Math for Investment Consultants

2nd Edition

Chapter 3: Fixed Income—Duration and Yield

The purpose of this session is to provide the conceptual and mechanical skills needed to perform bond calculations. Successful completion of this session will provide the ability to use a variety of essential equations in the analysis of both bonds and multibond portfolios.

Fundamentals of Bond Valuation

We learned that the price of any asset is the sum of the asset's discounted cash flows. There are three steps to valuing bonds:

1. Estimate cash flows, which include the annual or semiannual coupon payments and the recovery of principal at maturity.

2. Determine the appropriate discount rate, which generally is the yield to maturity of a bond.

3. Calculate the present value of the estimated cash flows.

Thus, the present value of a bond equals the present value of coupon payments plus the present value of the principal at maturity. This is expressed mathematically by the following formula:

$$PV_B = \frac{CF_1}{(1+i)^1} + \frac{CF_2}{(1+i)^2} + \dots + \frac{CF_n}{(1+i)^n} + \frac{PAR_n}{(1+i)^n}$$

Equation 3-1

Where, PV_B is the price of a bond; CF_t is the cash flow (coupon payment) in year t; where t goes from 1 to n; PAR_n is the par value of the bond; i is the current interest rate; n is the number of years to maturity. This essentially is the same equation we saw when solving for NPV, with the exception of including a terminal value (ending value). Let us consider an example. Find the price of a five-year bond with a 7-percent annual coupon and a face value of $1,000. The yield to maturity of competitive bonds with a similar length of time to maturity and credit risk is 5 percent. The value of the bonds is

$$PV_B = \frac{\$70}{(1+0.05)^1} + \frac{\$70}{(1+0.05)^2} + \frac{\$70}{(1+0.05)^3} + \frac{\$70}{(1+0.05)^4} + \frac{\$70}{(1+0.05)^5} + \frac{\$1000}{(1+0.05)^5}$$

$$= \$66.6667 + \$63.4921 + \$60.4686 + \$57.5892 + \$54.8468 + \$783.5262 = \$1086.59$$

This calculation also is simplified into PV of coupon payments (i.e., ordinary annuity) *plus* PV of face value. The calculation is

$$PV_B = \left[\frac{\left(1 - \frac{1}{(1+r)^n}\right)}{r} \times (\text{coupon payment}) \right] + \frac{PAR}{(1+i)^n}$$

Equation 3-2

$$= \left[\frac{\left(1 - \frac{1}{(1+0.05)^5}\right)}{0.05} \times (70) \right] + \frac{\$1,000}{(1+0.05)^5}$$

$$= \$303.0634 + \$783.5262 = \$1,086.59$$

Semiannual Compounding

In general, bonds pay twice a year rather than once a year. The equation previously presented is modified slightly by adjusting the total number of periods and the amount of each payment for semiannual payments. The total number of periods becomes 10 (2×5) and the amount of payment is \$35 (\$70/2). The yield to maturity of similar bonds is 5 percent, so the appropriate discount rate to use then becomes 2.5 percent if we divide the current market rate in half (semiannual compounding). The price of this bond is then \$1,087.52.

$$= \left[\frac{\left(1 - \frac{1}{(1+0.025)^{10}}\right)}{0.025} \times (35) \right] + \frac{\$1,000}{(1+0.025)^{10}}$$

$$= \$306.3222 + \$781.1984 = \$1,087.52$$

Pricing a Zero-Coupon Bond

A zero-coupon bond does not pay a coupon. Its only cash flow is the recovery of principal at maturity. The value of a zero-coupon bond is

$$\text{Zero-Coupon Value} = \frac{\text{Maturity Value}}{(1+i)^{\text{no. of years} \times 2}}$$

Equation 3-3

Where, *i* is the semiannual discount rate. It may be surprising that we use semiannual compounding when pricing a zero-coupon bond, but the use of six-month periods is required in order to have uniformity between the present value calculation of the maturity value for a coupon bond that pays semiannually and a zero-coupon bond.

