Math for Investment Consultants

2nd Edition

Chapter 2: Time Value of Money and Rates of Return

In this section we will review how changes in return will affect the return on an investment. It is important to distinguish between compounding and discounting. *Compounding* is the growth of money to a future value, and *discounting* is the reduction of money due in the future to a smaller amount, the present value, of the money today.

Simple interest is the interest that is earned only on the original principal amount. It does not account for compounding. The calculation is

$I = P \times R \times N$

Where, I is the simple interest, P is the principal amount, R is the rate of interest, and N is the time period for which the principal is invested or lent.

Thus, the simple interest on \$100,000 for 45 days at 10 percent is

$I = 100{,}000 \times 0.10 \times 45/365 = \$1{,}232.87$

The future value is \$100,000 + \$1,232.87 = \$101,232.87

Investments that grow in this fashion are linear (figure 2-1). This creates a linear equation (straight line) in which the rate at which the investment grows is the slope of the line. However, we usually are more concerned with the graph in figure 2-2 of the exponential curve, which describes compounded interest.

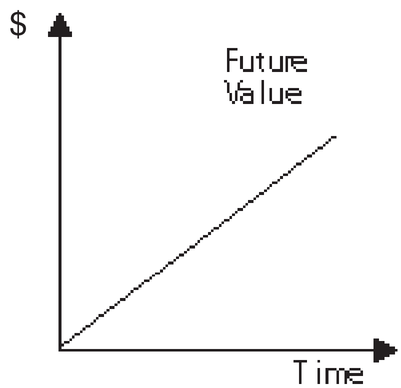


Figure 2-1

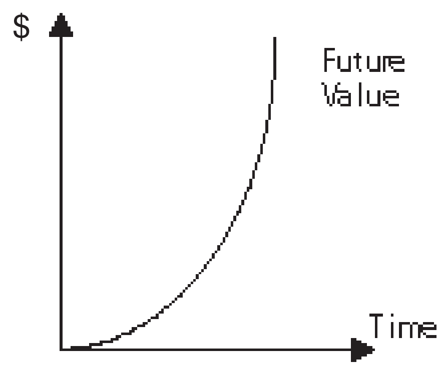


Figure 2-2

The slope of the line in figure 2-2 is not as easily calculated as that of figure 3-1. Calculus is required to determine the slope of a point on a curved line, which is beyond the scope of this text. The concept is important though. Unlike simple interest, compound interest is computed not only on the original principal but also on the interest already earned. This causes the well-known "miracle of compounding." As it applies to future value, it is apparent that an investment that grows exponentially is much preferred over the linear model.

Future Value Single Sum

The future value formula for compounding interest looks like this:

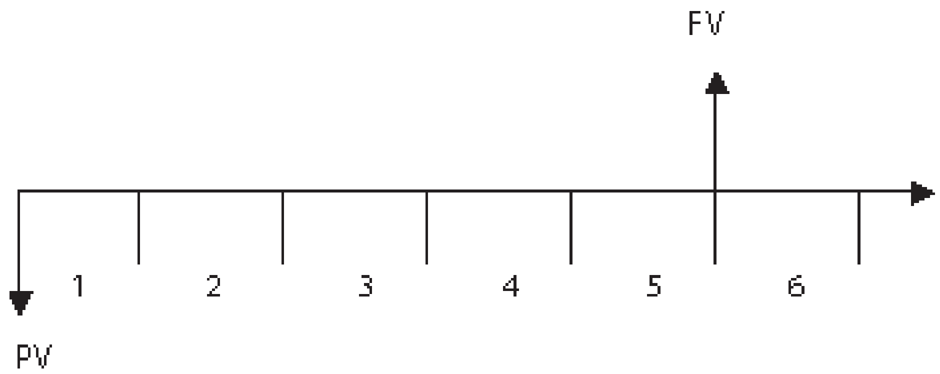
$FV = PV\,(1 + r)^n$ **Equation 2-1**

Where, FV is the future value of an investment, PV is the present value of an investment, r is the interest rate expressed as a decimal, and n is the number of time periods. The n is of particular importance because mathematically it represents exponential growth. Suppose that an individual invests $10 in an account that earns 5-percent compounded interest. After 50 years the account will have $115, in 50 more years the account will have $1,315. Notice how fast the money grows over time.

