(n+1)$$

$$= \tfrac{1}{4}n(n+1)(2n+2)$$

$$= \tfrac{1}{2}n(n+1)^2$$

(e)

$$\sum_{i=0}^{n}\sum_{j=0}^{m} 2^{i+j}$$

$$= \sum_{i=0}^{n}(\sum_{j=0}^{m} 2^i * 2^j)$$

$$= \sum_{i=0}^{n}(2^i \sum_{j=0}^{m} 2^j)$$

$$= \sum_{i=0}^{n}(2^i * (2^{m+1} - 1))$$

$$= (2^{m+1} - 1)\ \sum_{i=0}^{n} 2^i$$

$$= (2^{m+1} - 1)\ (2^{n+1} - 1)$$

Problem 9.37

	a	b	c	d	e	f
i	i $\in$ O(a)	b $\in$ O(i)	c $\in$ O(i)	i $\in$ O(d)	i $\in$ O(e)	i $\in$ O(f)
ii	both	ii $\in$ O(b)	ii $\in$ O(c)	ii $\in$ O(d)	both	ii $\in$ O(f)
iii	a $\in$ O(iii)	b $\in$ O(iii)	c $\in$ O(iii)	both	e $\in$ O(iii)	neither
iv	both	b $\in$ O(iv)	c $\in$ O(iv)	iv $\in$ O(d)	e $\in$ O(iv)	iv $\in$ O(f)
v	v $\in$ O(a)	both	c $\in$ O(v)	both	both	f $\in$ O(v)

Problem 9.44(a)

Give upper and lower bounds and the asymptotic (big-Theta) behavior for

$$\sum_{i=1}^{n} \frac{i^2}{i^3+1}$$

It is decreasing and approaching to 0

$$\int_{m-1}^{n} f(x)\,dx \;\geq\; \sum_{i=m}^{n} f(i) \;\geq\; \int_{m}^{n+1} f(x)\,dx$$

Let $u=i^3+1$, $du=3i^2\,dx$

f(x) dx = $\frac{1}{3u}$

$$\int_{0}^{n} \frac{1}{3u}\,du \;\geq \sum_{i=1}^{n} \frac{i^2}{i^3+1} \;\geq\; \int_{1}^{n+1} \frac{1}{3u}\,du$$

$$\int_{0}^{n} \tfrac{1}{3}ln|i^3+1| \;\geq \sum_{i=1}^{n} \frac{i^2}{i^3+1} \;\geq\; \int_{1}^{n+1} \tfrac{1}{3}ln|i^3+1|$$

$$\tfrac{1}{3}ln|n^3+1| \;\geq \sum_{i=1}^{n} \frac{i^2}{i^3+1} \;\geq\; \tfrac{1}{3}ln|(n+1)^3+1| - \tfrac{1}{3}ln2$$