To illustrate the calculation, find the value of a five-year zero-coupon bond with a maturity value of $1000 discounted at a 10-percent interest rate.

$$i = 0.05 \ (= 0.10/2)$$

$$= \frac{\$1000}{(1.05)^{5\times 2}} = \$613.91$$

Current Yield (CY)
The current yield is important to investors interested in income. Those investors that want a high yearly income want bonds with a high current yield. The current yield of a bond bought at par always will be its coupon rate. The equation for calculating current yield is

$$CY = \frac{i}{V}$$

Equation 3-4

Where CY is the current yield, *i* is the annual coupon payment of the bond, and *V* is the market value of the bond at the time of calculation. Find the current yield of a $1,000-par-value bond priced at $900 with a 10-percent semiannual coupon rate.

$$\frac{\$100}{\$900} = 0.111 \text{ or } 11.1\%$$

Current yield often can paint an incomplete picture of the actual economics involved in a particular bond or bond comparison. Most notably, CY is deficient in that it is not affected by maturity length or change in principal value between purchase price (premium or discount) and maturity value. As a result, when providing CY information to an investor, it always should be accompanied by information regarding yield to maturity (YTM).

Yield To Maturity (YTM)

The YTM is the percentage rate of return required to make the present value of the cash flows equal to the cost of the bond. Also referred to as the internal rate of return or IRR, it is the percentage rate that an investor would require in order to replace the benefits offered by the subject bond when held to maturity. The YTM accounts for such variables among bonds as purchase price, redemption value, time to maturity, coupon yield, and the amount of time between interest payments.

Where the bond is priced at a discount from its maturity value, the YTM will be greater than the bond's current yield (CY). YTM will be less than CY where the bond is valued at a premium.

Bond yield tables are useful in arriving at approximate YTM, but a programmable calculator is needed for fully accurate results. Another way to conceptualize YTM is that it is actually the *discount rate* applied to compute the present value of all future payments (including the maturity value). As a result, the total PV of all future payments will equal the current cost of the bond.

The YTM calculation can be expressed as follows:

$$PV_B = \frac{CF_1}{(1+YTM)^1} + \frac{CF_2}{(1+YTM)^2} + \frac{CF_3}{(1+YTM)^3} + \ldots + \frac{CF_t}{(1+YTM)^n}$$

Where, PV_B is the present value of a bond, CF_t is the cash flow in year t, and n is the number of years to maturity. In this formula, the last cash flow typically would include an interest payment as well as the principal repayment.

Solving for the YTM in a multipayment bond requires trial and error. The goal is to secure a rate of return that makes the PV equal to the price of the bond. To employ this trial-and-error process where the interest rate will make the PV equal to the bond price, simply (1) choose a beginning interest rate *guess*, (2) use that *guess* as the discount factor, calculate the PV of each payment, and (3) total the PV's from (2). Compare the total to the bond price and adjust the *guess* up or down, recalculating until you arrive at the PV that equals the bond price. This can be quite a labor-intensive process.

The good news is that the yield to maturity easily can be solved using a financial calculator. Assume a $1,000-par bond is priced at $950 with a 10-percent semiannual coupon payment. The bond matures in three years. The yield to maturity using a calculator is 12.03 percent (adjusting to an annual rate).

Notation Used on Most Calculators	Numerical Value for This Problem
N	6
PV	–950
FV	1,000
PMT	50
%i Compute (× 2)	

The significant assumption in YTM calculation is that interim interest payments can be reinvested at the calculated YTM rate. This time value of money is the major YTM assumption. As a result, *reinvestment risk* is inherent in YTM calculations.

Macaulay Duration

When comparing bonds with the same length to maturity but different coupons, the bond with the lower coupon is more volatile to changing interest rates (i.e., higher duration). For comparing bonds with the same coupons but different maturities, the bond with the longer maturity is more volatile (i.e., higher duration). However, how are bonds with different maturities and coupons compared? The most often used technique is finding the duration of a bond (or portfolio).

In 1938, Frederick Macaulay created a formula, which he termed *duration*, to assess the degree and impact of interest rate volatility in bond prices. His formula was based upon the weighted average term-to-maturity (WAT) of the bond's cash flow. Using the present value of each cash-. flow receipt as a percentage of the present value of total cash flows, he created a weighting system that provided a result that was expressed in terms of years.

By comparing the WAT of one bond versus another, an investor could assess relative price sensitivity to interest-. rate changes. Even better, by multiplying the WAT of a bond against a projected percentage change in interest rates, a rough figure of the expected percentage price change in a bond could be achieved. In more recent times, the Macaulay duration standard has been generally replaced by a more useful measure, known as *modified duration*.

