(**QFT_3qubits**)

$$\text{QF}[\text{v\_}] = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & v & v^2 & v^3 & v^4 & v^5 & v^6 & v^7 \\ 1 & v^2 & v^4 & v^6 & 1 & v^2 & v^4 & v^6 \\ 1 & v^3 & v^6 & v & v^4 & v^7 & v^2 & v^5 \\ 1 & v^4 & 1 & v^4 & 1 & v^4 & 1 & v^4 \\ 1 & v^5 & v^2 & v^7 & v^4 & v & v^6 & v^3 \\ 1 & v^6 & v^4 & v^2 & 1 & v^6 & v^4 & v^2 \\ 1 & v^7 & v^6 & v^5 & v^4 & v^3 & v^2 & v \end{pmatrix};$$

$$\text{QF}\left[e^{\text{I Pi}/4}\right]$$

$$\begin{aligned}\text{QF3} = \Big\{&\Big\{\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}\Big\},\\
&\Big\{\tfrac{1}{2\sqrt{2}}, \tfrac{e^{\frac{i\pi}{4}}}{2\sqrt{2}}, \tfrac{i}{2\sqrt{2}}, \tfrac{e^{\frac{3i\pi}{4}}}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, \tfrac{e^{-\frac{3i\pi}{4}}}{2\sqrt{2}}, -\tfrac{i}{2\sqrt{2}}, \tfrac{e^{-\frac{i\pi}{4}}}{2\sqrt{2}}\Big\},\\
&\Big\{\tfrac{1}{2\sqrt{2}}, \tfrac{i}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, -\tfrac{i}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, \tfrac{i}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, -\tfrac{i}{2\sqrt{2}}\Big\},\\
&\Big\{\tfrac{1}{2\sqrt{2}}, \tfrac{e^{\frac{3i\pi}{4}}}{2\sqrt{2}}, -\tfrac{i}{2\sqrt{2}}, \tfrac{e^{\frac{i\pi}{4}}}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, \tfrac{e^{-\frac{i\pi}{4}}}{2\sqrt{2}}, \tfrac{i}{2\sqrt{2}}, \tfrac{e^{-\frac{3i\pi}{4}}}{2\sqrt{2}}\Big\},\\
&\Big\{\tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}\Big\},\\
&\Big\{\tfrac{1}{2\sqrt{2}}, \tfrac{e^{-\frac{3i\pi}{4}}}{2\sqrt{2}}, \tfrac{i}{2\sqrt{2}}, \tfrac{e^{-\frac{i\pi}{4}}}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, \tfrac{e^{\frac{i\pi}{4}}}{2\sqrt{2}}, -\tfrac{i}{2\sqrt{2}}, \tfrac{e^{\frac{3i\pi}{4}}}{2\sqrt{2}}\Big\},\\
&\Big\{\tfrac{1}{2\sqrt{2}}, -\tfrac{i}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, \tfrac{i}{2\sqrt{2}}, \tfrac{1}{2\sqrt{2}}, -\tfrac{i}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, \tfrac{i}{2\sqrt{2}}\Big\},\\
&\Big\{\tfrac{1}{2\sqrt{2}}, \tfrac{e^{-\frac{i\pi}{4}}}{2\sqrt{2}}, -\tfrac{i}{2\sqrt{2}}, \tfrac{e^{-\frac{3i\pi}{4}}}{2\sqrt{2}}, -\tfrac{1}{2\sqrt{2}}, \tfrac{e^{\frac{3i\pi}{4}}}{2\sqrt{2}}, \tfrac{i}{2\sqrt{2}}, \tfrac{e^{\frac{i\pi}{4}}}{2\sqrt{2}}\Big\}\Big\};\end{aligned}$$

(*Defining the generators of SU(3) and indentifying the ones with non-zero contribution to QFT*)