Year	Amount at the End of the Year	Growth Between Each 50 Years
1	$11	N/A
50	$115	$105
100	$1,315	$1,200
150	$15,080	$13,765
200	$172,926	$157,846
250	$1,983,009	$1,810,084
300	$22,739,961	$20,756,952
350	$260,768,227	$238,028,266
400	$2,990,333,512	$2,729,565,285
450	$34,291,349,880	$31,301,016,368
500	$393,232,618,272	$358,941,268,392

We can see that although the interest rate is held constant at 5 percent, the compounding of that interest rate generates the wealth.

It is important to consider the length of time money is invested, or more accurately, whether there is one initial deposit (PV) that grows to a future value (FV) or a number of deposits made over time. The simplest case is illustrated in the timeline:



This timeline indicates that a deposit is made at the beginning of period 1. This initial deposit is the present value and traditionally is indicated by the arrow placed below the timeline. The future value of a withdrawal is noted by the arrow above the line and would be the amount in the account at the end of year 5 (beginning of year 6).

If we invest $10,000 at the beginning of year 1 at an interest rate of 3 percent compounded annually, at the end of year 5 (beginning of year 6), we would find:

$$FV = \$10{,}000(1.03)^5$$

$$FV = \$11{,}592.74$$

The future value calculating with simple interest at the end of five years is:

$$\$10{,}000 + (\$10{,}000 \times 0.03 \times 5) = \$11{,}500.00$$

Present Value Single Sum

Present value is what a future cash flow is worth today. The basic future value equation can be rewritten to solve for present value:

$$PV = \frac{FV}{(1+i)^n}$$

Equation 2-2

Where, *FV* is the future value of an investment, *PV* is the present value of an investment, *r* is the interest rate expressed as a decimal, and *n* is the number of time periods.

Determine the present value of $20,000 to be received in five years, given a discount rate of 10 percent compounded annually. The calculation is

$$PV = \frac{20{,}000}{(1+0.10)^5} = \$12{,}418.42$$

Compounding Frequencies

Normally we think of the interest rate in terms of annual compounding. If we put $1,000 in an account that earns 8 percent annually, at the end of one year the account will contain $1,080 [$1,000(1.08)]. What if the 8 percent was compounded quarterly? Then we would have 4 periods of interest per year and we would adjust the interest rate accordingly:

$$r = \frac{8\%}{4} = 2\%$$

So using the future value equation:

$$\$1{,}000(1+0.02)^4 = \$1{,}082.43$$

Using the same example, if the 8 percent was compounded semiannually:

$$\$1{,}000(1+0.04)^2 = \$1{,}081.60$$

Unless told otherwise you can assume the interest in this text is annual.

Rule of 72

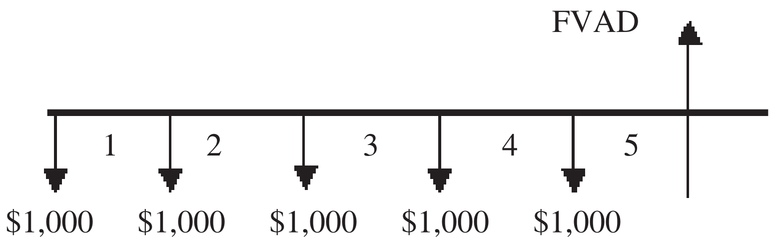
The rule of 72 provides a fast way to determine the time or interest rate required for an investment to double in value. This rule really is an estimate. In order to determine the number of years for an investment to double in value, 72 is divided by the annual interest rate. For example, if the interest rate is 7 percent, then the time required for an investment to double is 10.28 years (72/7). In contrast, we can find the interest rate required for an investment to double in 10.28 years by taking 72 divided by 10.28 years, which is 7 percent (72/10.28).