To understand the concept of modified duration, it is important to understand the foundation (and flaws) of Macaulay duration. The weighted-average term-to-maturity (WAT) of a bond or a portfolio reflects how soon an investor will recover his or her investment in the form of coupon and maturity payments. It specifically takes into consideration that a high-coupon bond will provide a faster return than a low-coupon bond.

The formula can be expressed as follows:

$$\text{Macaulay Duration} = \left(\frac{1 \times PVCF_1 + 2 \times PVCF_2 + \ldots + t \times PVCF_t}{k \times PVTCF} \right)$$

Equation 3-5

Where, k is the number of periods per year (k = 1 for an annual-pay bond, k = 2 for a semiannual-pay bond), t is the number of periods until maturity (number of years to maturity times k), $PVCF_t$ is the present value of cash flow in period t discounted at the yield to maturity (where t = 1, 2, …, n), and *PVTCF* is the sum of the present values of total cash flows. In narrative, this becomes the sum of one times the PV of CF from period 1, plus two times the PV of CF from period 2, plus three times the PV of CF from period 3, etc. Next, the resulting sum is divided by the PV of TCF, including the maturity payment.

Find the Macaulay duration calculation for a five-year $1,000 4-percent bond. The market interest rate on similar bonds is 5 percent. To simplify the example, we use annual payments rather than semiannual payments. The Macaulay duration is $4,420.30 / $956.71 = 4.62

Period (A)	CF (B)	PV of CF at 5% (C)	Periods × PV of CF (A) × (C)
1	$40	$40 / (1.05)^1 = $38.10	$38.10
2	$40	$40 / (1.05)^2 = $36.28	$72.56
3	$40	$40 / (1.05)^3 = $34.55	$103.65
4	$40	$40 / (1.05)^4 = $32.91	$131.64
5	$1,040	$1,040 / (1.05)^5 = $814.87	$4,074.35
Totals	$1,200	$956.71	$4,420.30

In the case of semiannual coupon payments, the CF is simply 1Ú2 of the annual amount. Note also, that the CF in period n is the final payment and includes both the coupon payment *and* the maturity value. Macaulay duration provides a result in terms of CF payments, so by multiplying by k ($k = 2$ for semiannual compounding) in the denominator, the figure can be converted into years.

Another way of looking at the Macaulay duration formula assumes that the market price of a bond accurately reflects the discounted present value of its future payments (i.e., PVTCF). Therefore, assuming the price of the bond is equal to the present value of all payments:

$$\text{Macaulay Duration} = \left(\frac{1 \times PVCF_1 + 2 \times PVCF_2 + \ldots + t \times PVCF_t}{k \times \text{Bond Price}} \right)$$

Equation 3-6

If we apply this calculation to the previous example, we find that Macaulay's duration is equal to 4.62 years. The only difference is that $956.71 in the denominator now represents the current bond price, which accurately reflects the present value of total cash flows.

Limitations of Macaulay's Duration

Because it accounts for income received before maturity (coupon payments), Macaulay's weighting formula is a more accurate measure of volatility than simple maturity, YTM, or other earlier measures that failed to incorporate PV factors.

In general, it can be said that the greater the Macaulay duration, the more volatility can be expected in periods of changing market interest rates. While Macaulay duration certainly is more helpful in assessing volatility than maturity alone, it fails to reflect or respond to changes in actual market interest rates and assumes that interest payments are continuously compounded at the same rate.

The deficiency in Macaulay duration is the assumption that YTM remains constant for a given bond, even as actual interest reinvestment rates change in the market place. The deficiency stems from the fixed-reinvestment-rate assumption used in calculating the bond's yield to maturity. It assumes that a bond's YTM will remain constant over its maturity period and ignores reinvestment risk.

In the real world, except in zero-coupon bonds, *reinvestment risk* means that beginning YTM can be expected to change with shifting interest rates. This is also the general case even in static interest-rate environments, since reinvestment opportunities change as a bond approaches maturity. For example, a five-year 4-percent bond becomes a four-year 4-percent bond in the following year, then three-year, two-year, and so on. Assuming a typical interest-rate market where shorter maturities command lower coupon rates, the interest payments from this bond will, as it approaches maturity, have to be reinvested in progressively shorter maturities and at lower yields.

