Math for Investment Consultants

2nd Edition

Chapter 4: Measuring Risk

The reading material in the appendixes is optional, you will not be tested on it.

Measuring risk is an important part of any consultant's or money manager's skills and talents. An investment asset is not considered as risky if its returns remain fairly stable over time, such as a Treasury bill or a savings account. If an investment asset's return fluctuates over a period of time, it is considered risky. The more the returns fluctuate, the riskier the asset is considered to be.

Standard Deviation and Variance

There are two common measures of volatility or risk: standard deviation and variance. These measures evaluate how much returns fluctuate from a mean or some target return. If one value is known, the other can be calculated. While expressed in a complex mathematical formula, the calculation of standard deviation is fairly straightforward.

1. Calculate the arithmetic mean rate of the returns.
2. Subtract this mean rate of return from each year's returns.
3. Square these differences.
4. Add the squares of the differences and find the arithmetic average of these sums. This value is the variance. The square root of this value is the standard deviation.

The equations look like this:

$$\text{Variance} = \sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}$$

Equation 4-1

Where, σ^2 is the variance, $\Sigma(x_i - \bar{x})^2$ is the sum of the squared deviations from the arithmetic mean return, and n is the total number of returns.

$$\text{Standard deviation} = \sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n}}$$

Equation 4-2

Where, σ is the standard deviation (i.e., square root of variance), $\Sigma(x_i - \bar{x})^2$ is the sum of the squared deviations from the arithmetic mean return, and n is the total number of returns.

While this sounds complicated, if you plot the steps on a grid, like the example below, it becomes fairly straightforward.

Table 4-1

	Return	Arithmetic Mean	Difference (Ret – Mean)	Squared Difference
Client #1	10.000	10.000	0.000	0.000
Client #2	10.250	10.000	0.250	0.063
Client #3	10.875	10.000	0.875	0.766
Client #4	9.750	10.000	–0.250	0.063
Client #5	9.125	10.000	–0.875	0.766
Sum	50.000			1.656
Average	10.000			0.331

0.331%^{2} is the variance of this portfolio. Standard deviation is the square root of variance or
$\sqrt{0.331} = 0.575\%$

Notice that the descriptive label of the variance is the "percent squared," but standard deviation is a "percent."

The standard deviation defines a band around the mean within which an investment's (or a portfolio's) returns tend to fall. The higher the standard deviation, the wider the band. One deviation from the mean indicates that the spread of returns falls between 0.575 percent above to 0.575 percent below the mean return. If we have a normal distribution of data, this creates a symmetrical bell curve and there is a 68-percent probability of falling within this range. There is a 95-percent probability of falling within two standard deviations of the mean. In this example, two standard deviations would be a range of 1.15 percent (0.575×2) above or below the mean return.

You may have seen these risk formulas using $n - 1$ instead of n in the denominator. Statistical theories suggest that to find the σ^2 and σ of a *population*, divide by the number (n) of data points, whereas the σ^2 and σ of a *sample* is found by dividing by the number of data points minus one ($n - 1$). This is called "*one degree of freedom*." It is important to recognize the different uses for n and $n - 1$; however, within the scope of this text the student should assume using n for the denominator unless otherwise instructed.

Semivariance and Downside Deviation

Standard deviation and variance measure variability, meaning that they measure both upside and downside risk. Downside variance and downside deviation are other measures of variability that focus only on the risk attributed to deviations below a minimum acceptable return.

The concepts are the same as variance and standard deviation, except we use only those values below our average or target return. Let's look at some slightly revised numbers for our table:

Table 4-2

	Return	Arithmetic Mean	Difference (Ret – Mean)	Squared Difference
Client #1	8.0%			
Client #2	–3.0%	2.4	–5.4	29.16
Client #3	1.0%	2.4	–1.4	1.96
Client #4	–1.0%	2.4	–3.4	11.56
Client #5	7.0%			
Sum	12.0%			42.68
Average	2.4%			8.54

Notice that we only use the returns that are below our average of 2.4 percent (i.e., our target return). In this example the semivariance is 8.54. (Notice that we still divide by 5—the total *n* rather than just 3.) The downside deviation is the square root of that, which is 2.92 percent.