Future Value Annuity Due

A more complicated case arises when many deposits are made over a period of time. The client either invests money at the beginning of each period (daily, weekly, monthly, etc.), or at the end of each period. If money is deposited at *the beginning* of each period, it is called an *annuity due*, and we can calculate the future value at some time *t*. The money accrued by compounding at the end of this time *t* is called the future value of an annuity due (FVAD).

Suppose the client deposits $1,000 in the bank today and at the beginning of each of the next four years. What is the FVAD at the end of the fifth year, given a 7-percent interest rate compounded annually? The timeline illustrates this case:

This changes our future value equation somewhat. The equation for FVAD is as follows:



$$\begin{aligned} FVAD &= (1+r)\left[\left(\frac{(1+r)^n - 1}{r}\right)\times(\text{initial deposit})\right] \\ &= (1.07)\left[\left(\frac{(1.07)^5 - 1}{0.07}\right)\times 1{,}000\right] \\ &= \$6{,}153.29 \end{aligned}$$

Equation 2-3

What the equation calculates really is a series of future value problems as shown below:

$FV = PV\,(1 + 0.07)^1$

$FV = \$1{,}000\ (1.07) = \$1{,}070.00$ at end of year one

For subsequent years:

$FV = \mathrm{PV}\,(1 + 0.07)^2$

$FV = \$1{,}000\ (1.1449) = \$1{,}144.90$ from year 2

$FV = \mathrm{PV}\,(1 + 0.07)^3$

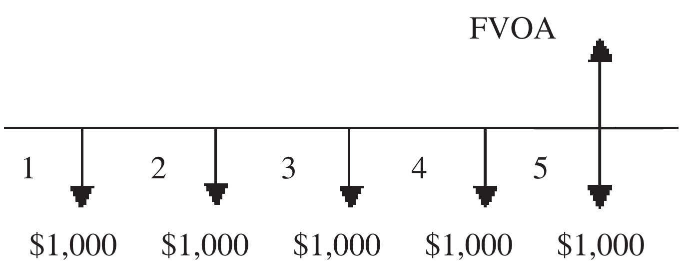
$FV = \$1{,}000\ (1.22504) = \$1{,}225.04$ from year 3

Year 4 would bring \$1,310.80 and year 5 = \$1,402.55

Adding these up we find the total value to be in agreement with our original answer of \$6,153.29 calculated by the FVAD formula.

Future Value Ordinary Annuity (FVOA)

An ordinary annuity assumes that the deposits were made at the end of the year rather than the beginning. The timeline would change as illustrated:



The equation changes by losing the $(1 + r_i)$ factor outside the bracket to indicate the loss of one year's compounding. Thus, the future value of an ordinary annuity (FVOA) is

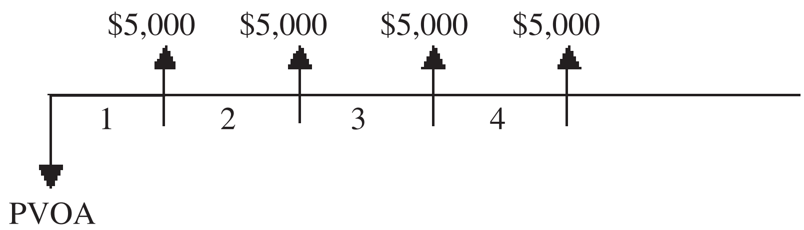
$$\begin{aligned} \mathrm{FVOA} &= \frac{(1+r)^n - 1}{r} \times (\text{initial deposit}) \\ &= \frac{(1.07)^5 - 1}{0.07} \times 1{,}000 \\ &= \$5{,}750.74 \end{aligned}$$

Equation 2-4

The difference is \$402.55 between depositing at the beginning of the year versus year-end.