Since YTM is dependent upon price, time to maturity, and coupon, as these variables change, the YTM will change. In the case of price, a higher price paid for a bond equates to a lower YTM. Since we know that the yield curve is not constant, an additional calculation (involving the impact of changes in the yield curve) can help provide a more accurate and useful measure of duration in the case of coupon bonds where reinvestment risk is a reality.

Note that changing market rates will not affect the YTM of a zero-coupon bond since it has no reinvestment risk. In coupon bonds, the higher the coupon rate, the greater the reinvestment risk but the lower the price volatility. Further, since longer maturities mean a longer exposure to reinvestment risk, as maturities lengthen, the greater the reinvestment risk and price volatility. It is important to note that the duration of the zero-coupon bond always will be its maturity since it has no reinvestment risk.

Modified Duration—The Current Standard to Assess Interest-Rate Risk

In modified duration calculations, Macaulay's calculation is modified to reflect coupon payment periods. Where Macaulay duration was meant to provide a figure that could be compared against other bond investments, modified duration is designed to provide a percentage factor by which the change in price can be computed.

The equation for modified duration in a semiannual pay bond, where *y* is yield divided by the interest payments per year, can be expressed as:

$$\text{Modified Duration} = \frac{1}{(1+\text{yield}/k)}\left(\frac{1\times \text{PVCF}_1 + 2\times \text{PVCF}_2 + \dots + t\times \text{PVCF}_t}{k\times \text{Price}}\right)$$

$$= \frac{\text{Macaulay Duration}}{(1+\text{yield}/k)}$$

Equation 3-7

Where, k is the number of periods per year (k = 1 for an annual-pay bond, k = 2 for a semiannual-pay bond, t is the number of periods until maturity (number of years to maturity times k), *yield* is the yield to maturity of the bond, and PVCF_t is the present value of cash flow in period t discounted at the yield to maturity where $t = 1, 2, \dots, n$.

We learned earlier that Macaulay's duration for a five-year $1,000 4-percent annual-pay bond was 4.62 years. Keep in mind that the yield to maturity is 5 percent. Thus, the calculation of modified duration is

$$\text{Modified Duration} = \frac{\text{Macaulay Duration}}{(1+\text{yield}/k)}$$

$$= \frac{4.62}{\left(1+\frac{0.05}{1}\right)} = 4.40$$

We can use modified duration to derive an estimate of the approximate percentage price change of a bond for a given change in interest rates.

Approximate % price change of a bond = (−)(modified duration)(change in interest rates)

Equation 3-8

What is the approximate percentage price change assuming a 1.5-percent upward change in market rates?

Approximate % price change of a bond = −(4.40) ×(1.5) = −6.60%

This makes sense, because a bond's price moves down when interest rates move up. In contrast, if there was a 1.5-percent downward change in market interest rates, the approximate percentage price change is

Approximate % price change of a bond = −(4.40) × (−1.5) = 6.60%

The dollar change in value = % change expressed as a decimal × current value

= 0.0660 × \$956.71 = \$63.14

Thus, the price of the bond after a 1.5-percent downward change in market interest rates is \$1,019.85 (\$956.71 + \$63.14). The bond then trades at a premium from previously trading at a discount. This makes sense because the YTM is above its coupon rate of 4 percent when at 5 percent; therefore, the bond must trade at a discount. In contrast, the YTM is below its coupon rate at 3.5 percent, which implies that the bond must trade at a premium.

Application of Duration to Portfolio Construction

Just like the average return of a portfolio is a weighted average of the assets in the portfolio, so is the duration of a bond portfolio a weighted average of the duration of the bonds in the portfolio. To determine the duration of a portfolio, it only is necessary to know the duration of the individual bonds and the relative percentage of the portfolio that each represents.

Mathematically, the equation for portfolio duration is:

$$\text{Portfolio duration} = w_1 D_1 + w_2 D_2 + w_3 D_3 + \ldots + w_k D_k$$

Equation 3-9

Where,

w_i = market value of bond i (PV) / market value of the portfolio (Total PV)

D_i = duration of bond i

K = number of bonds in the portfolio

To illustrate this calculation, consider the following three-bond portfolio:

Bond	Market Value	Duration
A	$5,000,000	4.251
B	$5,750,000	7.953
C	$1,500,000	9.355

The market value for the portfolio is $12,250,000. So, *K* is equal to 3 and the weights are:

W_A = $5,000,000/$12,250,000 = 0.408

W_B = $5,750,000/$12,250,000 = 0.469

W_C = $1,500,000/$12,250,000 = 0.122

The portfolio's duration is

$$0.408(4.251) + 0.469(7.953) + 0.122(9.355) = 6.61$$

A portfolio duration of 6.61 means that for a 100-basis-point change in the yield for each of the three bonds, the market value of the portfolio will change by approximately 6.61 percent. A critical assumption is worth mentioning. The yield for each bond must change by the same 100 basis points for the duration measure to be useful.