Let's practice all of this again using a more-detailed example:

Table 4-3

Col. 1	Col. 2	Col. 3	Col. 4	Col. 5	Col. 6
Year	Return of B	Arithmetic Mean	Difference from Mean	Deviation Squared	Downside Semivariance
1	8	5.5	2.5	6.25	
2	11	5.5	5.5	30.25	
3	14	5.5	8.5	72.25	
4	16	5.5	10.5	110.25	
5	12	5.5	6.5	42.25	
6	8	5.5	2.5	6.25	
7	4	5.5	–1.5	2.25	2.25
8	–2	5.5	–7.5	56.25	56.25
9	–7	5.5	–12.5	156.25	156.25
10	–9	5.5	–14.5	210.25	210.25
Sum:	**55**			**692.50**	**425.00**
Arith Mean:	**5.5**		**Variance:**	**69.25**	**42.5**

$$\text{Variance of B} = \sigma_B^2 = \frac{692.5}{10} = 69.25\%^2$$

$$\text{Standard deviation of B} = \sigma_B = \sqrt{69.25} = 8.32\%$$

The downside risk of semivariance and semistandard deviation are found as before by using the returns below our average return of 5.5 percent:

$$\text{Semivariance } = \sigma^2 \text{ downside B} = \frac{2.25 + 56.25 + 156.25 + 210.25}{10}$$

$$= \frac{425}{10} = 42.50\%^2$$

$$\text{Downside deviation } = \sigma \text{ downside B} = \sqrt{42.50} = 6.52\%$$

Let's look at one final example:

Table 4-4

Year	Return of M	Arithmetic Mean	Difference from Mean	Deviation Squared	Downside Semivariance
1	6	4.8	1.2	1.44	
2	8	4.8	3.2	10.24	
3	10	4.8	5.2	27.04	
4	12	4.8	7.2	51.84	
5	13	4.8	8.2	67.24	
6	4	4.8	–0.8	0.64	0.64
7	–1	4.8	–5.8	33.64	33.64
8	–3	4.8	–7.8	60.84	60.84
9	–4	4.8	–8.8	77.44	77.44
10	3	4.8	–1.8	3.24	3.24
Sum:	48			**333.60**	**175.80**
Arith Mean:	**4.8**		**Variance:**	**33.36**	**17.58**

Therefore our variance is $33.36\%^2$ and standard deviation is 5.78 percent. Our downside measures are 17.58 and 4.19 percent, respectively.

To summarize:

Step 1. Find the arithmetic mean by summing up the returns and dividing by n

Step 2. Find the difference or "deviation" for each return by subtracting the mean from the return.

Step 3. Square each of the differences.

Step 4. Sum up all the squared differences.

Step 5. Divide the sum of the squares by n; this will be the variance.

Step 6. Take the square root of the variance to find the standard deviation.

Finding the downside semivariance and downside deviation:

Step 1. Find the mean.

Step 2. Find the difference for each return (use only those returns below the average or target return)

Step 3. Square the differences.

Step 4: Sum up those squared differences.

Step 5. Divide by *n*. (Note that even though you aren't using the positive values for the calculation, you don't reduce *n*.)

Step 6. Take the square root of the semivariance to find the downside deviation.

Please see Appendix 4–1 "Value at Risk and Probability of Loss" for closer look at measuring downside risk and what that means to your clients.

Covariance and Correlation

Now let's look at two other aspects of risk: covariance and correlation. *Covariance* is a measure of how two investments (or portfolios) move together or apart. Covariance, however, may be a large number so we use the concept of *correlation* to compress the covariance into a range of +/– 1. A negative correlation indicates an inverse relationship (assets move in opposite directions) whereas a positive correlation is indicative of a direct relation (assets move in the same direction). Obviously if diversification is the goal, the more negative the covariance the better. While one investment is flourishing, the other will move in the opposite direction. Covariance and correlation, then, can be important tools when exploring the benefits of diversification and investment risk. However, it is worth addressing that the benefit gained through diversification is minimal if we have overall negative returns. Thus, it is just as important to have average positive returns along with diversification.

Correlation is denoted by the Greek letter for R which is ρ (pronounced *rho*). To illustrate the concept of covariance and correlation we will find the covariance of investment A and the market (M). It should seem logical that if you want to examine how the returns move together or separately you should examine their deviations. The equation for covariance is:

$$\text{Covariance} = \frac{\sum_{i=1}^{n}(X_A - \bar{X}_A)(X_M - \bar{X}_M)}{n}$$

Equation 4-3

In words, this says:

Step 1. Find the differences between each return and the arithmetic mean for that asset.

This is exactly what we did for standard deviation.

Step 2. Multiply the differences of the two sets of assets, for each period.

Step 3. Add those up and divide by *n*.

Let us find the covariance of asset A and the market M.

Step 1. Assume asset A has the following deviations (differences) from the mean over ten years: –2.6, –0.6, –6.6, –7.6, –0.6, 3.4, 4.4, 2.4, 1.4, and 6.4. We already calculated the differences from the mean for the market (table 4-5).