Present Value Ordinary Annuity (PVOA)

To find the present value of an annuity, we discount the cash flow stream back to the present. Suppose a client needs to deposit a lump sum in trust to provide future child support payments. The payments are to be $5,000 per year for four years, beginning one year from today. If the amount to be placed in trust is assumed to earn 6-percent interest compounded annually, how much should the client place in trust now? The timeline for this situation is



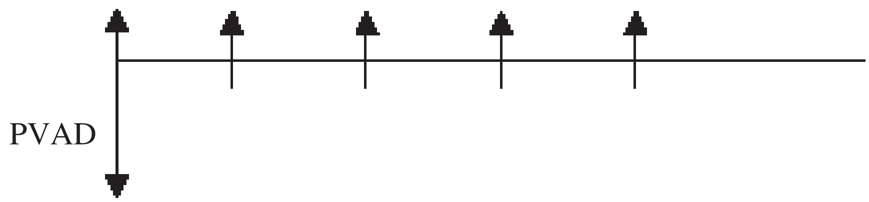
This timeline asks to find the **present value of an ordinary annuity** whose payments start at the end of year 1. The equation is as follows

$$\begin{aligned} PVOA &= \frac{\left(1 - \frac{1}{(1+r)^n}\right)}{r} \times (\text{initial deposit}) \\ &= \frac{\left(1 - \frac{1}{(1+0.06)^4}\right)}{0.06} \times 5{,}000 \\ &= \$17{,}325.53 \end{aligned}$$

Equation 2-5

Present Value Annuity Due (PVAD)

Alternatively, if the payments were paid at the beginning of each year, the timeline would be as follows:



This would be the PVAD and so the PVOA formula should be multiplied by (1.06) to account for the extra year:

$PVAD = (1.06)(17{,}325.53) = \$18{,}365.06$

The formula for adding the extra payment is similar to that of FVAD in that there is one extra period to pay:

$$PVAD = (1+r)\left[\frac{\left(1-\frac{1}{(1+r)^n}\right)}{r} \times (\text{initial deposit})\right]$$

$$= (1+0.06)\left[\frac{\left(1-\frac{1}{(1+0.06)^4}\right)}{0.06} \times 5{,}000\right]$$

$$= \$18{,}365.06$$

Equation 2-6

Holding-Period Rate of Return

The holding-period return is determined by taking the total return divided by the initial cost of the investment. It is

$$HPR = \frac{(P_1 - P_0) + D_1}{P_0} \times 100$$

Equation 2-7

Where, P_1 is the sale price (price at the end of the period), P_0 is the purchase price (initial price), and D_1 is the cash distribution received during period t. For example, an individual buys a stock for \$100, receives a \$1 dividend, and later sells it for \$115 dollars. The holding-period return is

$$HPR = \frac{\$115 - \$100 + \$1}{\$100} \times 100 = 16\%$$

There is a major weakness with using the holding period. It does not consider how long it took to earn the return. We learned that this problem is fixed through compounding calculations.

Effective Interest Rate

Earning a rate of 10 percent compounded annually is different than earning 10 percent compounded quarterly. The stated rate of interest is known as the nominal rate. The periodic rate of interest is the rate of interest earned over a single compounding period. For example, a rate of 10 percent compounded quarterly is a periodic rate of 2.5 percent. However, the true rate of interest is known as the effective rate and represents the rate that actually is earned.

The effective rate of interest is

Effective rate = $(1 + \text{periodic rate})^m - 1$ **Equation 2-8**

Where, m is the number of compounding periods in a year.

Compute the effective rate of 10 percent compounded quarterly:

Effective rate = $(1.025)^4 - 1 = 0.1038$ or 10.38%

Real Rate of Return

The real return is that rate of return that is above inflation. Investors must overcome any loss of purchasing power to achieve their financial goals. The real return is

$$\text{Real Return} = \left(\frac{1+\text{nominal rate}}{1+\text{inflation rate}} - 1\right) \times 100$$

Equation 2-9

Where, the nominal rate is the stated rate and inflation is the rate of inflation for the period. For example, assume the nominal rate over five years is 10 percent and the inflation rate is 3 percent. The real return is

$$\text{Real Return} = \left(\frac{1.10}{1.03} - 1\right) \times 100 = 6.79\%$$

Notice that the nominal rate of return is a derivation of the formula:

(1 + nominal rate) = (1 + real rate) × (1 + inflation rate)

After-tax Return

The after-tax return = pre-tax return (1 − marginal tax rate) **Equation 2-10**

For example, if an asset has a taxable return of 12 percent and an investor is in a 36-percent tax bracket, the after-tax return is

After-tax return = 0.12 (1 − 0.36) = 0.0768 or 7.68%

We can also solve for the taxable-equivalent return:

Taxable-equivalent return = after-tax return / (1 − marginal tax rate)

Internal Rate of Return (IRR)

Internal rate of return is the discount interest rate that makes the present value of the cash outflows equal to the present value of cash inflows. It also is known as the yield to maturity (YTM) for bonds.