Q: Duration measures the years until principal and interest will be repaid; agree or disagree?
A: Disagree. While duration is expressed in units of years, it is a theoretical measurement intended for use as a method of comparison of the susceptibility to change in value with changes in interest rates among bonds, bond funds, and bond managers.

Convexity
In viewing a chart of bond duration under varying interest rates, it quickly appears that there is a linear relationship between interest-rate changes and bond prices. If this concept is extrapolated, it becomes apparent that there is a problem with using duration to measure price change in bonds. The problem arises as the hypothetical market interest rate either becomes very high or very low.

As an example of the problem, consider the zero-coupon bond in the face of changing interest rates. If we use a 10-year maturity, we know that duration is 10. Multiplying duration by the change in market interest rates will allow us to assess the percentage change in the value of the zero coupon bond.

Using a 6-percent discount factor, the PV or current market value of a $1,000 ten-year zero-coupon bond would be $553.67 ($\$1,000 / (1 + 0.03)^{20}$). At this point, consider the impact of a 10-percent rise in market interest rates. We multiply the interest rate increase by the duration to learn the percentage change in bond prices.

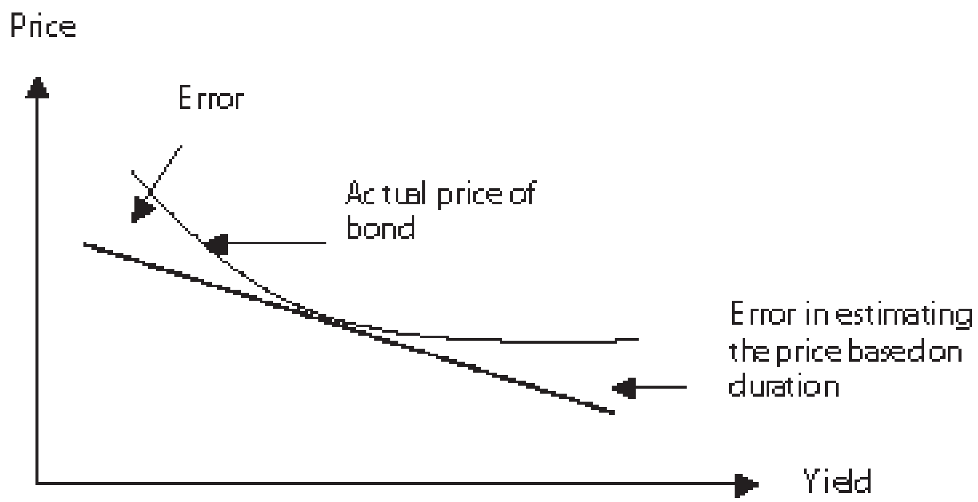
Thus, a 10-percent rate increase multiplied by a duration of 10 would equal a negative change in value of 100 percent, implying the bond's price would drop to $0. Now, we know that in ten years, the holder of the bond will receive its maturity value of $1,000 and we certainly can calculate the present value of that payment under a 16-percent interest-rate scenario to learn that the PV would be $214.55 ($214.55 ($1,000 / $(1 + 0.08)^{20}$). So then, how is it that our duration calculation seems to indicate that the change in price of this bond would reduce its value to zero? The problem is that bond values do not move linearly, although the duration calculation would seem to indicate so.

The converse also holds true where interest rates drop. This time let's say they drop by 5 percent. Now we must multiply the 5-percent change by the duration of 10 and arrive at a change in market value of 50 percent. A 50-percent increase in our $553.67 zero-coupon bond would indicate a value of $830.51 ($553.67 × 1.50), yet when we calculate the PV of $1,000 due in ten years with a 1-percent discount factor, we get a value of $905.06 (($1,000 / $(1 + 0.005)^{20}$).