Step 2. Multiply the differences of asset A and M for each period. See table 4-5 for the calculation up to this point

Step 3. Divide the sum by *n* (10) to get –16.98 as our covariance A and M.

Table 4-5

Year	Col. 4 Deviation from Mean A	Col. 4 Deviation from Mean M	A × M
1	–2.6	1.2	–3.12
2	–0.6	3.2	–1.92
3	–6.6	5.2	–34.32
4	–7.6	7.2	–54.72
5	–0.6	8.2	–4.92
6	3.4	–0.8	–2.72
7	4.4	–5.8	–25.52
8	2.4	–7.8	–18.72
9	1.4	–8.8	–12.32
10	6.4	–1.8	–11.52
		Sum>>	**–169.8**

As this is a negative value, investment A moves contrary to the market. When A is negative, M is positive and vice versa.

Covariance is also found as:

$$\sigma^2_{A,M} = \rho_{A,M}\sigma_A\sigma_M$$

Equation 4-4

Where, σ^2_{AM} is the covariance of asset A and M (i.e., market), ρ (called *rho*) is the correlation coefficient of assets A and M, σ_A is the standard deviation of asset A, and σ_M is the standard deviation of asset M.

Once the covariance has been found, the correlation is found by utilizing the following formula:

$$\text{Correlation}_{A,M} = \rho_{A,M} = \frac{\text{Covariance}_{A,M}}{\sigma_A \sigma_M}$$

Equation 4-5

Where, ρ is the correlation coefficient of assets A and M, σ_A is the standard deviation of asset A, and σ_M is the standard deviation of asset M.

To find the correlation, divide the covariance of the two investments you want to compare by the product (multiplication) of the standard deviations of the two investments.

Assume the standard deviation of asset A is 4.34. The standard deviation of the market (table 4-4) is 5.78. The calculation for the correlation of asset A and M is:

$$\rho_{A,M} = \frac{-16.98}{(4.34)(5.78)} = -0.677$$

Now let's use the data for investment B (table 4-3) to find the covariance of investment B and the market.

Table 4-6

Year	Col. 4 Deviation from Mean B	Col. 4 Deviation from Mean M	B × M
1	2.5	1.2	3.0
2	5.5	3.2	17.6
3	8.5	5.2	44.2
4	10.5	7.2	75.6
5	6.5	8.2	53.3
6	2.5	–0.8	–2.0
7	–1.5	–5.8	8.7
8	–7.5	–7.8	58.5
9	–12.5	–8.8	110.0
10	–14.5	–1.8	26.1
		Sum>>	**395.0**

$$\text{Covariance}_{B,M} = \frac{395}{10} = 39.5$$

$$\text{Correlation}_{B,M} = \rho_{B,M} = \frac{\text{Covariance}_{B,M}}{\sigma_B \sigma_M} = \frac{39.50}{(8.32)(5.78)}$$

$$= \rho_{B,M} = 0.821$$

Investment B and the market have a positive correlation. Their averages move together.

Finally, the covariance of A and B is:

Table 4-7

Year	Deviation from Mean B	Deviation from Mean A	B × A
1	2.5	–2.6	–6.5
2	5.5	–0.6	–3.3
3	8.5	–6.6	–56.1
4	10.5	–7.6	–79.8
5	6.5	–0.6	–3.9
6	2.5	3.4	8.5
7	–1.5	4.4	–6.6
8	–7.5	2.4	–18.0
9	–12.5	1.4	–17.5
10	–14.5	6.4	–92.8
		Sum>>	**–276.0**

$$\text{Covariance}_{B,A} = \frac{-276}{10} = -27.6$$

$$\text{Correlation}_{B,A} = \rho_{B,A} = \frac{\text{Covariance}_{B,A}}{\sigma_B \sigma_a} = \frac{-27.6}{(8.32)(4.34)}$$

$$= \rho_{B,A} = -0.764$$

Beta

Another measure of investment risk is beta (β). Beta is a measure of systematic or *market-related risk*. It cannot be diversified away because it is the risk of being invested in the market. The beta of an investment reflects both the risk of an investment relative to the market and also reflects the correlation with the market. It is, in some sense, a measure of the response of the security to market movement. The beta of an investment may be found by a number of methods, two of which are presented below. A summary of results from numbers that were given (on asset A) and from the previous examples:

	Results of Returns from A	B	M
Variance	18.84%	69.25%	33.36%
Standard deviation	4.34%	8.32%	5.78%
Semivariance	10.88%	42.50%	17.58%
Downside	3.30%	6.52%	4.19%
	AB	**AM**	**BM**
Covariance	–27.6	–16.98	39.5
Correlation	–0.764	–0.677	0.821

The equation for beta is:

$$\text{Beta}_{\text{Investment}} = \frac{\text{Covariance}_{\text{Investment,Market}}}{\text{Variance}_{\text{Market}}} = \frac{\text{Cov}_{I,M}}{\sigma_M^2}$$

$$\text{The Beta of A} = \beta_A = \frac{\text{Covariance}_{A,M}}{\sigma_M^2} = \frac{-16.98}{33.36} = -0.509$$

Equation 4-6

A second method:

$$\text{Beta}_{\text{Investment}} = \frac{(\text{Standard deviation}_{\text{Investment}})(\text{Correlation}_{\text{Investment,Market}})}{\text{Standard deviation}_{\text{Market}}} = \frac{\sigma_I \rho_{I,M}}{\sigma_M}$$

$$\text{The Beta of A} = \beta_A = \frac{\sigma_A \rho_{A,M}}{\sigma_M} = \frac{(4.34)(-0.677)}{5.78} = -0.509$$

Equation 4-7

The beta of B by both methods:

$$\beta_B = \frac{Cov_{B,M}}{\sigma_M^2} = \frac{39.50}{33.36} = 1.184$$

$$\beta_B = \frac{\sigma_B \rho_{B,M}}{\sigma_M} = \frac{(8.32)(0.821)}{5.78} = 1.182$$

The beta of the market is the standard in this example and therefore will equal 1.0. If the stock has a beta of 1.0, the implication is the stock moves exactly with the market. A beta of 1.3 is 30-percent more volatile than the market, and a beta of 0.7 is 30-percent less volatile than the market.

Capital Asset Pricing Model (CAPM)

The capital asset pricing model advances the relationship between risk and return in the efficient market context, and adds the possibility of earning a risk-free return. The concept is applied at the macro level, which specifies the relationship between risk and return on a portfolio, and the micro context, which specifies this relationship on a specific asset.

The macro aspect of CAPM is the development of the *capital market line*, where risk is measured by the portfolio's standard deviation. In effect, the capital market line becomes the efficient frontier. Portfolios on this line represent the best attainable combination of risk and return. Portfolio combinations range from no risk by earning the risk-free rate return to high-risk in which securities are bought on margin.

The equation for the capital market line is derived from the equation for a straight line:

$$Y = a + bX$$

Where, Y is the return on the portfolio (r_p); a, the intercept, is the risk-free rate (r_f), X is the risk premium, and b is the slope of the line.

The equation for the capital market line is

$$R_p = R_f + \left(\frac{R_m - R_f}{\sigma_m}\right)\sigma_p$$

Equation 4-8

The equation states that the return on the portfolio (R_p) is the sum of the return earned on the risk-free asset (R_f) such as a Treasury bill and a risk premium that is dependent upon (1) the extent to which the return on the market exceeds the risk-free return ($R_m - R_f$) and (2) the standard deviation of the portfolio (σ_p) relative to the standard deviation of the market (σ_m). We find that if the dispersion of the portfolio is greater than the dispersion of the market, the return will have to exceed the return associated with the market.

The micro aspect of CAPM is the development of the *security market line* (SML). In the security market line, risk is measured by a beta coefficient. Thus, at the micro level, the CAPM is the specification of the relationship between risk and return for the individual asset. The equation for the capital market line then becomes the security market line:

$$R_i = R_f + (R_m - R_f)\beta \quad \textbf{Equation 4-9}$$

Where, the return of an investment (R_i) is equal to the risk-free Treasury rate (R_f) plus the beta (β) of the investment times the difference of return of the market and the risk-free rate ($R_m - R_f$). The term ($R_m - R_f$) is the market risk premium.

This equilibrium equation demonstrates the combination of efficient asset selection and beta. The CAPM return can be thought of as style-adjusted return. It allows us to determine the effect of adding a security to a portfolio against a market index.

The equation for the security market line also is derived from the equation for a straight line (i.e., $Y = a + bX$). The difference is the measure of risk on the x-axis. The capital market line uses standard deviation as a measure of risk, and the security market line uses beta. The CAPM seeks to explain security returns. The CAPM also establishes a criterion for assessing portfolio performance.