Let us start first by considering a simple example. Assume the present value of an asset is $500 and the future value is $1,000 in five years. The IRR simply is found by using the present value equation and solving for *i*:

$$\$500 = \frac{\$1,000}{(1+i)^5}$$

$$(1+i)^5 = \frac{\$1,000}{\$500}$$

$$(1+i) = \$2^{1/5}$$

$$i = 1.1487 - 1 = 0.1487\% \text{ or } 14.87\%$$

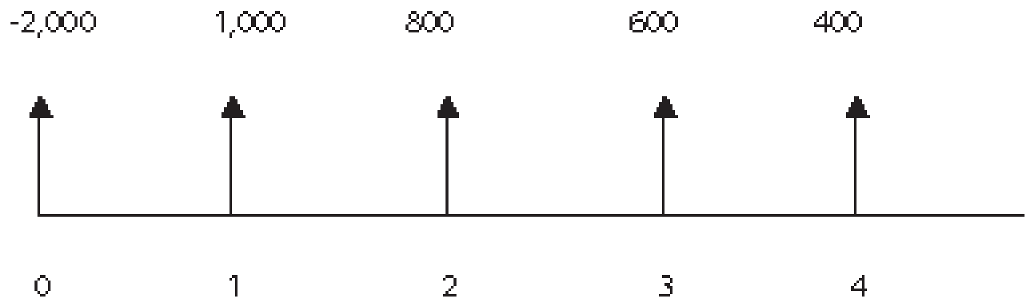
The above calculation does not factor in any cash flows between periods. If we introduce cash flows for each period, we find that these cash flows are nothing more than a stream of (annual) single cash flows, which then are summed up to find the PV or FV. We now can state that the IRR also is the rate of return for which the net present value (NPV) is zero.

$$NPV = 0 = CF_0 + \frac{CF_1}{(1+IRR)^1} + \frac{CF_2}{(1+IRR)^2} + \ldots + \frac{CF_n}{(1+IRR)^n}$$

Where, CF_n is the cash flow for *n* periods.

To calculate the IRR with uneven cash flows, we use trial and error. That is, we will need to keep picking values for IRR until the extensive formula works out. This is a mathematical challenge.The good news is that the calculation is easy with the use of a calculator. A calculator can find the NPV and IRR of an uneven cash-flow stream by using the cash-flow keys and the NPV function on most calculators.

Consider an investment cash-flow stream that changes each year. An example of such a stream of cash flows is



The IRR of the uneven cash flows is

$$0 = -2000 + \frac{1000}{(1+IRR)^1} + \frac{800}{(1+IRR)^2} + \frac{600}{(1+IRR)^3} + \frac{400}{(1+IRR)^4}$$

Trial and error gives us IRR = 17.80%. The good news is that the internal rate of return easily is determined by using a calculator. The common notation on most calculators is

Notation Used on Most Calculators	Numerical Value for This Problem
CF_0	–2000
CF_j	1000
CF_j	800
CF_j	600
CF_j	400
%i Compute	

The IRR has one large problem. It assumes all cash flows received during the period are reinvested at the IRR. This rarely is the case. We know that interest rates change over time.

Net Present Value (NPV)

Net present value is the present value of future cash flows discounted at the required rate of return (i.e., the company's cost of funds). NPV shows the amount of cash flow that a project or investment generates after repaying the invested capital and the required rate of return on the capital. If a project or asset as NPV > 0, then the project increases shareholders' wealth. If NPV < 0, then the project or asset decreases shareholders' wealth. The equation for NPV is

$$NPV = CF_0 + \frac{CF_1}{(1+i)^1} + \frac{CF_2}{(1+i)^2} + \ldots + \frac{CF_n}{(1+i)^n}$$ **Equation 2-11**

Where, CF_n is the cash flow for n periods.