$$\text{ss}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \text{ss}_2 = \begin{pmatrix} 0 & -\text{I} \\ \text{I} & 0 \end{pmatrix};\ \text{ss}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ \text{ss}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

```
f = {};
Do[Do[Do[S = KroneckerProduct[ss_i, ss_j, ss_k]; o = Tr[N[QF3].S]; If[Abs[o] > 0.01,
    Print[i, j, k]; f = Append[f, {i, j, k}]]], {i, 1, 4}], {j, 1, 4}], {k, 1, 4}]
```

```
311

221

331

341

212

322

232

242

113

413

133

433

143

443

114

414

134

434

144

444
```

```
Dimensions[f]
```

```
{20, 3}
```

```
(* finding the subgroup of the generators which enter in QFT*)
```

```
Com[a_, b_, c_, d_, e_, f_] :=
 (A = KroneckerProduct[ss_a, ss_b, ss_c];
  B = KroneckerProduct[ss_d, ss_e, ss_f]; Co = A.B - B.A;
  ff = {}; Do[Do[Do[S = KroneckerProduct[ss_i, ss_j, ss_k]; o = Abs[Tr[Co.S]];
      If[o > 0.01, ff = Append[ff, {i, j, k}]], {i, 1, 4}], {j, 1, 4}], {k, 1, 4}];
  Return[
   ff])
```

```
L = Dimensions[f][[1]]; HH = {};
Do[Do[H = Com[f[[i, 1]], f[[i, 2]], f[[i, 3]], f[[j, 1]], f[[j, 2]], f[[j, 3]]];
  If[H ≠ {}, HH = Append[HH, H[[1]]]], {i, j + 1, L}], {j, 1, L}]
```

```
VV = DeleteDuplicates[Join[HH, f]]
```

```
{{4, 2, 4}, {1, 2, 3}, {3, 4, 2}, {2, 2, 2}, {3, 1, 2}, {2, 4, 1}, {3, 2, 1}, {2, 1, 1},
 {1, 2, 4}, {4, 2, 3}, {3, 3, 2}, {2, 3, 1}, {3, 1, 1}, {2, 2, 1}, {3, 3, 1}, {3, 4, 1},
 {2, 1, 2}, {3, 2, 2}, {2, 3, 2}, {2, 4, 2}, {1, 1, 3}, {4, 1, 3}, {1, 3, 3}, {4, 3, 3},
 {1, 4, 3}, {4, 4, 3}, {1, 1, 4}, {4, 1, 4}, {1, 3, 4}, {4, 3, 4}, {1, 4, 4}, {4, 4, 4}}
```

```
Dimensions[VV]
```

{32, 3}

```
L = Dimensions[VV][[1]]; HH2 = {};
Do[Do[H = Com[VV[[i, 1]], VV[[i, 2]], VV[[i, 3]], VV[[j, 1]], VV[[j, 2]], VV[[j, 3]]];
  If[H ≠ {}, HH2 = Append[HH2, H[[1]]]], {i, j + 1, L}], {j, 1, L}]
```

```
VV2 = DeleteDuplicates[Join[HH2, f, HH]]
```

{{3, 3, 2}, {2, 3, 1}, {3, 1, 2}, {2, 1, 1}, {3, 3, 1}, {3, 1, 1}, {2, 3, 2}, {2, 1, 2},
 {1, 3, 3}, {4, 3, 3}, {1, 1, 3}, {4, 1, 3}, {1, 3, 4}, {4, 3, 4}, {1, 1, 4}, {4, 1, 4},
 {1, 2, 4}, {4, 2, 3}, {2, 2, 2}, {3, 2, 1}, {4, 4, 3}, {1, 4, 4}, {3, 4, 1}, {2, 4, 2},
 {3, 4, 2}, {2, 4, 1}, {2, 2, 1}, {3, 2, 2}, {4, 2, 4}, {1, 2, 3}, {1, 4, 3}, {4, 4, 4}}

```
Dimensions[VV2]
```

{32, 3}

```
(* I can conclude from the above that the subgroup has 32 elements*)
```

```
(* in the rest I am looking for a
 minimum set able to generate the whole subgroup*)
```

```
VV3 = {{2, 3, 1}, {3, 1, 2}, {1, 4, 4}, {3, 3, 1}, {4, 2, 4},
  {4, 3, 4}, {4, 1, 4}, {4, 4, 3}, {4, 3, 3}, {1, 1, 4}, {1, 2, 4},
  {3, 4, 1}, {3, 4, 2}, {4, 1, 3}, {1, 4, 3}, {4, 4, 4}}; Dimensions[VV3]
```

{16, 3}

```
L = Dimensions[VV3][[1]]; HH3 = {};
Do[Do[H = Com[VV3[[i, 1]], VV3[[i, 2]], VV3[[i, 3]], VV3[[j, 1]], VV3[[j, 2]],
    VV3[[j, 3]]]; If[H ≠ {}, HH3 = Append[HH3, H[[1]]]], {i, j + 1, L}], {j, 1, L}];
VV4 = DeleteDuplicates[Join[VV3, HH3]]; Dimensions[VV4]
```