Let us look at an example of calculating CAPM. Assume the risk-free Treasury rate is 2 percent, beta is 1.28, and the market return is 9 percent. The expected return of asset B is

$$E(R_B) = 2 + 1.28\,(9 - 2)$$
$$E(R_B) = 2 + 1.28\,(7)$$
$$E(R_B) = 2 + 8.96$$
$$E(R_B) = 10.96\%$$

This value would be consistent with what we would expect, as we know there is a positive correlation between asset B and the market, and the beta of asset B is higher than the market beta of 1.00.

Assume we have an expected return of 8 percent for asset C. Given the market return of 9 percent and the risk-free Treasury of 2 percent, the beta of asset C is

$$E(R_C) = R_F + \beta_C\,(R_M - R_F)$$
$$8 = 2 + \beta_C\,(9 - 2)$$
$$8 = 2 + \beta_C\,(7)$$
$$6 = \beta_C\,(7)$$
$$\beta_C = 0.857$$

A well-diversified portfolio's returns are proportional to its beta. So a portfolio with a beta of 1.5 has twice the risk of a portfolio with a beta of 0.75.

Appendix 4–1

APPENDIX C

Value at Risk and Probability of Loss

Overview

Whether investing in traditional securities portfolios or in hedge funds, an investor should have some idea of how much risk of capital loss may exist and how to quantify it. This risk of loss can be measured as the potential loss in the value of a portfolio for a stated probability over a particular time horizon, or it can be the probability of a specific loss over the time horizon. The first measure, amount of potential loss, is called *value at risk,* and the second measure is simply called *probability of loss.*

Value at risk, or VaR, is the "*V*" in the following statement: "Over the following time period, *t*, the portfolio will lose no more than *V*. I am *p* percent confident in this." VaR can be stated as a dollar amount, or as a percent of the portfolio. Stating it as a percent provides for comparability across portfolios of different sizes. To fully understand a particular VaR number you need to know the *t* and *p* above; in other words, you need to know the time period over which this potential future maximum loss is being estimated, and the level of confidence that the loss could be no worse.

Probability of loss is similar to VaR

In this case, you specify the *t*, or time period, and the *V*, or amount of loss you're concerned about, and the measure delivers the *p*, or probability. Both VaR and probability of loss can be calculated using the assumption that returns are normally distributed. With this assumption, the risk measures are calculated by first converting the loss into the number of standard deviations away from the mean, and then calculating the probability of exceeding this number in a normal distribution.

When seeking to measure VaR and probability of loss for hedged portfolios, however, the assumption of normally distributed returns often fails because of the presumption that most absolute return strategies largely eliminate downside volatility (hence, the word "hedge"). Instead, distributions for hedged portfolios can be empirically derived through the use of Monte Carlo simulations, including bootstrap, non-normal, and GARCH (general autoregressive conditional heteroschedascticity) methodologies. VaR can also be calculated using the Parametric Methodology or historical simulations. Input for these models are generally based on the actual historic returns of each of the hedged portfolios being evaluated, including the periodic returns of diversified Fund of Hedge Fund portfolios. Monte Carlo methodology is preferable when non-linear instruments, including derivatives, are used. It is also useful to estimate the historic correlation and volatilities for the underlying component investments.

Risk Budgeting

Risk budgeting is a control mechanism for keeping an overall portfolio within acceptable and pre-defined risk limits. Sometimes called *risk allocation* (in contrast to *asset allocation*) the idea is to monitor overall portfolio risk, rather than allocations to asset classes. If risk falls outside the acceptable range—above or below—assets are re-allocated among asset classes and managers. Particular attention is focused on contemporaneous estimates of future risk, usually estimated as future standard deviations, and, most importantly, future interactions, measured as correlations. It is this interaction piece that brings home the benefits of diversification. An asset class or investment approach that is very risky may not affect the risk budget much at all if it is uncorrelated with other assets in the portfolio. Instead, such an allocation may actually introduce elements of stability to the portfolio. Risk budgeting, as an investment management practice, is greatly enhanced when a large number of low or negatively-correlated investment options are available, especially if the focus of these strategies is absolute return.

Sometimes risk budgets are monitored by using both VaR and *marginal VaR*, which measures the marginal contribution of each manager to downside exposure. The idea here is to calculate VaR on the entire portfolio, and the portfolio without a particular manager. The difference is that manager's marginal VaR.

A detailed explanation of the uses of VaR and its calculation is beyond the scope of this appendix. A good source for those seeking to delve further is *Value at Risk: The New Benchmark for Managing Financial Risk* by Philippe Jorion.[1] IMCA

[1] Philippe Jorion, *Value at Risk: The New Benchmark for Managing Financial Risk*, 2nd ed., 2000, McGraw-Hill.