Despite the apparent paradox of a zero-coupon bond being valued as worthless with a rise in interest rates, duration still is a valuable measure of interest-rate risk. It is especially valuable during small changes in interest rates and only becomes radically misleading when large rate changes are contemplated. Recognition and understanding of this problem therefore becomes important to a full understanding of interest-rate risk and bond portfolio construction.

The percentage price-change calculations in the zero-coupon bonds above were inaccurate because they failed to account for *convexity* in bond pricing. Convexity is a measure of the amount of "bend" in the bond's price-yield curve (see chart below) as compared to the straight line indicated by a simple duration calculation. This bend in the bond price graph is named after the convex shape of the curve that results. Because of the points on the curve (bond prices), for a given change in market interest rates (up or down), the increase in bond value is more pronounced than the decline in value caused by an equal change (up or down) in market rates. This results in a modest "upside capture" and "downside protection" expectation that can be accounted for by an understanding of convexity.

To put the benefits of convexity to work, an explanation of the mathematical principles are in order. The value of convexity is that the more convex a bond is, the less its price will drop in



response to rises in market interest rates and the greater its price will rise in response to market interest rate declines. In other words, greater convexity means less negative volatility and greater positive volatility. The caution here is that bonds with the same PV can exhibit different convexities. In general, higher convexity is associated with lower coupons and longer maturities and lower convexity is found in bonds with higher coupons and shorter maturities.

The formula for convexity is based upon its derivation from duration. Thus, the convexity of a bond equals the second derivative (the rate of change of slope) of the price-yield curve divided by the bond price. The convexity of a bond with a maturity of *n* years making annual coupon payments is

$$\text{Convexity} = \frac{1}{P \times (y+1)^2} \sum_{i=1}^{n} \left[\frac{CF_i}{(y+1)^t} (t^2 + t) \right]$$

Equation 3-10

Where, CF_i is the cash flow paid to the bondholder at date t, y is the yield to maturity, and P is the current price; CF_i represents the final coupon payment plus par value at the maturity date, but can also represent the coupon payment before maturity. Most portfolio software calculates convexity.

Immunization: Putting It All Together

In constructing bond portfolios to meet known funding needs, it is important to be aware of both price risk and reinvestment risk. Price risk arises when bond maturity extends past the time when the money will be needed. Conversely, reinvestment risk is present when bond maturity occurs before the time the money is needed. In a perfect world, these risks might be avoided by simply matching maturity dates and coupon payments with needs. Unfortunately, it is too difficult or too expensive or simply undesirable to adequately match liabilities by creating a *dedicated portfolio* where payments come due at the same time that liabilities come due.

Instead, *immunization* is used to reduce price risk and reinvestment risk in portfolios that do not exactly match cash needs with bond payments. The focus in an immunized portfolio turns to matching the present value of the investments with the present value of the liabilities. As discussed above, the key to present value and the main variable is the *assumed discount* rate. By eliminating all variables other than the discount rate, the resulting cash flow has risk exposure only in the event of changes in interest rates between the time of PV matching and the time the funds are required.

The goal of immunization is to reduce or eliminate the risk that the portfolio's value may change if interest rates change. As noted earlier, reducing interest rate volatility first requires a method to measure that volatility. Toward that end, modified duration and convexity have been discussed with an eye to calculating total portfolio volatility. In a well-immunized portfolio, the fluctuations in the portfolio value occur in tandem with fluctuations in the value of the liabilities. In other words, the duration of the asset and the liability are equal.

Thus, the process of immunizing a bond portfolio to fund a liability is to select a bond (or a bond portfolio) with a duration equal to the duration of the liability, and set the present value of the bond (or bond portfolio) equal to the present value of the liability.

While the mathematics of immunization is not introduced in this section, it will be useful to provide an example of an immunized portfolio. In the first example, modified duration is used. As mentioned above, duration is useful in measuring small changes in interest rates, but fails in an environment of large interest-rate swings. Since duration is also very sensitive to maturity, it can be said that, over time, the benefits of a duration-immunized portfolio will diminish.

Immunizing a Portfolio
Assume a pension plan has a known retirement benefit of $150 per year for the next ten years and we have two assets, a two-year zero-coupon bond and a seven-year 12-percent coupon bond. With a 5-percent discount rate, in what proportions should these bonds be combined to immunize the portfolio (i.e., match the duration of the liability)?