We have seen this equation when solving for IRR; the notable difference is the interest rate, which now is the required rate of return. For example, an investment promises to pay $100 one year from today, $200 two years from today, and $400 three years from today. If the required rate of return is 12 percent, what is the value of the investment today?

$$NPV = 0 + \frac{\$100}{(1+0.12)} + \frac{\$200}{(1+0.12)^2} + \frac{\$400}{(1+0.12)^3} = \$533.44$$

Dollar-Weighted Return

The dollar-weighted return is the internal rate of return of an investment portfolio, taking into account all cash inflows and outflows.

Assume an investor buys one share of stock for $100 at the beginning of the first year, and buys another share for $110 at the end of the first year. The investor earns $1 in dividends in the first year and $2 in the second year. What is the dollar-weighted return if the shares are sold at the end of the second year for $115 each?

Step 1. Determine the timing of each cash flow

There are two cash outflows: $100 at time period $t = 0$ and $110 at time period $t = 1$.

There also are two cash inflows: $1 at time period $t = 1$ and $232 ($2 dividends plus 230 proceeds) at time period $t = 2$.

Step 2. Net out the cash flows for each time period and set the present value of cash inflows equal to the present value of cash outflows.

The cash flows are grouped by time. The $t = 0$ net cash flow is –100, and $t = 1$ net cash flow is –109 (–\$110 + 1), and the $t = 2$ net cash flow is \$232 (\$230 + \$2).

PV (outflows) = PV (inflows)

$$\$100 + \frac{\$110}{(1+r)} = \frac{\$1}{(1+r)} + \frac{\$232}{(1+r)^2}$$

Step 3. Solve for r to find the dollar-weighted rate of return. This can be done using trial and error, the IRR function on your financial calculator, or a spreadsheet. We learned earlier that the easiest way to calculate the IRR is by calculator. The IRR is 7.27 percent, found by using the cash-flow function on a calculator.

Notation Used on Most Calculators	Numerical Value for This Problem
CF_0	–100
CF_j	–109
CF_j	232
%i Compute	

Time-Weighted Return

The time-weighted return calculates the return for each period and takes the average of the results. It is the average of the holding periods. The geometric mean is commonly used when calculating the average.

Returning to our earlier example, an investor buys one share of stock for \$100 at the beginning of the first year and buys another share for \$110 at the end of the first year. The investor earns \$1 in dividends in the first year and \$2 in the second year. What is the time-weighted return if the shares are sold at the end of the second year for \$115 each?

Step 1. Break the measuring period into subperiods based on timing of cash flows.

Holding Period 1

Beginning Price = $100

Dividends Earned = $1

Ending Price = $110

Holding Period 2

Beginning Price = $220 (2 shares at $110)

Dividends Earned = $2

Ending Price = $230

Step 2. Calculate the holding period return (HPR) for each holding period.

The return for the first year is

$$HPR = \frac{\$110 - \$100 + 1}{\$100} = .11 = 11\%$$

The return for the second year is

$$HPR = \frac{\$230 - \$220 + 2}{\$220} = .0545 = 5.45\%$$

Step 3. Take the geometric mean of annual returns to find the annualized time-weighted rate of return of the time period.

The geometric mean return is

$$[(1.11)(1.0545)]^{1/2} - 1 = 0.0819 \text{ or } 8.19\%$$

In the investment management industry, *the time-weighted return is the preferred method of performance measurement because it is not affected by the timing of cash flows.* If a client adds funds to an investment portfolio at an unfavorable time, the dollar-weighted return is worse than expected. If funds are added at a favorable time, the dollar-weighted return looks better than expected. It is very important for an investment consultant to show a client both his/her portfolio's dollar-weighted and time-weighted returns over a given period of time. Furthermore the investment consultant should explain the difference between these two measurements to the client and why each is important.