{31, 3}

```
VV4
```

{{2, 3, 1}, {3, 1, 2}, {1, 4, 4}, {3, 3, 1}, {4, 2, 4}, {4, 3, 4}, {4, 1, 4}, {4, 4, 3},
 {4, 3, 3}, {1, 1, 4}, {1, 2, 4}, {3, 4, 1}, {3, 4, 2}, {4, 1, 3}, {1, 4, 3}, {4, 4, 4},
 {1, 2, 3}, {2, 1, 1}, {2, 2, 1}, {2, 3, 2}, {2, 4, 2}, {1, 3, 4}, {2, 1, 2}, {3, 3, 2},
 {3, 2, 2}, {3, 1, 1}, {2, 4, 1}, {3, 2, 1}, {4, 2, 3}, {1, 1, 3}, {2, 2, 2}, {1, 3, 3}}

```
(***Double Commutators***)
```

```
(* Here I can prove that VV3 is a good set but possibly this could be improved*)
```

```
VV3 = {{1, 4, 4}, {4, 2, 4}, {4, 3, 4}, {4, 1, 4}, {4, 4, 3}, {4, 3, 3}, {4, 1, 3},
   {1, 1, 4}, {1, 2, 4}, {3, 4, 2}, {1, 4, 3}, {4, 4, 4}}; Dimensions[VV3]
```

{12, 3}

```
L = Dimensions[VV3][[1]]; HH3 = {};
Do[Do[H = Com[VV3[[i, 1]], VV3[[i, 2]], VV3[[i, 3]], VV3[[j, 1]], VV3[[j, 2]],
     VV3[[j, 3]]]; If[H ≠ {}, HH3 = Append[HH3, H[[1]]]], {i, j + 1, L}],
  {j, 1, L}]; VV4 = DeleteDuplicates[Join[VV3, HH3]];
L = Dimensions[VV4][[1]]; HH4 = {}; Do[Do[H =
    Com[VV4[[i, 1]], VV4[[i, 2]], VV4[[i, 3]], VV4[[j, 1]], VV4[[j, 2]], VV4[[j, 3]]];
   If[H ≠ {}, HH4 = Append[HH4, H[[1]]]], {i, j + 1, L}], {j, 1, L}];
VV5 = DeleteDuplicates[Join[VV4, HH3, VV3]];
Dimensions[VV5];
L = Dimensions[VV5][[1]]; HH5 = {};
Do[Do[H = Com[VV5[[i, 1]], VV5[[i, 2]], VV5[[i, 3]], VV5[[j, 1]], VV5[[j, 2]],
     VV5[[j, 3]]]; If[H ≠ {}, HH5 = Append[HH5, H[[1]]]], {i, j + 1, L}], {j, 1, L}];
VV6 = DeleteDuplicates[Join[VV4, VV3, VV5, HH5]];
Dimensions[VV6];
L = Dimensions[VV6][[1]]; HH6 = {};
Do[Do[H = Com[VV6[[i, 1]], VV6[[i, 2]], VV6[[i, 3]], VV6[[j, 1]], VV6[[j, 2]],
     VV6[[j, 3]]]; If[H ≠ {}, HH6 = Append[HH6, H[[1]]]], {i, j + 1, L}], {j, 1, L}];
VV7 = DeleteDuplicates[Join[VV4, VV3, VV5, VV6, HH6]];
Dimensions[VV7]
```

```
{32, 3}
```

```
VV7
```

```
{{1, 4, 4}, {4, 2, 4}, {4, 3, 4}, {4, 1, 4}, {4, 4, 3}, {4, 3, 3}, {4, 1, 3}, {1, 1, 4},
 {1, 2, 4}, {3, 4, 2}, {1, 4, 3}, {4, 4, 4}, {2, 4, 2}, {1, 3, 4}, {4, 2, 3}, {3, 4, 1},
 {1, 2, 3}, {1, 1, 3}, {3, 3, 1}, {1, 3, 3}, {3, 1, 1}, {2, 1, 2}, {2, 2, 2}, {2, 4, 1},
 {2, 3, 1}, {2, 1, 1}, {3, 1, 2}, {3, 2, 2}, {2, 3, 2}, {3, 2, 1}, {3, 3, 2}, {2, 2, 1}}
```

```
VV
```

```
{{4, 2, 4}, {1, 2, 3}, {3, 4, 2}, {2, 2, 2}, {3, 1, 2}, {2, 4, 1}, {3, 2, 1}, {2, 1, 1},
 {1, 2, 4}, {4, 2, 3}, {3, 3, 2}, {2, 3, 1}, {3, 1, 1}, {2, 2, 1}, {3, 3, 1}, {3, 4, 1},
 {2, 1, 2}, {3, 2, 2}, {2, 3, 2}, {2, 4, 2}, {1, 1, 3}, {4, 1, 3}, {1, 3, 3}, {4, 3, 3},
 {1, 4, 3}, {4, 4, 3}, {1, 1, 4}, {4, 1, 4}, {1, 3, 4}, {4, 3, 4}, {1, 4, 4}, {4, 4, 4}}
```