First, we must calculate the modified duration of the liability. It is 4.86, as follows:

Period (in Years) (A)	CF (B)	PV of CF at 5% (C)	Period × PV of CF (A) × (C)
1	$150	$142.86	$142.86
2	$150	$136.05	$272.10
3	$150	$129.58	$388.74
4	$150	$123.41	$493.64
5	$150	$117.53	$587.65
6	$150	$111.93	$671.58
7	$150	$106.60	$746.20
8	$150	$101.53	$812.24
9	$150	$96.69	$870.21
10	$150	$92.09	$920.90
Totals	$1,500	$1,158.27	$5,906.12

$$\text{Duration of Liabilities} = \frac{(1)PVL_1 + (2)PVL_2 + \ldots + (m)PVL_m}{\text{Total present value of liabilities}}$$

Equation 3-11

Where, PVL_t is the present value of the liability at time t, and m is the time of the last liability payment.

Macaulay's duration = $5,906.12 / $1,158.27 = 5.1

In this case, the question is how to invest \$1,158.27 (PV) to yield 5 percent per year for ten years.

$$\text{Modified Duration} = \frac{\text{Macaulay Duration}}{(1+\text{yield}/k)}$$

$$= \frac{5.1}{\left(1+\frac{0.05}{1}\right)} = 4.86$$

Thus, your strategy should be to select a bond portfolio with a duration of 4.86 years, and purchase a bond portfolio with a present value of \$1,158.27. If portfolio duration is less than liability duration, the portfolio is exposed to reinvestment risk. Alternatively, if portfolio duration is more than liability duration, the portfolio is exposed to price risk.

Secondly, we must learn the duration of our bond portfolio investments. Assume the modified duration of a 12-percent bond using a 5-percent discount rate is 5.11, and the duration of a two-year zero-coupon bond is 2.

Finally, we must solve for the combination or portfolio weighting of these two bonds that will produce the same duration as the liability. In viewing the calculations, remember that the sum of the weighted duration of the bonds in the portfolio is equal to the duration of the portfolio.

The formula for immunization portfolio weighting where *w* = weighting of asset is

W1 + W2 = 1, and by subtracting W1 from both sides, we arrive at

W2 = 1 − W1, substituting for W2, we get

5.11(W1) + [(1 − W1)(2)] = 4.86 and by solving for W1, we get

5.11(W1) + 2 − 2(W1) = 4.86

5.11(W1) – 2(W1) = 4.86 − 2

3.11(W1) = 2.86

W1 = 2.86/3.11 = 0.919 and round up to 92%

W1 = 92%, which means that W2 is 8% (1 − 92%)

92% of the PV of \$1,158.27 is \$1,065.61, which is the 12% bond

8% of the PV is \$92.66, which is the zero-coupon bond

We find that the portfolio's duration equals the liability's duration:

Portfolio duration = 0.92(5.11) + 0.08(2) = 4.86

Portfolio duration of 4.86 = Liability duration of 4.86

Let us view the performance of the immunized portfolio with a 1/2 percent rise in market rates using modified duration to learn the change in values:

		Bonds		
	Pension Liability	7-Year, 12% Bond 92%	2-Year, Zero-Coupon Bond 8%	Sum Bonds
Beginning Value (5.0% discount)	$1,158.27	$1,065.61	$92.66	
New Value (5.5% discount)	$1,130.64	$1,040.19	$90.45	
Change in Value	**$(27.63)**	**($25.42)**	**($2.21)**	**($ 27.63)**

Note: Amounts may vary slightly due to rounding.

Notice that the 1/2-percent increase in rates drives down the liability and asset values by equal amounts. Recognize that duration-based immunization strategies only are useful for short periods of time and under small changes in the interest-rate scenario. Using convexity, we are able to craft immunization programs that are more useful over longer periods and better maintain their accuracy in a climate of larger rate swings.

In conclusion, interest-rate risk can be quantified through modified duration, a time-weighted average of a bond's cash flow. The duration of a portfolio, like the rate of return, is the weighted average of the durations of the component bonds. Duration is valuable in computing the effect on market value of small interest-rate changes over short periods of time. By matching the duration of a portfolio with the duration of the liability, interest-rate risk can be reduced. However, the usefulness of duration fails under long periods or major rate movements due to what is known as convexity. Convexity cushions the downside risk of interest-rate moves and enhances the upside reward in bonds and can be used to successfully immunize a multibond portfolio against interest-rate risk for longer periods and greater rate changes than duration.